THE MULTILINEAR HÖRMANDER MULTIPLIER THEOREM WITH A LORENTZ-SOBOLEV CONDITION

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Abstract. In this article, we provide a multilinear version of the Hörmander multiplier theorem with a Lorentz-Sobolev space condition. The work is motivated by the recent result of the first author and Slavíková [12] where an analogous version of classical Hörmander multiplier theorem was obtained; this version is sharp in many ways and reduces the number of indices that appear in the statement of the theorem. As a natural extension of the linear case, in this work, we prove that if $mn/2 < s < mn$, then

$$\big\| T_\sigma(f_1, \dots, f_m) \big\|_{L^p(\mathbb{R}^n)} \lesssim \sup_{k\in\mathbb{Z}} \big\| \sigma(2^k \vec{\cdot}\,)\widehat{\Psi^{(m)}} \big\|_{L_s^{mn/s,1}(\mathbb{R}^{mn})} \|f_1\|_{L^{p_1}(\mathbb{R}^n)} \cdots \|f_m\|_{L^{p_m}(\mathbb{R}^n)}$$

for certain $p, p_1, \dots, p_m$ with $1/p = 1/p_1 + \cdots + 1/p_m$. We also show that the above estimate is sharp, in the sense that the Lorentz-Sobolev space $L_s^{mn/s,1}$ cannot be replaced by $L_s^{r,q}$ for $r < mn/s$, $0 < q \le \infty$, or by $L_s^{mn/s,q}$ for $q > 1$.

1. Introduction

Let $\mathcal{S}(\mathbb{R}^n)$ denote the space of all Schwartz functions on $\mathbb{R}^n$. Given a bounded function σ on $\mathbb{R}^n$, we define a linear multiplier operator

$$T_\sigma f(x) := \int_{\mathbb{R}^n} \sigma(\xi)\widehat{f}(\xi) e^{2\pi i\langle x,\xi\rangle} d\xi$$

acting on $f \in \mathcal{S}(\mathbb{R}^n)$ where $\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-2\pi i\langle x,\xi\rangle}dx$ is the Fourier transform of f. One of important problems in harmonic analysis is to find optimal sufficient conditions on σ for the corresponding operator T_σ to admit an L^p-bounded extension for all $1 < p < \infty$. The classical theorem of Mikhlin [16] states that if the condition

$$\big|\partial_\xi^\alpha \sigma(\xi)\big| \lesssim_\alpha |\xi|^{-|\alpha|}, \quad \xi \neq 0$$

holds for all multi-indices α with $|\alpha| \le [n/2] + 1$, then T_σ extends to a bounded operator in L^p for $1 < p < \infty$. Hörmander [14] refined this result, using the weaker condition

$$\sup_{k\in\mathbb{Z}} \big\|\sigma(2^k\cdot)\widehat{\psi}\big\|_{L_s^2(\mathbb{R}^n)} < \infty \tag{1.1}$$

for $s > n/2$, where $L_s^2(\mathbb{R}^n)$ denotes the standard fractional Sobolev space on $\mathbb{R}^n$ and ψ is a Schwartz function on $\mathbb{R}^n$ whose Fourier transform is supported in the annulus $1/2 < |\xi| < 2$ and satisfies $\sum_{k\in\mathbb{Z}} \widehat{\psi}(\xi/2^k) = 1$ for $\xi \neq 0$. Calderón and Torchinsky [1] proved that if (1.1) holds for $s > n/p - n/2$, then T_σ is bounded in $H^p(\mathbb{R}^n)$ for $0 < p \le 1$. They also showed that L_s^2 in (1.1) can be replaced by L_s^r for the L^p-boundedness, $1 < p < \infty$, using a complex

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interpolation method, and the assumption in their result was weakened by Grafakos, He, Honzík, and Nguyen [6]. Recently, Grafakos and Slavíková [12] have improved the previous multiplier theorems by replacing L^r_s by the Lorentz-Sobolev space $L^{n/s,1}_s$.

We recall the definition of Lorentz spaces $L^{p,q}(\mathbb{R}^n)$ and Lorentz-Sobolev spaces $L^{p,q}_s(\mathbb{R}^n)$. For any measurable function f on $\mathbb{R}^n$, we let $d_f(s) := \big|\{x \in \mathbb{R}^n : |f(x)| > s\}\big|$ be the distribution function of f and

$$f^*(t) := \inf\big\{s > 0 : d_f(s) \le t\big\}, \qquad t > 0$$

be its decreasing rearrangement. We adopt the convention that the infimum of the empty set is ∞. For $0 < p, q \le \infty$ the quasi-norm on the Lorentz space $L^{p,q}(\mathbb{R}^n)$ is given by

$$\|f\|_{L^{p,q}(\mathbb{R}^n)} := \begin{cases} \Big(\int_0^\infty \big(t^{1/p} f^*(t)\big)^q \dfrac{dt}{t}\Big)^{1/q}, & q < \infty \\ \sup_{t>0} t^{1/p} f^*(t), & q = \infty. \end{cases}$$

For $s > 0$ let $(I - \Delta)^{s/2}$ be the inhomogeneous fractional Laplacian operator, explicitly defined by

$$(I-\Delta)^{s/2} f := \big((1 + 4\pi^2|\cdot|^2)^{s/2}\widehat{f}\big)^\vee$$

where $f^\vee(\xi) := \widehat{f}(-\xi)$ is the inverse Fourier transform of f. Then for $0 < p, q \le \infty$ and $s > 0$ we define

$$\|f\|_{L^{p,q}_s(\mathbb{R}^n)} := \big\|(I-\Delta)^{s/2} f\big\|_{L^{p,q}(\mathbb{R}^n)}.$$

Theorem A. ([12]) Let $1 < p < \infty$ and $|n/p - n/2| < s < n$. Then there exists $C > 0$ such that

$$\|T_\sigma f\|_{L^p(\mathbb{R}^n)} \le C \sup_{k\in\mathbb{Z}} \big\|\sigma(2^k\cdot)\widehat{\psi}\big\|_{L^{n/s,1}_s(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}. \tag{1.2}$$

We also refer to [9] for an extension of Theorem A to the Hardy space $H^p(\mathbb{R}^n)$ for $0 < p < \infty$. Note that for $0 < r_1 < r_2 < \infty$ and $0 < q_1, q_2 \le \infty$

$$\big\|\sigma(2^k\cdot)\widehat{\psi}\big\|_{L^{r_1,q_1}_s(\mathbb{R}^n)} \lesssim \big\|\sigma(2^k\cdot)\widehat{\psi}\big\|_{L^{r_2,q_2}_s(\mathbb{R}^n)} \quad \text{uniformly in } \; k, \tag{1.3}$$

which follows from Hölder's inequality with even integers s, complex interpolation technique, and a proper embedding theorem. Moreover, if $q_1 \ge q_2$, then the embedding $L^{r,q_2}_s(\mathbb{R}^n) \hookrightarrow L^{r,q_1}_s(\mathbb{R}^n)$ yields that

$$\big\|\sigma(2^k\cdot)\widehat{\psi}\big\|_{L^{r,q_1}_s(\mathbb{R}^n)} \lesssim \big\|\sigma(2^k\cdot)\widehat{\psi}\big\|_{L^{r,q_2}_s(\mathbb{R}^n)} \quad \text{uniformly in } \; k. \tag{1.4}$$

Thus, $L^{n/s,1}_s(\mathbb{R}^n)$ is bigger than $L^{r,q}_s(\mathbb{R}^n)$ for $r > n/s$ when $0 < q \le \infty$ and than $L^{n/s,q}_s(\mathbb{R}^n)$ when $0 < q < 1$; the spaces $L^r_s(\mathbb{R}^n) = L^{r,r}_s(\mathbb{R}^n)$ with $r > n/s$ appeared in previous versions of the Hörmander multiplier theorem. Moreover, it was shown in [9] that the parameters $r = n/s$ and $q = 1$ in Theorem A are sharp, i.e., boundedness in (1.2) fails if n/s is replaced $r < n/s$ or if 1 is replaced by $q > 1$.

We now turn our attention to multilinear multiplier theory, which is the focus of this paper. Let m be a positive integer greater than 1, which will serve as the degree of the multilinearity of operators. For a bounded function σ on $\mathbb{R}^{mn}$ we define the corresponding m-linear multiplier operator T_σ by

$$T_\sigma\big(f_1, \dots, f_m\big)(x) := \int_{\mathbb{R}^{mn}} \sigma(\vec{\boldsymbol{\xi}}) \Big(\prod_{j=1}^m \widehat{f}_j(\xi_j)\Big) e^{2\pi i \langle x, \sum_{j=1}^m \xi_j\rangle} d\vec{\boldsymbol{\xi}}, \qquad x \in \mathbb{R}^n$$

for $f_j \in \mathcal{S}(\mathbb{R}^n)$ where $\vec{\boldsymbol{\xi}} := (\xi_1, \dots, \xi_m) \in (\mathbb{R}^n)^m$ and $d\vec{\boldsymbol{\xi}} := d\xi_1 \cdots d\xi_m$. As a multilinear extension of Mikhlin's result, Coifman and Meyer [2] proved that if L is sufficiently large and σ satisfies

$$\big|\partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_m}^{\alpha_m} \sigma(\vec{\boldsymbol{\xi}})\big| \lesssim_{\alpha_1,\dots,\alpha_m} \big(|\xi_1| + \cdots + |\xi_m|\big)^{-(|\alpha_1|+\cdots+|\alpha_m|)}, \quad \vec{\boldsymbol{\xi}} \neq \vec{\mathbf{0}}$$

for $\xi_1, \dots, \xi_m \in \mathbb{R}^n$ and multi-indices $\alpha_1, \dots, \alpha_m \in \mathbb{Z}^n$ with $|\alpha_1| + \cdots + |\alpha_m| \le L$, then T_σ is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ to L^p for all $1 < p < \infty$ and $1 < p_1, \dots, p_m \le \infty$ satisfying $1/p = 1/p_1 + \cdots + 1/p_m$. This result was extended to $p \le 1$ by Kenig and Stein [15] with $1 < p_1, \dots, p_m < \infty$ and by Grafakos and Torres [13] in which L^{p_j} should be replaced by L_c^∞ if some $p_j = \infty$. Here L_c^∞ is the space of compactly supported functions in L^∞.

Let $\Psi^{(m)}$ be the m-linear counterpart of ψ. That is, $\Psi^{(m)}$ is a Schwartz function on $\mathbb{R}^{mn}$ having the properties:

$$\mathrm{Supp}(\widehat{\Psi^{(m)}}) \subset \big\{\vec{\boldsymbol{\xi}} \in \mathbb{R}^{mn} : 1/2 \le |\vec{\boldsymbol{\xi}}| \le 2\big\}, \qquad \sum_{k\in\mathbb{Z}} \widehat{\Psi^{(m)}}(\vec{\boldsymbol{\xi}}/2^k) = 1, \quad \vec{\boldsymbol{\xi}} \neq \vec{\mathbf{0}}.$$

Let $(\vec{I} - \vec{\Delta})^{s/2}$ denote the inhomogeneous fractional Laplacian operator acting on functions on $\mathbb{R}^{mn}$. For $s \ge 0$ and $0 < r < \infty$ the Sobolev norm of f is defined as

$$\|f\|_{L_s^r(\mathbb{R}^{mn})} := \big\|(\vec{I} - \vec{\Delta})^{s/2} f\big\|_{L^r(\mathbb{R}^{mn})}.$$

Tomita [20] obtained an $L^{p_1} \times \cdots \times L^{p_m}$ to L^p boundedness for T_σ in the range $1 < p, p_1, \dots, p_m < \infty$ under a condition analogous to (1.1):

Theorem B. ([20]) Let $1 < p, p_1, \dots, p_m < \infty$ satisfy $1/p = 1/p_1 + \cdots + 1/p_m$. Suppose $s > mn/2$. Then there exists $C > 0$ such that

$$\big\|T_\sigma\big(f_1, \dots, f_m\big)\big\|_{L^p(\mathbb{R}^n)} \le C \sup_{k\in\mathbb{Z}} \big\|\sigma(2^k\,\vec{\cdot}\,)\widehat{\Psi^{(m)}}\big\|_{L_s^2(\mathbb{R}^{mn})} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)}$$

for $f_1, \dots, f_m \in \mathcal{S}(\mathbb{R}^n)$.

Grafakos and Si [11] extended Theorem B to $p \le 1$ using L^r-based Sobolev norms of σ for $1 < r \le 2$:

Theorem C. ([11]) Let $1 < r \le 2$, $0 < p < \infty$, $r \le p_1, \dots, p_m < \infty$, and $1/p_1 + \cdots + 1/p_m = 1/p$. Suppose $s > mn/r$. Then there exists $C > 0$ such that

$$\big\|T_\sigma\big(f_1, \dots, f_m\big)\big\|_{L^p(\mathbb{R}^n)} \le C \sup_{k\in\mathbb{Z}} \big\|\sigma(2^k\,\vec{\cdot}\,)\widehat{\Psi^{(m)}}\big\|_{L_s^r(\mathbb{R}^{mn})} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)}$$

for $f_1, \dots, f_m \in \mathcal{S}(\mathbb{R}^n)$.

Note that Theorem C provides a broader range of p's but requires stronger assumptions on s, while, under the same condition $s > mn/2$ (when $r = 2$), the estimate in Theorem C is contained in Theorem B. We also refer to [4, 5, 7, 8, 10, 17, 18, 20] for further results.

The aim of this paper is to provide a multilinear extension of Theorem A, which also provides a sharp version of Theorem B and C. In order to state our main results, we first define two open sets $\mathcal{Q}_l$ and $\mathcal{P}$ in $\mathbb{R}^m$ as follows

$$\mathcal{Q}_l := \big\{(r_1, \dots, r_m) \in \mathbb{R}^m : 0 < r_j < l,\ 1 \le j \le m\big\},$$

$$\mathcal{P} := \big\{(r_1, \dots, r_m) \in \mathbb{R}^m : 0 < r_1 + \cdots + r_m < 1\big\},$$

and denote by $hull\big(\mathcal{Q}_l,\mathcal{P}\big)$ the convex hull containing both $\mathcal{Q}_l$ and $\mathcal{P}$. Then our first main result is

Theorem 1.1. *Let $0<p,p_1,\dots,p_m<\infty$ satisfy $1/p=1/p_1+\cdots+1/p_m$. Suppose $mn/2<s<mn$ and*

$$(1/p_1,\dots,1/p_m)\in hull\big(\mathcal{Q}_{\frac{s}{mn}},\mathcal{P}\big).$$

Then there exists $C>0$ such that

$$\big\|T_\sigma\vec{\boldsymbol{f}}\big\|_{L^p(\mathbb{R}^n)}\le C\sup_{k\in\mathbb{Z}}\big\|\sigma(2^k\,\vec{\cdot}\,)\widehat{\Psi^{(m)}}\big\|_{L_s^{mn/s,1}(\mathbb{R}^{mn})}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\mathbb{R}^n)} \tag{1.5}$$

for $f_1,\dots,f_m\in\mathcal{S}(\mathbb{R}^n)$.

Figure 1 shows the range of indices p_1,p_2 for which boundedness holds in the bilinear case $m=2$. Note that only two cases $(1/p_1,\dots,1/p_m)\in\mathcal{Q}_{\frac{s}{mn}}$ and $(1/p_1,\dots,1/p_m)\in\mathcal{P}$ will be treated in the proof as the desired result follows immediately via interpolation. The first case is equivalent to $mn/s<p_1,\dots,p_m<\infty$ for which the proof is based on the Littlewood-Paley theory and the pointwise estimate in Lemma 3.1 below. Since $mn/s<2$, the first one contains the result for $2\le p_1,\dots,p_m<\infty$ and then a method of transposes of T_σ and duality arguments will be applied to the case $1<p,p_1,\dots,p_m<\infty$ that coincides with the second part.

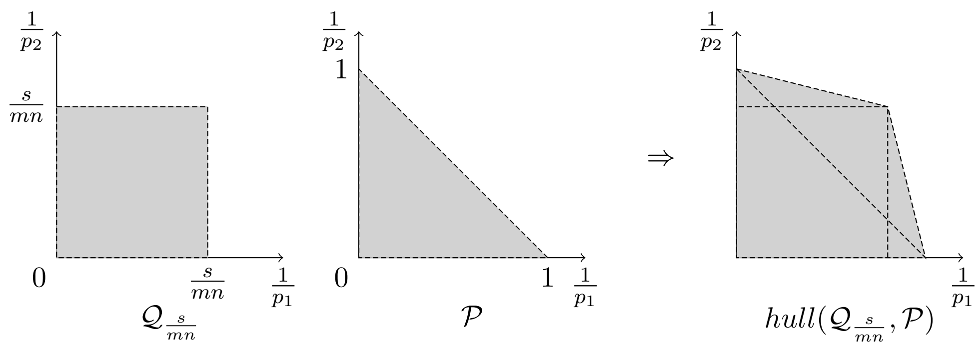

FIGURE 1. $L^{p_1}\times L^{p_2}\to L^p$ boundedness of T_σ, $m=2$.

As in the linear case, using in (1.3) and (1.4), we may replace $L_s^{mn/s,1}$ in (1.5) by $L_s^{r,q}$ for $r>mn/s$ and $0<q\le\infty$ or by $L_s^{mn/s,q}$ for $0<q<1$. We remark that Theorem 1.1 clearly improves Theorem B and C in view of $L_s^r=L_s^{r,r}$.

Our second main result is the sharpness of the parameters r,q. That is, $r=mn/s$ cannot be replaced by a smaller number, and if $r=mn/s$, then $q=1$ is the largest number for (1.5) to hold. This is contained in the following theorem:

Theorem 1.2. *Let $0<p<\infty$ and $0<p_1,\dots,p_m\le\infty$ satisfy $1/p=1/p_1+\cdots+1/p_m$. Suppose $0<s<mn$.*

(1) For any $0<r<mn/s$ and $0<q\le\infty$, there exists σ satisfying

$$\sup_{k\in\mathbb{Z}}\big\|\sigma(2^k\,\vec{\cdot}\,)\widehat{\Psi^{(m)}}\big\|_{L_s^{r,q}(\mathbb{R}^{mn})}<\infty \tag{1.6}$$

such that T_σ is not bounded from $L^{p_1}\times\cdots\times L^{p_m}$ to L^p.

(2) For $q>1$, there exists σ satisfying

$$\sup_{k\in\mathbb{Z}} \big\| \sigma(2^k\, \vec{\cdot}\,)\widehat{\Psi^{(m)}}\big\|_{L_s^{mn/s,q}(\mathbb{R}^{mn})} < \infty$$

such that T_σ is not bounded from $L^{p_1}\times\cdots\times L^{p_m}$ to L^p.

The key ingredients in the proof of Theorem 1.2 are a variant of Bessel potential estimates introduced by Grafakos and Park [9], and the scaling arguments used in [10, 19].

Remark. Theorem 1.2 proves that (1.6) with $r\ge mn/s$ and $q\le 1$ is a necessary condition for the boundedness of T_σ for all $0<p<\infty$ and $0<p,p_1,\dots,p_m\le\infty$ satisfying $1/p=1/p_1+\cdots+1/p_m$. Since our techniques are not applicable to the case $(1/p_1,\dots,1/p_m)\notin hull(\mathcal{Q}_{\frac{s}{mn}},\mathcal{P})$ in Theorem 1.1, an alternative argument will be needed in the case. A similar question arises in terms of the parameter s. That is, we need to verify that estimate (1.5) holds for $0<s\le mn/2$.

2. Preliminaries : Inequalities in Lorentz spaces

In this section we review several inequalities that will be useful in the proof of the main results, and refer the reader to [9, 12]. We fix $N\in\mathbb{N}$ and discuss inequalities of functions on the N dimensional space $\mathbb{R}^N$.

For a locally integrable function f defined on $\mathbb{R}^N$, let

$$\mathcal{M}^{(N)}f(x):=\sup_{Q:x\in Q}\frac{1}{|Q|}\int_Q |f(y)|dy$$

be the Hardy-Littlewood maximal function of f where the supremum is taken over all cubes in $\mathbb{R}^N$ containing x, and $\mathcal{M}_r^{(N)}f(x):=\big(\mathcal{M}^{(N)}\big(|f|^r\big)(x)\big)^{1/r}$ for $0<r<\infty$. Then the Fefferman-Stein vector-valued maximal inequality [3] says that for $0<r<p,q<\infty$

$$\big\|\big\{\mathcal{M}_r^{(N)}f_k\big\}_{k\in\mathbb{Z}}\big\|_{L^p(\ell^q)} \lesssim \|\{f_k\}_{k\in\mathbb{Z}}\|_{L^p(\ell^q)}. \tag{2.1}$$

Moreover, (2.1) holds for $0<p\le\infty$ and $q=\infty$.

We now recall some inequalities in Lorentz spaces. Most of them are consequences of a real interpolation technique and inequalities in Lebesgue spaces.

Lemma 2.1. [9, Lemma 2.1] *Let $1<p\le r<\infty$, $1\le q<r$, and $0<t\le\infty$ satisfy $1/r+1=1/p+1/q$. Then*

$$\|f*g\|_{L^{r,t}(\mathbb{R}^N)} \le \|f\|_{L^{p,t}(\mathbb{R}^N)}\|g\|_{L^q(\mathbb{R}^N)}.$$

Lemma 2.2. [9, Lemma 2.2] *Let $2<p<\infty$ and $0<r\le\infty$. Then*

$$\|\widehat{f}\,\|_{L^{p,r}(\mathbb{R}^N)} \le \|f\|_{L^{p',r}(\mathbb{R}^N)},$$

where $1/p+1/p'=1$.

Lemma 2.3. [9, Lemma 2.3] *Let $1<p<\infty$, $0<r\le\infty$, and $s>0$. For any $\vartheta\in S(\mathbb{R}^N)$, we have*

$$\|\vartheta\cdot f\|_{L_s^{p,r}(\mathbb{R}^N)} \lesssim_{N,s,p,r,\vartheta} \|f\|_{L_s^{p,r}(\mathbb{R}^N)}.$$

Lemma 2.4. [9, Lemma 2.5] *Let $1<p<\infty$ and $1\le q\le\infty$. Then*

$$\int_{\mathbb{R}^N}\big|f(x)g(x)\big|dx \le \|f\|_{L^{p,q}(\mathbb{R}^N)}\|g\|_{L^{p',q'}(\mathbb{R}^N)}$$

where $1/p+1/p'=1/q+1/q'=1$.

A significant role is played in the proof of the main theorem by the following lemma, whose proof can be found in [12].

Lemma 2.5. [12, Lemma 2.1] *Let $0 < s < N$, and $q > N/s$. Then for any measurable function f on $\mathbb{R}^N$ and $k \in \mathbb{Z}$, there exists $C > 0$ such that*

$$\Big\| \frac{f(x - \cdot/2^k)}{(1+4\pi^2|\cdot|^2)^{s/2}} \Big\|_{L^{N/s,\infty}(\mathbb{R}^N)} \le C\mathcal{M}_q^{(N)} f(x) \qquad \textit{uniformly in } \ k.$$

3. Proof of Theorem 1.1

3.1. **The case $mn/s < p_1, \dots, p_m < \infty$.** Let $\Theta^{(m)}$ be a Schwartz function on $\mathbb{R}^{mn}$ such that

$$\widehat{\Theta^{(m)}}(\vec{\boldsymbol{\xi}}) = 1 \quad \text{ for } \ 2^{-2}m^{-1/2} \le |\vec{\boldsymbol{\xi}}| \le 2^2 m^{1/2},$$
$$\mathrm{Supp}(\widehat{\Theta^{(m)}}) \subset \big\{ \vec{\boldsymbol{\xi}} \in \mathbb{R}^{mn} : 2^{-3}m^{-1/2} \le |\vec{\boldsymbol{\xi}}| \le 2^3 m^{1/2} \big\}.$$

Using the fact that $\sum_{k\in\mathbb{Z}} \widehat{\Psi^{(m)}}(\vec{\boldsymbol{\xi}}/2^k) = 1$ for $\vec{\boldsymbol{\xi}} \neq \vec{\boldsymbol{0}}$, a triangle inequality, and Lemma 2.3, we see that

$$\sup_{k\in\mathbb{Z}} \big\| \sigma(2^k \,\vec{\cdot}\,)\widehat{\Theta^{(m)}} \big\|_{L_s^{mn/s,1}(\mathbb{R}^{mn})} \lesssim \sup_{k\in\mathbb{Z}} \big\| \sigma(2^k \,\vec{\cdot}\,)\widehat{\Psi^{(m)}} \big\|_{L_s^{mn/s,1}(\mathbb{R}^{mn})}. \tag{3.1}$$

Therefore it suffices to show that

$$\big\| T_\sigma \vec{\boldsymbol{f}} \big\|_{L^p(\mathbb{R}^n)} \lesssim \sup_{k\in\mathbb{Z}} \big\| \sigma(2^k \,\vec{\cdot}\,)\widehat{\Theta^{(m)}} \big\|_{L_s^{mn/s,1}(\mathbb{R}^{mn})} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)}. \tag{3.2}$$

Recall that ψ is a Schwartz function on $\mathbb{R}^n$ generating Littlewood-Paley functions with $\mathrm{Supp}(\widehat{\psi}) \subset \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\}$ and $\sum_{k\in\mathbb{Z}} \widehat{\psi}(\xi/2^k) = 1$ for $\xi \neq 0$. Letting $\psi_k := 2^{kn}\psi(2^k\cdot)$ and $\phi_k := 2^{kn}\phi(2^k\cdot)$ where

$$\widehat{\phi}(\xi) := \begin{cases} \sum_{j\le 0} \widehat{\psi_j}, & \xi \neq 0 \\ 1, & \xi = 0 \end{cases},$$

we define the convolution operators Q_k, P_k by

$$Q_k f := \psi_k * f, \qquad P_k f := \phi_k * f.$$

Then $T_\sigma \vec{\boldsymbol{f}}$ can be written as

$$T_\sigma \vec{\boldsymbol{f}} = \sum_{k_1,\dots,k_m \in \mathbb{Z}} T_\sigma\big(Q_{k_1}f_1, \dots, Q_{k_m}f_m\big) = T_\sigma^{(1)}\vec{\boldsymbol{f}} + \cdots + T_\sigma^{(m)}\vec{\boldsymbol{f}}$$

where

$$T_\sigma^{(\mu)}\vec{\boldsymbol{f}} := \sum_{k\in\mathbb{Z}} T_\sigma\big(P_{k-1}f_1, \dots, P_{k-1}f_{\mu-1}, Q_k f_\mu, P_k f_{\mu+1}, \dots, P_k f_m\big).$$

Therefore, the proof of (3.3) can be reduced to the inequalities

$$\big\| T_\sigma^{(\mu)} \vec{\boldsymbol{f}} \big\|_{L^p(\mathbb{R}^n)} \lesssim \sup_{k\in\mathbb{Z}} \big\| \sigma(2^k \,\vec{\cdot}\,)\widehat{\Theta^{(m)}} \big\|_{L_s^{mn/s,1}(\mathbb{R}^{mn})} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)}, \qquad 1 \le \mu \le m. \tag{3.3}$$

We are only concerned with the case $\mu = 1$ as the others follow via symmetry. We first observe that the summand in $T_\sigma^{(1)}$ is expressed in the form

$$\big[\sigma \cdot \big(\widehat{Q_k f_1} \otimes \widehat{P_k f_2} \otimes \cdots \otimes \widehat{P_k f_m}\big)\big]^\vee(x, \cdots, x), \tag{3.4}$$

and since $\widehat{\Theta^{(m)}}(\vec{\boldsymbol{\xi}}/2^k) = 1$ for $2^{k-1} \le |\xi_1| \le 2^{k+1}$ and $|\xi_j| \le 2^{k+1}$ for $2 \le j \le m$,

$$\sigma_k(\vec{\boldsymbol{\xi}}) := \sigma(\vec{\boldsymbol{\xi}})\widehat{\Theta^{(m)}}(\vec{\boldsymbol{\xi}}/2^k) \tag{3.5}$$

can replace σ in (3.4). Note that for each $l \in \mathbb{N}$

$$P_k f = P_{k-l} f + \sum_{j=k-l+1}^{k} Q_j f.$$

Using this, we write

$$\begin{aligned}
&T_{\sigma_k}\big(Q_k f_1, P_k f_2, \dots, P_k f_m\big)\\
&= T_{\sigma_k}\big(Q_k f_1, P_{k,m} f_2, \dots, P_{k,m} f_m\big)\\
&+ \sum_{k-4-\lfloor \log_2 m\rfloor \le k_2 \le k} T_{\sigma_k}\big(Q_k f_1, Q_{k_2} f_2, P_{k_2} f_3, \dots, P_{k_2} f_m\big)\\
&+ \sum_{k-4-\lfloor \log_2 m\rfloor \le k_3 \le k} T_{\sigma_k}\big(Q_k f_1, P_{k_3-1} f_2, Q_{k_3} f_3, P_{k_3} f_4, \dots, P_{k_3} f_m\big)\\
&\quad \vdots\\
&+ \sum_{k-4-\lfloor \log_2 m\rfloor \le k_m \le k} T_{\sigma_k}\big(Q_k f_1, P_{k_m-1} f_2, \dots, P_{k_m-1} f_{m-1}, Q_{k_m} f_m\big)
\end{aligned}$$

where $P_{k,m} f := P_{k-5-\lfloor \log_2 m\rfloor} f$. We will actually consider the first two terms as a symmetric argument is applicable in the remaining cases. Our claim is that

$$\begin{aligned}
&\Big\| \sum_{k\in\mathbb{Z}} T_{\sigma_k}\big(Q_k f_1, P_{k,m} f_2, \dots, P_{k,m} f_m\big)\Big\|_{L^p(\mathbb{R}^n)}\\
&\lesssim \sup_{k\in\mathbb{Z}} \big\|\sigma_k(2^k\,\vec{\cdot}\,)\big\|_{L^{mn/s,1}_s(\mathbb{R}^{mn})} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^n)}
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
&\Big\| \sum_{k\in\mathbb{Z}} \sum_{k-4-\lfloor \log_2 m\rfloor \le k_2 \le k} T_{\sigma_k}\big(Q_k f_1, Q_{k_2} f_2, P_{k_2} f_3, \dots, P_{k_2} f_m\big)\Big\|_{L^p(\mathbb{R}^n)}\\
&\lesssim \sup_{k\in\mathbb{Z}} \big\|\sigma_k(2^k\,\vec{\cdot}\,)\big\|_{L^{mn/s,1}_s(\mathbb{R}^{mn})} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^n)}.
\end{aligned} \tag{3.7}$$

Here, we note that

$$\big\|\sigma_k(2^k\,\vec{\cdot}\,)\big\|_{L^{mn/s,1}_s(\mathbb{R}^{mn})} = \big\|\sigma(2^k\,\vec{\cdot}\,)\widehat{\Theta^{(m)}}\big\|_{L^{mn/s,1}_s(\mathbb{R}^{mn})}.$$

If σ is compactly supported, like σ_k in (3.5), then $\sigma^\vee$ exists, and $T_\sigma\big(f_1, \dots, f_m\big)(x)$ can be written as the convolution $\sigma^\vee * \big(f_1 \otimes \cdots \otimes f_m\big)(x, \dots, x)$. Then we may use the following lemma whose assertion is analogous to the key estimate in the proof of Theorem A in [12].

Lemma 3.1. *Let σ be a bounded function on $\mathbb{R}^{mn}$ such that $\sigma^\vee$ exists. Suppose that $mn/2 < s < mn$ and $q > mn/s$. Then we have*

$$\big|\sigma^\vee * \big(f_1 \otimes \cdots \otimes f_m\big)(\vec{\boldsymbol{x}})\big| \lesssim \big\|\sigma(2^k\,\vec{\cdot}\,)\big\|_{L^{mn/s,1}_s(\mathbb{R}^{mn})} \mathcal{M}_q^{(n)} f_1(x_1) \cdots \mathcal{M}_q^{(n)} f_m(x_m). \tag{3.8}$$

Proof. Let $F(\vec{\boldsymbol{x}}) := f_1(x_1)\cdots f_m(x_m)$. Then the left-hand side of (3.8) is

$$\big|\sigma^\vee * F(\vec{\boldsymbol{x}})\big| \le \int_{\mathbb{R}^{mn}} \big(1+4\pi^2|\vec{\boldsymbol{y}}|^2\big)^{s/2}\big|\big(\sigma(2^k\,\vec{\cdot}\,)\big)^\vee(\vec{\boldsymbol{y}})\big|\frac{|F(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}/2^k)|}{(1+4\pi^2|\vec{\boldsymbol{y}}|^2)^{s/2}}d\vec{\boldsymbol{y}}$$

and this is bounded by

$$\begin{aligned}&\big\|\big(1+4\pi^2|\,\vec{\cdot}\,|^2\big)^{s/2}\big(\sigma(2^k\,\vec{\cdot}\,)\big)^\vee\big\|_{L^{(mn/s)',1}(\mathbb{R}^{mn})}\Big\|\frac{F(\vec{\boldsymbol{x}}-\,\vec{\cdot}\,/2^k)}{(1+4\pi^2|\,\vec{\cdot}\,|^2)^{s/2}}\Big\|_{L^{mn/s,\infty}(\mathbb{R}^{mn})}\\ &\lesssim \big\|\sigma(2^k\,\vec{\cdot}\,)\big\|_{L^{mn/s,1}_s(\mathbb{R}^{mn})}\mathcal{M}_q^{(mn)}F(\vec{\boldsymbol{x}}),\end{aligned}$$

by applying Lemma 2.2 with $mn/s > 2$ and Lemma 2.4 with $s < mn$ and $q > mn/s$. Note that every cube Q in $\mathbb{R}^{mn}$ containing $\vec{\boldsymbol{x}}$ can be written as the product of m cubes $Q_1,\dots,Q_m$ in $\mathbb{R}^n$ such that $x_j \in Q_j$ for $1 \le j \le m$, and $|Q| = |Q_1|\times\cdots\times|Q_m|$. This implies that

$$\mathcal{M}_q^{(mn)}F(\vec{\boldsymbol{x}}) \le \mathcal{M}_q^{(n)}f_1(x_1)\cdots\mathcal{M}_q^{(n)}f_m(x_m)$$

and therefore (3.8) follows. □

3.1.1. *Proof of (3.6).* The proof relies on the fact that if $\widehat{g_k}$ is supported in $\{\xi \in \mathbb{R}^n : C^{-1}2^k \le |\xi| \le C2^k\}$ for $C > 1$ then

$$\Big\|\Big\{\psi_j * \Big(\sum_{k=j-h}^{j+h} g_k\Big)\Big\}_{j\in\mathbb{Z}}\Big\|_{L^p(\ell^q)} \lesssim_{h,C} \big\|\{g_k\}_{k\in\mathbb{Z}}\big\|_{L^p(\ell^q)} \tag{3.9}$$

for $h \in \mathbb{N}$. The proof of (3.9) is elementary and standard, so it is omitted here. Just use the estimate that for any $r > 0$ and $j-h \le k \le j+h$,

$$|\psi_j * g_k(x)| \lesssim_r 2^{j(n/r-n)}\Big(\int_{\mathbb{R}^n}\big|\phi_j(x-y)\big|^r\big|g_k(y)\big|^r dy\Big)^{1/r} \lesssim \mathcal{M}_r^{(n)}g_k(x)$$

where Bernstein's inequality is applied in the first inequality, and apply the maximal inequality (2.1) with $r < \min(p,q)$. See [21, Theorem 3.6] for details.

We see that the Fourier transform of $T_{\sigma_k}\big(Q_kf_1, P_{k,m}f_2,\dots,P_{k,m}f_m\big)$ is supported in $\big\{\xi \in \mathbb{R}^n : 2^{k-2} \le |\xi| \le 2^{k+2}\big\}$ and thus

$$\psi_j * \Big(\sum_{k\in\mathbb{Z}} T_{\sigma_k}\big(Q_kf_1, P_{k,m}f_2,\dots,P_{k,m}f_m\big)\Big) = \psi_j * \Big(\sum_{k=j-3}^{j+3}\cdots\Big).$$

Now the Littlewood-Paley theory and (3.9) yield that the left-hand side of (3.6) is less than a constant times

$$\begin{aligned}&\Big\|\Big\{\psi_j * \Big(\sum_{k=j-3}^{j+3} T_{\sigma_k}\big(Q_kf_1, P_{k,m}f_2,\dots,P_{k,m}f_m\big)\Big)\Big\}_{j\in\mathbb{Z}}\Big\|_{L^p(\ell^2)}\\ &\lesssim \big\|\big\{T_{\sigma_k}\big(Q_kf_1, P_{k,m}f_2,\dots,P_{k,m}f_m\big)\big\}_{k\in\mathbb{Z}}\big\|_{L^p(\ell^2)}.\end{aligned} \tag{3.10}$$

Applying Lemma 3.1,

$$\begin{aligned}&\big|T_{\sigma_k}\big(Q_kf_1, P_{k,m}f_2,\dots,P_{k,m}f_m\big)(x)\big|\\ &= \big|\sigma_k^\vee * \big(Q_kf_1\otimes P_{k,m}f_2\otimes\cdots\otimes P_{k,m}f_m\big)(x,\dots,x)\big|\\ &\lesssim \big\|\sigma_k(2^k\,\vec{\cdot}\,)\big\|_{L^{mn/s,1}_s(\mathbb{R}^{mn})}\mathcal{M}_q^{(n)}Q_kf_1(x)\prod_{j=2}^m\mathcal{M}_q^{(n)}P_{k,m}f_j(x)\end{aligned}$$

for $mn/s < q < \min(2, p_1, \ldots, p_m)$. Therefore, the right-hand side of the inequality (3.10) is estimated by

$$\sup_{k\in\mathbb{Z}} \big\|\sigma_k(2^k\,\vec{\cdot}\,)\big\|_{L^{mn/s,1}_s(\mathbb{R}^{mn})} \Big\| \Big\{ \mathcal{M}_q^{(n)} Q_k f_1 \prod_{j=2}^m \mathcal{M}_q^{(n)} P_{k,m} f_j \Big\}_{k\in\mathbb{Z}} \Big\|_{L^p(\ell^2)}.$$

We now apply Hölder's inequality and (2.1) to show that

$$\begin{aligned}
&\Big\| \Big\{ \mathcal{M}_q^{(n)} Q_k f_1 \prod_{j=2}^m \mathcal{M}_q^{(n)} P_{k,m} f_j \Big\}_{k\in\mathbb{Z}} \Big\|_{L^p(\ell^2)} \\
&\lesssim \big\| \big\{ \mathcal{M}_q^{(n)} Q_k f_1 \big\}_{k\in\mathbb{Z}} \big\|_{L^{p_1}(\ell^2)} \prod_{j=2}^m \big\| \big\{ \mathcal{M}_q^{(n)} P_{k,m} f_j \big\}_{k\in\mathbb{Z}} \big\|_{L^{p_j}(\ell^\infty)} \\
&\lesssim \big\| \big\{ Q_k f_1 \big\}_{k\in\mathbb{Z}} \big\|_{L^{p_1}(\ell^2)} \prod_{j=2}^m \big\| \big\{ P_{k,m} f_j \big\}_{k\in\mathbb{Z}} \big\|_{L^{p_j}(\ell^\infty)}.
\end{aligned}$$

The well known equivalences

$$\big\| \big\{ Q_k f_1 \big\}_{k\in\mathbb{Z}} \big\|_{L^{p_1}(\ell^2)} \approx \|f_1\|_{L^{p_1}(\mathbb{R}^n)} \qquad \text{for } \ 1 < p_1 < \infty$$

and

$$\big\| \big\{ P_{k,m} f_j \big\}_{k\in\mathbb{Z}} \big\|_{L^{p_j}(\ell^\infty)} \approx_m \|f_j\|_{H^{p_j}(\mathbb{R}^n)} \approx \|f_j\|_{L^{p_j}(\mathbb{R}^n)} \qquad \text{for } \ 1 < p_j < \infty$$

conclude the proof of (3.6).

3.1.2. *Proof of (3.7).* Since the sum over k_2 in the left-hand side of (3.7) is a finite sum over k_2 near k, we may consider only the case $k_2 = k$ and thus we need to prove

(3.11)

$$\Big\| \sum_{k\in\mathbb{Z}} T_{\sigma_k}\big(Q_k f_1, Q_k f_2, P_k f_3, \ldots, P_k f_m\big) \Big\|_{L^p(\mathbb{R}^n)} \lesssim \sup_{k\in\mathbb{Z}} \big\|\sigma(2^k\,\vec{\cdot}\,)\big\|_{L^{mn/s,1}_s(\mathbb{R}^{mn})} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)}.$$

To prove the validity of (3.11) we express, as before,

$$\begin{aligned}
&\big| T_{\sigma_k}\big(Q_k f_1, Q_k f_2, P_k f_3, \ldots, P_k f_m\big)(x) \big| \\
&= \big| \sigma_k^\vee * \big(Q_k f_1 \otimes Q_k f_2 \otimes P_k f_3 \otimes \cdots \otimes P_k f_m\big)(x, \ldots, x) \big|,
\end{aligned}$$

and apply Lemma 3.1 with $mn/s < q < \min(2, p_1, \ldots, p_m)$. Then the preceding expression is dominated by a constant multiple of

$$\big\|\sigma_k(2^k\,\vec{\cdot}\,)\big\|_{L^{mn/s,1}_s(\mathbb{R}^{mn})} \mathcal{M}_q^{(n)} Q_k f_1(x) \mathcal{M}_q^{(n)} Q_k f_2(x) \prod_{j=3}^m \mathcal{M}_q^{(n)} P_k f_j(x),$$

and this yields that the left-hand side of (3.11) is controlled by

$$\sup_{k\in\mathbb{Z}} \big\|\sigma_k(2^k\,\vec{\cdot}\,)\big\|_{L^{mn/s,1}_s(\mathbb{R}^{mn})} \Big\| \sum_{k\in\mathbb{Z}} \mathcal{M}_q^{(n)} Q_k f_1 \mathcal{M}_q^{(n)} Q_k f_2 \Big(\prod_{j=3}^m \mathcal{M}_q^{(n)} P_k f_j \Big) \Big\|_{L^p(\mathbb{R}^n)}$$

$$\lesssim \sup_{k\in\mathbb{Z}} \big\|\sigma_k(2^k\,\vec{\cdot}\,)\big\|_{L^{mn/s,1}_s(\mathbb{R}^{mn})} \Big(\prod_{i=1}^2 \big\| \{Q_k f_i\}_{k\in\mathbb{Z}} \big\|_{L^{p_i}(\ell^2)} \Big) \Big(\prod_{j=3}^m \big\| \{P_k f_j\}_{k\in\mathbb{Z}} \big\|_{L^{p_j}(\ell^\infty)} \Big)$$

$$\approx \sup_{k\in\mathbb{Z}} \big\|\sigma_k(2^k\,\vec{\cdot}\,)\big\|_{L^{mn/s,1}_s(\mathbb{R}^{mn})} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)}$$

in view of Hölder's inequality, (2.1) and the equivalences that used in the proof of (3.6). This completes the proof of (3.11).

3.2. **The case $1 < p, p_1, \dots, p_m < \infty$.** Let T_σ^{*j} be the jth transpose of T_σ, defined as the unique operator satisfying

$$\big\langle T_\sigma^{*j}(f_1, \dots, f_m), h \big\rangle := \big\langle T_\sigma(f_1, \dots, f_{j-1}, h, f_{j+1}, \dots, f_m), f_j \big\rangle$$

for $f_1, \dots, f_m, h \in \mathcal{S}(\mathbb{R}^n)$. Observe that $T_\sigma^{*j} = T_{\sigma^{*j}}$ where

$$\sigma^{*j}(\xi_1, \dots, \xi_m) = \sigma\big(\xi_1, \dots, \xi_{j-1}, -(\xi_1 + \cdots + \xi_m), \xi_{j+1}, \dots, \xi_m\big)$$

and we claim that for any $1 \le j \le m$

$$\sup_{k\in\mathbb{Z}} \big\|\sigma^{*j}(2^k\,\vec{\cdot}\,)\widehat{\Psi^{(m)}}\big\|_{L^{mn/s,1}_s(\mathbb{R}^{mn})} \lesssim \sup_{k\in\mathbb{Z}} \big\|\sigma(2^k\,\vec{\cdot}\,)\widehat{\Psi^{(m)}}\big\|_{L^{mn/s,1}_s(\mathbb{R}^{mn})}. \tag{3.12}$$

To see this, we need the following lemma:

Lemma 3.2. *Let $1 < p < \infty$, $0 < q \le \infty$, and $s \ge 0$. Let $f \in \mathcal{S}(\mathbb{R}^{mn})$ and for each $1 \le j \le m$ let*

$$T^j f(x_1, \dots, x_m) := f(x_1, \dots, x_{j-1}, -(x_1 + \cdots + x_m), x_{j+1}, \dots, x_m)$$

for $x_1, \dots, x_m \in \mathbb{R}^n$. Then every T^j satisfies the estimate

$$\big\|T^j f\big\|_{L^{p,q}_s(\mathbb{R}^{mn})} \lesssim \big\|f\big\|_{L^{p,q}_s(\mathbb{R}^{mn})}. \tag{3.13}$$

Proof. It is enough to deal only with the case $j = 1$ because the other cases will follow from a symmetric argument.

Step 1. We claim that for $k \in \{0, 1, 2, \dots\}$

$$\big\|\big(\vec{I} - \vec{\Delta}\big)^k T^1 f\big\|_{L^p(\mathbb{R}^{mn})} \lesssim \big\|\big(\vec{I} - \vec{\Delta}\big)^k f\big\|_{L^p(\mathbb{R}^{mn})}. \tag{3.14}$$

Using Leibniz's rule we write

$$\begin{aligned} \big|\big(\vec{I} - \vec{\Delta}\big)^k T^1 f(x_1, \dots, x_m)\big| &\approx \Big| \sum_{l=0}^k c_l (-\vec{\Delta})^l T^1 f(x_1, \dots, x_m) \Big| \\ &\approx \Big| \sum_{l=0}^k d_l \big[(-\vec{\Delta})^l f\big]\big(-(x_1 + \cdots + x_m), x_2, \dots, x_m \big) \Big| \\ &\approx \big| \big[\big(\vec{I} - \vec{\Delta}\big)^k f\big]\big(-(x_1 + \cdots + x_m), x_2, \dots, x_m \big) \big|, \end{aligned}$$

for some constants c_l, d_l. Then (3.14) can be achieved through a change of variables in L^p.

Step 2. From Step 1, T^1 is a linear operator

$$T^1 : L^p_{2k}(\mathbb{R}^{mn}) \to L^p_{2k}(\mathbb{R}^{mn})$$

for all $1 < p < \infty$ and $k \in \{0, 1, 2, \dots\}$. We perform a complex interpolation method with the fact that $\big(L^p_{s_0}(\mathbb{R}^{mn}), L^p_{s_1}(\mathbb{R}^{mn})\big)_\theta = L^p_s(\mathbb{R}^{mn})$ for $s = (1-\theta)s_0 + \theta s_1$, and then obtain that

$$\big\|T^1 f\big\|_{L^p_s(\mathbb{R}^{mn})} \lesssim \|f\|_{L^p_s(\mathbb{R}^{mn})} \tag{3.15}$$

for all $s \geq 0$ and $1 < p < \infty$.

Step 3. Let $0 < q \leq \infty$ and $s \geq 0$. We define a linear operator $T^{1,s}$ by

$$T^{1,s} f(x_1, \dots, x_m) := \big(\vec{I} - \vec{\Delta}\big)^{s/2}\big[T^1\big(\vec{I} - \vec{\Delta}\big)^{-s/2} f\big](x_1, \dots, x_m).$$

Then (3.15) implies that for all $1 < p < \infty$

$$\big\|T^{1,s} f\big\|_{L^p(\mathbb{R}^{mn})} \lesssim \|f\|_{L^p(\mathbb{R}^{mn})}.$$

Using real interpolation with $\big(L^{p_0}(\mathbb{R}^{mn}), L^{p_1}(\mathbb{R}^{mn})\big)_{\theta,q} = L^{p,q}(\mathbb{R}^{mn})$ for $1/p = (1-\theta)/p_0 + \theta/p_1$, we obtain

$$\big\|T^{1,s} f\big\|_{L^{p,q}(\mathbb{R}^{mn})} \lesssim \|f\|_{L^{p,q}(\mathbb{R}^{mn})},$$

which is equivalent to (3.13). □

Now let us prove (3.12). Lemma 3.2 yields that

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}} \big\|\sigma^{*j}(2^k\,\vec{\cdot}\,)\widehat{\Psi^{(m)}}\big\|_{L^{mn/s,1}_s(\mathbb{R}^{mn})} \\
&\quad\lesssim \sup_{k\in\mathbb{Z}} \big\|\sigma(2^k\vec{\boldsymbol{\xi}})\widehat{\Psi^{(m)}}\big(\xi_1, \dots, \xi_{j-1}, -(\xi_1 + \dots + \xi_m), \xi_{j+1}, \dots, \xi_m\big)\big\|_{L^{mn/s,1}_s(\vec{\boldsymbol{\xi}})}.
\end{aligned}$$

Since

$$\frac{1}{\sqrt{3}}|\vec{\boldsymbol{\xi}}| \leq \big(|\xi_1|^2 + \dots + |\xi_{j-1}|^2 + |\xi_1 + \dots + \xi_m|^2 + |\xi_{j+1}|^2 + \dots + |\xi_m|^2\big)^{1/2} \leq \sqrt{3}|\vec{\boldsymbol{\xi}}|,$$

the preceding expression can be written as

$$\sup_{k\in\mathbb{Z}} \Big\|\sigma(2^k\vec{\boldsymbol{\xi}})\widehat{\Lambda^{(m)}}(\vec{\boldsymbol{\xi}})\widehat{\Psi^{(m)}}\big(\xi_1, \dots, \xi_{j-1}, -(\xi_1 + \dots + \xi_m), \xi_{j+1}, \dots, \xi_m\big)\Big\|_{L^{mn/s,1}_s(\vec{\boldsymbol{\xi}})}$$

where $\Lambda^{(m)}$ is a Schwartz function on $\mathbb{R}^{mn}$ having the properties that $\widehat{\Lambda^{(m)}}$ is supported in the annulus $2^{-2} \leq |\vec{\boldsymbol{\xi}}| \leq 2^2$ and $\widehat{\Lambda^{(m)}}(\vec{\boldsymbol{\xi}}) = 1$ for $\frac{1}{2\sqrt{3}} \leq |\vec{\boldsymbol{\xi}}| \leq 2\sqrt{3}$. Using Lemma 2.3, the supremum is controlled by a constant multiple of

$$\sup_{k\in\mathbb{Z}} \big\|\sigma(2^k\,\vec{\cdot}\,)\widehat{\Lambda^{(m)}}\big\|_{L^{mn/s,1}_s(\mathbb{R}^{mn})}$$

and we obtain (3.12) in the same way as (3.1).

Now we complete the proof. Assume $1 < p \leq \min(p_1, \dots, p_m) < 2$ (otherwise, we are done from Section 3.1). Observe that only one of p_j could be less than 2 because $1/p = 1/p_1 + \dots + 1/p_m < 1$, and we will actually look at the case $1 < p_1 < 2 \leq p_2, \dots, p_m$. Let $2 < p', p_1' < \infty$ be the Hölder conjugates of p, p_1, respectively. That is, $1/p + 1/p' =$

$1/p_1+1/p_1'=1$ and accordingly, $1/p_1'=1/p'+1/p_2+\cdots+1/p_m$ and $2\le p',p_2,\dots,p_m<\infty$. Finally, we have

$$\begin{aligned}\big\|T_\sigma(\vec{\boldsymbol f})\big\|_{L^p(\mathbb R^n)} &= \sup_{\|h\|_{L^{p'}(\mathbb R^n)}=1}\big|\big\langle T_{\sigma^{*1}}(h,f_2,\dots,f_m),f_1\big\rangle\big| \\ &\le \|f_1\|_{L^{p_1}(\mathbb R^n)}\sup_{\|h\|_{L^{p'}(\mathbb R^n)}=1}\big\|T_{\sigma^{*1}}(h,f_2,\dots,f_m)\big\|_{L^{p_1'}(\mathbb R^n)} \\ &\lesssim \sup_{k\in\mathbb Z}\big\|\sigma(2^k\,\vec\cdot\,)\Psi^{(m)}\big\|_{L_s^{mn/s,1}(\mathbb R^{mn})}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\mathbb R^n)}\end{aligned}$$

where (3.12) is applied.

4. Proof of Theorem 1.2

For any $0<t,\gamma<\infty$ we define

$$\mathcal H_{(t,\gamma)}(\vec{\boldsymbol x}):=\frac{1}{(1+4\pi^2|\vec{\boldsymbol x}|^2)^{t/2}}\frac{1}{(1+\ln(1+4\pi^2|\vec{\boldsymbol x}|^2))^{\gamma/2}}.$$

We first see that

$$\|\mathcal H_{(t,\gamma)}\|_{L^r(\mathbb R^{mn})}<\infty \quad\text{if and only if}\quad t>mn/r \quad\text{or}\quad t=mn/r,\gamma>2/r. \tag{4.1}$$

Moreover, it was shown in [9] that

$$\big|\widehat{\mathcal H_{(t,\gamma)}}(\vec{\boldsymbol\xi})\big|\lesssim_{t,\gamma,n,m}e^{-|\vec{\boldsymbol\xi}|/2}\quad\text{for}\quad|\vec{\boldsymbol\xi}|>1$$

and when $0<t<mn$,

$$\big|\widehat{\mathcal H_{(t,\gamma)}}(\vec{\boldsymbol\xi})\big|\approx_{t,\gamma,n,m}|\vec{\boldsymbol\xi}|^{-(mn-t)}(1+2\ln|\vec{\boldsymbol\xi}|^{-1})^{-\gamma/2}\quad\text{for}\quad|\vec{\boldsymbol\xi}|\le 1.$$

The estimates imply that

$$\big\|\widehat{\mathcal H_{(t,\gamma)}}\big\|_{L^{r,q}(\mathbb R^{mn})}<\infty\ \text{ if and only if }\ t>mn-mn/r\quad\text{or}\quad t=mn-mn/r,\gamma>2/q. \tag{4.2}$$

Based on the properties of $\mathcal H_{t,\gamma}$, let us construct counter examples to prove Theorem 1.2. Let Γ denote a Schwartz function on $\mathbb R^{mn}$ such that $\mathrm{Supp}(\widehat\Gamma)\subset\{\vec{\boldsymbol\xi}\in\mathbb R^{mn}:\frac{99}{100}\le|\vec{\boldsymbol\xi}|\le\frac{101}{100}\}$ and $\widehat\Gamma(\vec{\boldsymbol\xi})=1$ for $\frac{999}{1000}\le|\vec{\boldsymbol\xi}|\le\frac{1001}{1000}$. Let Φ be a Schwartz function on $\mathbb R^{mn}$ whose Fourier transform is equal to 1 on the ball $\{\vec{\boldsymbol\xi}\in\mathbb R^{mn}:|\vec{\boldsymbol\xi}|\le 1\}$ and is supported in a larger ball. Let N be a sufficiently large positive integer and $\Phi_N:=N^{mn}\Phi(N\vec\cdot)$. We define

$$\mathcal H^{(N)}_{(t,\gamma)}(\vec{\boldsymbol x}):=\mathcal H_{(t,\gamma)}(\vec{\boldsymbol x})\widehat{\Phi_N}(\vec{\boldsymbol x}),\qquad \vec{\boldsymbol x}\in\mathbb R^{mn}$$

and

$$\sigma^{(N)}(\vec{\boldsymbol\xi}):=\widehat{\mathcal H^{(N)}_{(t,\gamma)}}(\vec{\boldsymbol\xi})\widehat\Gamma(\vec{\boldsymbol\xi}),\qquad\vec{\boldsymbol\xi}\in\mathbb R^{mn}.$$

Then $\sigma^{(N)}$ is supported in $\{\vec{\boldsymbol\xi}\in\mathbb R^{mn}:\frac{99}{100}\le|\vec{\boldsymbol\xi}|\le\frac{101}{100}\}$ in view of the support of $\widehat\Gamma$, and this implies that $\sigma^{(N)}(2^k\vec{\boldsymbol\xi})\widehat{\Psi^{(m)}}(\vec{\boldsymbol\xi})$ vanishes unless $-1\le k\le 1$. Therefore, using Lemma 2.3 and a scaling argument, we have

$$\begin{aligned}\sup_{k\in\mathbb Z}\big\|\sigma^{(N)}(2^k\,\vec\cdot\,)\widehat{\Psi^{(m)}}\big\|_{L_s^{r,q}(\mathbb R^{mn})} &= \max_{-1\le k\le 1}\big\|\sigma^{(N)}(2^k\,\vec\cdot\,)\widehat{\Psi^{(m)}}\big\|_{L_s^{r,q}(\mathbb R^{mn})} \\ &\lesssim \max_{-1\le k\le 1}\big\|\sigma^{(N)}(2^k\,\vec\cdot\,)\big\|_{L_s^{r,q}(\mathbb R^{mn})}\lesssim\|\sigma^{(N)}\|_{L_s^{r,q}(\mathbb R^{mn})}.\end{aligned}$$

This can be further estimated, using Lemma 2.3, by a constant times

$$\big\|\widehat{\mathcal{H}^{(N)}_{(t,\gamma)}}\big\|_{L^{r,q}_s(\mathbb{R}^{mn})} = \big\|\Phi_N * \widehat{\mathcal{H}_{(t-s,\gamma)}}\big\|_{L^{r,q}(\mathbb{R}^{mn})}$$

where the equality follows from fact that

$$(\vec{I}-\vec{\Delta})^{s/2}\widehat{\mathcal{H}^{(N)}_{(t,\gamma)}}(\vec{\boldsymbol{\xi}}) = \widehat{\mathcal{H}^{(N)}_{(t-s,\gamma)}}(\vec{\boldsymbol{\xi}}) = \Phi_N * \widehat{\mathcal{H}_{(t-s,\gamma)}}(\vec{\boldsymbol{\xi}}).$$

Finally, Lemma 2.1 yields that

$$\sup_{k\in\mathbb{Z}}\big\|\sigma^{(N)}(2^k\,\vec{\cdot}\,)\widehat{\Psi^{(m)}}\big\|_{L^{r,q}_s(\mathbb{R}^{mn})} \lesssim \big\|\widehat{\mathcal{H}_{(t-s,\gamma)}}\big\|_{L^{r,q}(\mathbb{R}^{mn})}, \qquad \text{uniformly in } N. \tag{4.3}$$

On the other hand, for $0<\epsilon<1/100$ and for each $0<p_j\le\infty$, let

$$f_1^{(\epsilon)}(x) := \epsilon^{n/p_1}\theta(\epsilon x)e^{2\pi i\langle x,e_1\rangle}, \qquad f_j^{(\epsilon)}(x) := \epsilon^{n/p_j}\theta(\epsilon x), \qquad 2\le j\le m$$

where $e_1 := (1,0,\dots,0)\in\mathbb{Z}^n$ and θ is a Schwartz function on $\mathbb{R}^n$ with $\mathrm{Supp}(\widehat{\theta})\subset\{\xi\in\mathbb{R}^n : \frac{1}{2000\sqrt{m}}\le|\xi|\le\frac{1}{1000\sqrt{m}}\}$. Clearly, we have

$$\|f_j^{(\epsilon)}\|_{L^{p_j}(\mathbb{R}^n)} = \|\theta\|_{L^{p_j}(\mathbb{R}^n)} \lesssim_{p_j,n} 1 \qquad \text{uniformly in } \epsilon. \tag{4.4}$$

In addition,

$$T_{\sigma^{(N)}}\big(f_1^{(\epsilon)},\dots,f_m^{(\epsilon)}\big)(x) = \epsilon^{n/p}\mathcal{H}^{(N)}_{(t,\gamma)} * \big(\theta(\epsilon\cdot)e^{2\pi i\langle\cdot,e_1\rangle}\otimes\cdots\otimes\theta(\epsilon\cdot)\big)(x,\dots,x)$$

since $\widehat{\Gamma}=1$ on the support of the Fourier transform of $\theta(\epsilon\cdot)e^{2\pi i\langle\cdot,e_1\rangle}\otimes\cdots\otimes\theta(\epsilon\cdot)$. By taking the L^p norm and using a scaling, we see that

$$\begin{aligned}&\big\|T_{\sigma^{(N)}}\big(f_1^{(\epsilon)},\dots,f_m^{(\epsilon)}\big)\big\|_{L^p(\mathbb{R}^n)}\\ &= \Big(\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^{mn}}\mathcal{H}^{(N)}_{(t,\gamma)}(\vec{\boldsymbol{y}})\theta(x-\epsilon y_1)\cdots\theta(x-\epsilon y_m)e^{-2\pi i\langle y_1,e_1\rangle}d\vec{\boldsymbol{y}}\Big|^p dx\Big)^{1/p}.\end{aligned}$$

We now apply (4.4) and Fatou's lemma to obtain

$$\begin{aligned}&\|T_{\sigma^{(N)}}\|_{L^{p_1}\times\cdots\times L^{p_m}\to L^p}\\ &\gtrsim \liminf_{\epsilon\to0}\big\|T_{\sigma^{(N)}}\big(f_1^{(\epsilon)},\dots,f_m^{(\epsilon)}\big)\big\|_{L^p(\mathbb{R}^n)}\\ &\ge \Big(\int_{\mathbb{R}^n}\Big|\liminf_{\epsilon\to0}\int_{\mathbb{R}^{mn}}\mathcal{H}^{(N)}_{(t,\gamma)}(\vec{\boldsymbol{y}})\theta(x-\epsilon y_1)\cdots\theta(x-\epsilon y_m)e^{-2\pi i\langle y_1,e_1\rangle}d\vec{\boldsymbol{y}}\Big|^p dx\Big)^{1/p}.\end{aligned} \tag{4.5}$$

Since

$$\big|\mathcal{H}^{(N)}_{(t,\gamma)}(\vec{\boldsymbol{y}})\theta(x-\epsilon y_1)\cdots\theta(x-\epsilon y_m)e^{-2\pi i\langle y_1,e_1\rangle}\big| \lesssim \big|\mathcal{H}^{(N)}_{(t,\gamma)}(\vec{\boldsymbol{y}})\big| \qquad \text{uniformly in } \epsilon>0,\ x\in\mathbb{R}^n$$

and

$$\big\|\mathcal{H}^{(N)}_{(t,\gamma)}\big\|_{L^1(\mathbb{R}^{mn})} \le \|\widehat{\Phi_N}\|_{L^1(\mathbb{R}^{mn})} \lesssim N^{mn} < \infty,$$

we can utilize the Lebesgue dominated convergence theorem and then

$$(4.5) = \big\||\theta|^m\big\|_{L^p(\mathbb{R}^n)}\big\|\mathcal{H}^{(N)}_{(t,\gamma)}\big\|_{L^1(\mathbb{R}^{mn})} \approx \big\|\mathcal{H}_{(t,\gamma)}\widehat{\Phi_N}\big\|_{L^1(\mathbb{R}^{mn})}.$$

By taking $\liminf_{N\to\infty}$, we finally obtain that

$$\liminf_{N\to\infty}\|T_{\sigma^{(N)}}\|_{L^{p_1}\times\cdots\times L^{p_m}\to L^p} \gtrsim \|\mathcal{H}_{(t,\gamma)}\|_{L^1(\mathbb{R}^{mn})} \tag{4.6}$$

where the monotone convergence theorem is applied.

If $0 < r < mn/s$ and $0 < q \le \infty$, then choose t satisfying

$$mn - (mn/r - s) < t < mn,$$

which is equivalent to $mn - mn/r < t - s < mn$ and $t < mn$. It follows from (4.3) and (4.2) that

$$\limsup_{N\to\infty} \sup_{k\in\mathbb{Z}} \big\| \sigma^{(N)}(2^k\cdot)\widehat{\Psi^{(m)}} \big\|_{L^{r,q}_s(\mathbb{R}^{mn})} < \infty,$$

and (4.6) and (4.1) yield that

$$\liminf_{N\to\infty} \|T_{\sigma^{(N)}}\|_{L^{p_1}\times\cdots\times L^{p_m}\to L^p} = \infty,$$

which proves the first assertion of Theorem 1.2

If $r = mn/s$ and $q > 1$, then let $t = mn$ and γ be a positive number with $2/q < \gamma \le 2$. Then thanks to (4.1), (4.2), (4.3), and (4.6), we have

$$\limsup_{N\to\infty} \sup_{k\in\mathbb{Z}} \big\| \sigma^{(N)}(2^k\cdot)\widehat{\Psi^{(m)}} \big\|_{L^{mn/s,q}_s(\mathbb{R}^{mn})} < \infty$$

and

$$\liminf_{N\to\infty} \|T_{\sigma^{(N)}}\|_{L^{p_1}\times\cdots\times L^{p_m}\to L^p} = \infty,$$

which completes the proof of Theorem 1.2.

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