

MULTILINEAR FOURIER MULTIPLIERS WITH MINIMAL SOBOLEV REGULARITY, I

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ABSTRACT. We find optimal conditions on m -linear Fourier multipliers that give rise to bounded operators from products of Hardy spaces H^{p_k} , $0 < p_k \leq 1$, to Lebesgue spaces L^p . These conditions are expressed in terms of L^2 -based Sobolev spaces with sharp indices within the classes of multipliers we consider. Our results extend those obtained in the linear case ($m = 1$) by Calderón and Torchinsky [1] and in the bilinear case ($m = 2$) by Miyachi and Tomita [15]. We also prove a coordinate-type Hörmander integral condition which we use to obtain certain endpoint cases.

1. INTRODUCTION

Let σ be a bounded function on \mathbb{R}^n . We denote by T_σ the linear Fourier multiplier operator, whose action on Schwartz functions is given by

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \sigma(\xi) \widehat{f}(\xi) e^{2\pi i x \xi} d\xi. \quad (1.1)$$

Mikhlin's [14] classical result states that the T_σ admits an L^p -bounded extension for $1 < p < \infty$, whenever

$$|\partial_\xi^\alpha \sigma(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}, \quad \xi \neq 0 \quad (1.2)$$

for all multi-indices α with $|\alpha| \leq [\frac{n}{2}] + 1$. This result was refined by Hörmander [12] who proved that (1.2) can be replaced by the Sobolev-norm condition

$$\sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{W^s} < \infty, \quad (1.3)$$

for some $s > \frac{n}{2}$, where $\widehat{\psi}$ is a smooth function supported in $\frac{1}{2} \leq |\xi| \leq 2$ that satisfies

$$\sum_{j \in \mathbb{Z}} \widehat{\psi}(2^{-j} \xi) = 1$$

for all $\xi \neq 0$. Here $\|g\|_{W^s} = \|(I - \Delta)^{s/2} g\|_{L^2}$, where I is the identity operator and $\Delta = \sum_{j=1}^n \partial_j^2$, is the Laplacian on \mathbb{R}^n .

Calderón and Torchinsky [1] showed that the Fourier multiplier operator in (1.1) admits a bounded extension from the Hardy space H^p to H^p with $0 < p \leq 1$ if

$$\sup_{t > 0} \left\| \sigma(t \cdot) \widehat{\psi} \right\|_{W^s} < \infty$$

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and $s > \frac{n}{p} - \frac{n}{2}$. Here the index $s = \frac{n}{p} - \frac{n}{2}$ is critical in the sense that the boundedness of T_σ on H^p does not hold if $s \leq \frac{n}{p} - \frac{n}{2}$. This was pointed out later by Miyachi and Tomita [15].

The bilinear counterpart of the Fourier multiplier theory has been rather similar in the formulation of results, but substantially more complicated in its proofs. The theory of multilinear operators, and in particular that of multilinear multiplier operators, originated in the work of Coifman and Meyer [2], [3], [13] and resurfaced in the work of Grafakos and Torres [11]. Multilinear Fourier multipliers are bounded functions σ on $\mathbb{R}^{mn} = \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ associated with the m -linear Fourier multiplier operator in the following way

$$T_\sigma(f_1, \dots, f_m)(x) = \int_{\mathbb{R}^{mn}} e^{2\pi i x \cdot (\xi_1 + \cdots + \xi_m)} \sigma(\xi_1, \dots, \xi_m) \widehat{f}_1(\xi_1) \cdots \widehat{f}_m(\xi_m) d\vec{\xi}, \quad (1.4)$$

where f_j are in the Schwartz space of \mathbb{R}^n and $d\vec{\xi} = d\xi_1 \cdots d\xi_m$.

Tomita [17] obtained $L^{p_1} \times \cdots \times L^{p_m} \rightarrow L^p$ boundedness ($1 < p_1, \dots, p_m, p < \infty$) for multilinear multiplier operators under a condition analogous to (1.3). Grafakos and Si [10] extended Tomita's results to the case $p \leq 1$ by using L^r -based Sobolev norms for σ with $1 < r \leq 2$. Fujita and Tomita [4] provided weighted extensions of these results but also noticed that the Sobolev space W^s in (1.3) can be replaced by a product-type Sobolev space $W^{(s_1, \dots, s_m)}$ when $p > 2$. Grafakos, Miyachi, and Tomita [8] extended the range of p in [4] to $p > 1$ and obtained boundedness even in the endpoint case where all but one indices p_j are equal to infinity. Miyachi and Tomita [15] provided extensions of the Calderón and Torchinsky results [1] for Hardy spaces in the bilinear case; note that in [15] it was pointed out that the conditions on the indices are sharp, even in the linear case, i.e., in the Calderón and Torchinsky theorem.

Following this stream of work, we are interested in finding conditions analogous to those in [15] in the multilinear setting, i.e., when $m \geq 3$. Our work is inspired by that of Calderón and Torchinsky [1], Grafakos and Kalton [7], and certainly of Miyachi and Tomita [15]. As in [15], we find necessary and sufficient conditions, which coincide with those in [15] when $m = 2$, that imply boundedness for multilinear multiplier operators on a products of Hardy spaces. One important aspect of this work is an appropriate regularization of the multilinear multiplier operator which allows the interchange of its action with infinite sums of H^{p_j} atoms (see Section 3). In this article we restrict attention to the case where the domain is a product of Hardy spaces. We study the case where the domain is a mix of Lebesgue and Hardy spaces in a subsequent article.

We introduce the Sobolev spaces that will be used throughout this paper. First, for $x \in \mathbb{R}^n$ we set $\langle x \rangle = \sqrt{1 + |x|^2}$. For $s_1, \dots, s_m > 0$, we denote by $W^{(s_1, \dots, s_m)}$ the Sobolev space (of product type) consisting all functions f on \mathbb{R}^{mn} such that

$$\|f\|_{W^{(s_1, \dots, s_m)}} := \left(\int_{\mathbb{R}^{mn}} \left| \widehat{f}(y_1, \dots, y_m) \langle y_1 \rangle^{s_1} \cdots \langle y_m \rangle^{s_m} \right|^2 dy_1 \cdots dy_m \right)^{\frac{1}{2}} < \infty.$$

Notice that $W^{(s_1, \dots, s_m)}$ is a subspace of L^2 .

Let ψ be a smooth function on \mathbb{R}^{mn} whose Fourier transform $\widehat{\psi}$ is supported in $\frac{1}{2} \leq |\xi| \leq 2$ and satisfies

$$\sum_{j \in \mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1, \quad \xi \neq 0.$$

For $0 < p < \infty$ we denote by H^p the Lebesgue space L^p if $p > 1$ and the Hardy space H^p if $p \leq 1$. The following is the main result of this paper.

Theorem 1.1. *Let $\frac{n}{2} < s_1, \dots, s_m < \infty$, $0 < p_1, \dots, p_m \leq 1$, $0 < p \leq 1$ such that*

$$\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p},$$

and that

$$\sum_{k \in J} \left(\frac{s_k}{n} - \frac{1}{p_k} \right) > -\frac{1}{2} \quad (1.5)$$

for every subset $J \subset \{1, 2, \dots, m\}$. If the function σ defined on \mathbb{R}^{mn} satisfies

$$A := \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} < \infty, \quad (1.6)$$

then T_σ is bounded from $H^{p_1} \times \dots \times H^{p_m} \rightarrow L^p$ with constant at most a multiple of A . Moreover, the set of $2^m - 1$ conditions (1.5) is optimal.

Remark 1.2. Conditions (1.5) imply that $s_k > \frac{n}{2}$ whenever $0 < p_k \leq 1$ for all $1 \leq k \leq m$. Moreover, the condition in (1.6) is sufficient to guarantee that σ lies in $L^\infty(\mathbb{R}^{mn})$. Indeed, suppose that σ is a function on \mathbb{R}^{mn} that satisfies (1.6). It is easy to see that $\widehat{\psi}(\frac{1}{2}x) + \widehat{\psi}(x) + \widehat{\psi}(2x) = 1$ for all $1 \leq x \leq 2$. Now we want to verify that $|\sigma(2^{j_0}x)|$ is uniformly bounded in $j_0 \in \mathbb{Z}$ for a.e. $1 \leq |x| \leq 2$. Applying the Cauchy-Schwarz inequality and using the conditions $s_k > \frac{n}{2}$, we write

$$\begin{aligned} & |\sigma(2^{j_0}x)| \\ &= \left| \sum_{|l| \leq 1} \sigma(2^{j_0}x) \widehat{\psi}(2^l x) \right| \\ &\leq \sum_{|l| \leq 1} \left| \int_{\mathbb{R}^{mn}} (\sigma(2^{j_0-l} \cdot) \widehat{\psi})^\vee(\xi) e^{2^{l+1} \pi i x \xi} d\xi \right| \\ &\leq \sum_{|l| \leq 1} \int_{\mathbb{R}^{mn}} \prod_{k=1}^m (1 + |\xi_k|^2)^{-\frac{s_k}{2}} \left| \prod_{k=1}^m (1 + |\xi_k|^2)^{\frac{s_k}{2}} (\sigma(2^{j_0-l} \cdot) \widehat{\psi})^\vee(\xi_1, \dots, \xi_m) \right| d\xi_1 \cdots d\xi_m \\ &\leq \sum_{|l| \leq 1} C(s_1, \dots, s_m, n) \left\| \sigma_{j_0-l} \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} \\ &\leq 3C(s_1, \dots, s_m, n) \sup_{j \in \mathbb{Z}} \left\| \sigma_j \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}}, \end{aligned}$$

for almost all x satisfying $1 \leq |x| \leq 2$. Here we set $\sigma_j(\vec{\xi}) = \sigma(2^j \vec{\xi})$. Thus

$$\|\sigma\|_{L^\infty(\mathbb{R}^{mn})} \leq 3C(s_1, \dots, s_m, n) \sup_{j \in \mathbb{Z}} \left\| \sigma_j \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} < \infty.$$

The structure of this paper is as follows: Section 2 contains preliminaries and known results. In Section 3, we regularize the multiplier to be able to work with a nicer operator and thus facilitate the passage of infinite sums in and out the operator in the proof of the main result given in Section 4. In section 5, we construct examples to justify the minimality of conditions (1.5) claimed in the main theorem. Section 6 will present some results about the boundedness of our operator in the endpoint cases where we need the coordinate-type Hörmander integral conditions.

The last section contains the detail proof of some technical lemmas using through the paper.

We use the notation $A \lesssim B$ to indicate that $A \leq CB$, where the constant C is independent of any essential parameters, and $A \approx B$ if both $A \lesssim B$ and $B \lesssim A$ hold simultaneously.

2. PRELIMINARIES AND KNOWN RESULTS

Now fix $0 < p < \infty$ and a Schwartz function Φ with $\widehat{\Phi}(0) \neq 0$. Then the Hardy space H^p contains all tempered distributions f on \mathbb{R}^n such that

$$\|f\|_{H^p} := \left\| \sup_{0 < t < \infty} |\Phi_t * f| \right\|_{L^p} < \infty.$$

It is well known that the definition of the Hardy space does not depend on the choice of the function Φ . Note that $H^p = L^p$ for all $p > 1$. When $0 < p \leq 1$, one of nice features of Hardy spaces is the atomic decomposition. More precisely, any function $f \in H^p$ ($0 < p \leq 1$) can be decomposed as $f = \sum_j \lambda_j a_j$, where a_j 's are L^∞ -atoms for H^p supported in cubes Q_j such that $\|a_j\|_{L^\infty} \leq |Q_j|^{-\frac{1}{p}}$ and $\int x^\gamma a_j(x) dx = 0$ for all $|\gamma| \leq [n(\frac{1}{p} - 1)] + 1$, and the coefficients λ_j satisfy $\sum_j |\lambda_j|^p \leq 2^p \|f\|_{H^p}^p$.

The following two lemmas are essentially contained in [15] modulo some minor modifications.

Lemma 2.1 ([15]). *Let k, l be positive integers, $0 < s_1, \dots, s_{k+l} < \infty$, and let $1 < \rho < \infty$. Assume that σ is a bounded function defined on $\mathbb{R}^{kn} \times \mathbb{R}^{ln}$, supported in $\{(x, y) \in \mathbb{R}^{kn} \times \mathbb{R}^{ln} : |x|^2 + |y|^2 \leq 4\}$, where we denote $x = (x_1, \dots, x_k), y = (y_1, \dots, y_l)$ with $x_1, \dots, x_k, y_1, \dots, y_l \in \mathbb{R}^n$, and set $K = \sigma^\vee$, the inverse Fourier transform of σ . Then there exists a constant $C > 0$ such that*

$$\|\langle y_1 \rangle^{s_1} \cdots \langle y_l \rangle^{s_l} K(x, y)\|_{L^\infty(\mathbb{R}^{ln}, dy)} \leq C \|\langle y_1 \rangle^{s_1} \cdots \langle y_l \rangle^{s_l} K(x, y)\|_{L^\rho(\mathbb{R}^{ln}, dy)}$$

for all $x \in \mathbb{R}^{kn}$.

Proof. Take φ a Schwartz function on \mathbb{R}^{ln} such that $\widehat{\varphi}(y) = 1$ for all $y \in \mathbb{R}^{ln}$, $|y| \leq 2$. Then we have $\sigma(x, y) = \sigma(x, y)\widehat{\varphi}(y)$. Using the inverse Fourier transform we have

$$\begin{aligned} K(x, y) &= \left(K * (\delta_0 \otimes \varphi) \right)(x, y) = \int_{\mathbb{R}^{kn} \times \mathbb{R}^{ln}} K(x - u, y - v) \delta_0(u) \varphi(v) du dv \\ &= \int_{\mathbb{R}^{ln}} K(x, y - v) \varphi(v) dv, \end{aligned}$$

where δ_0 is the Dirac distribution. Therefore,

$$\begin{aligned} & \|\langle y_1 \rangle^{s_1} \cdots \langle y_l \rangle^{s_l} |K(x, y)|\|_{L^\infty(\mathbb{R}^{ln}, dy)} \\ &= \|\langle y_1 \rangle^{s_1} \cdots \langle y_l \rangle^{s_l} \left| \int_{\mathbb{R}^{ln}} K(x_1, \dots, x_k, y_1 - v_1, \dots, y_l - v_l) \varphi(v) dv \right|\|_{L^\infty(\mathbb{R}^{ln}, dy)} \\ &\lesssim \int_{\mathbb{R}^{ln}} \left(\prod_{j=1}^l \langle y_j - v_j \rangle^{s_j} \right) |K(x_1, \dots, x_k, y_1 - v_1, \dots, y_l - v_l)| \langle v_1 \rangle^{s_1} \cdots \langle v_l \rangle^{s_l} |\varphi(v)| dv \\ &\leq C_1 \|\langle y_1 \rangle^{s_1} \cdots \langle y_l \rangle^{s_l} K(x, y)\|_{L^\rho(\mathbb{R}^{ln}, dy)} \|\langle v_1 \rangle^{s_1} \cdots \langle v_l \rangle^{s_l} |\varphi(v)|\|_{L^{\rho'}(\mathbb{R}^{ln}, dv)} \\ &\leq C_2 \|\langle y_1 \rangle^{s_1} \cdots \langle y_l \rangle^{s_l} K(x, y)\|_{L^\rho(\mathbb{R}^{ln}, dy)}, \end{aligned}$$

where we used Hölder's inequality in the second to last line. \square

Lemma 2.2 ([15]). *Let $s_k > \frac{n}{2}$ for $1 \leq k \leq m$, and let $\widehat{\zeta}$ be a smooth function which is supported in an annulus centered at zero. Suppose that Φ is a smooth function away from zero that satisfies the estimates*

$$|\partial_\xi^\alpha \Phi(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}$$

for all $\xi \in \mathbb{R}^{mn}$, $x \neq 0$ and for all multi-indices α . Then there exists a constant C such that

$$\sup_{j \in \mathbb{Z}} \left\| \sigma(2^j(\cdot)) \Phi(2^j(\cdot)) \widehat{\zeta} \right\|_{W^{(s_1, \dots, s_m)}} \leq C \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j(\cdot)) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}}.$$

Adapting the Calderón and Torchinsky interpolation techniques in the multilinear setting (for details on this we refer to [8, p. 318]) allows us to interpolate between two endpoint estimates for multilinear multiplier operators from a product of some Hardy spaces to Lebesgue spaces.

Theorem 2.3 ([8]). *Let $0 < p_1, p_2, p_{1k}, p_{2k} \leq \infty$ and $\frac{n}{2} < s_{1k}, s_{2k} < \infty$ and $1 \leq k \leq m$. For $0 < \theta < 1$, set $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, $\frac{1}{p_k} = \frac{1-\theta}{p_{1k}} + \frac{\theta}{p_{2k}}$, and $s_k = (1-\theta)s_{1k} + \theta s_{2k}$. Assume that the multilinear operator T_σ defined in (1.4) satisfies the estimates*

$$\|T_\sigma\|_{H^{p_{11}} \times \dots \times H^{p_{lm}} \rightarrow L^{p_l}} \leq C_l \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j(\cdot)) \widehat{\psi} \right\|_{W^{(s_{l1}, \dots, s_{lm})}}, \quad (l = 1, 2).$$

Then

$$\|T_\sigma\|_{H^{p_1} \times \dots \times H^{p_m} \rightarrow L^p} \leq C \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j(\cdot)) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}}.$$

The following result is due to Fujita and Tomita [4] for $2 < p < \infty$, while the extension to $p > 1$ and the endpoint case where all but one indices are equal to infinity is due to Grafakos, Miyachi and Tomita [8].

Theorem 2.4 ([4, 8]). *Let $1 < p_1, \dots, p_m \leq \infty$, $1 < p < \infty$ and $\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p}$. If σ satisfies (1.6), then the multilinear multiplier operator T_σ is bounded from $L^{p_1} \times \dots \times L^{p_m} \rightarrow L^p$ with constant at most a multiple of A .*

Finally, we will need the following lemma from [7].

Lemma 2.5 ([7, Lemma 2.1]). *Let $0 < p \leq 1$ and let $(f_Q)_{Q \in \mathcal{J}}$ be a family of nonnegative integrable functions with $\text{supp}(f_Q) \subset Q$ for all $Q \in \mathcal{J}$, where \mathcal{J} is a family of finite or countable cubes in \mathbb{R}^n . Then we have*

$$\left\| \sum_{Q \in \mathcal{J}} f_Q \right\|_{L^p} \lesssim \left\| \sum_{Q \in \mathcal{J}} \left(\frac{1}{|Q|} \int_Q f_Q(x) dx \right) \chi_{Q^*} \right\|_{L^p},$$

with the implicit constant depending only on p . Here Q^* is a dimensional dilate of the cube Q .

3. REGULARIZATION THE MULTIPLIER

In this section, we show that the operator defined in (1.1) with enough smoothness of the multiplier can be approximated by a family of very nice operators.

Theorem 3.1. *Let σ be a function on \mathbb{R}^{mn} and $s_k > \frac{n}{2}$ for $1 \leq k \leq m$ satisfying (1.6). Then there exists a family of functions $(\sigma^\epsilon)_{0 < \epsilon < \frac{1}{2}}$ such that $K^\epsilon := (\sigma^\epsilon)^\vee$ is smooth and compactly supported for every $0 < \epsilon < \frac{1}{2}$; also*

$$\sup_{0 < \epsilon < \frac{1}{2}} \sup_{j \in \mathbb{Z}} \left\| \sigma^\epsilon(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} \lesssim \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}}, \quad (3.1)$$

and

$$\lim_{\epsilon \rightarrow 0} \|T_\epsilon(f_1, \dots, f_m) - T_\sigma(f_1, \dots, f_m)\|_{L^2} = 0 \quad (3.2)$$

for all functions $f_k \in L^{2m}$, $1 \leq k \leq m$, where T_ϵ are multilinear singular integral operators of convolution type associated to K^ϵ .

The following lemma, whose proof will be given in the last section, is the first step in constructing such a family of functions σ^ϵ as stated in Theorem 3.1.

Lemma 3.2. *Let φ be a Schwartz function. Suppose σ is a function on \mathbb{R}^{mn} satisfying (1.6) for $s_k > \frac{n}{2}$. Then we have*

$$\sup_{\epsilon > 0} \sup_{j \in \mathbb{Z}} \left\| [(\varphi_\epsilon * \sigma)(2^j \cdot)] \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} \lesssim \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}},$$

where $\varphi_\epsilon(x_1, \dots, x_m) = \epsilon^{-mn} \varphi(\epsilon^{-1}x_1, \dots, \epsilon^{-1}x_m)$ for all $x_k \in \mathbb{R}^n$, $1 \leq k \leq m$.

We now start the proof of Theorem 3.1.

Proof of Theorem 3.1. Fix $0 < \epsilon < \frac{1}{2}$. Choose a smooth function φ such that $\widehat{\varphi}$ is supported in the unit ball and $\widehat{\varphi}(0) = 1$. Denote by $\sigma^\epsilon = \varphi_\epsilon * (\sigma\phi^\epsilon)$, where $\phi^\epsilon = \theta(\epsilon^{-1}\cdot) - \theta(\cdot)$, and θ is a smooth function satisfying $\theta(x) = 0$ for all $|x| \leq 1$ and $\theta(x) = 1$ for all $|x| \geq 2$. We note that these functions are suitable regularized versions of the multiplier in Theorem 3.1. Indeed, let $K^\epsilon = (\sigma^\epsilon)^\vee = (\sigma\phi^\epsilon)^\vee \widehat{\varphi}(\epsilon \cdot)$; then, K^ϵ are smooth functions with compact support for all $0 < \epsilon < \frac{1}{2}$.

Using the fact that

$$|\partial^\alpha \phi^\epsilon(\xi)| \leq C_{\alpha, \theta} |\xi|^{-\alpha}, \quad \xi \neq 0, \quad 0 < \epsilon < \frac{1}{2},$$

Lemma 3.2 applied to the function $\sigma\phi^\epsilon$ combined with Lemma 2.2 gives

$$\begin{aligned} \sup_{0 < \epsilon < \frac{1}{2}} \sup_{j \in \mathbb{Z}} \left\| \sigma^\epsilon(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} &\lesssim \sup_{0 < \epsilon < \frac{1}{2}} \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \phi^\epsilon(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} \\ &\lesssim \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}}, \end{aligned}$$

which yields (3.1). Thus, we are left with establishing (3.2). For $\epsilon > 0$, now recall

$$\begin{aligned} T_\epsilon(f_1, \dots, f_m)(x) &= \int K^\epsilon(x - y_1, \dots, x - y_m) f_1(y_1) \cdots f_m(y_m) dy \\ &= \int \sigma^\epsilon(\xi_1, \dots, \xi_m) \widehat{f}_1(\xi_1) \cdots \widehat{f}_m(\xi_m) e^{2\pi i x(\xi_1 + \cdots + \xi_m)} d\xi. \end{aligned}$$

Invoking estimate (3.1) and Theorem 2.4, we can see that T_σ and T_ϵ are uniformly bounded from $L^{2m} \times \cdots \times L^{2m} \rightarrow L^2$ for all $0 < \epsilon < \frac{1}{2}$. By density, it suffices to verify (3.2) for all functions in the Schwartz class.

Now fix Schwartz functions f_k , for $1 \leq k \leq m$. The Fourier transform of $T_\sigma(f_1, \dots, f_m)$ can be written by

$$\int_{\mathbb{R}^{n(m-1)}} \sigma(\xi_1, \dots, \xi_{m-1}, \xi - \sum_{l=1}^{m-1} \xi_l) \widehat{f_1}(\xi_1) \cdots \widehat{f_{m-1}}(\xi_{m-1}) \widehat{f_m}(\xi - \sum_{l=1}^{m-1} \xi_l) d\xi_1 \cdots d\xi_{m-1}.$$

Similarly, the Fourier transform of $T_\epsilon(f_1, \dots, f_m)$ is

$$\int_{\mathbb{R}^{n(m-1)}} \sigma^\epsilon(\xi_1, \dots, \xi_{m-1}, \xi - \sum_{l=1}^{m-1} \xi_l) \widehat{f_1}(\xi_1) \cdots \widehat{f_{m-1}}(\xi_{m-1}) \widehat{f_m}(\xi - \sum_{l=1}^{m-1} \xi_l) d\xi_1 \cdots d\xi_{m-1}.$$

We now claim that σ^ϵ converges pointwise to σ . Take this claim for granted, we have

$$\left(T_\epsilon(f_1, \dots, f_m) \right)^\wedge(\xi) \longrightarrow \left(T_\sigma(f_1, \dots, f_m) \right)^\wedge(\xi), \quad \epsilon \rightarrow 0$$

for a.e. $\xi \in \mathbb{R}^n$. Notice that

$$\|T_\epsilon(f_1, \dots, f_m) - T_\sigma(f_1, \dots, f_m)\|_{L^2} = \left\| \left(T_\epsilon(f_1, \dots, f_m) \right)^\wedge - \left(T_\sigma(f_1, \dots, f_m) \right)^\wedge \right\|_{L^2}.$$

Since $\|\sigma^\epsilon\|_{L^\infty} \lesssim \|\sigma\|_{L^\infty} < \infty$ for all $\epsilon > 0$, Lebesgue's dominated convergence theorem implies that

$$\left(T_\epsilon(f_1, \dots, f_m) \right)^\wedge \longrightarrow \left(T_\sigma(f_1, \dots, f_m) \right)^\wedge \quad \text{as } \epsilon \rightarrow 0$$

in L^2 , and this establishes (3.2).

It remains to prove the above claim about pointwise convergence of σ^ϵ as $\epsilon \rightarrow 0$. Now fix $j_0 \in \mathbb{Z}$, we want to show that $\sigma^\epsilon(x) \rightarrow \sigma(x)$ for a.e. $2^{j_0} \leq |x| \leq 2^{j_0+1}$. Indeed, let $0 < \epsilon < \min\{2^{2j_0-2}, 2^{-|j_0|-2}\}$ be an arbitrarily small positive number. Then we have

$$\begin{aligned} |\sigma^\epsilon(x) - \sigma(x)| &\leq \int_{|y| \leq \sqrt{\epsilon}} |\varphi_\epsilon(y)| |\sigma(x-y)| \sup_{2^{j_0} \leq |x| \leq 2^{j_0+1}} |\phi^\epsilon(x-y) - 1| dy \\ &\quad + \int_{|y| \leq \sqrt{\epsilon}} |\varphi_\epsilon(y)| |\sigma(x-y) - \sigma(x)| dy \\ &\quad + \int_{|y| > \sqrt{\epsilon}} |\varphi_\epsilon(y)| |\sigma(x-y) \phi^\epsilon(x-y) - \sigma(x)| dy. \end{aligned}$$

The first integral vanishes since $\phi^\epsilon(x) = 1$ for all $2\epsilon \leq |x| \leq \frac{1}{\epsilon}$. To estimate the second integral, we denote

$$\widehat{\Psi}(x) = \sum_{|l| \leq 2} \widehat{\psi}(2^{-l}x).$$

Then $\widehat{\Psi}(x) = 1$ for all $\frac{1}{4} \leq |x| \leq 4$. Therefore we have

$$\widehat{\Psi}(2^{-j_0}(x-y)) = \widehat{\Psi}(2^{-j_0}x) = 1$$

for all $2^{j_0} \leq |x| \leq 2^{j_0+1}$ and $|y| \leq 2^{j_0-1}$. Now recall $\sigma_j(x) = \sigma(2^j x)$ and estimate

$$\begin{aligned} &\int_{|y| \leq \sqrt{\epsilon}} |\varphi_\epsilon(y)| |\sigma(x-y) - \sigma(x)| dy \\ &= \int_{|y| \leq \sqrt{\epsilon}} |\varphi_\epsilon(y)| \left| \sigma(x-y) \widehat{\Psi}(2^{-j_0}(x-y)) - \sigma(x) \widehat{\Psi}(2^{-j_0}x) \right| dy \end{aligned}$$

$$\begin{aligned}
&\leq \|\varphi\|_{L^1} \sup_{|y| \leq \sqrt{\epsilon}} \left\| \sigma(\cdot - y) \widehat{\Psi}(2^{-j_0}(\cdot - y)) - \sigma \widehat{\Psi}(2^{-j_0} \cdot) \right\|_{L^\infty} \\
&\leq \|\varphi\|_{L^1} \sum_{j=j_0-2}^{j_0+2} \sup_{|y| \leq \sqrt{\epsilon}} \left\| (\sigma_j \widehat{\psi})(\cdot - 2^{-j}y) - (\sigma_j \widehat{\psi}) \right\|_{L^\infty} \\
&= \|\varphi\|_{L^1} \sum_{j=j_0-2}^{j_0+2} \sup_{|y| \leq 2^{-j} \sqrt{\epsilon}} \left\| (\sigma_j \widehat{\psi})(\cdot - y) - (\sigma_j \widehat{\psi}) \right\|_{L^\infty}.
\end{aligned}$$

We would like to show

$$\lim_{\epsilon \rightarrow 0} \sup_{|y| \leq 2^{-j} \sqrt{\epsilon}} \left\| (\sigma_j \widehat{\psi})(\cdot - y) - (\sigma_j \widehat{\psi}) \right\|_{L^\infty} = 0.$$

The preceding limit apparently converges to 0 as $\epsilon \rightarrow 0$ because $\sigma_j \widehat{\psi} \in W^{(s_1, \dots, s_m)}$ for $s_k > \frac{n}{2}$, $1 \leq k \leq m$. The last term is majorized by

$$C \|\sigma\|_{L^\infty} \int_{|y| \geq \frac{1}{\sqrt{\epsilon}}} |\varphi(y)| dy,$$

which tends to 0 when $\epsilon \rightarrow 0$.

Thus $\sigma^\epsilon(x) \rightarrow \sigma(x)$ as $\epsilon \rightarrow 0$ for a.e. $2^{j_0} \leq |x| \leq 2^{j_0+1}$. Hence, σ^ϵ converges to σ pointwise on \mathbb{R}^{mn} . Also $\|\sigma^\epsilon\|_{L^\infty(\mathbb{R}^{mn})} \lesssim \|\sigma\|_{L^\infty(\mathbb{R}^{mn})}$ uniformly for all $\epsilon > 0$. The proof of Theorem 3.1 is now complete. \square

For a Schwartz function K we denote the multilinear singular integral operator of convolution type associated with the kernel K by

$$T^K(f_1, \dots, f_m)(x) = \int_{\mathbb{R}^{mn}} K(x - y_1, \dots, x - y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m.$$

Proposition 3.3. *Let K be a smooth function on \mathbb{R}^{mn} with compact support. Then we have*

$$\|T^K\|_{H^{p_1} \times \dots \times H^{p_m} \rightarrow L^p} \leq C_K < \infty$$

for all $0 < p_1, \dots, p_m, p < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, where T^K is the multilinear singular integral operator of convolution type associated with the kernel K .

Proof. The boundedness of the operator T^K can be deduced from [5, Lemma 4.2], which provides the estimate (for some sufficiently large integer N)

$$|T^K(f_1, \dots, f_m)(x)| \lesssim \prod_{k=1}^m \mathcal{M}_N(f_k)(x), \quad (3.3)$$

for all $f_k \in L^2 \cap H^{p_k}$, in which

$$\mathcal{M}_N(f)(x) = \sup_{\varphi \in \mathfrak{F}_N} \sup_{t > 0} \sup_{y \in B(x, t)} |(\varphi_t * f)(y)|$$

is the grand maximal function with respect to N , and

$$\mathfrak{F}_N := \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |x|)^N \sum_{|\alpha| \leq N+1} |\partial^\alpha \varphi(x)| dx \leq 1 \right\}.$$

Taking the L^p quasinorm, applying Holder's inequality to (3.3), and using the quasi-norm equivalence of some maximal functions [6, Theorem 6.4.4] yields

$$\|T^K(f_1, \dots, f_m)\|_{L^p} \lesssim \prod_{k=1}^m \|\mathcal{M}_N(f_k)\|_{L^{p_k}} \leq C_K \prod_{k=1}^m \|f_k\|_{H^{p_k}}.$$

□

Working with smooth kernels K with compact support comes handy when dealing with infinite sums of atoms, since we are able to freely interchange the action of T^K with infinite sums of atoms. Precisely, a consequence of the boundedness of T^K , given in Proposition 3.3, is the following result.

Proposition 3.4. *Let $0 < p_1, \dots, p_m \leq 1$ and let K be a smooth function with compact support. Then for every $f_k \in H^{p_k}$ with atomic representation $f_k = \sum_{j_k} \lambda_{k,j_k} a_{k,j_k}$, where a_{k,j_k} are L^∞ -atoms for H^{p_k} and $\sum_{j_k} |\lambda_{k,j_k}|^{p_k} \leq 2^{p_k} \|f_k\|_{H^{p_k}}^{p_k}$ for $1 \leq k \leq m$. Then*

$$T^K(f_1, \dots, f_m)(x) = \sum_{j_1} \cdots \sum_{j_m} \lambda_{1,j_1} \cdots \lambda_{m,j_m} T^K(a_{1,j_1}, \dots, a_{m,j_m})(x)$$

for a.e. $x \in \mathbb{R}^n$.

Proof. Let $0 < p < \infty$ be number such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. For any positive integers N_1, \dots, N_m , Proposition 3.3 gives the estimate

$$\begin{aligned} & \left\| T^K(f_1, \dots, f_m) - \sum_{j_1=1}^{N_1} \cdots \sum_{j_m=1}^{N_m} \lambda_{1,j_1} \cdots \lambda_{m,j_m} T^K(a_{1,j_1}, \dots, a_{m,j_m}) \right\|_{L^p} \\ & \leq C_K \sum_{k=1}^m \left\| f_k - \sum_{j_k=1}^{N_k} \lambda_{k,j_k} a_{k,j_k} \right\|_{H^{p_k}} \prod_{l \neq k} \|f_l\|_{H^{p_l}}. \end{aligned}$$

Now passing to the limit, we obtain

$$T^K(f_1, \dots, f_m)(x) = \sum_{j_1=1}^{\infty} \cdots \sum_{j_m=1}^{\infty} \lambda_{1,j_1} \cdots \lambda_{m,j_m} T^K(a_{1,j_1}, \dots, a_{m,j_m})(x)$$

for a.e. $x \in \mathbb{R}^n$. □

4. THE PROOF OF THE MAIN RESULT

In this section, we prove the main theorem. To do so, we first consider the case where σ is smooth such that its Fourier's transform is compactly supported, then, by regularization, we can improve the result for any multiplier σ in general.

We now start the proof of Theorem 1.1.

Proof of the main theorem. By regularization, we may assume that the inverse Fourier transform of σ is smooth and compactly supported. If this case is established, then Theorem 3.1 yields the existence of a family of multilinear multiplier operators $(T_\epsilon)_{0 < \epsilon < \frac{1}{2}}$ associated with a family of multipliers $(\sigma^\epsilon)_{0 < \epsilon < \frac{1}{2}}$ such that $K^\epsilon = (\sigma^\epsilon)^\vee$ are smooth functions with compact supports for all $0 < \epsilon < \frac{1}{2}$, and that (3.1), (3.2)

hold. Fix $f_k \in H^{p_k} \cap L^{2m}$, ($1 \leq k \leq m$). The L^2 convergence in (3.2) implies that we can find a sequence of positive numbers $(\epsilon_j)_j$ convergent to 0 such that

$$\lim_{j \rightarrow \infty} T_{\epsilon_j}(f_1, \dots, f_m)(x) = T_\sigma(f_1, \dots, f_m)(x)$$

for a.e. $x \in \mathbb{R}^n$. Fatou's lemma connecting with (3.1) gives us

$$\begin{aligned} \|T_\sigma(f_1, \dots, f_m)\|_{L^p} &\leq \liminf_{j \rightarrow \infty} \|T_{\epsilon_j}(f_1, \dots, f_m)\|_{L^p} \\ &\lesssim \sup_{0 < \epsilon < \frac{1}{2}} \|T_\epsilon(f_1, \dots, f_m)\|_{L^p} \\ &\lesssim \sup_{0 < \epsilon < \frac{1}{2}} \sup_{j \in \mathbb{Z}} \left\| \sigma^\epsilon(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} \|f_1\|_{H^{p_1}} \cdots \|f_m\|_{H^{p_m}} \\ &\lesssim \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} \|f_1\|_{H^{p_1}} \cdots \|f_m\|_{H^{p_m}}, \end{aligned}$$

thus establishing the claimed estimate for a general multiplier σ .

In view of this deduction, we suppose σ^\vee is smooth and compactly supported. The aim is to show that

$$\|T_\sigma(f_1, \dots, f_m)\|_{L^p} \lesssim \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} \|f_1\|_{H^{p_1}} \cdots \|f_m\|_{H^{p_m}}. \quad (4.1)$$

Fix functions $f_k \in H^{p_k}$. Using atomic representations for H^{p_k} -functions, write

$$f_k = \sum_{j_k \in \mathbb{Z}} \lambda_{k, j_k} a_{k, j_k}, \quad (1 \leq k \leq m),$$

where a_{k, j_k} are L^∞ -atoms for H^{p_k} satisfying

$$\text{supp}(a_{k, j_k}) \subset Q_{k, j_k}, \quad \|a_{k, j_k}\|_{L^\infty} \leq |Q_{k, j_k}|^{-\frac{1}{p_k}}, \quad \int_{Q_{k, j_k}} x^\alpha a_{k, j_k}(x) dx = 0$$

for all $|\alpha|$ large enough, and $\sum_{j_k} |\lambda_{k, j_k}|^{p_k} \leq 2^{p_k} \|f_k\|_{H^{p_k}}^{p_k}$.

For the cube Q , denote by Q^* the dilation of the cube Q with factor $2\sqrt{n}$. Since $K = \sigma^\vee$ is smooth and compactly supported, Proposition 3.4 yields that

$$T_\sigma(f_1, \dots, f_m)(x) = \sum_{j_1} \cdots \sum_{j_m} \lambda_{1, j_1} \cdots \lambda_{m, j_m} T_\sigma(a_{1, j_1}, \dots, a_{m, j_m})(x)$$

for a.e. $x \in \mathbb{R}^n$. Now we can split $T_\sigma(f_1, \dots, f_m)$ into two parts and estimate

$$|T_\sigma(f_1, \dots, f_m)(x)| \leq G_1(x) + G_2(x),$$

where

$$G_1(x) = \sum_{j_1} \cdots \sum_{j_m} |\lambda_{1, j_1}| \cdots |\lambda_{m, j_m}| |T_\sigma(a_{1, j_1}, \dots, a_{m, j_m})| \chi_{Q_{1, j_1}^* \cap \dots \cap Q_{m, j_m}^*}(x)$$

and

$$G_2(x) = \sum_{j_1} \cdots \sum_{j_m} |\lambda_{1, j_1}| \cdots |\lambda_{m, j_m}| |T_\sigma(a_{1, j_1}, \dots, a_{m, j_m})| \chi_{(Q_{1, j_1}^* \cap \dots \cap Q_{m, j_m}^*)^c}(x).$$

First we estimate the L^p -norm of G_1 , in which we repeat the arguments in [7] for the sake of completeness. Without loss of generality, suppose $Q_{1, j_1}^* \cap \dots \cap Q_{m, j_m}^* \neq \emptyset$ and Q_{1, j_1} has the smallest length among $Q_{1, j_1}, \dots, Q_{m, j_m}$. Since Q_{k, j_k}^* , $1 \leq k \leq m$, have the non-empty intersection, we can pick a cube R_{j_1, \dots, j_m} such that

$$Q_{1, j_1}^* \cap \dots \cap Q_{m, j_m}^* \subset R_{j_1, \dots, j_m} \subset R_{j_1, \dots, j_m}^* \subset Q_{1, j_1}^\sharp \cap \dots \cap Q_{m, j_m}^\sharp$$

and $|Q_{1,j_1}| \lesssim |R_{j_1, \dots, j_m}|$, where the implicit constant depends only on n and Q_{k,j_k}^\sharp denotes for a suitable dilation of Q_{k,j_k} . For $s_k > n/2$, it was showed in [8] that

$$\|T_\sigma\|_{L^2 \times L^\infty \times \dots \times L^\infty \rightarrow L^2} \lesssim A.$$

Therefore, by the Cauchy-Schwarz inequality we have

$$\begin{aligned} \int_{R_{j_1, \dots, j_m}} |T_\sigma(a_{1,j_1}, \dots, a_{m,j_m})(x)| dx &\leq \|T_\sigma(a_{1,j_1}, \dots, a_{m,j_m})\|_{L^2} |R_{j_1, \dots, j_m}|^{\frac{1}{2}} \\ &\lesssim A |R_{j_1, \dots, j_m}|^{\frac{1}{2}} \|a_{1,j_1}\|_{L^2} \prod_{k=2}^m \|a_{k,j_k}\|_{L^\infty} \\ &\lesssim A |R_{j_1, \dots, j_m}|^{\frac{1}{2}} |Q_{1,j_1}|^{\frac{1}{2}} \prod_{k=1}^m |Q_{k,j_k}|^{-\frac{1}{p_k}} \\ &\lesssim A |R_{j_1, \dots, j_m}| \prod_{k=1}^m |Q_{k,j_k}|^{-\frac{1}{p_k}}. \end{aligned}$$

The last inequality implies that

$$\frac{1}{|R_{j_1, \dots, j_m}|} \int_{R_{j_1, \dots, j_m}} |T_\sigma(a_{1,j_1}, \dots, a_{m,j_m})(x)| dx \lesssim A \prod_{k=1}^m |Q_{k,j_k}|^{-\frac{1}{p_k}}.$$

Now the trivial estimate

$$G_1(x) \leq \sum_{j_1} \dots \sum_{j_m} |\lambda_{1,j_1}| \dots |\lambda_{m,j_m}| |T_\sigma(a_{1,j_1}, \dots, a_{m,j_m})| \chi_{R_{j_1, \dots, j_m}}(x)$$

combined with Lemma 2.5 yields

$$\begin{aligned} \|G_1\|_{L^p} &\leq \left\| \sum_{j_1} \dots \sum_{j_m} |\lambda_{1,j_1}| \dots |\lambda_{m,j_m}| |T_\sigma(a_{1,j_1}, \dots, a_{m,j_m})| \chi_{R_{j_1, \dots, j_m}} \right\|_{L^p} \\ &\lesssim A \left\| \sum_{j_1} \dots \sum_{j_m} |\lambda_{1,j_1}| \dots |\lambda_{m,j_m}| \left(\prod_{k=1}^m |Q_{k,j_k}|^{-\frac{1}{p_k}} \right) \chi_{R_{j_1, \dots, j_m}}^* \right\|_{L^p} \\ &\leq A \left\| \sum_{j_1} \dots \sum_{j_m} |\lambda_{1,j_1}| \dots |\lambda_{m,j_m}| \prod_{k=1}^m \left(|Q_{k,j_k}|^{-\frac{1}{p_k}} \chi_{Q_{k,j_k}^\sharp} \right) \right\|_{L^p} \\ &= A \left\| \prod_{k=1}^m \left(\sum_{j_k} |\lambda_{k,j_k}| |Q_{k,j_k}|^{-\frac{1}{p_k}} \chi_{Q_{k,j_k}^\sharp} \right) \right\|_{L^p} \\ &\leq A \prod_{k=1}^m \left\| \sum_{j_k} |\lambda_{k,j_k}| |Q_{k,j_k}|^{-\frac{1}{p_k}} \chi_{Q_{k,j_k}^\sharp} \right\|_{L^{p_k}} \\ &\lesssim A \prod_{k=1}^m \|f_k\|_{H^{p_k}}. \end{aligned}$$

Thus

$$\|G_1\|_{L^p} \lesssim A \|f_1\|_{H^{p_1}} \dots \|f_m\|_{H^{p_m}}. \quad (4.2)$$

Now for the harder part, $G_2(x)$, we first restrict $x \in (\cap_{k \notin J} Q_{k,j_k}^*) \setminus (\cup_{k \in J} Q_{k,j_k}^*)$ for some nonempty subset $J \subset \{1, 2, \dots, m\}$. To continue, we need the following lemma whose proof will be given in the last section.

Lemma 4.1 (The key lemma). *Let $n/2 < s_1, \dots, s_m < \infty$, $0 < p_1, \dots, p_m, p \leq 1$ be numbers and let σ be a function satisfying (1.5) and (1.6). Suppose a_k are atoms supported in the cube Q_k , ($k = 1, \dots, m$) such that*

$$\|a_k\|_{L^\infty} \leq |Q_k|^{-\frac{1}{p_k}}, \quad \int_{Q_k} x^\alpha a_k(x) dx = 0,$$

for all $|\alpha| \leq N_k$ with $N_k = \lceil n(\frac{1}{p_k} - 1) \rceil + 1$. Fix a non-empty subset $J_0 \subset \{1, \dots, m\}$. Then there exist positive functions b_1, \dots, b_m such that

$$|T_\sigma(a_1, \dots, a_m)(x)| \lesssim A b_1(x) \cdots b_m(x) \quad (4.3)$$

for all $x \in (\cap_{k \notin J_0} Q_k^*) \setminus (\cup_{k \in J_0} Q_k^*)$, and $\|b_k\|_{L^{p_k}} \lesssim 1$ for all $1 \leq k \leq m$.

Lemma 4.1 guarantees the existence of positive functions $b_{1,j_1}^J, \dots, b_{m,j_m}^J$ depending on $Q_{1,j_1}, \dots, Q_{m,j_m}$ respectively, such that

$$|T_\sigma(a_{1,j_1}, \dots, a_{m,j_m})| \lesssim A b_{1,j_1}^J \cdots b_{m,j_m}^J \quad (4.4)$$

for all $x \in (\cap_{k \notin J} Q_{k,j_k}^*) \setminus (\cup_{k \in J} Q_{k,j_k}^*)$ and $\|b_{k,j_k}^J\|_{L^{p_k}} \lesssim 1$. Now set

$$b_{k,j_k} = \sum_{\emptyset \neq J \subset \{1, 2, \dots, m\}} b_{k,j_k}^J.$$

Then

$$|T_\sigma(a_{1,j_1}, \dots, a_{m,j_m})| \chi_{(Q_{1,j_1}^* \cap \dots \cap Q_{m,j_m}^*)^c} \lesssim A b_{1,j_1} \cdots b_{m,j_m} \quad (4.5)$$

and $\|b_{k,j_k}\|_{L^{p_k}} \lesssim 1$. Estimate (4.5) yields

$$G_2(x) \lesssim A \prod_{k=1}^m \left(\sum_{j_k} |\lambda_{k,j_k}| b_{k,j_k}(x) \right).$$

Then we apply Hölder's inequality to deduce that

$$\|G_2\|_{L^p} \lesssim A \|f_1\|_{H^{p_1}} \cdots \|f_m\|_{H^{p_m}}. \quad (4.6)$$

Combining (4.2) and (4.6) yields (4.1) as needed. The proof of Theorem 1.1 is now complete. \square

5. MINIMALITY OF CONDITIONS

In this section we will show that conditions (1.5) and $s_k > \frac{n}{2}$ are minimal in general that guarantee boundedness for multilinear multiplier operators. We fix a smooth function ψ whose Fourier transform is supported in $\{2^{-\frac{3}{4}} \leq |\xi| \leq 2^{\frac{3}{4}}\}$, it satisfies $\widehat{\psi}(\xi) = 1$ for all $2^{-\frac{1}{4}} \leq |\xi| \leq 2^{\frac{1}{4}}$, and for some nonzero constant c

$$\sum_{j \in \mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = c, \quad \xi \neq 0.$$

Now we have the following theorem:

Theorem 5.1. *Let $0 < p_k \leq \infty$, $0 < p < \infty$, and $s_k > 0$ for $1 \leq k \leq m$. Suppose that the estimate*

$$\|T_\sigma(f_1, \dots, f_m)\|_{L^p} \lesssim \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} \prod_{k=1}^m \|f_k\|_{H^{p_k}}$$

holds for all $f_k \in H^{p_k}$ and $\sigma \in L^\infty$ such that $\sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} < \infty$.

The following conditions are then necessary:

$$s_k \geq \frac{n}{2}, \quad \forall 1 \leq k \leq m, \quad (5.1)$$

and

$$\sum_{k \in J} \left(\frac{s_k}{n} - \frac{1}{p_k} \right) \geq -\frac{1}{2} \quad (5.2)$$

for every nonempty subset $J \subset \{1, \dots, m\}$.

The following lemma is obvious by changing variables, so its proof is omitted.

Lemma 5.2. *Let φ be a nontrivial Schwartz function and $s > 0$. Then*

$$\left(\int |\varphi(\epsilon y)|^2 (1 + |y|^2)^s dy \right)^{\frac{1}{2}} \approx \epsilon^{-\frac{n}{2} - s}$$

for all $0 < \epsilon \leq 1$.

Proof of Theorem 5.1. We show first the necessary conditions (5.1) for $1 \leq k \leq m$. Without loss of generality, we will show $s_1 \geq \frac{n}{2}$. To establish this inequality, we need to construct some functions σ^ϵ , ($0 < \epsilon \ll 1$), and $f_k \in H^{p_k}$ such that $\|f_k\|_{H^{p_k}} = 1$ for all $1 \leq k \leq m$, and $\|T_{\sigma^\epsilon}(f_1, \dots, f_m)\|_{L^p} \approx 1$, and further that

$$\sup_{j \in \mathbb{Z}} \left\| \sigma^\epsilon(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} \lesssim \epsilon^{\frac{n}{2} - s_1}.$$

Once these functions are constructed, one observes that

$$1 \approx \|T_{\sigma^\epsilon}(f_1, \dots, f_m)\|_{L^p} \lesssim \sup_{j \in \mathbb{Z}} \left\| \sigma^\epsilon(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} \prod_{k=1}^m \|f_k\|_{H^{p_k}} \lesssim \epsilon^{\frac{n}{2} - s_1}$$

for all $0 < \epsilon \ll 1$. Therefore we must have $s_1 \geq \frac{n}{2}$.

Let φ be a nontrivial Schwartz function such that $\widehat{\varphi}$ is supported in the unit ball, and let $\phi_2 = \dots = \phi_{m-1}$ be Schwartz functions whose Fourier transforms, $\widehat{\phi}_2$, is supported in an annulus $\frac{1}{17m} \leq |\xi| \leq \frac{1}{13m}$, and identical to 1 on $\frac{1}{16m} \leq |\xi| \leq \frac{1}{14m}$. Similarly, fix a Schwartz function ϕ_m with $\widehat{\phi}_m \subset \{\xi \in \mathbb{R}^n : \frac{12}{13} \leq |\xi| \leq \frac{14}{13}\}$ and $\widehat{\phi}_m \equiv 1$ on an annulus $\frac{25}{26} \leq |\xi| \leq \frac{27}{26}$. Take $a, b \in \mathbb{R}^n$ with $|a| = \frac{1}{15m}$ and $|b| = 1$.

For $0 < \epsilon < \frac{1}{240m}$, set

$$\sigma^\epsilon(\xi_1, \dots, \xi_m) = \widehat{\varphi}\left(\frac{\xi_1 - a}{\epsilon}\right) \widehat{\phi}_2(\xi_2) \cdots \widehat{\phi}_m(\xi_m).$$

It is easy to check that $\text{supp } \sigma^\epsilon \subset \{2^{-\frac{1}{4}} \leq |\xi| \leq 2^{\frac{1}{4}}\}$; hence, $\sigma^\epsilon(2^j \cdot) \widehat{\psi} = \sigma^\epsilon$ for $j = 0$ and $\sigma^\epsilon(2^j \cdot) \widehat{\psi} = 0$ for $j \neq 0$. This directly implies that

$$\sup_{j \in \mathbb{Z}} \left\| \sigma^\epsilon(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} = \|\sigma^\epsilon\|_{W^{(s_1, \dots, s_m)}}.$$

Taking the inverse Fourier transform of σ^ϵ gives

$$(\sigma^\epsilon)^\vee(x_1, \dots, x_m) = \epsilon^n e^{2\pi i a \cdot x_1} \varphi(\epsilon x_1) \phi_2(x_2) \cdots \phi_m(x_m).$$

Now apply Lemma 5.2 to have

$$\|\sigma^\epsilon\|_{W^{(s_1, \dots, s_m)}} \lesssim \epsilon^{\frac{n}{2} - s_1}.$$

Thus

$$\sup_{j \in \mathbb{Z}} \left\| \sigma^\epsilon(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} \lesssim \epsilon^{\frac{n}{2} - s_1}.$$

Now choose $\widehat{f}_k(\xi) = \epsilon^{\frac{n}{p_k} - n} \widehat{\varphi}\left(\frac{\xi - a}{\epsilon}\right)$ for $1 \leq k \leq m-1$, and $\widehat{f}_m(\xi) = \epsilon^{\frac{n}{p_m} - n} \widehat{\varphi}\left(\frac{\xi - b}{\epsilon}\right)$. Then we will show that these functions are what we needed to construct.

In the following estimates, we will use the fact, its proof can be done by using the Littlewood-Paley characterization for Hardy spaces, that if f is a function whose Fourier transform is supported in a fixed annulus centered at the origin, then $\|f\|_{H^p} \approx \|f\|_{L^p}$ for $0 < p < \infty$, (cf. [4, Remark 7.1]).

Indeed, using the above fact and checking that each \widehat{f}_k is supported in an annulus centered at zero and not depending on ϵ allow us to estimate H^p -norms via L^p -norms, namely

$$\|f_k\|_{H^{p_k}} \approx \|f_k\|_{L^{p_k}} = 1, \quad (1 \leq k \leq m).$$

Thus, we are left with showing that $\|T_{\sigma^\epsilon}(f_1, \dots, f_m)\|_{L^p} \approx 1$. Notice that $\widehat{\phi}_k(\xi) = 1$ on the support of \widehat{f}_k for $2 \leq k \leq m$. Therefore,

$$\begin{aligned} T_{\sigma^\epsilon}(f_1, \dots, f_m)(x) &= \left(\widehat{\varphi}\left(\frac{\cdot - a}{\epsilon}\right) \epsilon^{\frac{n}{p_1} - n} \widehat{\varphi}\left(\frac{\cdot - a}{\epsilon}\right) \right)^\vee(x) \left(\widehat{\phi}_2 \widehat{f}_2 \right)^\vee(x) \cdots \left(\widehat{\phi}_m \widehat{f}_m \right)^\vee(x) \\ &= \left(\widehat{\varphi}\left(\frac{\cdot - a}{\epsilon}\right) \epsilon^{\frac{n}{p_1} - n} \widehat{\varphi}\left(\frac{\cdot - a}{\epsilon}\right) \right)^\vee(x) \left(\widehat{f}_2 \right)^\vee(x) \cdots \left(\widehat{f}_m \right)^\vee(x) \\ &= \epsilon^{\frac{n}{p_1} + \dots + \frac{n}{p_m}} e^{2\pi i[(m-1)a+b] \cdot x} (\varphi * \varphi)(\epsilon x) [\varphi(\epsilon x)]^{m-1} \\ &= \epsilon^{\frac{n}{p}} e^{2\pi i[(m-1)a+b] \cdot x} (\varphi * \varphi)(\epsilon x) [\varphi(\epsilon x)]^{m-1}, \end{aligned}$$

which obviously gives $\|T_{\sigma^\epsilon}(f_1, \dots, f_m)\|_{L^p} \approx 1$. So far, we have proved that $s_1 \geq \frac{n}{2}$; hence, by symmetry, we have $s_k \geq \frac{n}{2}$ for all $1 \leq k \leq m$.

It now remains to show (5.2). By symmetry, we just only need to prove that

$$\sum_{k=1}^r \left(\frac{s_k}{n} - \frac{1}{p_k} \right) \geq -\frac{1}{2} \quad (5.3)$$

for some fixed $1 \leq r \leq m$. To achieve our goal, we construct a multiplier σ^ϵ such that

$$\sup_{j \in \mathbb{Z}} \left\| \sigma^\epsilon(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} \lesssim \epsilon^{\frac{n}{2} - s_1 - \dots - s_r}$$

for $0 < \epsilon \ll 1$ and functions f_k satisfying $\|f_k\|_{H^{p_k}} \approx 1$ for $1 \leq k \leq m$ and

$$\|T_{\sigma^\epsilon}(f_1, \dots, f_m)\|_{L^p} \approx \epsilon^{n - \frac{n}{p_1} - \dots - \frac{n}{p_r}}.$$

Then inequalities

$$\begin{aligned} \epsilon^{n - \frac{n}{p_1} - \dots - \frac{n}{p_r}} &\approx \|T_{\sigma^\epsilon}(f_1, \dots, f_m)\|_{L^p} \\ &\leq \sup_{j \in \mathbb{Z}} \left\| \sigma^\epsilon(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} \prod_{k=1}^m \|f_k\|_{H^{p_k}} \\ &\lesssim \epsilon^{\frac{n}{2} - s_1 - \dots - s_r}, \end{aligned}$$

for all small positive numbers ϵ yield (5.3).

We construct functions that give us enough ingredients to establish the multiplier σ^ϵ and functions f_k , ($1 \leq k \leq m$), as mentioned above. Take two smooth functions

φ, ϕ such that $\varphi(0) \neq 0$, $\widehat{\varphi}$ is supported in $\{\xi \in \mathbb{R}^n : |\xi| \leq \frac{1}{19mr}\}$ and $\widehat{\varphi}(\xi) = 1$ for all $|\xi| \leq \frac{1}{30mr}$, and that $\widehat{\phi}$ is supported in an annulus $\frac{1}{23m} \leq |\xi| \leq \frac{1}{19m}$ and $\widehat{\phi}(\xi) = 1$ for all $\frac{1}{22m} \leq |\xi| \leq \frac{1}{20m}$. Fix $a, b \in \mathbb{R}^n$ such that $|a| = r^{-\frac{1}{2}}$, $|b| = \frac{1}{21m}$.

For $0 < \epsilon < \frac{1}{462m}$, for $r > 1$, define

$$\begin{aligned} & \sigma^\epsilon(\xi_1, \dots, \xi_m) \\ &= \widehat{\varphi}\left(\frac{1}{r\epsilon} \sum_{l=1}^r (\xi_l - a)\right) \widehat{\varphi}\left(\frac{1}{r} \sum_{l=1}^r (\xi_l - \xi_2)\right) \cdots \widehat{\varphi}\left(\frac{1}{r} \sum_{l=1}^r (\xi_l - \xi_r)\right) \widehat{\phi}(\xi_{r+1}) \cdots \widehat{\phi}(\xi_m). \end{aligned}$$

Note that in the case $r = 1$ we will replace the above by the following function

$$\sigma^\epsilon(\xi_1, \dots, \xi_m) = \widehat{\varphi}\left(\frac{1}{\epsilon}(\xi_1 - a)\right) \widehat{\phi}(\xi_2) \cdots \widehat{\phi}(\xi_m).$$

Once again, we have $\text{supp } \sigma^\epsilon \subset \{\xi \in \mathbb{R}^n : 2^{-\frac{1}{4}} \leq |\xi| \leq 2^{\frac{1}{4}}\}$, which, as in the previous case, implies that

$$\sup_{j \in \mathbb{Z}} \left\| \sigma^\epsilon(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} = \|\sigma^\epsilon\|_{W^{(s_1, \dots, s_m)}}.$$

We give a detailed computations for the inverse Fourier transform of σ^ϵ . Note that we have to consider $r > 1$ for this case. The case $r = 1$ needs to be dealt separately with a small modification. Indeed, we have

$$\begin{aligned} (\sigma^\epsilon)^\vee(x_1, \dots, x_m) &= \int \widehat{\varphi}\left(\frac{1}{r\epsilon} \sum_{l=1}^r (\xi_l - a)\right) \widehat{\varphi}\left(\frac{1}{r} \sum_{l=1}^r (\xi_l - \xi_2)\right) \cdots \widehat{\varphi}\left(\frac{1}{r} \sum_{l=1}^r (\xi_l - \xi_r)\right) \times \\ &\quad \times \widehat{\phi}(\xi_{r+1}) \cdots \widehat{\phi}(\xi_m) e^{2\pi i(x_1 \xi_1 + \cdots + x_r \xi_r + x_{r+1} \xi_{r+1} + \cdots + x_m \xi_m)} d\xi_1 \cdots d\xi_m \end{aligned}$$

Now we set

$$\begin{cases} y_1 = & \frac{1}{r} \sum_{l=1}^r (\xi_l - a), \\ y_2 = & \frac{1}{r} \sum_{l=1}^r (\xi_l - \xi_2), \\ \dots & \dots \\ y_r = & \frac{1}{r} \sum_{l=1}^r (\xi_l - \xi_r), \\ y_{r+1} = & \xi_{r+1}, \\ \dots & \dots \\ y_m = & \xi_m. \end{cases}$$

Solving the above system of equations, we obtain

$$\begin{cases} \xi_1 = & y_1 + \cdots + y_r + a, \\ \xi_2 = & y_1 - y_2 + a, \\ \dots & \dots \\ \xi_r = & y_1 - y_r + a, \\ \xi_{r+1} = & y_{r+1}, \\ \dots & \dots \\ \xi_m = & y_m. \end{cases}$$

The Jacobian of this changing variables is r and then we have

$$\begin{aligned} (\sigma^\epsilon)^\vee(x_1, \dots, x_m) &= r \int \widehat{\varphi}\left(\frac{1}{\epsilon}y_1\right) \widehat{\varphi}(y_2) \cdots \widehat{\varphi}(y_r) \widehat{\phi}(y_{r+1}) \cdots \widehat{\phi}(y_m) e^{2\pi i y_1(x_1 + \cdots + x_r)} \\ &\quad \times e^{2\pi i[(y_2(x_1 - x_2) + \cdots + y_r(x_1 - x_r) + x_{r+1}y_{r+1} + \cdots + x_m y_m)]} \\ &\quad \times e^{2\pi i a(x_1 + \cdots + x_r)} dy_1 \cdots dy_m \end{aligned}$$

By changing variables above, we can obtain the inverse Fourier transform of σ^ϵ as follows

$$\begin{aligned} (\sigma^\epsilon)^\vee(x_1, \dots, x_m) &= r e^{2\pi i a \sum_{l=1}^r x_l} \epsilon^n \varphi\left(\epsilon \sum_{l=1}^r x_l\right) \varphi(x_1 - x_2) \cdots \varphi(x_1 - x_r) \phi(x_{r+1}) \cdots \phi(x_m). \end{aligned}$$

Taking Sobolev norm deduces

$$\|\sigma^\epsilon\|_{W^{(s_1, \dots, s_m)}} = C \epsilon^n \left(\int_{\mathbb{R}^{nr}} \left| \varphi\left(\epsilon \sum_{l=1}^r x_l\right) \prod_{l=2}^r \varphi(x_1 - x_l) \right|^2 \prod_{l=1}^r (1 + |x_l|^2)^{s_l} dx_1 \cdots dx_r \right)^{\frac{1}{2}},$$

where $C = r \|\phi\|_{W^{s_{r+1}}} \cdots \|\phi\|_{W^{s_m}}$.

Next, we show that

$$\begin{aligned} &\int_{\mathbb{R}^{nr}} \left| \varphi\left(\epsilon \sum_{l=1}^r x_l\right) \varphi(x_1 - x_2) \cdots \varphi(x_1 - x_r) \right|^2 \prod_{l=1}^r (1 + |x_l|^2)^{s_l} dx_1 \cdots dx_r \\ &\lesssim \epsilon^{-n-2(s_1 + \cdots + s_r)}. \end{aligned} \quad (5.4)$$

In fact, changing variables in the above integral together with Lemma 5.2 yields

$$\begin{aligned} &\int_{\mathbb{R}^{nr}} \left| \varphi\left(\epsilon \sum_{l=1}^r x_l\right) \varphi(x_1 - x_2) \cdots \varphi(x_1 - x_r) \right|^2 \prod_{l=1}^r (1 + |x_l|^2)^{s_l} dx_1 \cdots dx_r \\ &= \frac{1}{r} \int_{\mathbb{R}^{nr}} |\varphi(\epsilon y_1) \varphi(y_2) \cdots \varphi(y_r)|^2 \left(1 + \frac{1}{r^2} \left|\sum_{l=1}^r y_l\right|^2\right)^{s_1} \\ &\quad \prod_{l=2}^r \left(1 + \left| -y_l + \frac{1}{r} \sum_{l=1}^r y_l \right|^2\right)^{s_l} dy_1 \cdots dy_r \\ &\lesssim \int_{\mathbb{R}^{nr}} |\varphi(\epsilon y_1) \varphi(y_2) \cdots \varphi(y_r)|^2 \prod_{l=1}^r (1 + |y_l|^2)^{s_1 + \cdots + s_r} dy_1 \cdots dy_r \\ &\lesssim \int_{\mathbb{R}^{nr}} |\varphi(\epsilon y_1)|^2 (1 + |y_1|^2)^{s_1 + \cdots + s_r} dy_1 \\ &\lesssim \epsilon^{-n-2s_1 - \cdots - 2s_r}, \end{aligned}$$

where the implicit constants do not depend on ϵ . Inequality (5.4) gives us

$$\sup_{j \in \mathbb{Z}} \left\| \sigma^\epsilon(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} = \|\sigma^\epsilon\|_{W^{(s_1, \dots, s_m)}} \lesssim \epsilon^{\frac{n}{2} - s_1 - \cdots - s_r}.$$

To construct functions f_k , we fix a smooth function ζ such that $\widehat{\zeta}$ is supported in $\{\xi \in \mathbb{R}^n : |\xi - a| \leq \frac{1}{3m}\}$ and is identical to 1 on $\{\xi \in \mathbb{R}^n : |\xi - a| \leq \frac{3}{19m}\}$. Now

set $f_1 = \dots = f_r = \zeta$ and $\widehat{f}_k(\xi) = \epsilon^{\frac{n}{p_k} - n} \widehat{\varphi}(\frac{\xi - b}{\epsilon})$ for $r + 1 \leq k \leq m$. It is clear that

$$\|f_k\|_{H^{p_k}} \approx \|f_k\|_{L^{p_k}} \approx 1, \quad 1 \leq k \leq m.$$

Moreover, $\widehat{f}_1(\xi_1) \cdots \widehat{f}_r(\xi_r) = 1$ on the support of the function

$$\widehat{\varphi}\left(\frac{1}{r\epsilon} \sum_{l=1}^r (\xi_l - a)\right) \widehat{\varphi}\left(\frac{1}{r} \sum_{l=1}^r (\xi_l - \xi_2)\right) \cdots \widehat{\varphi}\left(\frac{1}{r} \sum_{l=1}^r (\xi_l - \xi_r)\right)$$

and also $\widehat{\phi}(\xi) = 1$ on the support of the functions \widehat{f}_k for all $r + 1 \leq k \leq m$. Therefore we have

$$T_{\sigma^\epsilon}(f_1, \dots, f_m)(x) = r e^{2\pi i (ra + (m-r)b) \cdot x} \epsilon^n \varphi(\epsilon r x) [\varphi(0)]^{r-1} \epsilon^{\frac{n}{p_{r+1}} + \dots + \frac{n}{p_m}} [\varphi(\epsilon x)]^{m-r}.$$

Take L^p -norm, we get

$$\|T_{\sigma^\epsilon}(f_1, \dots, f_m)\|_{L^p} \approx \epsilon^{n - \frac{n}{p_1} - \dots - \frac{n}{p_r}},$$

which is the last thing we want to obtain for our construction. Notice that the above argument also works for $p_k = \infty$. \square

6. ENDPOINT ESTIMATES

In this section we consider two endpoint estimates for multilinear singular integral operators. In the first case all indices are equal to infinity and in the second case one index is 1 and the others are equal to infinity.

For $x \in \mathbb{R}^n$ and $1 \leq k \leq m$, define

$$\Gamma_x^k = \{(y_1, \dots, y_m) \in \mathbb{R}^{mn} : |y_k| > 2|x|\}.$$

We say that a locally integrable function $K(y_1, \dots, y_m)$ on $\mathbb{R}^{mn} \setminus \{0\}$ satisfies a *coordinate-type Hörmander condition* if for some finite constant A we have

$$\sum_{k=1}^m \int_{\Gamma_x^k} |K(y_1, \dots, y_{k-1}, x - y_k, y_{k+1}, \dots, y_m) - K(y_1, \dots, y_m)| d\vec{y} \leq A \quad (6.1)$$

for all $0 \neq x \in \mathbb{R}^n$. Another type of (bi)-linear Hörmander condition of geometric nature appeared in Pérez and Torres [16].

Denote by $\Lambda_p = \{(p, \infty, \dots, \infty), (\infty, p, \infty, \dots, \infty), \dots, (\infty, \dots, \infty, p)\}$ the set of all m -tuples with $(m-1)$ entries equal to infinity and only one entry equal to p . The following result provides a version of the classical multilinear Calderón-Zygmund theorem in which the kernel satisfies a coordinate-type Hörmander condition under the initial assumption that the operator is bounded on Lebesgue spaces with indices in Λ_2 . We denote by L_c^∞ the space of all compactly supported bounded functions.

Theorem 6.1. *Suppose that an m -linear singular integral operator of convolution type T with kernel K is bounded from $L^{q_1} \times \dots \times L^{q_m}$ to L^2 with norm at most B for all $(q_1, \dots, q_m) \in \Lambda_2$. If K satisfies the coordinate-type Hörmander condition (6.1), then*

$$\|T(f_1, \dots, f_m)\|_{BMO} \lesssim (A + B) \|f_1\|_{L^\infty} \cdots \|f_m\|_{L^\infty} \quad (6.2)$$

for all f_j in L_c^∞ . Moreover, T has a bounded extension which satisfies

$$\|T(f_1, \dots, f_m)\|_{L^{1,\infty}} \lesssim (A + B) \|f_l\|_{L^1} \prod_{\substack{l=1 \\ k \neq l}}^m \|f_k\|_{L^\infty} \quad (6.3)$$

for all $1 \leq k \leq m$, $f_l \in L^1$, and $f_k \in L_c^\infty$ for $k \neq l$.

Proof. Fix a cube Q . To prove (6.2) we show that there exists a constant C_Q such that

$$\frac{1}{|Q|} \int_Q |T(f_1, \dots, f_m)(x) - C_Q| dx \lesssim (A+B) \|f_1\|_{L^\infty} \cdots \|f_m\|_{L^\infty}. \quad (6.4)$$

We decompose each function $f_l = f_l^0 + f_l^1$, where $f_l^0 = f_l \chi_{Q^c}$ and $f_l^1 = f_l \chi_{(Q^*)^c}$. Let F be the set of the 2^m sequences of length m consisting of zeros and ones. We claim that for each sequence $\vec{k} = (k_1, \dots, k_m)$ in F there is a constant $C_{\vec{k}}$ such that

$$\frac{1}{|Q|} \int_Q |T(f_1^{k_1}, \dots, f_m^{k_m})(x) - C_{\vec{k}}| dx \lesssim (A+B) \|f_1\|_{L^\infty} \cdots \|f_m\|_{L^\infty}. \quad (6.5)$$

Assuming the validity of the preceding claim we obtain (6.4) with $C_Q = \sum_{\vec{k} \in F} C_{\vec{k}}$.

Next, we want to establish (6.5) for each $\vec{k} \in F$. If $\vec{k} = (k_1, \dots, k_m)$ has at least one zero entry we pick $C_{\vec{k}} = 0$. Without loss of generality, we may assume that $k_1 = 0$. Since T maps $L^2 \times L^\infty \times \cdots \times L^\infty$ to L^2 , we have

$$\begin{aligned} \frac{1}{|Q|} \int_Q |T(f_1^{k_1}, \dots, f_m^{k_m})(x)| dx &\leq \left(\frac{1}{|Q|} \int_Q |T(f_1^{k_1}, \dots, f_m^{k_m})(x)|^2 dx \right)^{\frac{1}{2}} \\ &\leq \left(\frac{1}{|Q|} \int_{\mathbb{R}^n} |T(f_1^{k_1}, \dots, f_m^{k_m})(x)|^2 dx \right)^{\frac{1}{2}} \\ &\leq B |Q|^{-\frac{1}{2}} \|f_1^0\|_{L^2} \|f_2^{k_2}\|_{L^\infty} \cdots \|f_m^{k_m}\|_{L^\infty} \\ &\leq B |Q|^{-\frac{1}{2}} |Q^*|^{\frac{1}{2}} \|f_1\|_{L^\infty} \cdots \|f_m\|_{L^\infty} \\ &\lesssim B \|f_1\|_{L^\infty} \cdots \|f_m\|_{L^\infty}. \end{aligned}$$

Now suppose that $\vec{k} = (1, \dots, 1)$. Set $C_{\vec{k}} = T(f_1^{k_1}, \dots, f_m^{k_m})(x_Q)$, where x_Q is the center of the cube Q . Then, by the coordinate-type Hörmander condition (6.1), we have

$$\begin{aligned} &\frac{1}{|Q|} \int_Q |T(f_1^1, \dots, f_m^1)(x) - C_{\vec{k}}| dx \\ &\leq \frac{1}{|Q|} \int_Q \int_{\mathbb{R}^{nm}} |K(x-y_1, \dots, x-y_m) - K(x_Q-y_1, \dots, x_Q-y_m)| \prod_{k=1}^m |f_k^1(y_k)| dy dx \\ &\leq \frac{\prod_{k=1}^m \|f_k\|_{L^\infty}}{|Q|} \int_Q \sum_{k=1}^m \int_{\Gamma_{x-x_Q}^k} |K(y_1, \dots, (x-x_Q)-y_k, \dots, y_m) - K(y_1, \dots, y_m)| dy dx \\ &\lesssim A \|f_1\|_{L^\infty} \cdots \|f_m\|_{L^\infty}. \end{aligned}$$

This completes the proof of (6.2) and we are left with establishing (6.3). Fix $\lambda > 0$. It is enough to show that

$$|\{x \in \mathbb{R}^n : |T(f_1, \dots, f_m)(x)| > 2\lambda\}| \lesssim (A+B) \frac{1}{\lambda} \|f_1\|_{L^1} \|f_2\|_{L^\infty} \cdots \|f_m\|_{L^\infty}.$$

By scaling, we may assume that $\|f_1\|_{L^1} = \|f_2\|_{L^\infty} = \cdots = \|f_m\|_{L^\infty} = 1$. Let δ be a positive number chosen later and let $f_1 = g_1 + b_1$ be the Calderón-Zygmund decomposition at height $\delta\lambda$, and $b_1 = \sum_j b_{1,j}$, where $b_{1,j}$ are functions supported in the (pairwise disjoint) cubes Q_j such that

$$\text{supp}(b_{1,j}) \subset Q_j, \quad \int b_{1,j}(x) dx = 0,$$

$$\begin{aligned}\|b_{1,j}\|_{L^1} &\leq 2^{n+1}\delta\lambda|Q_j|, \quad \sum_j |Q_j| \leq \frac{1}{\delta\lambda}, \\ \|g_1\|_{L^\infty} &\leq 2^n\delta\lambda, \quad \|g_1\|_{L^1} \leq 1.\end{aligned}$$

Now we can estimate

$$\begin{aligned}|\{x \in \mathbb{R}^n : |T(f_1, \dots, f_m)(x)| > 2\lambda\}| &\leq |\{x \in \mathbb{R}^n : |T(g_1, \dots, f_m)(x)| > \lambda\}| \\ &\quad + |\{x \in \mathbb{R}^n : |T(b_1, \dots, f_m)(x)| > \lambda\}|.\end{aligned}$$

Since T maps $L^2 \times L^\infty \times \dots \times L^\infty$ to L^2 , the first part can be controlled by

$$\begin{aligned}|\{x \in \mathbb{R}^n : |T(g_1, \dots, f_m)(x)| > \lambda\}| &\leq \frac{1}{\lambda^2} \int_{\mathbb{R}^n} |T(g_1, f_2, \dots, f_m)(x)|^2 dx \\ &\leq \frac{B^2}{\lambda^2} \|g_1\|_{L^2}^2 \|f_2\|_{L^\infty}^2 \cdots \|f_m\|_{L^\infty}^2 \\ &\leq \frac{2^n B^2 \delta}{\lambda}.\end{aligned}$$

To estimate the second part, we set $G = \cup_j Q_j^*$. Then we have

$$\begin{aligned}|\{x \in \mathbb{R}^n : |T(b_1, \dots, f_m)(x)| > \lambda\}| &\leq |G| + |\{x \in G^c : |T(b_1, \dots, f_m)(x)| > \lambda\}| \\ &\leq |G| + \frac{1}{\lambda} \sum_j \int_{(Q_j^*)^c} |T(b_{1,j}, \dots, f_m)(x)| dx.\end{aligned}$$

Notice that

$$|G| \leq \sum_j |Q_j^*| \lesssim \sum_j |Q_j| \leq \frac{1}{\delta\lambda}.$$

Denote by c_j the center of the cube Q_j . Invoking condition (6.1) yields

$$\begin{aligned}&\int_{(Q_j^*)^c} |T(b_{1,j}, \dots, f_m)(x)| dx \\ &\leq \int_{(Q_j^*)^c} \left| \int K(x - y_1, \dots, x - y_m) b_{1,j}(y_1) f_2(y_2) \cdots f_m(y_m) d\vec{y} dx \right| \\ &\leq \int_{(Q_j^*)^c} \left| \int [K(x - y_1, y_2, \dots, y_m) - K(x - c_j, y_2, \dots, y_m)] b_{1,j}(y_1) \prod_{k=2}^m f_k(x - y_k) d\vec{y} dx \right| \\ &\leq \prod_{k=2}^m \|f_k\|_{L^\infty} \int_{(Q_j^*)^c} \int |K(x - y_1, y_2, \dots, y_m) - K(x - c_j, y_2, \dots, y_m)| |b_{1,j}(y_1)| d\vec{y} dx \\ &\leq \int_{Q_j} \left\{ \int_{\Gamma_{y_1 - c_j}^1} |K(y_1 - z_1, z_2, \dots, z_m) - K(z_1, z_2, \dots, z_m)| d\vec{z} \right\} |b_{1,j}(y_1)| dy_1 \\ &\leq A \|b_{1,j}\|_{L^1}.\end{aligned}$$

Therefore

$$\frac{1}{\lambda} \sum_j \int_{(Q_j^*)^c} |T(b_{1,j}, \dots, f_m)(x)| dx \leq \frac{A}{\lambda} \sum_j \|b_{1,j}\|_{L^1} \leq \frac{2^{n+1}A}{\lambda}.$$

Choosing $\delta = B^{-1}$ and combining the preceding inequalities we obtain

$$|\{x \in \mathbb{R}^n : |T(f_1, \dots, f_m)(x)| > 2\lambda\}| \leq \frac{1}{\lambda} (2^n B + B + 2^{n+1}A) \leq 2^{n+1} (A + B) \frac{1}{\lambda},$$

which yields (6.3). \square

This result allows us to obtain intermediate estimates between the results in [4] (in which $2 < p_k < \infty$ and $2 < p < \infty$) and the results in [8] (in which $1 < p_k \leq \infty$ and $1 < p \leq 2$).

Corollary 6.2. *Let $1 < p_k \leq \infty$ and $1 < p < \infty$ satisfy $1/p_1 + \dots + 1/p_m = 1/p$. Assume that (1.6) holds for a function σ on \mathbb{R}^{mn} where $s_k > n/2$ for all k . Then the multilinear Fourier multiplier operator T_σ maps $L^{p_1} \times \dots \times L^{p_m}$ to L^p .*

Proof. Note that Sobolev condition (1.6) for σ implies Hörmander condition (6.1) for $K = \sigma^\vee$. The proof of this implication is standard in the linear case and similarly, in the m -linear case, it follows by freezing all but one variable (in the bilinear case it is contained in [15]). We are now able to apply Theorem 6.1 to T_σ , and hence Corollary 6.3 follows. Interpolating between (6.2) and (6.3) yields that T_σ maps $L^p \times L^\infty \times \dots \times L^\infty$ to L^p for all $1 < p < \infty$. By symmetry, we deduce that T_σ is bounded from $L^{q_1} \times \dots \times L^{q_m}$ to L^p for all $(q_1, \dots, q_m) \in \Lambda_p$ and $1 < p < \infty$. Once again, by interpolation, we have that T_σ maps from $L^{p_1} \times \dots \times L^{p_m}$ to L^p for all $1 < p_1, \dots, p_m \leq \infty$ and $1 < p < \infty$ such that $\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p}$ with norm at most a multiple of A . \square

Corollary 6.3. *Let σ be a bounded function on $\mathbb{R}^{mn} \setminus \{0\}$ which satisfies (1.6) with $s_k > n/2$ for all $k = 1, \dots, m$. Then we have the estimate*

$$\|T_\sigma(f_1, \dots, f_m)\|_{BMO} \lesssim A \|f_1\|_{L^\infty} \dots \|f_m\|_{L^\infty} \quad (6.6)$$

for all functions $f_k \in L_c^\infty$.

Proof. As before condition (1.6) for σ implies (6.1) for $K = \sigma^\vee$. Applying Theorem 6.1 to T_σ , Corollary 6.3 follows. \square

7. PROOFS OF SOME TECHNICAL LEMMAS

In this section, we will give the detail proofs of some lemmas that were used in previous sections.

7.1. The proof of Lemma 3.2. For $\rho \in \mathbb{Z}, \rho \geq 2$ denote

$$F_\rho = \{y \in \mathbb{R}^{mn} : 2^{\rho-1} - 2 \leq |y| \leq 2^{\rho+1} + 2\}.$$

Fix $x = (x_1, \dots, x_m) \in \mathbb{R}^{mn}$. Then we have

$$\begin{aligned} (\varphi_\epsilon * \sigma)(2^j x) \widehat{\psi}(x) &= \left\{ \int \epsilon^{-mn} \varphi(\epsilon^{-1} y) \sigma(2^j x - y) dy \right\} \widehat{\psi}(x) \\ &= \left\{ \int \epsilon^{-mn} 2^{jmn} \varphi(\epsilon^{-1} 2^j y) \sigma(2^j(x - y)) dy \right\} \widehat{\psi}(x) \\ &= \sum_{\rho \in \mathbb{Z}} \left\{ \int \varphi_{\epsilon 2^{-j}}(y) \sigma(2^j(x - y)) \widehat{\psi}(2^{-\rho}(x - y)) dy \right\} \widehat{\psi}(x) \\ &= \sum_{\rho \leq -3} \left\{ \int \varphi_{\epsilon 2^{-j}}(x - y) \sigma(2^j y) \widehat{\psi}(2^{-\rho} y) dy \right\} \widehat{\psi}(x) \quad (7.1) \end{aligned}$$

$$+ \sum_{|\rho| \leq 2} \int \varphi_{\epsilon 2^{-j}}(y) \sigma(2^j(x - y)) \widehat{\psi}(2^{-\rho}(x - y)) \widehat{\psi}(x) dy \quad (7.2)$$

$$+ \sum_{\rho \geq 3} \left\{ \int \varphi_{\epsilon 2^{-j}}(x-y) \sigma(2^j y) \widehat{\psi}(2^{-\rho} y) dy \right\} \widehat{\psi}(x). \quad (7.3)$$

The $W^{(s_1, \dots, s_m)}$ norm of term (7.2) can be estimated easily by

$$\begin{aligned} & \sum_{|\rho| \leq 2} \int |\varphi_{\epsilon 2^{-j}}(y)| \left\| \sigma(2^j(\cdot - y)) \widehat{\psi}(2^{-\rho}(\cdot - y)) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} dy \\ & \leq \sum_{|\rho| \leq 2} \int |\varphi_{\epsilon 2^{-j}}(y)| \left\| \sigma(2^j(\cdot - y)) \widehat{\psi}(2^{-\rho}(\cdot - y)) \right\|_{W^{(s_1, \dots, s_m)}} \left\| \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} dy \\ & \lesssim \sum_{|\rho| \leq 2} \left\| \sigma(2^{j+\rho} \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} \int |\varphi_{\epsilon 2^{-j}}(y)| dy \lesssim \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}}, \end{aligned}$$

in which the second last inequality follows from the fact [4, Proposition A.2] that

$$\|fg\|_{W^{(s_1, \dots, s_m)}} \lesssim \|f\|_{W^{(s_1, \dots, s_m)}} \|g\|_{W^{(s_1, \dots, s_m)}},$$

when $f, g \in W^{(s_1, \dots, s_m)}$ for $s_1, \dots, s_m > \frac{n}{2}$.

Now fix integer numbers $N_k \geq s_k$, ($1 \leq k \leq m$) and set $N = N_1 + \dots + N_m$. Since $\|f\|_{W^{(s_1, \dots, s_m)}} \leq \|f\|_{W^N}$, the Sobolev norm of the term in (7.1) is bounded by

$$\begin{aligned} & \sum_{\rho \leq -3} \left\| \left\{ \int \varphi_{\epsilon 2^{-j}}(\cdot - y) \sigma(2^j y) \widehat{\psi}(2^{-\rho} y) dy \right\} \widehat{\psi} \right\|_{W^N} \\ & \lesssim \sum_{\rho \leq -3} \sum_{|\alpha| + |\beta| \leq N} \left\| \left\{ \int (\epsilon 2^{-j})^{-|\alpha|} (\partial^\alpha \varphi)_{\epsilon 2^{-j}}(\cdot - y) \sigma(2^j y) \widehat{\psi}(2^{-\rho} y) dy \right\} \partial^\beta \widehat{\psi} \right\|_{L^2} \\ & = \sum_{\rho \leq -3} \sum_{|\alpha| + |\beta| \leq N} \left\| \left\{ \int_{\frac{1}{4} \leq |y| \leq \frac{9}{4}} \frac{(\partial^\alpha \varphi)_{\epsilon 2^{-j}}(y)}{(\epsilon 2^{-j})^{|\alpha|}} \sigma(2^j(\cdot - y)) \widehat{\psi}(2^{-\rho}(\cdot - y)) dy \right\} \partial^\beta \widehat{\psi} \right\|_{L^2} \\ & \lesssim \sum_{\rho \leq -3} \sum_{|\alpha| \leq N} \int \left(\frac{|y|}{\epsilon 2^{-j}} \right)^{|\alpha|} |(\partial^\alpha \varphi)_{\epsilon 2^{-j}}(y)| \left\| \sigma(2^j(\cdot - y)) \widehat{\psi}(2^{-\rho}(\cdot - y)) \right\|_{L^2} dy \\ & \lesssim \sum_{\rho \leq -3} 2^{\frac{\rho n}{2}} \left\| \sigma(2^{j+\rho} \cdot) \widehat{\psi} \right\|_{L^2} \sum_{|\alpha| \leq N} \int \left(\frac{|y|}{\epsilon 2^{-j}} \right)^{|\alpha|} |(\partial^\alpha \varphi)_{\epsilon 2^{-j}}(y)| dy \\ & \lesssim \sum_{\rho \leq -3} 2^{\frac{\rho n}{2}} \left\| \sigma(2^{j+\rho} \cdot) \widehat{\psi} \right\|_{W^s} \sum_{|\alpha| \leq N} \int |y|^{|\alpha|} |(\partial^\alpha \varphi)(y)| dy \\ & \lesssim \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}}. \end{aligned}$$

Finally, we deal with term (7.3). We have

$$\begin{aligned} & \left\| \sum_{\rho \geq 3} \left\{ \int \varphi_{\epsilon 2^{-j}}(\cdot - y) \sigma(2^j y) \widehat{\psi}(2^{-\rho} y) dy \right\} \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} \\ & \leq \sum_{\rho \geq 3} \left\| \left\{ \int \varphi_{\epsilon 2^{-j}}(\cdot - y) \sigma(2^j y) \widehat{\psi}(2^{-\rho} y) dy \right\} \widehat{\psi} \right\|_{W^N} \\ & \lesssim \sum_{\rho \geq 3} \sum_{|\alpha| + |\beta| \leq N} \left\| \left\{ \int (\epsilon 2^{-j})^{-|\alpha|} (\partial^\alpha \varphi)_{\epsilon 2^{-j}}(\cdot - y) \sigma(2^j y) \widehat{\psi}(2^{-\rho} y) dy \right\} \partial^\beta \widehat{\psi} \right\|_{L^2} \end{aligned}$$

$$\begin{aligned}
&= \sum_{\rho \geq 3} \sum_{|\alpha|+|\beta| \leq N} \left\| \left\{ \int_{F_\rho} (\epsilon 2^{-j})^{-|\alpha|} (\partial^\alpha \varphi)_{\epsilon 2^{-j}}(y) \sigma(2^j(\cdot - y)) \widehat{\psi}(2^{-\rho}(\cdot - y)) dy \right\} \partial^\beta \widehat{\psi} \right\|_{L^2} \\
&\lesssim \sum_{\rho \geq 3} \sum_{|\alpha|+|\beta| \leq N} \int_{F_\rho} \left(\frac{|y|}{2^\rho \epsilon 2^{-j}} \right)^{|\alpha|} |(\partial^\alpha \varphi)_{\epsilon 2^{-j}}(y)| \left\| \sigma(2^j(\cdot - y)) \widehat{\psi}(2^{-\rho}(\cdot - y)) \partial^\beta \widehat{\psi} \right\|_{L^2} dy \\
&= \sum_{\rho \geq 3} \sum_{|\alpha|+|\beta| \leq N} \int_{F_\rho} \left(\frac{|y|}{2^\rho \epsilon 2^{-j}} \right)^{|\alpha|} |(\partial^\alpha \varphi)_{\epsilon 2^{-j}}(y)| \left\| \sigma(2^j \cdot) \widehat{\psi}(2^{-\rho} \cdot) (\partial^\beta \widehat{\psi})(\cdot + y) \right\|_{L^2} dy \\
&\leq \sum_{\rho \geq 3} \sum_{|\alpha|+|\beta| \leq N} \int_{F_\rho} 2^{\frac{\rho n}{2}} \left(\frac{|y|}{\epsilon 2^{-j}} \right)^{|\alpha|} |(\partial^\alpha \varphi)_{\epsilon 2^{-j}}(y)| \left\| \sigma(2^{j+\rho} \cdot) \widehat{\psi}(\partial^\beta \widehat{\psi})(2^\rho \cdot + y) \right\|_{L^2} dy \\
&\leq \sum_{\rho \geq 3} \sum_{|\alpha| \leq N} \int_{F_\rho} 2^{\frac{\rho n}{2}} \left(\frac{|y|}{\epsilon 2^{-j}} \right)^{|\alpha|} |(\partial^\alpha \varphi)_{\epsilon 2^{-j}}(y)| \left\| \sigma(2^{j+\rho} \cdot) \widehat{\psi} \right\|_{L^2(B(-2^{-\rho}y, 2^{1-\rho}))} dy \\
&\lesssim \sum_{|\alpha| \leq N} \sum_{\rho \geq 3} \left\| \sigma(2^{j+\rho} \cdot) \widehat{\psi} \right\|_{L^\infty} \int_{F_\rho} \left(\frac{|y|}{\epsilon 2^{-j}} \right)^{|\alpha|} |(\partial^\alpha \varphi)_{\epsilon 2^{-j}}(y)| dy \\
&\leq \sum_{|\alpha| \leq N} \sum_{\rho \geq 3} \left\| \sigma(2^{j+\rho} \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} \int_{F_\rho} \left(\frac{|y|}{\epsilon 2^{-j}} \right)^{|\alpha|} |(\partial^\alpha \varphi)_{\epsilon 2^{-j}}(y)| dy \\
&\lesssim \sup_{\rho \in \mathbb{Z}} \left\| \sigma(2^{j+\rho} \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} \sum_{|\alpha| \leq N} \sum_{\rho \geq 3} \int_{F_\rho} \left(\frac{|y|}{\epsilon 2^{-j}} \right)^{|\alpha|} |(\partial^\alpha \varphi)_{\epsilon 2^{-j}}(y)| dy \\
&\lesssim \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}}.
\end{aligned}$$

The proof of the lemma is now complete.

7.2. The proof of Lemma 4.1. Before verifying Lemma 4.1, we mention approaches that were used by other authors. First, with assumption on the kernel

$$|\partial_{y_0, \dots, y_m}^\alpha K(y_0, \dots, y_m)| \leq A \left(\sum_{k, l=0}^m |y_k - y_l| \right)^{-mn - |\alpha|}$$

for all $|\alpha| \leq N$, Grafakos and Kalton [7] showed that estimate (4.3) holds for the corresponding multilinear singular integral operator with

$$b_k(x) = \frac{|Q_k|^{1 - \frac{1}{p_k} + \frac{N+1}{mn}}}{\left(|x - c_k| + \ell(Q_k) \right)^{n + \frac{N+1}{m}}}.$$

Miyachi and Tomita [15] constructed functions b_k satisfying Lemma 4.1 in the bilinear case. We adapt these techniques to prove the key lemma in the multilinear setting.

Now we start the proof of Lemma 4.1. We may assume that $J_0 = \{1, \dots, r\}$ for some $1 \leq r \leq m$. Fix

$$x \in \left(\bigcap_{k=r+1}^m Q_k^* \right) \setminus \bigcup_{k=1}^r Q_k^*$$

(when $r = m$, just fix $x \in \mathbb{R}^n \setminus \bigcup_{k=1}^m Q_k^*$). Now we rewrite $T_\sigma(a_1, \dots, a_m)(x)$ as

$$T_\sigma(a_1, \dots, a_m)(x) = \sum_{j \in \mathbb{Z}} g_j(x),$$

where

$$g_j(x) = \int_{\mathbb{R}^{mn}} 2^{jmn} K_j(2^j(x - y_1), \dots, 2^j(x - y_m)) a_1(y_1) \cdots a_m(y_m) dy_1 \cdots dy_m$$

with $K_j = (\sigma(2^j \cdot) \widehat{\psi})^\vee$. Let c_k be the center of the cube Q_k ($1 \leq k \leq m$). For $1 \leq k \leq r$, since $x \notin Q_k^*$ and $y_k \in Q_k$, $|x - c_k| \approx |x - y_k|$. Fix $1 \leq l \leq r$. Using Lemma 2.1 with $s_k > \frac{n}{2}$ and applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} & \prod_{k=1}^r \langle 2^j(x - c_k) \rangle^{s_k} |g_j(x)| \\ & \lesssim 2^{jmn} \int_{Q_1 \times \cdots \times Q_m} \left(\prod_{k=1}^r \langle 2^j(x - y_k) \rangle^{s_k} \right) \\ & \quad |K_j(2^j(x - y_1), \dots, 2^j(x - y_m))| \prod_{k=1}^m \|a_k\|_{L^\infty} d\vec{y} \\ & \leq 2^{jmn} \int_{Q_1 \times \cdots \times Q_m} \left(\prod_{k=1}^r \langle 2^j(x - y_k) \rangle^{s_k} \right) \\ & \quad |K_j(2^j(x - y_1), \dots, 2^j(x - y_m))| \prod_{k=1}^m |Q_k|^{-\frac{1}{p_k}} d\vec{y} \\ & \leq 2^{jrn} \prod_{k=1}^m |Q_k|^{-\frac{1}{p_k}} \int_{Q_1 \times \cdots \times Q_r \times \mathbb{R}^{(m-r)n}} \left(\prod_{k=1}^r \langle 2^j(x - y_k) \rangle^{s_k} \right) \times \\ & \quad \times |K_j(2^j(x - y_1), \dots, 2^j(x - y_r), y_{r+1}, \dots, y_m)| dy_1 \cdots dy_r dy_{r+1} \cdots dy_m \\ & \leq 2^{jrn} \prod_{k=1}^r |Q_k|^{1-\frac{1}{p_k}} \prod_{k=r+1}^m |Q_k|^{-\frac{1}{p_k}} \int_{\mathbb{R}^{(m-r)n}} \int_{Q_l} |Q_l|^{-1} \langle 2^j(x - y_l) \rangle^{s_l} \times \\ & \quad \times \left\| \left(\prod_{\substack{k=1 \\ k \neq l}}^r \langle y_k \rangle^{s_k} \right) K_j(y_1, \dots, y_{l-1}, 2^j(x - y_l), y_{l+1}, \dots, y_m) \right\|_{L^\infty(dy_1 \cdots \widehat{dy}_l \cdots dy_r)} dy_l dy_{r+1} \cdots dy_m \\ & \lesssim 2^{jrn} \prod_{k=1}^r |Q_k|^{1-\frac{1}{p_k}} \prod_{k=r+1}^m |Q_k|^{-\frac{1}{p_k}} \int_{\mathbb{R}^{(m-r)n}} \int_{Q_l} |Q_l|^{-1} \langle 2^j(x - y_l) \rangle^{s_l} \times \\ & \quad \times \left\| \left(\prod_{\substack{k=1 \\ k \neq l}}^r \langle y_k \rangle^{s_k} \right) K_j(y_1, \dots, y_{l-1}, 2^j(x - y_l), y_{l+1}, \dots, y_m) \right\|_{L^2(dy_1 \cdots \widehat{dy}_l \cdots dy_r)} dy_l dy_{r+1} \cdots dy_m \\ & \lesssim 2^{jrn} \prod_{k=1}^r |Q_k|^{1-\frac{1}{p_k}} \prod_{k=r+1}^m |Q_k|^{-\frac{1}{p_k}} \int_{Q_l} |Q_l|^{-1} \langle 2^j(x - y_l) \rangle^{s_l} \times \end{aligned}$$

$$\begin{aligned}
& \times \left\| \left(\prod_{\substack{k=1 \\ i \neq k}}^m \langle y_k \rangle^{s_i} \right) K_j(y_1, \dots, y_{l-1}, 2^j(x - y_l), y_{l+1}, \dots, y_m) \right\|_{L^2(dy_1 \cdots \widehat{dy}_l \cdots dy_m)} dy_l \\
& = 2^{jrn} \left(\prod_{k=1}^r |Q_k|^{1 - \frac{1}{p_k}} \right) h_j^{(l,0)}(x) \prod_{k=r+1}^m b_k(x) \quad (7.4)
\end{aligned}$$

for all $x \in (\cap_{k=r+1}^m Q_k^*) \setminus (\cup_{k=1}^r Q_k^*)$, where

$$\begin{aligned}
h_j^{(l,0)}(x) &= \frac{1}{|Q_l|} \int_{Q_l} \langle 2^j(x - y_l) \rangle^{s_l} \\
& \times \left\| \left(\prod_{\substack{k=1 \\ k \neq l}}^m \langle y_k \rangle^{s_k} \right) K_j(y_1, \dots, y_{l-1}, 2^j(x - y_l), y_{l+1}, \dots, y_m) \right\|_{L^2(dy_1 \cdots \widehat{dy}_l \cdots dy_m)} dy_l
\end{aligned}$$

and $b_k(x) = |Q_k|^{-\frac{1}{p_k}} \chi_{Q_k^*}(x)$ for $r+1 \leq k \leq m$. A direct computation gives

$$\left\| h_j^{(l,0)} \right\|_{L^2} \leq 2^{-\frac{jn}{2}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}} = A 2^{-\frac{jn}{2}}.$$

Using the vanishing moment condition of a_k and Taylor's formula, we write

$$\begin{aligned}
g_j(x) &= 2^{jmn} \sum_{|\alpha|=N_l} C_\alpha \int_{\mathbb{R}^{mn}} \left\{ \int_0^1 (1-t)^{N_l-1} \right. \\
& \quad \times \partial_{y_l}^\alpha K_j(2^j(x - y_1), \dots, 2^j(x - c_l - t(y_l - c_l)), \dots, 2^j(x - y_m)) \\
& \quad \left. \times (2^j(y_l - c_l))^\alpha a_1(y_1) \cdots a_m(y_m) dt \right\} dy_1 \cdots dy_m.
\end{aligned}$$

Repeat the preceding argument to obtain

$$\prod_{k=1}^r \langle 2^j(x - c_k) \rangle^{s_k} |g_j(x)| \lesssim 2^{jrn} \left(\prod_{k=1}^r |Q_k|^{1 - \frac{1}{p_k}} \right) h_j^{(l,1)}(x) \prod_{k=r+1}^m b_k(x) \quad (7.5)$$

for all $x \in (\cap_{k=r+1}^m Q_k^*) \setminus (\cup_{k=1}^r Q_k^*)$, where $b_k(x) = |Q_k|^{-\frac{1}{p_k}} \chi_{Q_k^*}(x)$ for $r+1 \leq k \leq m$ and

$$\begin{aligned}
h_j^{(l,1)}(x) &= \sum_{|\alpha|=N_l} \int_{Q_l} \left\{ \int_0^1 \langle 2^j x_{c_l, y_l}^t \rangle^{s_l} \right. \\
& \quad \times \left\| \left(\prod_{\substack{k=1 \\ k \neq l}}^m \langle y_k \rangle^{s_k} \right) \partial_{y_l}^\alpha K_j(y_1, \dots, y_{l-1}, 2^j x_{c_l, y_l}^t, y_{l+1}, \dots, y_m) \right\|_{L^2(dy_1 \cdots \widehat{dy}_l \cdots dy_m)} \\
& \quad \left. \times (2^j \ell(Q_l))^{N_l} |Q_l|^{-1} dt \right\} dy_l,
\end{aligned}$$

with $x_{c_l, y_l}^t = x - c_l - t(y_l - c_l)$. Applying Minkowski's inequality together with Lemma 2.2 implies that

$$\left\| h_j^{(l,1)} \right\|_{L^2} \leq A 2^{-\frac{jn}{2}} (2^j \ell(Q_l))^{N_l}.$$

Combine inequalities (7.4) and (7.5), we get

$$\begin{aligned} \prod_{k=1}^r \langle 2^j(x - c_k) \rangle^{s_k} |g_j(x)| \\ \leq 2^{jrn} \left(\prod_{k=1}^r |Q_k|^{1 - \frac{1}{p_k}} \right) \min \left\{ h_j^{(l,0)}(x), h_j^{(l,1)}(x) \right\} \prod_{k=r+1}^m b_k(x) \end{aligned} \quad (7.6)$$

for all $1 \leq l \leq r$. The inequalities in (7.6) imply that

$$\begin{aligned} |g_j(x)| \\ \leq 2^{jrn} \prod_{k=1}^r |Q_k|^{1 - \frac{1}{p_k}} \prod_{k=1}^r \langle 2^j(x - c_k) \rangle^{-s_k} \min_{1 \leq l \leq r} \left\{ h_j^{(l,0)}(x), h_j^{(l,1)}(x) \right\} \prod_{k=r+1}^m b_k(x) \end{aligned} \quad (7.7)$$

for all $x \in (\cap_{k=r+1}^m Q_k^*) \setminus (\cup_{k=1}^r Q_k^*)$.

Now we need to construct functions u_j^k ($1 \leq k \leq r$) such that

$$g_j(x) \lesssim A \prod_{k=1}^r u_j^k(x) \prod_{k=r+1}^m b_k(x)$$

for all $x \in (\cap_{k=r+1}^m Q_k^*) \setminus (\cup_{k=1}^r Q_k^*)$ and that $\left\| \sum_j u_j^k \right\|_{L^{p_k}} \lesssim 1$ for all $1 \leq k \leq r$.

Then the lemma follows by taking $b_k = \sum_j u_j^k$ ($1 \leq k \leq r$) and $b_k = |Q_k|^{-\frac{1}{p_k}} \chi_{Q_k^*}$ ($r+1 \leq k \leq m$).

Indeed, we can choose $0 < \lambda_k < \min \left\{ \frac{1}{2}, \frac{s_k}{n} - \frac{1}{p_k} + \frac{1}{2} \right\}$ such that

$$\sum_{k=1}^r \lambda_k = \frac{r-1}{2}.$$

This is suitable since conditions (1.5) implies that

$$\sum_{k=1}^r \min \left\{ \frac{1}{2}, \frac{s_k}{n} - \frac{1}{p_k} + \frac{1}{2} \right\} > \frac{r-1}{2}.$$

Set $\alpha_k = \frac{1}{p_k} - \frac{1}{2} + \lambda_k$ and $\beta_k = 2(\frac{1}{p_k} - \alpha_k)$. Then we have

$$\sum_{k=1}^r \alpha_k = \sum_{k=1}^r \frac{1}{p_k} - \frac{1}{2},$$

$\beta_k > 0$ and $\beta_1 + \dots + \beta_r = 1$. Now define

$$u_j^k = A^{-\beta_k} 2^{jn} |Q_k|^{1 - \frac{1}{p_k}} \langle 2^j(\cdot - c_k) \rangle^{-s_k} \chi_{(Q_k^*)^c} \min \left\{ h_j^{(k,0)}, h_j^{(k,1)} \right\}^{\beta_k}, \quad 1 \leq k \leq r.$$

Then, from (7.7), it is easy to see that

$$g_j(x) \lesssim A \prod_{k=1}^r u_j^k(x) \prod_{k=r+1}^m b_k(x)$$

for all $x \in (\cap_{k=r+1}^m Q_k^*) \setminus (\cup_{k=1}^r Q_k^*)$. It remains to check that $\sum_j \int_{\mathbb{R}^n} |u_j^k(x)|^{p_k} dx \lesssim 1$.

Since $\frac{1}{p_k} = \alpha_k + \frac{\beta_k}{2}$, setting $\frac{1}{p'_k} = 1 - \frac{1}{p_k}$, Hölder's inequality gives

$$\|u_j^k\|_{L^{p_k}} \leq A^{-\beta_k} 2^{jn} |Q_k|^{\frac{1}{p'_k}} \left\| \langle 2^j(\cdot - c_k) \rangle^{-s_k} \chi_{(Q_k^*)^c} \right\|_{L^{\frac{1}{\alpha_k}}} \left\| \min \{h_j^{(k,0)}, h_j^{(k,1)}\}^{\beta_k} \right\|_{L^{\frac{2}{\beta_k}}}$$

for all $1 \leq k \leq r$. Notice that $n < \frac{s_k}{\alpha_k}$, we have

$$\left\| \langle 2^j(\cdot - c_k) \rangle^{-s_k} \chi_{(Q_k^*)^c} \right\|_{L^{1/\alpha_k}} \lesssim 2^{-jn\alpha_k} \min \{1, (2^j \ell(Q_k))^{\alpha_k n - s_k}\}$$

and

$$\begin{aligned} \left\| \left(\min \{h_j^{(k,0)}, h_j^{(k,1)}\} \right)^{\beta_k} \right\|_{L^{2/\beta_k}} &\leq \min \left\{ \left\| h_j^{(k,0)} \right\|_{L^2}^{\beta_k}, \left\| h_j^{(k,1)} \right\|_{L^2}^{\beta_k} \right\} \\ &\lesssim \left(A 2^{-jn/2} \min \{1, (2^j \ell(Q_k))^{N_k}\} \right)^{\beta_k}. \end{aligned}$$

Therefore

$$\begin{aligned} \|u_j^k\|_{L^{p_k}} &\leq 2^{jn} |Q_k|^{1 - \frac{1}{p_k}} 2^{-\frac{jn}{p'_k}} \min \{1, (2^j \ell(Q_k))^{\alpha_k n - s_k}\} \min \{1, (2^j \ell(Q_k))^{N_k \beta_k}\} \\ &\lesssim (2^j \ell(Q_k))^{n - \frac{n}{p_k}} \min \{1, (2^j \ell(Q_k))^{\alpha_k n - s_k}\} \min \{1, (2^j \ell(Q_k))^{N_k \beta_k}\}. \end{aligned}$$

This inequality is enough to establish what we needed $\sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} |u_j^k(x)|^{p_k} dx \lesssim 1$. The proof of Lemma 4.1 is complete.

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