MULTILINEAR FOURIER MULTIPLIERS WITH MINIMAL SOBOLEV REGULARITY, II

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Abstract. We provide characterizations for boundedness of multilinear Fourier operators on Hardy or Lebesgue spaces with symbols locally in Sobolev spaces. Let $H^q(\mathbb{R}^n)$ denote the Hardy space when $0 < q \le 1$ and the Lebesgue space $L^q(\mathbb{R}^n)$ when $1 < q \le \infty$. We find optimal conditions on m-linear Fourier multiplier operators to be bounded from $H^{p_1} \times \cdots \times H^{p_m}$ to L^p when $1/p = 1/p_1 + \cdots + 1/p_m$ in terms of local L^2-Sobolev space estimates for the symbol of the operator. Our conditions provide multilinear analogues of the linear results of Calderón and Torchinsky [1] and of the bilinear results of Miyachi and Tomita [17]. The extension to general m is significantly more complicated both technically and combinatorially; the optimal Sobolev space smoothness required of the symbol depends on the Hardy-Lebesgue exponents and is constant on various convex simplices formed by configurations of $m2^{m-1} + 1$ points in $[0, \infty)^m$.

1. Introduction

We denote by T_σ the linear Fourier multiplier operator, acting on Schwartz functions f, defined by

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \sigma(\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}\, d\xi, \tag{1.1}$$

where σ is a bounded function on $\mathbb{R}^n$ and $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx$ denotes the Fourier transform of f. Hörmander [15] proved that T_σ is bounded from $L^p(\mathbb{R}^n)$ to itself for $1 < p < \infty$ if

$$\sup_{j\in\mathbb{Z}} \left\| \sigma(2^j\cdot)\widehat{\psi} \right\|_{W^s} < \infty \tag{1.2}$$

for some $s > \frac{n}{2}$, where $\widehat{\psi}$ is a smooth function supported in $\frac{1}{2} \le |\xi| \le 2$ that satisfies

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1$$

for all $\xi \ne 0$. In this paper, W^s denotes the Sobolev space with norm

$$\|g\|_{W^s} = \|(I-\Delta)^{s/2}g\|_{L^2},$$

where I is the identity operator and $\Delta = \sum_{j=1}^n \partial_j^2$ is the Laplacian on $\mathbb{R}^n$. Hörmander's result strengthens an earlier result of Mikhlin [19].

Throughout this work, $H^p(\mathbb{R}^n)$ denotes the real-variable Hardy space of Fefferman and Stein [4], for $0 < p \le \infty$. This space coincides with the Lebesgue space $L^p(\mathbb{R}^n)$ when $1 < p \le \infty$. Calderón and Torchinsky [1] provided an extension of Hörmander's result to $H^p(\mathbb{R}^n)$ for $p \le 1$. They showed that the Fourier multiplier operator in (1.1) admits a bounded extension from the Hardy space $H^p(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$ with $0 < p \le 1$ if

$$\sup_{t>0} \left\| \sigma(t\cdot)\widehat{\psi} \right\|_{W^s} < \infty$$

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and $s > \frac{n}{p} - \frac{n}{2}$. Moreover, the boundedness of T_σ on H^p may not hold if $s < \frac{n}{p} - \frac{n}{2}$; in other words, the Calderón and Torchinsky condition $s > \frac{n}{p} - \frac{n}{2}$ is sharp (for this, see for instance [17, Remark 1.3]).

In this work we study analogues of these results for multilinear multipliers defined on products of Hardy or Lebesgue spaces on the entire range of indices $0 < p \le \infty$. Multilinear multiplier operators were studied by Coifman and Meyer [2, 3, 16] and more recently by Grafakos and Torres [14]. Multilinear Fourier multiplier is a bounded function σ on $\mathbb{R}^{mn} = \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ associated with the m-linear Fourier multiplier operator

$$T_\sigma(f_1, \dots, f_m)(x) = \int_{\mathbb{R}^{mn}} e^{2\pi i x\cdot(\xi_1+\cdots+\xi_m)} \sigma(\xi_1, \dots, \xi_m) \widehat{f_1}(\xi_1) \cdots \widehat{f_m}(\xi_m)\, d\vec{\xi}, \tag{1.3}$$

where f_j are in the Schwartz space of $\mathbb{R}^n$ and $d\vec{\xi} = d\xi_1 \cdots d\xi_m$.

A short history of the known results concerning multilinear multipliers with minimal smoothness is as follows: Tomita [22] obtained $L^{p_1} \times \cdots \times L^{p_m} \to L^p$ boundedness ($1 < p_1, \dots, p_m, p < \infty$) for multilinear multiplier operators under a condition (1.2). Grafakos and Si [12] extended Tomita's result to the case $p \le 1$ using L^r-based Sobolev spaces with $1 < r \le 2$. Fujita and Tomita [6] provided weighted extensions of these results and also noticed that the Sobolev space W^s in (1.2) can be replaced by a product-type Sobolev space $W^{(s_1,\dots,s_m)}$ when $p > 2$. Grafakos, Miyachi, and Tomita [10] extended the range of p in [6] to $p > 0$ and obtained the boundedness even in the endpoint case where all but one indices p_j are equal to infinity. Miyachi and Tomita [17] provided sharp conditions on the entire range of indices ($0 < p_j \le \infty$), extending the Calderón and Torchinsky [1] result to the case $m = 2$.

In this work we provide extensions of the result of Calderón and Torchinsky [1] ($m = 1$) and of Miyachi and Tomita [17] ($m = 2$) to the cases $m \ge 3$. We point out that the complexity of the problem increases significantly as m increases. In fact, the main difficulty concerns the case where $1 < p_j < 2$, in which the boundedness holds exactly in the interior of a convex simplex in $\mathbb{R}^m$. This simplex has $m2^{m-1} + 1$ vertices but it is not enough to obtain the corresponding estimates for the vertices of the simplex, since interpolation between the vertices does not yield minimal smoothness in the interior. We overcome this difficulty by establishing estimates for all the points inside the simplex being arbitrarily close to those $m2^{m-1} + 1$ points without losing smoothness.

Before stating our main result we introduce some notation. First, for $x \in \mathbb{R}^n$ we set $\langle x \rangle = \sqrt{1 + |x|^2}$. For $s_1, \dots, s_m > 0$, we denote by $W^{(s_1,\dots,s_m)}$ the product-type-Sobolev space consisting of all functions f on $\mathbb{R}^{mn}$ such that

$$\|f\|_{W^{(s_1,\dots,s_m)}} := \left(\int_{\mathbb{R}^{mn}} \left| \widehat{f}(y_1, \dots, y_m) \langle y_1 \rangle^{s_1} \cdots \langle y_m \rangle^{s_m} \right|^2 dy_1 \cdots dy_m \right)^{\frac{1}{2}} < \infty.$$

Notice that $W^{(s_1,\dots,s_m)}$ is a subspace of L^2.

We also denote by ψ a smooth function on $\mathbb{R}^{mn}$ whose Fourier transform $\widehat{\psi}$ is supported in $\frac{1}{2} \le |\xi| \le 2$ and satisfies

$$\sum_{j \in \mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1, \qquad \xi \ne 0.$$

The following is the main result of this paper. It concerns boundedness of operators of the form (1.3) on products of Hardy spaces in the full range of indices.

Theorem 1.1. *Let $0 < p_1, \dots, p_m \le \infty$, $0 < p < \infty$, $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$, $s_1, \dots, s_m > n/2$, and suppose*

$$\sum_{k \in J} \left(\frac{s_k}{n} - \frac{1}{p_k} \right) > -\frac{1}{2} \tag{1.4}$$

for every nonempty subset $J \subset \{1,2,\dots,m\}$. If σ satisfies

$$A := \sup_{j\in\mathbb{Z}} \left\| \sigma(2^j\cdot)\widehat{\psi} \right\|_{W^{(s_1,\dots,s_m)}} < \infty, \tag{1.5}$$

then we have

$$\|T_\sigma\|_{H^{p_1}\times\cdots\times H^{p_m}\longrightarrow L^p} \lesssim A. \tag{1.6}$$

Moreover, this result is optimal in the sense that if (1.5) *and* (1.6) *are valid then we must necessarily have $s_1,\dots,s_m \geq n/2$ and*

$$\sum_{k\in J}\Big(\frac{s_k}{n}-\frac{1}{p_k}\Big) \geq -\frac{1}{2} \tag{1.7}$$

for every nonempty subset J of $\{1,2,\dots,m\}$.

Remark 1.2. This paper is a sequel of [13] for the following reasons:

(1) The case $p_i \leq 1$ for all $1\leq i\leq m$ is contained in [13].
(2) The endpoint case of Theorem 1.1 in the case where $p_i = p = \infty$ for all $i \in \{1,\dots,m\}$ is proved in [13]:

$$\|T_\sigma(f_1,\dots,f_m)\|_{BMO} \lesssim \sup_{j\in\mathbb{Z}} \left\| \sigma(2^j\cdot)\widehat{\psi} \right\|_{W^{(s_1,\dots,s_m)}} \prod_{i=1}^m \|f_i\|_{L^\infty}$$

for $s_1,\dots,s_m > n/2$.
(3) The necessity of the conditions $s_i \geq n/2$ and (1.7) was shown in [13, Theorem 5.1] for the entire range of indices $0 < p_j \leq \infty$, $0<p<\infty$.

We will consistently use the notation $A \lesssim B$ to indicate that $A \leq CB$ for some constant $C>0$, and $A\approx B$ if $A\lesssim B$ and $B \lesssim A$ simultaneously.

The paper is structured as follows. Section 2 contains preliminaries and known results. In Section 3, we give the proof of the main result by considering four cases. In Section 4, we present detailed proofs of the lemmas used in Section 3. In the last section, Section 5, we give a result concerning the space L^1 and weak type estimate.

2. Preliminaries and known results

Now fix $0<p\leq\infty$ and a Schwartz function Φ with $\widehat{\Phi}(0)\neq 0$. Then the Hardy space H^p contains all tempered distributions f on $\mathbb{R}^n$ such that

$$\|f\|_{H^p} := \left\| \sup_{0<t<\infty} |\Phi_t * f| \right\|_{L^p} < \infty.$$

It is well known that the definition of the Hardy space does not depend on the choice of the function Φ. Note that $H^p = L^p$ for all $p>1$. When $0<p\leq 1$, one of nice features of Hardy spaces is the atomic decomposition. More precisely, any function $f\in H^p$ $(0<p\leq 1)$ can be decomposed as $f=\sum_k \lambda_k a_k$, where a_k's are L^∞-atoms for H^p supported in cubes Q_k such that $\|a_k\|_{L^\infty} \leq |Q_k|^{-\frac{1}{p}}$ and $\int x^\gamma a_k(x)dx = 0$ for all $|\gamma|<N$, and the coefficients λ_k satisfy $\sum_k |\lambda_k|^p \leq 2^p \|f\|_{H^p}^p$. The order N of the moment condition can be taken arbitrarily large.

A fundamental L^2 estimate for T_σ is given in the following theorem.

Theorem 2.1 ([10])**.** *If $s_1,\dots,s_m > n/2$, then*

$$\|T_\sigma\|_{L^2\times L^\infty\times\cdots\times L^\infty\longrightarrow L^2} \leq C\sup_{j\in\mathbb{Z}} \left\| \sigma(2^j\cdot)\widehat{\psi} \right\|_{W^{(s_1,\dots,s_m)}}.$$

The following two lemmas are essentially contained in [17], modulo a few minor modifications.

Lemma 2.2 ([17]). *Let m be a positive integer, σ be a function defined on $\mathbb{R}^{mn}$, and $K = \sigma^{\vee}$, the inverse Fourier transform of σ. Suppose that σ is supported in the ball $\{y \in \mathbb{R}^{mn} : |y| \le 2\}$ and suppose $1 \le l \le n$, $s_i \ge 0$ for $1 \le i \le m$ and $1 \le p \le q \le \infty$. Then for each multi-index α there exists a constant C_α such that*

$$\left\| \langle y_1 \rangle^{s_1} \cdots \langle y_l \rangle^{s_l} \, \partial_y^\alpha K(y) \right\|_{L^q(\mathbb{R}^{ml},\, dy_1 \cdots dy_l)} \le C_\alpha \left\| \langle y_1 \rangle^{s_1} \cdots \langle y_l \rangle^{s_l} \, K(y) \right\|_{L^p(\mathbb{R}^{ml},\, dy_1 \cdots dy_l)},$$

where $y = (y_1, \dots, y_m)$ with $y_j \in \mathbb{R}^n$.

Lemma 2.3 ([17]). *Let $s_i > \frac{n}{2}$ for $1 \le i \le m$, and let $\widehat{\zeta}$ be a smooth function which is supported in an annulus centered at zero. Suppose that Φ is a smooth function away from zero that satisfies the estimates*

$$\left| \partial_\xi^\alpha \Phi(\xi) \right| \le C_\alpha |\xi|^{-|\alpha|}$$

for all $\xi \in \mathbb{R}^{mn}$, $\xi \ne 0$, and for all multi-indices α. Then there exists a constant C such that

$$\sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \Phi(2^j \cdot) \widehat{\zeta} \right\|_{W^{(s_1, \dots, s_m)}} \le C \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}}.$$

Adapting the Calderón and Torchinsky interpolation techniques in the multilinear setting (for details on this we refer to [10, p. 318]) allows us to interpolate between two estimates for multilinear multiplier operators from a product of some Hardy spaces or Lebesgue spaces to Lebesgue spaces.

Theorem 2.4 ([10]). *Let $0 < p_i, p_{i,k} \le \infty$ and $s_{i,k} > 0$ for $i = 1, 2$ and $1 \le k \le m$. For $0 < \theta < 1$, set $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, $\frac{1}{p_k} = \frac{1-\theta}{p_{1,k}} + \frac{\theta}{p_{2,k}}$, and $s_k = (1-\theta) s_{1,k} + \theta s_{2,k}$. Assume that the multilinear operator T_σ defined in (1.3) satisfies the estimates*

$$\| T_\sigma \|_{H^{p_{i,1}} \times \cdots \times H^{p_{i,m}} \longrightarrow L^{p_i}} \le C_i \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_{i,1}, \dots, s_{i,m})}}, \qquad i = 1, 2,$$

where L^{p_i} should be replaced by BMO if $p_i = \infty$. Then

$$\| T_\sigma \|_{H^{p_1} \times \cdots \times H^{p_m} \longrightarrow L^p} \le C \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}},$$

where L^p should be replaced by BMO if $p = \infty$.

Fix a Schwartz function K. We denote the multilinear operator of convolution type associated with the kernel K by

$$T^K(f_1, \dots, f_m)(x) = \int_{\mathbb{R}^{mn}} K(x - y_1, \dots, x - y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m.$$

The following result can be verified with a very similar argument as showed in [13, Proposition 3.4].

Proposition 2.5. *Let $0 < p_1, \dots, p_l \le 1$ and $1 < p_{l+1}, \dots, p_m \le \infty$. Let K be a smooth function with compact support. Suppose $f_i \in H^{p_i}$, $1 \le i \le l$, has atomic representation $f_i = \sum_{k_i} \lambda_{i,k_i} a_{i,k_i}$, where a_{i,k_i} are L^∞-atoms for H^{p_i} and $\sum_{k_i} |\lambda_{i,k_i}|^{p_i} \le 2^{p_i} \|f_i\|_{H^{p_i}}^{p_i}$. Suppose $f_i \in L^{p_i}$ for $l+1 \le i \le m$. Then*

$$T^K(f_1, \dots, f_m)(x) = \sum_{k_1} \cdots \sum_{k_l} \lambda_{1,k_1} \cdots \lambda_{l,k_l} T^K(a_{1,k_1}, \dots, a_{l,k_l}, f_{l+1}, \dots, f_m)(x)$$

for almost all $x \in \mathbb{R}^n$.

We also use the following lemmas.

Lemma 2.6 ([9, Lemma 2.1]). *Let $0 < p \le 1$ and let $\Big(f_Q\Big)_{Q\in\mathcal{J}}$ be a family of nonnegative integrable functions with* $\operatorname{supp}(f_Q) \subset Q$ *for all $Q \in \mathcal{J}$, where $\mathcal{J}$ is a family of finite or countable cubes in $\mathbb{R}^n$. Then we have*

$$\Big\|\sum_{Q\in\mathcal{J}} f_Q\Big\|_{L^p} \lesssim \Big\|\sum_{Q\in\mathcal{J}}\Big(\frac{1}{|Q|}\int_Q f_Q(x)\,dx\Big)\chi_Q\Big\|_{L^p},$$

where the constant of the inequality depends only on p.

Lemma 2.7 ([10, Lemma 3.3]). *Let $s > n/2$,* $\max\{1, n/s\} < q < 2$*, and*

$$\zeta_j(x) = 2^{jn}(1 + |2^j x|)^{-sq}, \quad j \in \mathbb{Z}, \quad x \in \mathbb{R}^n.$$

Suppose $\sigma \in W^{(s,\dots,s)}(\mathbb{R}^{mn})$ and $\operatorname{supp}\sigma \subset \{|\xi| \le 2^{j+1}\}$ *for some $j \in \mathbb{Z}$. Then there exists a constant $C > 0$ depending only on m, n, s, and q such that*

$$|T_\sigma(f_1,\dots,f_m)(x)| \le C\|\sigma(2^j\cdot)\|_{W^{(s,\dots,s)}}(\zeta_j * |f_1|^q)(x)^{1/q}\dots(\zeta_j * |f_m|^q)(x)^{1/q}$$

for all $x \in \mathbb{R}^n$.

Lemma 2.8 ([10, Lemma 3.2]). *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\varphi(0) = 0$, and set*

$$\Delta_j f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\,\varphi(2^{-j}\xi)\widehat{f}(\xi)\,d\xi, \quad j \in \mathbb{Z}. \tag{2.1}$$

Let $\epsilon > 0$ and $\zeta_j(x) = 2^{jn}(1 + |2^j x|)^{-n-\epsilon}$, $j \in \mathbb{Z}$, $x \in \mathbb{R}^n$. Then the following inequalities hold for each $0 < q < 2$:

$$\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n} |\Delta_j f(x)|^2\,dx \le C\|f\|_{L^2}^2, \tag{2.2}$$

$$\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n} (\zeta_j * |f|^q)(x)^{2/q}(\zeta_j * |\Delta_j g|^q)(x)^{2/q}\,dx \le C_q\|f\|_{L^2}^2\|g\|_{BMO}^2. \tag{2.3}$$

Lemma 2.9. *Suppose $\{F_j\} \subset \mathcal{S}'(\mathbb{R}^n)$ and suppose there exists a constant $B > 1$ such that* $\operatorname{supp}\widehat{F_j} \subset \{\zeta \in \mathbb{R}^n \mid B^{-1}2^j \le |\zeta| \le B2^j\}$ *for all $j \in \mathbb{Z}$. Then, for each $0 < p < \infty$,*

$$\Big\|\sum_j F_j\Big\|_{H^p} \lesssim \Big\|\Big(\sum_j |F_j|^2\Big)^{1/2}\Big\|_{L^p}.$$

The preceding lemma is well known in the Littlewood-Paley theory, see for example [23, 5.2.4] and [8, Lemma 7.5.2].

3. The proof of the main result

In this section, we prove the main theorem by considering four cases.

3.1. **The first case: $0 < p_i \le 1$, $1 \le i \le m$.** This case is a consequence of the following result established in [13]:

Theorem 3.1 ([13]). *Let $\frac{n}{2} < s_1,\dots,s_m < \infty$, $0 < p_1,\dots,p_m \le 1$, and $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$. Suppose* (1.4) *holds for every nonempty subset J of $\{1,2,\dots,m\}$. Then* (1.6) *holds.*

3.2. **The second case:** $0 < p_i \le 1$ **or** $p_i = \infty$**.**

Theorem 3.2. *Let* $\frac{n}{2} < s_1, \dots, s_m < \infty$, $0 < p_1, \dots, p_l \le 1$, $1 \le l < m$, *and* $\frac{1}{p_1} + \cdots + \frac{1}{p_l} = \frac{1}{p}$. *Suppose* (1.4) *holds for every nonempty subset* J *of* $\{1, 2, \dots, l\}$*. Then*

$$\|T_\sigma\|_{H^{p_1}\times\cdots\times H^{p_l}\times L^\infty\times\cdots\times L^\infty\longrightarrow L^p} \lesssim \sup_{j\in\mathbb{Z}} \left\|\sigma(2^j\cdot)\widehat{\psi}\right\|_{W^{(s_1,\dots,s_m)}}. \tag{3.1}$$

Proof. The proof of this theorem is similar to the proof of Theorem 3.1 given in [13].

By regularization (see [13, Section 3]), we can always assume that the inverse Fourier transform of σ is smooth and compactly supported. The aim is to show that

$$\|T_\sigma(f_1, \dots, f_m)\|_{L^p} \lesssim A \|f_1\|_{H^{p_1}} \cdots \|f_l\|_{H^{p_l}} \prod_{i=l+1}^{m} \|f_i\|_{L^\infty}. \tag{3.2}$$

Fix functions $f_i \in H^{p_i}$. Using atomic representations for H^{p_i}-functions, write

$$f_i = \sum_{k_i\in\mathbb{Z}} \lambda_{i,k_i} a_{i,k_i}, \quad 1 \le i \le l,$$

where a_{i,k_i} are L^∞-atoms for H^{p_i} satisfying

$$\operatorname{supp}(a_{i,k_i}) \subset Q_{i,k_i}, \quad \|a_{i,k_i}\|_{L^\infty} \le |Q_{i,k_i}|^{-\frac{1}{p_i}}, \quad \int_{Q_{i,k_i}} x^\alpha a_{i,k_i}(x)dx = 0$$

for $|\alpha| < N_i$ with N_i large enough, and $\sum_{k_i} |\lambda_{i,k_i}|^{p_i} \le 2^{p_i} \|f_i\|_{H^{p_i}}^{p_i}$.

For a cube Q we denote by Q^* its dilation by the factor $2\sqrt{n}$. Since $K = \sigma^\vee$ is smooth and compactly supported, Proposition 2.5 yields that

$$T_\sigma(f_1, \dots, f_m)(x) = \sum_{k_1} \cdots \sum_{k_l} \lambda_{1,k_1} \dots \lambda_{l,k_l} T_\sigma(a_{1,k_1}, \dots, a_{l,k_l}, f_{l+1}, \dots, f_m)(x)$$

for a.e. $x \in \mathbb{R}^n$. Now we can split $T_\sigma(f_1, \dots, f_m)$ into two parts and estimate

$$|T_\sigma(f_1, \dots, f_m)(x)| \le G_1(x) + G_2(x),$$

where

$$G_1 = \sum_{k_1} \cdots \sum_{k_l} |\lambda_{1,k_1}| \cdots |\lambda_{l,k_l}| |T_\sigma(a_{1,k_1}, \dots, a_{l,k_l}, f_{l+1}, \dots, f_m)| \chi_{Q^*_{1,k_1}\cap\cdots\cap Q^*_{l,k_l}}$$

and

$$G_2 = \sum_{k_1} \cdots \sum_{k_l} |\lambda_{1,k_1}| \cdots |\lambda_{l,k_l}| |T_\sigma(a_{1,k_1}, \dots, a_{l,k_l}, f_{l+1}, \dots, f_m)| \chi_{(Q^*_{1,k_1}\cap\cdots\cap Q^*_{l,k_l})^c}.$$

The first part $G_1(x)$ can be dealt via the argument in [9] (reprised more clearly in [13]). Suppose the cubes $Q^*_{1,k_1}, \dots, Q^*_{l,k_l}$ satisfy $Q^*_{1,k_1} \cap \cdots \cap Q^*_{l,k_l} \neq \emptyset$. From these cubes, choose a cube that has the minimum sidelength, and denote it by $R_{k_1,\dots,k_l}$. Then

$$Q^*_{1,k_1} \cap \cdots \cap Q^*_{l,k_l} \subset R_{k_1,\dots,k_l} \subset Q^{**}_{1,k_1} \cap \cdots \cap Q^{**}_{l,k_l},$$

where Q^{**}_{i,k_i} denotes a suitable dilation of Q^*_{i,k_i}. We shall prove

$$\begin{aligned} &\frac{1}{|R_{k_1,\dots,k_l}|} \int_{R_{k_1,\dots,k_l}} |T_\sigma(a_{1,k_1}, \dots, a_{l,k_l}, f_{l+1}, \dots, f_m)(x)| \, dx \\ &\lesssim A \prod_{i=1}^{l} |Q_{i,k_i}|^{-\frac{1}{p_i}} \prod_{i=l+1}^{m} \|f_i\|_{L^\infty}. \end{aligned} \tag{3.3}$$

To show this, assume without loss of generality $R_{k_1,\dots,k_l} = Q^*_{1,k_1}$. Theorem 2.1 gives us

$$\int_{R_{k_1,\dots,k_l}} |T_\sigma(a_{1,k_1}, \dots, a_{l,k_l}, f_{l+1}, \dots, f_m)(x)| dx$$

$$
\begin{aligned}
&\le \|T_\sigma(a_{1,k_1},\dots,a_{l,k_l},f_{l+1},\dots,f_m)\|_{L^2}\,|R_{k_1,\dots,k_l}|^{\frac12}\\
&\lesssim A|R_{k_1,\dots,k_l}|^{\frac12}\,\|a_{1,k_1}\|_{L^2}\prod_{i=2}^{l}\|a_{i,k_i}\|_{L^\infty}\prod_{i=l+1}^{m}\|f_i\|_{L^\infty}\\
&\le A|R_{k_1,\dots,k_l}|^{\frac12}|Q_{1,k_1}|^{\frac12}\prod_{i=1}^{l}|Q_{i,k_i}|^{-\frac{1}{p_i}}\prod_{i=l+1}^{m}\|f_i\|_{L^\infty}\\
&\le A|R_{k_1,\dots,k_l}|\prod_{i=1}^{l}|Q_{i,k_i}|^{-\frac{1}{p_i}}\prod_{i=l+1}^{m}\|f_i\|_{L^\infty}\,,
\end{aligned}
$$

which implies (3.3). Now using Lemma 2.6, the estimate (3.3), and Hölder's inequality, we obtain

$$
\begin{aligned}
\|G_1\|_{L^p} &\lesssim \Bigg\|\sum_{k_1}\cdots\sum_{k_m}\Big(\prod_{i=1}^{l}|\lambda_{i,k_i}|\Big)\frac{1}{|R_{k_1,\dots,k_l}|}\chi_{R_{k_1,\dots,k_l}}\times\\
&\qquad\times\int_{R_{k_1,\dots,k_l}}|T_\sigma(a_{1,k_1},\dots,a_{l,k_l},f_{l+1},\dots,f_m)(x)|\,dx\Bigg\|_{L^p}\\
&\lesssim A\Bigg\|\sum_{k_1}\cdots\sum_{k_m}\Big(\prod_{i=1}^{l}|\lambda_{i,k_i}|\,|Q_{i,k_i}|^{-\frac{1}{p_i}}\,\chi_{Q^{**}_{i,k_i}}\Big)\prod_{i=l+1}^{m}\|f_i\|_{L^\infty}\Bigg\|_{L^p}\\
&= A\Bigg\|\prod_{i=1}^{l}\Big(\sum_{k_i}|\lambda_{i,k_i}|\,|Q_{i,k_i}|^{-\frac{1}{p_i}}\,\chi_{Q^{**}_{i,k_i}}\Big)\Bigg\|_{L^p}\prod_{i=l+1}^{m}\|f_i\|_{L^\infty}\\
&\le A\prod_{i=1}^{l}\Bigg\|\sum_{k_i}|\lambda_{i,k_i}|\,|Q_{i,k_i}|^{-\frac{1}{p_i}}\,\chi_{Q^{**}_{i,k_i}}\Bigg\|_{L^{p_i}}\prod_{i=l+1}^{m}\|f_i\|_{L^\infty}\\
&\lesssim A\prod_{i=1}^{l}\|f_i\|_{H^{p_i}}\prod_{i=l+1}^{m}\|f_i\|_{L^\infty}\,.
\end{aligned}
$$

Thus we have

$$
\|G_1\|_{L^p}\lesssim A\,\|f_1\|_{H^{p_1}}\cdots\|f_m\|_{H^{p_m}}\,. \tag{3.4}
$$

Now for the more difficult part, $G_2(x)$, we first restrict $x\in(\bigcap_{i\notin J}Q^*_{i,k_i})\setminus(\bigcup_{i\in J}Q^*_{i,k_i})$ for some nonempty subset J of $\{1,2,\dots,l\}$. To continue, we need the following lemma whose proof is given in Section 4.

Lemma 3.3. *Let* $\frac{n}{2}<s_1,\dots,s_m<\infty$, $0<p_1,\dots,p_l\le1$, $1\le l<m$, *and suppose* (1.4) *holds for all* $J\subset\{1,\dots,l\}$. *Let* σ *be a function satisfying* (1.5). *Suppose* a_i, $1\le i\le l$, *are atoms supported in the cube* Q_i *such that*

$$
\|a_i\|_{L^\infty}\le|Q_i|^{-\frac{1}{p_i}},\qquad \int_{Q_i}x^\alpha a_i(x)dx=0
$$

for all $|\alpha|<N_i$ *with* N_i *sufficiently large. Fix a non-empty subset* J_0 *of* $\{1,\dots,l\}$. *Then there exist positive functions* $b_1,\dots,b_l$ *such that* b_i *depends only on* m, n, $(s_i)_{i=1,\dots,m}$, $(p_i)_{i=1,\dots,m}$, σ, J_0, N_i, *and* Q_i, *and*

$$
|T_\sigma(a_1,\dots,a_l,f_{l+1},\dots,f_m)(x)|\lesssim A\,b_1(x)\cdots b_l(x)\,\|f_{l+1}\|_{L^\infty}\cdots\|f_m\|_{L^\infty}
$$

for all $x\in(\bigcap_{i\notin J_0}Q^*_i)\setminus(\bigcup_{i\in J_0}Q^*_i)$, *and* $\|b_i\|_{L^{p_i}}\lesssim1$, $1\le i\le l$.

For each nonempty subset J of $\{1,2,\dots,l\}$, Lemma 3.3 guarantees the existence of positive functions $b^J_{1,k_1},\dots,b^J_{l,k_l}$ depending on $Q_{1,k_1},\dots,Q_{l,k_l}$ respectively, such that

$$|T_\sigma(a_{1,k_1},\dots,a_{l,k_l},f_{l+1},\dots,f_m)| \lesssim A\, b^J_{1,k_1}\cdots b^J_{l,k_l} \prod_{i=l+1}^{\infty} \|f_i\|_{L^\infty}$$

for all $x\in(\bigcap_{i\notin J} Q^*_{i,k_i})\setminus(\bigcup_{i\in J} Q^*_{i,k_i})$ and $\big\|b^J_{i,k_i}\big\|_{L^{p_i}}\lesssim 1$. Now set

$$b_{i,k_i} = \sum_{\emptyset\neq J\subset\{1,2,\dots,l\}} b^J_{i,k_i}.$$

Then

$$|T_\sigma(a_{1,k_1},\dots,a_{l,k_l},f_{l+1},\dots,f_m)|\chi_{(Q^*_{1,k_1}\cap\dots\cap Q^*_{l,k_l})^c} \lesssim A\, b_{1,k_1}\cdots b_{l,k_l}\prod_{i=l+1}^{\infty}\|f_i\|_{L^\infty} \tag{3.5}$$

and $\|b_{i,k_i}\|_{L^{p_i}}\lesssim 1$. Estimate (3.5) yields

$$G_2(x)\lesssim A\prod_{i=1}^{l}\left(\sum_{k_i}|\lambda_{i,k_i}|b_{i,k_i}(x)\right)\prod_{i=l+1}^{\infty}\|f_i\|_{L^\infty}.$$

Then apply Hölder's inequality to deduce that

$$\|G_2\|_{L^p}\lesssim A\,\|f_1\|_{H^{p_1}}\cdots\|f_l\|_{H^{p_l}}\prod_{i=l+1}^{\infty}\|f_i\|_{L^\infty}. \tag{3.6}$$

Combining (3.4) and (3.6), we obtain (3.2) as needed. This completes the proof. □

3.3. **The third case:** $0<p_i\le 1$ **or** $2\le p_i\le\infty$.

Theorem 3.4. *Let* $\frac{n}{2}<s_1,\dots,s_m<\infty$, $p_1,\dots,p_m\in(0,1]\cup[2,\infty]$, $0<p<\infty$, *and* $\frac{1}{p_1}+\dots+\frac{1}{p_m}=\frac{1}{p}$. *Assume there exists at least one index i such that* $p_i\in(0,1]$ *and also assume the condition* (1.4) *holds for every nonempty subset J of* $\{1,2,\dots,m\}$. *Then* (1.6) *holds.*

Proof. In addition to the assumptions of the theorem, we also assume there exists at least one i such that $p_i\in[2,\infty)$, since otherwise the claim is already covered by Theorems 3.1 or 3.2. Thus without loss of generality, we may assume that $0<p_1,\dots,p_l\le 1$, $2\le p_{l+1},\dots,p_\rho<\infty$, $p_{\rho+1}=\dots=p_m=\infty$, $1\le l<\rho\le m$, and $\frac{1}{p_1}+\dots+\frac{1}{p_\rho}=\frac{1}{p}$. Our goal is to establish the estimate

$$\|T_\sigma\|_{H^{p_1}\times\dots\times H^{p_l}\times L^{p_{l+1}}\times\dots\times L^{p_\rho}\times L^\infty\times\dots\times L^\infty\longrightarrow L^p}\lesssim\sup_{j\in\mathbb{Z}}\left\|\sigma(2^j\cdot)\widehat{\psi}\right\|_{W^{(s_1,\dots,s_m)}}. \tag{3.7}$$

Assume momentarily the validity of the following estimate

$$\|T_\sigma\|_{H^{p_1}\times\dots\times H^{p_l}\times\underbrace{L^2\times\dots\times L^2}_{(\rho-l)-\text{ times}}\times\underbrace{L^\infty\times\dots\times L^\infty}_{(m-\rho)-\text{ times}}\longrightarrow L^p}\lesssim\sup_{j\in\mathbb{Z}}\left\|\sigma(2^j\cdot)\widehat{\psi}\right\|_{W^{(s_1,\dots,s_m)}}. \tag{3.8}$$

Then using Theorem 2.4 to interpolate between (3.8) and (3.1), we obtain the estimate (3.7) as required. (In fact, since the condition (1.4) with $(p_i)_{i=1,\dots,m}$ in the estimates of (3.1), (3.7), and (3.8) gives the same restriction on $(s_i)_{i=1,\dots,m}$, in order to deduce (3.7) from (3.8) and (3.1), we may fix $(s_i)_{i=1,\dots,m}$ and could use the usual real or complex interpolation for linear operators.) Thus it suffices to prove (3.8). In the rest of the proof, we assume $p_{l+1}=\dots=p_\rho=2$.

Before we proceed to the proof of (3.8), we shall see that it is sufficient to consider σ that has support in some cone. To see this, for $\eta=(\eta_1,\dots,\eta_m)\in\mathbb{R}^{mn}$, consider the $m+1$ vectors $\eta_1,\dots,\eta_m$, $\eta_{m+1}=\sum_{i=1}^m\eta_i\in\mathbb{R}^n$. If η belongs to the unit sphere $\Sigma=\{\eta\in\mathbb{R}^{mn}\ :\ |\eta|=1\}$, then at least two of these $m+1$ vectors are not zero. Hence, by the compactness of Σ, there exists a constant $a > 0$ such that Σ is covered by the $\binom{m+1}{2}$ open sets

$$V(k_1, k_2) = \{\eta \in \Sigma : \ |\eta_{k_1}| > a, \ |\eta_{k_2}| > a\}, \quad 1 \le k_1 < k_2 \le m+1.$$

We take a smooth partition of unity $\{\varphi_{k_1,k_2}\}$ on Σ such that $\operatorname{supp} \varphi_{k_1,k_2} \subset V(k_1,k_2)$ and decompose the multiplier σ as

$$\sigma(\xi) = \sum_{1\le k_1<k_2\le m+1} \sigma(\xi)\varphi_{k_1,k_2}(\xi/|\xi|) = \sum_{1\le k_1<k_2\le m+1} \sigma_{k_1,k_2}(\xi).$$

Then

$$\operatorname{supp} \sigma_{k_1,k_2} \subset \Gamma(V(k_1,k_2)) = \{\xi \in \mathbb{R}^{mn} \setminus \{0\} : \ \xi/|\xi| \in V(k_1,k_2)\}$$

and Lemma 2.3 gives

$$\sup_{j\in\mathbb{Z}} \left\| \sigma_{k_1,k_2}(2^j \cdot)\widehat{\psi} \right\|_{W^{(s_1,\dots,s_m)}} \lesssim \sup_{j\in\mathbb{Z}} \left\| \sigma(2^j \cdot)\widehat{\psi} \right\|_{W^{(s_1,\dots,s_m)}}.$$

The estimate (1.6) follows if we prove it with σ_{k_1,k_2} in place of σ. This means that it is sufficient to prove (1.6) under the additional assumption that

$$\operatorname{supp} \sigma \subset \Gamma(V(k_1,k_2)) \tag{3.9}$$

for some $1 \le k_1 < k_2 \le m+1$.

To simplify notation, we also assume

$$\sup_{j\in\mathbb{Z}} \left\| \sigma(2^j \cdot)\widehat{\psi} \right\|_{W^{(s_1,\dots,s_m)}} = 1 \tag{3.10}$$

and write

$$\sigma = \sum_{j\in\mathbb{Z}} \sigma_j, \quad \sigma_j(\xi) = \sigma(\xi)\widehat{\psi}(2^{-j}\xi). \tag{3.11}$$

We shall divide the proof into two cases. First case: σ satisfies (3.9) with $1 \le k_1 < k_2 \le m$. Second case: σ satisfies (3.9) with $1 \le k_1 \le m$ and $k_2 = m+1$. In the first case, we shall directly prove the estimate

$$\|T_\sigma(f_1,\dots,f_m)\|_{L^p} \lesssim \prod_{i=1}^m \|f_i\|_{H^{p_i}}. \tag{3.12}$$

In the second case, we shall use a Littlewood-Paley function. Notice that, in the second case, the support of the Fourier transform of $T_{\sigma_j}(f_1,\dots,f_m)$ is included in the annulus $\{\xi \in \mathbb{R}^n : \ B^{-1}2^j \le |\xi| \le B2^j\}$ with some constant $B > 1$. Hence, by Lemma 2.9, we have

$$\|T_\sigma(f_1,\dots,f_m)\|_{H^p} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} \left|T_{\sigma_j}(f_1,\dots,f_m)\right|^2 \right)^{1/2} \right\|_{L^p}. \tag{3.13}$$

Thus, in the second case, we shall consider the function

$$GT_\sigma(f_1,\dots,f_m) = \left(\sum_{j\in\mathbb{Z}} \left|T_{\sigma_j}(f_1,\dots,f_m)\right|^2 \right)^{1/2}$$

and prove the estimate

$$\|GT_\sigma(f_1,\dots,f_m)\|_{L^p} \lesssim \prod_{i=1}^m \|f_i\|_{H^{p_i}}, \tag{3.14}$$

which combined with (3.13) implies (3.12).

The essential part of the proofs of (3.12) and (3.14) are given in the following two lemmas.

Lemma 3.5. *Let* $\frac{n}{2} < s_1, \dots, s_m < \infty$, $0 < p_1, \dots, p_l \le 1$, $p_{l+1} = \cdots = p_\rho = 2$, $p_{\rho+1} = \cdots = p_m = \infty$, $1 \le l < \rho \le m$, *and suppose* (1.4) *holds for every nonempty subset* J *of* $\{1, \dots, l\}$. *Let* a_i, $1 \le i \le l$, *be* H^{p_i} *atoms such that*

$$\operatorname{supp} a_i \subset Q_i, \quad \|a_i\|_{L^\infty} \le |Q_i|^{-1/p_i}, \quad \int a_i(x) x^\alpha \, dx = 0$$

for $|\alpha| < N_i$, *where* N_i *is a sufficiently large positive integer and* Q_i *is a cube. Let* $f_{l+1}, \dots, f_\rho \in L^2$ *and* $f_{\rho+1}, \dots, f_m \in L^\infty$. *Finally suppose* σ *satisfies* (3.10) *and* (3.9) *with some* $1 \le k_1 < k_2 \le m$. *Then there exist functions* $b_1, \dots, b_l$ *and* $\widetilde{f}_{l+1}, \dots, \widetilde{f}_\rho$ *such that*

$$|T_\sigma(a_1, \dots, a_l, f_{l+1}, \dots, f_m)(x)| \lesssim \prod_{i=1}^{l} b_i(x) \cdot \prod_{i=l+1}^{\rho} \widetilde{f}_i(x) \cdot \prod_{i=\rho+1}^{m} \|f_i\|_{L^\infty} ; \tag{3.15}$$

the function b_i *depends only on* m, n, $(s_i)_{i=1,\dots,m}$, $(p_i)_{i=1,\dots,m}$, σ, i, a_i, *and* $(f_i)_{i=\rho+1,\dots,m}$; *the function* $\widetilde{f}_i$ *depends only on* m, n, $(s_i)_{i=1,\dots,m}$, i, f_i, *and* $(f_i)_{i=\rho+1,\dots,m}$; *and they satisfy the estimates* $\|b_i\|_{L^{p_i}} \lesssim 1$ *and* $\|\widetilde{f}_i\|_{L^2} \lesssim \|f_i\|_{L^2}$.

Lemma 3.6. *Let* s_i, p_i, a_i, *and* f_i *be the same as in Lemma 3.5. Suppose* σ *satisfies* (3.10) *and* (3.9) *with some* $1 \le k_1 \le m$ *and* $k_2 = m+1$. *Then there exist functions* $b_1, \dots, b_l$ *and* $\widetilde{f}_{l+1}, \dots, \widetilde{f}_\rho$ *that satisfy*

$$GT_\sigma(a_1, \dots, a_l, f_{l+1}, \dots, f_m)(x) \lesssim \prod_{i=1}^{l} b_i(x) \cdot \prod_{i=l+1}^{\rho} \widetilde{f}_i(x) \cdot \prod_{i=\rho+1}^{m} \|f_i\|_{L^\infty}$$

and have the same properties as in Lemma 3.5.

The proofs of these lemmas will be given in Section 4. We shall continue the proof of Theorem 3.4. To utilize the above lemmas, we decompose $f_i \in H^{p_i}$, $1 \le i \le l$, into atoms as $f_i = \sum_{k_i \in \mathbb{Z}} \lambda_{i,k_i} a_{i,k_i}$ with λ_{i,k_i}, a_{i,k_i}, and the cubes Q_{i,k_i} being the same as in the proof of Theorem 3.2.

Consider the first case where σ satisfies (3.9) with $1 \le k_1 < k_2 \le m$. In this case, Lemma 3.5 yields functions b_{i,k_i} $(1 \le i \le l,\ k_i \in \mathbb{Z})$ and $\widetilde{f}_i$ $(l+1 \le i \le \rho)$ such that

$$|T_\sigma(a_{1,k_1}, \dots, a_{l,k_l}, f_{l+1}, \dots, f_m)(x)| \lesssim \prod_{i=1}^{l} b_{i,k_i}(x) \cdot \prod_{i=l+1}^{\rho} \widetilde{f}_i(x) \cdot \prod_{i=\rho+1}^{m} \|f_i\|_{L^\infty}$$

and $\|b_{i,k_i}\|_{L^{p_i}} \lesssim 1$ and $\|\widetilde{f}_i\|_{L^2} \lesssim \|f_i\|_{L^2}$. Notice that b_{i,k_i} do not depend on k_j with $j \ne i$ and $\widetilde{f}_i$ do not depend on $k_1, \dots, k_l$. Hence, by the multilinear property of the operator T_σ, we have

$$\begin{aligned} |T_\sigma(f_1, \dots, f_m)(x)| &\lesssim \sum_{k_1} \cdots \sum_{k_l} |\lambda_{1,k_1} \dots \lambda_{l,k_l}| \prod_{i=1}^{l} b_{i,k_i}(x) \cdot \prod_{i=l+1}^{\rho} \widetilde{f}_i(x) \cdot \prod_{i=\rho+1}^{m} \|f_i\|_{L^\infty} \\ &= \prod_{i=1}^{l} \Big(\sum_{k_i} |\lambda_{i,k_i}| \, b_{i,k_i}(x) \Big) \cdot \prod_{i=l+1}^{\rho} \widetilde{f}_i(x) \cdot \prod_{i=\rho+1}^{m} \|f_i\|_{L^\infty} . \end{aligned}$$

(We omit a necessary limiting argument to treat the infinite sum, which could be achieved with the aid of Proposition 2.5.) For $1 \le i \le l$, we have

$$\Big\| \sum_{k_i} |\lambda_{i,k_i}| \, b_{i,k_i} \Big\|_{L^{p_i}}^{p_i} \le \sum_{k_i} |\lambda_{i,k_i}|^{p_i} \|b_{i,k_i}\|_{L^{p_i}}^{p_i} \lesssim \sum_{k_i} |\lambda_{i,k_i}|^{p_i} \lesssim \|f_i\|_{H^{p_i}}^{p_i} .$$

The above pointwise inequality and Hölder's inequality now give (3.12).

Next consider the second case where σ satisfies (3.9) with $1 \le k_1 \le m$ and $k_2 = m+1$. By the sublinear property of square function, we have

$$GT_\sigma(f_1,\dots,f_m)(x) \le \sum_{k_1}\cdots\sum_{k_l} |\lambda_{1,k_1}\cdots\lambda_{l,k_l}|\, GT_\sigma(a_{1,k_1},\dots,a_{l,k_l},f_{l+1},\dots,f_m)(x).$$

(Again we omit the necessary limiting argument.) Hence, using Lemma 3.6 and arguing in the same way as in the first case, we obtain (3.14). Thus the proof of Theorem 3.4 is reduced to Lemmas 3.5 and 3.6. □

3.4. **The last case: $0 < p_i \le \infty$.** In this subsection, we shall prove the estimate (1.6) for the entire range $0 < p_i \le \infty$. Since the necessity of the conditions $s_i \ge n/2$ and (1.7) has already been shown in [13, Theorem 5.1], this will complete the proof of Theorem 1.1. To simplify notation, we use the letters $\mathbf{s}$ and $\mathbf{p}$ to denote $(s_1,\dots,s_m)$ and $(p_1,\dots,p_m)$, respectively.

We shall slightly change the formulation of the claim of Theorem 1.1. We assume $0 < p_1,\dots,p_m \le \infty$,

$$\infty > s_1,\dots,s_m \ge n/2, \tag{3.16}$$

and assume they satisfy (1.7) for every nonempty subset J of $\{1,\dots,m\}$. We shall prove the estimate

$$\|T_\sigma\|_{H^{p_1}\times\cdots\times H^{p_m}\to L^p} \lesssim \sup_{j\in\mathbb{Z}} \left\|\sigma(2^j\cdot)\widehat{\psi}\right\|_{W^{(s_1+\epsilon,\cdots,s_m+\epsilon)}} \tag{3.17}$$

holds for every $\epsilon > 0$, where $1/p = 1/p_1+\cdots+1/p_m$ and the space L^p should be replaced by BMO if $p_1=\cdots=p_m=p=\infty$. This is equivalent to the estimate given in Theorem 1.1. The proof will be given in two steps.

In the first step, we fix $\mathbf{s}$ satisfying (3.16) and consider the set $\Delta(\mathbf{s})$ that consists of all $(1/p_1,\dots,1/p_m) \in [0,\infty)^m$ such that the condition (1.7) holds for every nonempty subset J of $\{1,\dots,m\}$. We prove the following lemma.

Lemma 3.7. *If $\mathbf{s}$ satisfies (3.16), then $\Delta(\mathbf{s})$ is the convex hull of the point $(0,\dots,0)$ and the points $(1/p_1,\dots,1/p_m)$ that satisfy*

$$1/p_i = 0 \text{ or } 1/p_i = s_i/n \text{ or } 1/p_i = s_i/n + 1/2 \text{ for all } i, \tag{3.18}$$

and

$$1/p_i = s_i/n + 1/2 \text{ for exactly one } i. \tag{3.19}$$

Proof. Fix $\mathbf{s} = (s_1,\dots,s_m)$ such that $s_i \ge \frac{n}{2}$ for all $1 \le i \le m$. Condition (1.7) gives a clearer presentation of the set $\Delta(\mathbf{s})$ as

$$\Delta(m,\mathbf{s}) = \left\{ \Big(\frac{1}{p_1},\dots,\frac{1}{p_m}\Big) \in \mathbb{R}^m :\ 0 \le \frac{1}{p_i} \le \frac{s_i}{n}+\frac{1}{2},\ \sum_{i\in J}\frac{1}{p_i} \le \sum_{i\in J}\frac{s_i}{n}+\frac{1}{2} \right\},$$

where J runs over all non-empty subsets of $\{1,\dots,m\}$. We let H denote the convex hull of $(0,\dots,0)$ and of all the points $(1/p_1,\dots,1/p_m)$ that satisfy (3.18) and (3.19). We will show that $\Delta(m,\mathbf{s}) = H$ by induction in m.

The case when $m=2$ is trivial because $\Delta(2,\mathbf{s})$ is the convex hull of the following points $(0,0)$, $(\frac{s_1}{n}+\frac{1}{2},0)$, $(\frac{s_1}{n}+\frac{1}{2},\frac{s_2}{n})$, $(0,\frac{s_2}{n}+\frac{1}{2})$ and $(\frac{s_1}{n},\frac{s_2}{n}+\frac{1}{2})$; hence, the statement of Lemma 3.7 holds obviously in this case.

Now fix an $m > 2$ and suppose that the statement of the lemma is true for $m-1$. For $1 \le k \le m$, denote

$$\Delta^k(m,\mathbf{s}) = \left\{ \Big(\frac{1}{p_1},\dots,\frac{1}{p_m}\Big) \in \Delta(m,\mathbf{s}) :\ 0 \le \frac{1}{p_k} \le \frac{s_k}{n} \right\},$$

$$F_0^k(m,\mathbf{s}) = \left\{ \Big(\frac{1}{p_1},\dots,\frac{1}{p_m}\Big) \in \Delta(m,\mathbf{s}) :\ \frac{1}{p_k} = 0 \right\},$$

$$F_1^k(m,\mathbf{s}) = \left\{ \Big(\frac{1}{p_1},\dots,\frac{1}{p_m}\Big) \in \Delta(m,\mathbf{s}) : \ \frac{1}{p_k} = \frac{s_k}{n} \right\},$$

and

$$\Delta^0(m,\mathbf{s}) = \left\{ \Big(\frac{1}{p_1},\dots,\frac{1}{p_m}\Big) \in \Delta(m,\mathbf{s}) : \ \frac{s_i}{n} \le \frac{1}{p_i} \le \frac{s_i}{n} + \frac{1}{2}, \ \forall\, 1 \le i \le m \right\}.$$

It is easy to see that $\Delta(m,\mathbf{s}) = \cup_{k=0}^m \Delta^k(m,\mathbf{s})$. We observe that H is a subset of $\Delta(m,\mathbf{s})$, since each vertex of H obviously sits inside the convex set $\Delta(m,\mathbf{s})$. Thus, it suffices to prove that $\Delta^k(m,\mathbf{s})$ is a subset of H for every $0 \le k \le m$.

We first consider $\Delta^k(m,\mathbf{s})$ for $1 \le k \le m$. By induction, the face $F_0^k(m,\mathbf{s})$ is the convex hull of the following points $(0,\dots,0)$ and $(\frac{1}{p_1},\dots,\frac{1}{p_m})$, where $\frac{1}{p_k} = 0$, $\frac{1}{p_i} \in \big\{0, \frac{s_i}{n}, \frac{s_i}{n} + \frac12\big\}$ for $i \ne k$, and there exists exactly one $i \ne k$ such that $\frac{1}{p_i} = \frac{s_i}{n} + \frac12$. Similarly, the face $F_1^k(m,\mathbf{s})$ is determined by the same constraints for all variables $\frac{1}{p_i}, i \ne k$ as those for $F_0^k(m,\mathbf{s})$. Therefore, by induction, we have that $F_1^k(m,\mathbf{s})$ is the convex hull of the points $(0,\dots,0,\frac{s_k}{n},0,\dots,0)$ and $(\frac{1}{p_1},\dots,\frac{1}{p_m})$, where $\frac{1}{p_k} = \frac{s_k}{n}$, $\frac{1}{p_i} \in \big\{0, \frac{s_i}{n}, \frac{s_i}{n} + \frac12\big\}$ for $i \ne k$, and there exists exactly one $i \ne k$ such that $\frac{1}{p_i} = \frac{s_i}{n} + \frac12$. Note that the point $(0,\dots,0,\frac{s_k}{n},0,\dots,0)$ belongs to the line segment that joins the origin $(0,\dots,0)$ with $(0,\dots,0,\frac{s_k}{n}+\frac12,0,\dots,0)$. Thus $F_0^k(m,\mathbf{s})$ and $F_1^k(m,\mathbf{s})$ are contained in H, and hence, $\Delta^k(m,\mathbf{s})$ is a subset of H since $\Delta^k(m,\mathbf{s})$ is a convex hull of two faces $F_0^k(m,\mathbf{s})$ and $F_1^k(m,\mathbf{s})$.

It remains to check that $\Delta^0(m,\mathbf{s}) \subset H$. In this case, we note that the constraints $0 \le \frac{1}{p_i} - \frac{s_i}{n} \le \frac12$, $\forall\, 1 \le i \le m$ and

$$\sum_{i=1}^m \left(\frac{1}{p_i} - \frac{s_i}{n} \right) \le \frac12$$

imply that $\Delta^0(m,\mathbf{s})$ is a standard m-simplex with vertices $(\frac{s_1}{n},\dots,\frac{s_m}{n})$ and $(\frac{1}{p_1},\dots,\frac{1}{p_m})$, where $\frac{1}{p_i} \in \big\{\frac{s_i}{n}, \frac{s_i}{n}+\frac12\big\}$ for $1 \le i \le m$, and there exists exactly one i such that $\frac{1}{p_i} = \frac{s_i}{n} + \frac12$, which implies $\Delta^0(m,\mathbf{s}) \subset H$ with noting that the point $(\frac{s_1}{n},\dots,\frac{s_m}{n}) \in F_1^m(m,\mathbf{s}) \subset H$. ☐

By virtue of Lemma 3.7 and Theorem 2.4, to prove the estimate (3.17) under the assumptions (3.16) and (1.7), it is sufficient to show it for $\mathbf{p} = (\infty,\dots,\infty)$ and for $\mathbf{p}$ satisfying (3.18) and (3.19). For $\mathbf{p} = (\infty,\dots,\infty)$, the estimate (3.17) with BMO in place of L^p is established in [13, Corollary 6.3]. Thus it is sufficient to consider the latter points.

In the second step, we shall prove the following lemma, which will complete the proof of Theorem 1.1.

Lemma 3.8. *Estimate* (3.17) *holds if* $\mathbf{s}$ *and* $\mathbf{p}$ *satisfy* (3.16)*,* (3.18)*, and* (3.19)*.*

Proof. For $\mathbf{p} \in (0,\infty]^m$, we define $\ell(\mathbf{p})$ to be the number of the indices $i \in \{1,\dots,m\}$ such that $1 < p_i < 2$. We shall prove the claim by induction on $\ell(\mathbf{p})$.

The conditions (3.16) and (3.19) imply in particular that there exists at least one i such that $p_i \le 1$. Hence if $\ell(\mathbf{p}) = 0$ then the claim directly follows from Theorem 3.4.

Assume $\ell_0 \ge 1$ and assume the claim holds if $\ell(\mathbf{p}) < \ell_0$. Let

$$(\mathbf{p}^0,\mathbf{s}^0) = (p_1^0,\dots,p_m^0,s_1^0,\dots,s_m^0)$$

be a point that satisfies the conditions (3.16), (3.18), and (3.19), and satisfies $\ell(\mathbf{p}^0) = \ell_0$. There exists an index i such that $1 < p_i^0 < 2$. Notice that $1/p_i^0 = s_i^0/n$ for this index i. Without loss of generality, we assume $1 > 1/p_1^0 = s_1^0/n > 1/2$. Then the condition (3.19) implies that there exists exactly one i such that $2 \le i \le m$ and $1/p_i^0 = s_i^0/n + 1/2$. Consider the following two points:

$$(\mathbf{p}',\mathbf{s}') = (1,p_2^0,\dots,p_m^0,n,s_2^0,\dots,s_m^0),$$

$$(\mathbf{p}'', \mathbf{s}'') = (2, p_2^0, \ldots, p_m^0, n/2, s_2^0, \ldots, s_m^0).$$

Both $(\mathbf{p}', \mathbf{s}')$ and $(\mathbf{p}'', \mathbf{s}'')$ satisfy the conditions (3.16), (3.18), and (3.19), and $\ell(\mathbf{p}') = \ell(\mathbf{p}'') = \ell_0 - 1$. Hence by the induction hypothesis the estimate (3.17) holds for $(\mathbf{p}', \mathbf{s}')$ and $(\mathbf{p}'', \mathbf{s}'')$. Then, by Theorem 2.4, it follows that the estimate (3.17) also holds for $(\mathbf{p}^0, \mathbf{s}^0)$. This completes the proof of Lemma 3.8. □

4. Proofs of the key lemmas

Proof of Lemma 3.3. Without loss of generality, we assume that $J_0 = \{1, \ldots, r\}$ for some $1 \le r \le l$, and $\|f_i\|_{L^\infty} = 1$ for all $l+1 \le i \le m$. Fix

$$x \in \Big(\bigcap_{i=r+1}^{l} Q_i^* \Big) \setminus \bigcup_{i=1}^{r} Q_i^*$$

(when $r = l$, just fix $x \in \mathbb{R}^n \setminus \bigcup_{i=1}^{l} Q_i^*$). Now we can write

$$T_\sigma(a_1, \ldots, a_l, f_{l+1}, \ldots, f_m)(x) = \sum_{j \in \mathbb{Z}} g_j(x),$$

where $g_j(x)$ is the function

$$\int_{\mathbb{R}^{mn}} 2^{jmn} K_j(2^j(x-y_1), \ldots, 2^j(x-y_m)) a_1(y_1) \cdots a_l(y_l) f_{l+1}(y_{l+1}) \cdots f_m(y_m)\, d\vec{y}$$

with $K_j = \big(\sigma(2^j \cdot)\widehat{\psi}\big)^\vee$. Let c_i be the center of the cube Q_i $(1 \le i \le l)$. For $1 \le i \le r$, since $x \notin Q_i^*$, we have $|x - c_i| \approx |x - y_i|$ for all $y_i \in Q_i$. Fix $1 \le k \le r$. Using Lemma 2.2 and applying the Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
&\prod_{i=1}^{r} \big\langle 2^j(x-c_i) \big\rangle^{s_i} |g_j(x)| \\
&\lesssim 2^{jmn} \int\limits_{Q_1 \times \cdots \times Q_l \times \mathbb{R}^{(m-l)n}} \prod_{i=1}^{r} \big\langle 2^j(x-y_i) \big\rangle^{s_i} \big| K_j(2^j(x-y_1), \ldots, 2^j(x-y_m)) \big| \prod_{i=1}^{l} \|a_i\|_{L^\infty}\, d\vec{y} \\
&\le 2^{jmn} \prod_{i=1}^{l} |Q_i|^{-\frac{1}{p_i}} \int\limits_{Q_1 \times \cdots \times Q_r \times \mathbb{R}^{(m-r)n}} \prod_{i=1}^{r} \big\langle 2^j(x-y_i) \big\rangle^{s_i} \big| K_j(2^j(x-y_1), \ldots, 2^j(x-y_m)) \big|\, d\vec{y} \\
&= 2^{jrn} \prod_{i=1}^{l} |Q_i|^{-\frac{1}{p_i}} \int\limits_{Q_1 \times \cdots \times Q_r \times \mathbb{R}^{(m-r)n}} \prod_{i=1}^{r} \big\langle 2^j(x-y_i) \big\rangle^{s_i} \\
&\quad \times \big| K_j(2^j(x-y_1), \ldots, 2^j(x-y_r), y_{r+1}, \ldots, y_m) \big|\, dy_1 \cdots dy_r dy_{r+1} \cdots dy_m \\
&\le 2^{jrn} \prod_{i=1}^{r} |Q_i|^{1-\frac{1}{p_i}} \prod_{i=r+1}^{l} |Q_i|^{-\frac{1}{p_i}} \int_{\mathbb{R}^{(m-r)n}} \int\limits_{Q_k} |Q_k|^{-1} \big\langle 2^j(x-y_k) \big\rangle^{s_k} \times \\
&\quad \times \Bigg\| \prod_{\substack{i=1 \\ i \ne k}}^{r} \langle y_i \rangle^{s_i} K_j(y_1, \ldots, y_{k-1}, 2^j(x-y_k), y_{k+1}, \ldots, y_m) \Bigg\|_{L^\infty(dy_1 \cdots \widehat{dy_k} \cdots dy_r)} dy_k dy_{r+1} \cdots dy_m \\
&\lesssim 2^{jrn} \prod_{i=1}^{r} |Q_i|^{1-\frac{1}{p_i}} \prod_{i=r+1}^{l} |Q_i|^{-\frac{1}{p_i}} \int_{\mathbb{R}^{(m-r)n}} \int\limits_{Q_k} |Q_k|^{-1} \big\langle 2^j(x-y_k) \big\rangle^{s_k} \times
\end{aligned}
$$

$$\times \left\| \prod_{\substack{i=1\\ i\neq k}}^{r} \langle y_i\rangle^{s_i} K_j(y_1,\dots,y_{k-1},2^j(x-y_k),y_{k+1},\dots,y_m)\right\|_{L^2(dy_1\cdots\widehat{dy_k}\cdots dy_r)} dy_k dy_{r+1}\cdots dy_m$$

$$\lesssim 2^{jrn}\prod_{i=1}^{r}|Q_i|^{1-\frac{1}{p_i}}\prod_{i=r+1}^{l}|Q_i|^{-\frac{1}{p_i}}\int_{Q_k}|Q_k|^{-1}\left\langle 2^j(x-y_k)\right\rangle^{s_k}\times$$

$$\times \left\| \prod_{\substack{i=1\\ i\neq k}}^{m} \langle y_i\rangle^{s_i} K_j(y_1,\dots,y_{k-1},2^j(x-y_k),y_{k+1},\dots,y_m)\right\|_{L^2(dy_1\cdots\widehat{dy_k}\cdots dy_m)} dy_k$$

(4.1)

$$= 2^{jrn}\Big(\prod_{i=1}^{r}|Q_i|^{1-\frac{1}{p_i}}\Big)\Big(\prod_{i=r+1}^{l}b_i(x)\Big)h_j^{(k,0)}(x),$$

where

$$h_j^{(k,0)}(x)=\frac{1}{|Q_k|}\int_{Q_k}\left\langle 2^j(x-y_k)\right\rangle^{s_k}$$

$$\times \left\| \prod_{\substack{i=1\\ i\neq k}}^{m} \langle y_i\rangle^{s_i} K_j(y_1,\dots,y_{k-1},2^j(x-y_k),y_{k+1},\dots,y_m)\right\|_{L^2(dy_1\cdots\widehat{dy_k}\cdots dy_m)} dy_k$$

and $b_i(x)=|Q_i|^{-\frac{1}{p_i}}\chi_{Q_i^*}(x)$ for $r+1\le i\le l$. The functions b_i, $r+1\le i\le l$, obviously satisfy the estimate $\|b_i\|_{L^{p_i}}\lesssim 1$. Minkowski's inequality gives

$$\left\|h_j^{(k,0)}\right\|_{L^2}\le 2^{-\frac{jn}{2}}\left\|\sigma(2^j\cdot)\widehat{\psi}\right\|_{W^{(s_1,\dots,s_m)}}\le A2^{-\frac{jn}{2}}.$$

Using the vanishing moment condition of a_k and Taylor's formula, we write

$$\begin{aligned} g_j(x) = {} & 2^{jmn}\sum_{|\alpha|=N_k}C_\alpha\int_{\mathbb{R}^{mn}}\Bigg\{\int_0^1(1-t)^{N_k-1}\\ & \times\partial_k^\alpha K_j\Big(2^j(x-y_1),\dots,2^jx^t_{c_k,y_k},\dots,2^j(x-y_m)\Big)\\ & \times(2^j(y_k-c_k))^\alpha a_1(y_1)\cdots a_l(y_l)f_{l+1}(y_{l+1})\cdots f_m(y_m)\,dt\Bigg\}\,dy_1\cdots dy_m,\end{aligned}$$

where $x^t_{c_k,y_k}=x-c_k-t(y_k-c_k)$ and $\partial_k^\alpha K_j(z_1,\dots,z_m)=\partial_{z_k}^\alpha K_j(z_1,\dots,z_m)$. Notice that $\left|x^t_{c_k,y_k}\right|\approx|x-c_k|$ for $x\notin Q_k^*$, $y_k\in Q_k$, and $0<t<1$. Repeating the preceding argument, we obtain

$$\prod_{i=1}^{r}\left\langle 2^j(x-c_i)\right\rangle^{s_i}|g_j(x)|\lesssim 2^{jrn}\Big(\prod_{i=1}^{r}|Q_i|^{1-\frac{1}{p_i}}\Big)\Big(\prod_{i=r+1}^{l}b_i(x)\Big)h_j^{(k,1)}(x), \tag{4.2}$$

where $b_i(x)$ are the same as above and

$$h_j^{(k,1)}(x)=(2^j\ell(Q_k))^{N_k}|Q_k|^{-1}\sum_{|\alpha|=N_k}\int_{Q_k}\Bigg\{\int_0^1\left\langle 2^jx^t_{c_k,y_k}\right\rangle^{s_k}$$

$$\times\left\| \prod_{\substack{i=1\\ i\neq k}}^{l} \langle y_i\rangle^{s_i}\partial_k^\alpha K_j(y_1,\dots,y_{k-1},2^jx^t_{c_k,y_k},y_{k+1},\dots,y_m)\right\|_{L^2(dy_1\cdots\widehat{dy_k}\cdots dy_m)} dt\Bigg\}\,dy_k$$

($\ell(Q_k)$ denotes the sidelength of the cube Q_k). Minkowski's inequality and Lemma 2.2 imply that

$$\left\| h_j^{(k,1)} \right\|_{L^2} \lesssim A 2^{-\frac{jn}{2}} (2^j \ell(Q_k))^{N_k}.$$

Combining inequalities (4.1) and (4.2), we obtain

$$\begin{aligned} &\left(\prod_{i=1}^{r} \langle 2^j (x-c_i) \rangle^{s_i} \right) |g_j(x)| \\ &\lesssim 2^{jrn} \left(\prod_{i=1}^{r} |Q_i|^{1-\frac{1}{p_i}} \right) \left(\prod_{i=r+1}^{l} b_i(x) \right) \min \left\{ h_j^{(k,0)}(x), h_j^{(k,1)}(x) \right\} \end{aligned} \tag{4.3}$$

for all $1 \le k \le r$. The inequalities in (4.3) imply that

$$\begin{aligned} &|g_j(x)| \\ &\le 2^{jrn} \left(\prod_{i=1}^{r} |Q_i|^{1-\frac{1}{p_i}} \langle 2^j (x-c_i) \rangle^{-s_i} \right) \left(\prod_{i=r+1}^{l} b_i(x) \right) \min_{1 \le k \le r} \left\{ h_j^{(k,0)}(x), h_j^{(k,1)}(x) \right\} \end{aligned} \tag{4.4}$$

for all $x \in (\bigcap_{i=r+1}^{l} Q_i^*) \setminus (\bigcup_{i=1}^{r} Q_i^*)$.

Now we need to construct functions u_j^k ($1 \le k \le r$) such that

$$|g_j(x)| \lesssim A \prod_{k=1}^{r} u_j^k(x) \prod_{i=r+1}^{l} b_i(x)$$

for all $x \in (\bigcap_{i=r+1}^{l} Q_i^*) \setminus (\bigcup_{i=1}^{r} Q_i^*)$ and that $\left\| \sum_j u_j^k \right\|_{L^{p_k}} \lesssim 1$. Then the lemma follows by taking $b_k = \sum_j u_j^k$ ($1 \le k \le r$).

For this, we choose λ_k, $1 \le k \le r$, such that

$$0 \le \lambda_k < \frac{1}{2}, \quad \frac{s_k}{n} > \frac{1}{p_k} - \frac{1}{2} + \lambda_k, \quad \sum_{k=1}^{r} \lambda_k = \frac{r-1}{2}.$$

This is possible since (1.4) implies

$$\sum_{k=1}^{r} \min \left\{ \frac{1}{2}, \frac{s_k}{n} - \frac{1}{p_k} + \frac{1}{2} \right\} > \frac{r-1}{2}.$$

We set $\alpha_k = \frac{1}{p_k} - \frac{1}{2} + \lambda_k$ and $\beta_k = 1 - 2\lambda_k$. Then we have $\alpha_k > 0$, $\beta_k > 0$, and $\sum_{k=1}^{r} \beta_k = 1$. We set

$$u_j^k = A^{-\beta_k} 2^{jn} |Q_k|^{1-\frac{1}{p_k}} \langle 2^j (\cdot - c_k) \rangle^{-s_k} \chi_{(Q_k^*)^c} \min \left\{ h_j^{(k,0)}, h_j^{(k,1)} \right\}^{\beta_k}, \quad 1 \le k \le r.$$

Then, from (4.4), it is easy to see that

$$g_j(x) \lesssim A \prod_{k=1}^{r} u_j^k(x) \prod_{i=r+1}^{l} b_i(x)$$

for all $x \in (\cap_{i=r+1}^{l} Q_i^*) \setminus (\cup_{i=1}^{r} Q_i^*)$. It remains to check that $\sum_j \int_{\mathbb{R}^n} |u_j^k(x)|^{p_k} dx \lesssim 1$. Since $\frac{1}{p_k} = \alpha_k + \frac{\beta_k}{2}$, Hölder's inequality gives

$$\left\| u_j^k \right\|_{L^{p_k}} \le A^{-\beta_k} 2^{jn} |Q_k|^{1-\frac{1}{p_k}} \left\| \langle 2^j (\cdot - c_k) \rangle^{-s_k} \chi_{(Q_k^*)^c} \right\|_{L^{\frac{1}{\alpha_k}}} \left\| \min \left\{ h_j^{(k,0)}, h_j^{(k,1)} \right\}^{\beta_k} \right\|_{L^{\frac{2}{\beta_k}}}.$$

Since $\frac{s_k}{\alpha_k} > n$, we have

$$\left\| \langle 2^j (\cdot - c_k) \rangle^{-s_k} \chi_{(Q_k^*)^c} \right\|_{L^{1/\alpha_k}} \approx 2^{-jn\alpha_k} \min \left\{ 1, (2^j \ell(Q_k))^{\alpha_k n - s_k} \right\}.$$

The estimates of L^2-norms of $h_j^{(k,0)}$ and $h_j^{(k,1)}$ given above imply

$$\begin{aligned}\left\| \left(\min\left\{ h_j^{(k,0)}, h_j^{(k,1)} \right\} \right)^{\beta_k} \right\|_{L^{2/\beta_k}} &\le \min\left\{ \left\| h_j^{(k,0)} \right\|_{L^2}^{\beta_k}, \left\| h_j^{(k,1)} \right\|_{L^2}^{\beta_k} \right\} \\ &\lesssim \Big(A 2^{-jn/2} \min\left\{1, (2^j \ell(Q_k))^{N_k} \right\} \Big)^{\beta_k}.\end{aligned}$$

Therefore

$$\begin{aligned}\left\| u_j^k \right\|_{L^{p_k}} &\le 2^{jn} |Q_k|^{1-\frac{1}{p_k}} 2^{-jn(\alpha_k+\beta_k/2)} \min\left\{1, (2^j\ell(Q_k))^{\alpha_k n - s_k}\right\} \min\left\{1, (2^j\ell(Q_k))^{N_k\beta_k}\right\} \\ &= \begin{cases} (2^j\ell(Q_k))^{n-n/p_k+N_k\beta_k}, & \text{if} \quad 2^j\ell(Q_k) \le 1 \\ (2^j\ell(Q_k))^{n-n/p_k+\alpha_k n - s_k}, & \text{if} \quad 2^j\ell(Q_k) > 1. \end{cases}\end{aligned}$$

This inequality is enough to establish what we needed $\sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \left| u_j^k(x) \right|^{p_k} dx \lesssim 1$. The proof of Lemma 3.3 is complete. □

Proof of Lemma 3.5. We use the following notations:

$$\mathrm{I} = \{1, \dots, l\}, \quad \mathrm{II} = \{l+1, \dots, \rho\}, \quad \mathrm{III} = \{\rho+1, \dots, m\}, \quad \Lambda = \{1, \dots, m\}.$$

Recall that we are assuming $\mathrm{I} \neq \emptyset$ and $\mathrm{II} \neq \emptyset$ (the set III might be empty). For a subset $B = \{i_1, \dots, i_k\}$ of $\{1, \dots, m\}$, we write $y_B = (y_{i_1}, \dots, y_{i_k})$ and $dy_B = dy_{i_1} \cdots dy_{i_k}$. We take a smooth function φ on $\mathbb{R}^n$ such that $\operatorname{supp} \varphi \subset \{\xi \in \mathbb{R}^n \mid 4^{-1}a < |\xi| < 4\}$ and $\varphi(\xi) = 1$ on $2^{-1}a \le |\xi| \le 2$, where a is the constant in the definition of $V(k_1, k_2)$, and define Δ_j, $j \in \mathbb{Z}$, by (2.1). We set $s = \min\{s_1, \dots, s_m\}$ and take a number q such that

$$\max\{1, n/s\} < q < 2;$$

this is possible since $s_1, \dots, s_m > n/2$.

Let a_i ($i \in \mathrm{I}$) and f_i ($i \in \mathrm{II} \cup \mathrm{III}$) be functions as mentioned in the lemma. Without loss of generality, we may assume $\|f_i\|_{L^\infty} = 1$ for $i \in \mathrm{III}$. We use the decomposition (3.11) and write

$$g = T_\sigma(a_1, \dots, a_l, f_{l+1}, \dots, f_m) = \sum_{j\in\mathbb{Z}} g_j,$$

where $g_j = T_{\sigma_j}(a_1, \dots, a_l, f_{l+1}, \dots, f_m)$.

To prove the pointwise estimate (3.15), we divide $\mathbb{R}^n$ as $\mathbb{R}^n = \bigcup_{J\subset \mathrm{I}} E_J$, where J runs over all subsets of I and E_J is defined by

$$E_J = \bigcap_{i\in J} (Q_i^*)^c \cap \bigcap_{i\in \mathrm{I}\setminus J} Q_i^*.$$

In order to prove (3.15), it is sufficient to construct functions b_i^J ($i \in \mathrm{I}$) and $\widetilde{f}_i^J$ ($i \in \mathrm{II}$), for each $J \subset \mathrm{I}$, such that

$$|g(x)|\, \chi_{E_J}(x) \lesssim b_1^J(x) \dots b_l^J(x) \widetilde{f}_{l+1}^J(x) \dots \widetilde{f}_\rho^J(x), \tag{4.5}$$

where the function b_i^J depends only on m, n, $(s_i)_{i\in\Lambda}$, $(p_i)_{i\in\Lambda}$, σ, J, i, a_i, and $(f_i)_{i\in\mathrm{III}}$; the function $\widetilde{f}_i^J$ depends only on m, n, $(s_i)_{i\in\Lambda}$, J, i, f_i, and $(f_i)_{i\in\mathrm{III}}$; and they satisfy the estimates

$$\left\| b_i^J \right\|_{L^{p_i}} \lesssim 1, \tag{4.6}$$

$$\left\| \widetilde{f}_i^J \right\|_{L^2} \lesssim \|f_i\|_{L^2}. \tag{4.7}$$

In fact, if this is proved, then the desired functions can be obtained by $b_i = \sum_{J\subset \mathrm{I}} b_i^J$ and $\widetilde{f}_i = \sum_{J\subset\mathrm{I}} \widetilde{f}_i^J$.

First, we shall prove the estimate (4.5) for $J = \emptyset$, $E_\emptyset = Q_1^* \cap \cdots \cap Q_l^*$. The argument to be given below will show the estimate (4.5) with some combination of the following choices of $b_i^\emptyset$ and $\widetilde{f}_i^\emptyset$:

$$b_i^\emptyset(x) = M_q(a_i)(x)\chi_{Q_i^*}(x), \tag{4.8}$$

$$b_i^\emptyset(x) = \Big(\sum_{j\in\mathbb{Z}} M_q(\Delta_j a_i)(x)^2 \Big)^{1/2} \chi_{Q_i^*}(x), \tag{4.9}$$

$$b_i^\emptyset(x) = \Big(\sum_{j\in\mathbb{Z}} (\zeta_j * |a_i|^q)(x)^{2/q} (\zeta_j * |\Delta_j f_k|^q)(x)^{2/q} \Big)^{1/2} \chi_{Q_i^*}(x), \quad k \in \mathrm{III}, \tag{4.10}$$

$$\widetilde{f}_i^\emptyset(x) = M_q(f_i)(x), \tag{4.11}$$

$$\widetilde{f}_i^\emptyset(x) = \Big(\sum_{j\in\mathbb{Z}} M_q(\Delta_j f_i)(x)^2 \Big)^{1/2}, \tag{4.12}$$

$$\widetilde{f}_i^\emptyset(x) = \Big(\sum_{j\in\mathbb{Z}} (\zeta_j * |f_i|^q)(x)^{2/q} (\zeta_j * |\Delta_j f_k|^q)(x)^{2/q} \Big)^{1/2}, \quad k \in \mathrm{III}, \tag{4.13}$$

where $\zeta_j(x) = 2^{jn}(1+\left|2^j x\right|)^{-sq}$ is the function in Lemma 2.7 and M_q denotes the maximal operator defined by

$$M_q(f)(x) = \sup_{r>0} \Big(\frac{1}{r^n} \int_{|x-y|<r} |f(y)|^q \, dy \Big)^{1/q}.$$

The above functions $b_i^\emptyset$ and $\widetilde{f}_i^\emptyset$ depend on other things as mentioned in the lemma. We shall see that they also satisfy the estimates (4.6) and (4.7). For $\widetilde{f}_i^\emptyset$ given by (4.11) or (4.12), the L^2-boundedness of M_q, $q < 2$, and Lemma 2.8 (2.2) give the L^2-estimate (4.7). For $\widetilde{f}_i^\emptyset$ given by (4.13), Lemma 2.8 (2.3) yields the same L^2-estimate since $\|f_k\|_{BMO} \lesssim \|f_k\|_{L^\infty} = 1$ for $k \in \mathrm{III}$. For $b_i^\emptyset$ given by (4.8), the L^2-estimate $\|M_q(a_i)\|_{L^2} \lesssim \|a_i\|_{L^2}$ and Hölder's inequality give the estimate (4.6):

$$\left\|b_i^\emptyset\right\|_{L^{p_i}} \le \|M_q(a_i)\|_{L^2} |Q_i^*|^{1/p_i - 1/2} \lesssim \|a_i\|_{L^2} |Q_i|^{1/p_i-1/2} \le 1.$$

For $b_i^\emptyset$ given by (4.9) or (4.10), the same estimate is proved in a similar way.

We divide the proof of (4.5) for $J = \emptyset$ into the following six cases, (1)–(6), depending on the indices k_1 and k_2 involved in assumption (3.9).

(1) $k_1, k_2 \in \mathrm{I}$. In this case, without loss of generality, we assume $\{k_1, k_2\} = \{1, 2\} \subset \mathrm{I}$. Then, by the assumption (3.9), it follows that $2^{j-1}a \le |\xi_1| \le 2^{j+1}$ and $2^{j-1}a \le |\xi_2| \le 2^{j+1}$ for all $\xi \in \operatorname{supp}\sigma_j$, and hence $\varphi(2^{-j}\xi_1) = \varphi(2^{-j}\xi_2) = 1$ on $\operatorname{supp}\sigma_j$. We write

$$g_j = T_{\sigma_j}(\Delta_j a_1, \Delta_j a_2, a_3, \dots, a_l, f_{l+1}, \dots, f_\rho, \dots, f_m).$$

By Lemma 2.7, we have the pointwise estimate

$$\begin{aligned} |g_j| &\lesssim (\zeta_j * |\Delta_j a_1|^q)^{1/q} (\zeta_j * |\Delta_j a_2|^q)^{1/q} (\zeta_j * |a_3|^q)^{1/q} \cdots (\zeta_j * |a_l|^q)^{1/q} \\ &\quad \times (\zeta_j * |f_{l+1}|^q)^{1/q} \cdots (\zeta_j * |f_\rho|^q)^{1/q} \cdots (\zeta_j * |f_m|^q)^{1/q} \\ &\lesssim M_q(\Delta_j a_1) M_q(\Delta_j a_2) M_q(a_3) \cdots M_q(a_l) M_q(f_{l+1}) \cdots M_q(f_\rho). \end{aligned}$$

(Notice that the inequality $(\zeta_j * |f|^q)^{1/q} \lesssim M_q(f)$ holds because $sq > n$.) Summing over $j \in \mathbb{Z}$ and using the Cauchy-Schwarz inequality, we obtain

$$|g| \lesssim \Big(\sum_{j\in\mathbb{Z}} \{M_q(\Delta_j a_1)\}^2 \Big)^{\frac12} \Big(\sum_{j\in\mathbb{Z}} \{M_q(\Delta_j a_2)\}^2 \Big)^{\frac12} M_q(a_3) \cdots M_q(a_l) M_q(f_{l+1}) \cdots M_q(f_\rho).$$

This implies (4.5) for $J=\emptyset$ with $b_i^{\emptyset}$ of (4.9) for $i=1,2$, with $b_i^{\emptyset}$ of (4.8) for $3\le i\le l$, and with $\widetilde{f}_i^{\emptyset}$ of (4.11) for $l+1\le i\le \rho$.

(2) $k_1,k_2\in$ II. In this case, without loss of generality, we assume $\{k_1,k_2\}=\{l+1,l+2\}\subset$ II. Then we can write

$$g_j=T_{\sigma_j}(a_1,\dots,a_l,\Delta_j f_{l+1},\Delta_j f_{l+2},f_{l+3},\dots,f_\rho,\dots,f_m).$$

Hence, by Lemma 2.7,

$$|g_j|\lesssim M_q(a_1)\cdots M_q(a_l)M_q(\Delta_j f_{l+1})M_q(\Delta_j f_{l+2})M_q(f_{l+3})\cdots M_q(f_\rho).$$

Taking sum over $j\in\mathbb{Z}$ and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|g|\lesssim M_q(a_1)\cdots M_q(a_l)&\Big(\sum_{j\in\mathbb{Z}}\{M_q(\Delta_j f_{l+1})\}^2\Big)^{1/2}\Big(\sum_{j\in\mathbb{Z}}\{M_q(\Delta_j f_{l+2})\}^2\Big)^{1/2}\\ &\times M_q(f_{l+3})\cdots M_q(f_\rho).\end{aligned}$$

This implies (4.5) for $J=\emptyset$ with $b_i^{\emptyset}$ of (4.8) for $1\le i\le l$, with $\widetilde{f}_i^{\emptyset}$ of (4.12) for $i=l+1,l+2$, and with $\widetilde{f}_i^{\emptyset}$ of (4.11) for $l+3\le i\le\rho$.

(3) $k_1,k_2\in$ III. Without loss of generality, we assume $\{k_1,k_2\}=\{\rho+1,\rho+2\}\subset$ III. Then g_j can be written as

$$g_j=T_{\sigma_j}(a_1,\dots,a_l,f_{l+1},\dots,f_\rho,\Delta_j f_{\rho+1},\Delta_j f_{\rho+2},f_{\rho+3},\dots,f_m)$$

and Lemma 2.7 yields

$$\begin{aligned}|g_j|\lesssim&(\zeta_j*|a_1|^q)^{1/q}M_q(a_2)\cdots M_q(a_l)\\ &\times(\zeta_j*|f_{l+1}|^q)^{1/q}M_q(f_{l+2})\cdots M_q(f_\rho)(\zeta_j*|\Delta_j f_{\rho+1}|^q)^{1/q}(\zeta_j*|\Delta_j f_{\rho+2}|^q)^{1/q}.\end{aligned}$$

Taking sum over $j\in\mathbb{Z}$ and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|g|\lesssim&\Big(\sum_{j\in\mathbb{Z}}(\zeta_j*|a_1|^q)^{2/q}(\zeta_j*|\Delta_j f_{\rho+1}|^q)^{2/q}\Big)^{1/2}M_q(a_2)\cdots M_q(a_l)\\ &\times\Big(\sum_{j\in\mathbb{Z}}(\zeta_j*|f_{l+1}|^q)^{2/q}(\zeta_j*|\Delta_j f_{\rho+2}|^q)^{2/q}\Big)^{1/2}M_q(f_{l+2})\cdots M_q(f_\rho).\end{aligned}$$

This implies (4.5) for $J=\emptyset$ with the following functions: $b_1^{\emptyset}$ is (4.10) with $i=1$ and $k=\rho+1$; $b_i^{\emptyset}$ is (4.8) for $2\le i\le l$; $\widetilde{f}_{l+1}^{\emptyset}$ is (4.13) with $i=l+1$ and $k=\rho+2$; and $\widetilde{f}_i^{\emptyset}$ is (4.11) for $l+2\le i\le\rho$.

(4) $k_1\in$ I and $k_2\in$ II. Without loss of generality, we assume $k_1=1$ and $k_2=l+1$. Then

$$g_j=T_{\sigma_j}(\Delta_j a_1,a_2,\dots,a_l,\Delta_j f_{l+1},f_{l+2},\dots,f_\rho,\dots,f_m)$$

and Lemma 2.7 yields

$$|g_j|\lesssim M_q(\Delta_j a_1)M_q(a_2)\cdots M_q(a_l)M_q(\Delta_j f_{l+1})M_q(f_{l+2})\cdots M_q(f_\rho).$$

Taking sum over $j\in\mathbb{Z}$ and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|g|\lesssim&\Big(\sum_{j\in\mathbb{Z}}\{M_q(\Delta_j a_1)\}^2\Big)^{1/2}M_q(a_2)\cdots M_q(a_l)\\ &\times\Big(\sum_{j\in\mathbb{Z}}\{M_q(\Delta_j f_{l+1})\}^2\Big)^{1/2}M_q(f_{l+2})\cdots M_q(f_\rho).\end{aligned}$$

This implies (4.5) for $J=\emptyset$ with $b_i^{\emptyset}$ of (4.9) for $i=1$, $b_i^{\emptyset}$ of (4.8) for $2\le i\le l$, with $\widetilde{f}_i^{\emptyset}$ of (4.12) for $i=l+1$, and with $\widetilde{f}_i^{\emptyset}$ of (4.11) for $l+2\le i\le\rho$.

(5) $k_1 \in \mathrm{II}$ and $k_2 \in \mathrm{III}$. Without loss of generality, we assume $k_1 = l+1$ and $k_2 = \rho+1$. Then we have

$$g_j = T_{\sigma_j}(a_1, \dots, a_l, \Delta_j f_{l+1}, f_{l+2}, \dots, f_\rho, \Delta_j f_{\rho+1}, f_{\rho+2}, \dots, f_m)$$

and Lemma 2.7 yields

$$\begin{aligned}|g_j| &\lesssim (\zeta_j * |a_1|^q)^{1/q} M_q(a_2) \cdots M_q(a_l) \\ &\quad \times M_q(\Delta_j f_{l+1}) M_q(f_{l+2}) \cdots M_q(f_\rho)(\zeta_j * |\Delta_j f_{\rho+1}|^q)^{1/q}.\end{aligned}$$

Taking sum over $j \in \mathbb{Z}$ and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|g| &\lesssim \Big(\sum_{j\in\mathbb{Z}} (\zeta_j * |a_1|^q)^{2/q} (\zeta_j * |\Delta_j f_{\rho+1}|^q)^{2/q} \Big)^{1/2} M_q(a_2) \cdots M_q(a_l) \\ &\quad \times \Big(\sum_{j\in\mathbb{Z}} \{ M_q(\Delta_j f_{l+1}) \}^2 \Big)^{1/2} M_q(f_{l+2}) \cdots M_q(f_\rho).\end{aligned}$$

This implies (4.5) for $J = \emptyset$ with the following functions: $b_1^\emptyset$ is (4.10) with $i = 1$ and $k = \rho + 1$; $b_i^\emptyset$ is (4.8) for $2 \le i \le l$; $\widetilde{f}_i^\emptyset$ is (4.12) for $i = l + 1$; and $\widetilde{f}_i^\emptyset$ is (4.11) for $l + 2 \le i \le \rho$.

(6) $k_1 \in \mathrm{I}$ and $k_2 \in \mathrm{III}$. Without loss of generality, we assume $k_1 = 1$ and $k_2 = \rho + 1$. Then g_j can be written as

$$g_j = T_{\sigma_j}(\Delta_j a_1, a_2, \dots, a_l, f_{l+1}, \dots, f_\rho, \Delta_j f_{\rho+1}, f_{\rho+2}, \dots, f_m)$$

and Lemma 2.7 yields

$$\begin{aligned}|g_j| &\lesssim M_q(\Delta_j a_1) M_q(a_2) \cdots M_q(a_l) \\ &\quad \times (\zeta_j * |f_{l+1}|^q)^{1/q} M_q(f_{l+2}) \cdots M_q(f_\rho)(\zeta_j * |\Delta_j f_{\rho+1}|^q)^{1/q}.\end{aligned}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|g| &\lesssim \Big(\sum_{j\in\mathbb{Z}} \{ M_q(\Delta_j a_1) \}^2 \Big)^{1/2} M_q(a_2) \cdots M_q(a_l) \\ &\quad \times \Big(\sum_{j\in\mathbb{Z}} (\zeta_j * |f_{l+1}|^q)^{2/q} (\zeta_j * |\Delta_j f_{\rho+1}|^q)^{2/q} \Big)^{1/2} M_q(f_{l+2}) \cdots M_q(f_\rho).\end{aligned}$$

This implies (4.5) for $J = \emptyset$ with the following functions: $b_i^\emptyset$ is (4.9) for $i = 1$; $b_i^\emptyset$ is (4.8) for $2 \le i \le l$; $\widetilde{f}_{l+1}^\emptyset$ is (4.13) with $i = l+1$ and $k = \rho+1$; and $\widetilde{f}_i^\emptyset$ is (4.11) for $l+2 \le i \le \rho$. Thus we have proved (4.5) for $J = \emptyset$.

Next we shall prove (4.5) for $J \neq \emptyset$. Here we will not use the assumption (3.9). We fix a nonempty subset J of I. We shall prove that there exist functions $u_{k,j}^J$, $k \in J$, $j \in \mathbb{Z}$, such that

$$|g_j(x)| \, \chi_{E_J}(x) \lesssim \prod_{k\in J} u_{k,j}^J(x) \cdot \prod_{i\in \mathrm{I}\setminus J} |Q_i|^{-1/p_i} \, \chi_{Q_i^*}(x) \cdot \prod_{i\in \mathrm{II}} M_q(f_i)(x) \tag{4.14}$$

for all $j \in \mathbb{Z}$ and all $x \in \mathbb{R}^n$; the function $u_{k,j}^J$ depends only on m, n, $(s_i)_{i\in\Lambda}$, $(p_i)_{i\in\Lambda}$, σ, J, k, j, N_k, and Q_k, and satisfies the estimate

$$\big\| u_{k,j}^J \big\|_{L^{p_k}} \lesssim \min\{ (2^j \ell(Q_k))^{\gamma_k}, \, (2^j \ell(Q_k))^{-\delta_k} \}, \tag{4.15}$$

where γ_k and δ_k are positive constants that will be given in terms of n, k, J, $(s_i)_{i\in J}$, $(p_i)_{i\in J}$, and N_k. If we have these functions $u_{k,j}^J$, then we have (4.5) with the functions

$$b_k^J = \sum_{j\in\mathbb{Z}} u_{k,j}^J \quad \text{for} \quad k \in J,$$

$$b_i^J = |Q_i|^{-1/p_i}\, \chi_{Q_i^*} \quad \text{for} \quad i \in \mathrm{I} \setminus J,$$
$$\widetilde{f}_i^J = M_q(f_i) \quad \text{for} \quad i \in \mathrm{II}.$$

In fact, b_k^J, $k \in J$, depends only on m, n, $(s_i)_{i\in J}$, $(p_i)_{i\in J}$, σ, J, k, N_k, and Q_k, and the estimate (4.6) follows from (4.15). The estimate (4.6) for b_i^J with $i \in \mathrm{I}\setminus J$ is obvious and the estimate (4.7) for $\widetilde{f}_i$ with $i \in \mathrm{II}$ holds by the L^2-boundedness of M_q, $q < 2$. Thus it is sufficient to construct the functions $u_{k,j}^J$.

Before we proceed to the construction of $u_{k,j}^J$, we observe that it is sufficient to treat only the case $j = 0$. In fact, if we have (4.14)-(4.15) for $j = 0$, then the case of general $j \in \mathbb{Z}$ can be derived by the use of the dilation formula

$$T_{\sigma_j}(f_1, \dots, f_m)(x) = T_{\sigma_j(2^j\cdot)}(f_1(2^{-j}\cdot), \dots, f_m(2^{-j}\cdot))(2^j x)$$

and by a simple computation.

Thus we shall consider $g_0(x)$. Using $K_0 = (\sigma_0)^\vee$ (the inverse Fourier transform of σ_0), we write

$$g_0(x) = \int_{\mathbb{R}^{mn}} K_0(x - y_1, \dots, x - y_m) \prod_{i\in\mathrm{I}} a_i(y_i) \cdot \prod_{i\in\mathrm{II}\cup\mathrm{III}} f_i(y_i)\, dy_1 \cdots dy_m. \tag{4.16}$$

We write c_i to denote the center of the cube Q_i. Since $|x - y_i| \approx |x - c_i|$ for $x \notin Q_i^*$ and $y_i \in Q_i$, from (4.16) we see that the following inequalities hold for $x \in E_J$:

$$\begin{aligned}
&\prod_{i\in J} \langle x - c_i\rangle^{s_i} \cdot |g_0(x)| \\
&\lesssim \int_{\mathbb{R}^{mn}} \prod_{i\in J} \langle x - y_i\rangle^{s_i} \cdot |K_0(x - y_1, \dots, x - y_m)| \prod_{i\in\mathrm{I}} |a_i(y_i)| \cdot \prod_{i\in\mathrm{II}\cup\mathrm{III}} |f_i(y_i)|\, dy_1 \cdots dy_m \\
&\le \int_{\mathbb{R}^{mn}} \prod_{i\in J} \langle x - y_i\rangle^{s_i} \cdot |K_0(x - y_1, \dots, x - y_m)| \\
&\qquad\qquad \times \prod_{i\in\mathrm{I}} |Q_i|^{-1/p_i}\, \chi_{Q_i}(y_i) \cdot \prod_{i\in\mathrm{II}} |f_i(y_i)|\, dy_1 \cdots dy_m.
\end{aligned}$$

We now fix a $k \in J$ and estimate the last integral by

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big\| \prod_{i\in J\cup\mathrm{II}} \langle x - y_i\rangle^{s_i} \cdot K_0(x - y_1, \dots, x - y_m) \Big\|_{L^\infty(y_{J\setminus\{k\}}) L^1(y_{\mathrm{I}\setminus J}) L^{q'}(y_{\mathrm{II}}) L^1(y_{\mathrm{III}})} \\
&\times \Big\| \prod_{i\in J} |Q_i|^{-1/p_i}\, \chi_{Q_i}(y_i) \Big\|_{L^1(y_{J\setminus\{k\}})} \Big\| \prod_{i\in\mathrm{I}\setminus J} |Q_i|^{-1/p_i}\, \chi_{Q_i}(y_i) \Big\|_{L^\infty(y_{\mathrm{I}\setminus J})} \\
&\times \Big\| \prod_{i\in\mathrm{II}} \langle x - y_i\rangle^{-s_i} f_i(y_i) \Big\|_{L^q(y_{\mathrm{II}})}\, dy_k,
\end{aligned}$$

where we used the following notation for mixed norm and its obvious generalization:

$$\|F(z_1, z_2)\|_{L^p(z_1)L^q(z_2)} = \left[\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |F(z_1, z_2)|^p\, dz_1 \right)^{q/p} dz_2 \right]^{1/q}.$$

Recall that the mixed norms satisfy

$$\|F(z_1, z_2)\|_{L^p(z_1)L^q(z_2)} \le \|F(z_1, z_2)\|_{L^q(z_2)L^p(z_1)} \quad \text{if} \quad p < q. \tag{4.17}$$

Since $s_i > n/2$, the Cauchy-Schwarz inequality gives

$$\|F(x-y_1,\dots,x-y_m)\|_{L^1(y_B)} \lesssim \left\| \prod_{i\in B} \langle x-y_i\rangle^{s_i} \cdot F(x-y_1,\dots,x-y_m)\right\|_{L^2(y_B)}. \tag{4.18}$$

Now repeated applications of (4.17), (4.18), and Lemma 2.2 yield

$$\begin{aligned}
&\left\| \prod_{i\in J\cup \mathrm{II}} \langle x-y_i\rangle^{s_i} \cdot K_0(x-y_1,\dots,x-y_m)\right\|_{L^\infty(y_{J\setminus\{k\}})L^1(y_{\mathrm{I}\setminus J})L^{q'}(y_{\mathrm{II}})L^1(y_{\mathrm{III}})}\\
&\lesssim \left\| \prod_{i\in \Lambda} \langle x-y_i\rangle^{s_i} \cdot K_0(x-y_1,\dots,x-y_m)\right\|_{L^\infty(y_{J\setminus\{k\}})L^2(y_{\mathrm{I}\setminus J})L^{q'}(y_{\mathrm{II}})L^2(y_{\mathrm{III}})}\\
&\lesssim \left\| \prod_{i\in \Lambda} \langle x-y_i\rangle^{s_i} \cdot K_0(x-y_1,\dots,x-y_m)\right\|_{L^2(y_{\Lambda\setminus\{k\}})}\\
&= \left\| \langle x-y_k\rangle^{s_k} \prod_{i\in \Lambda\setminus\{k\}} \langle z_i\rangle^{s_i} \cdot K_0(z_1,\dots,x-y_k,\dots,z_m)\right\|_{L^2(z_{\Lambda\setminus\{k\}})}.
\end{aligned}$$

Since $s_i q > n$ by our choice of q, we have

$$\left\| \prod_{i\in \mathrm{II}} \langle x-y_i\rangle^{-s_i} f_i(y_i)\right\|_{L^q(y_{\mathrm{II}})} \lesssim \prod_{i\in\mathrm{II}} M_q(f_i)(x).$$

Combining the above inequalities, we obtain the following estimate for $x \in E_J$:

$$\begin{aligned}
&\prod_{i\in J} \langle x-c_i\rangle^{s_i} \cdot |g_0(x)|\\
&\lesssim h^{(k,0)}(x) \prod_{i\in J} |Q_i|^{-1/p_i+1} \cdot \prod_{i\in \mathrm{I}\setminus J} |Q_i|^{-1/p_i} \cdot \prod_{i\in\mathrm{II}} M_q(f_i)(x),
\end{aligned} \tag{4.19}$$

where

$$\begin{aligned}
&h^{(k,0)}(x)\\
&= |Q_k|^{-1} \int_{Q_k} \left\| \langle x-y_k\rangle^{s_k} \prod_{i\in \Lambda\setminus\{k\}} \langle z_i\rangle^{s_i} \cdot K_0(z_1,\dots,x-y_k,\dots,z_m)\right\|_{L^2(z_{\Lambda\setminus\{k\}})} dy_k.
\end{aligned}$$

We have

$$\begin{aligned}
&\left\| h^{(k,0)}\right\|_{L^2(\mathbb{R}^n)}\\
&\le |Q_k|^{-1} \int_{Q_k} \left\| \langle x-y_k\rangle^{s_k} \prod_{i\in \Lambda\setminus\{k\}} \langle z_i\rangle^{s_i} \cdot K_0(z_1,\dots,x-y_k,\dots,z_m)\right\|_{L^2(z_{\Lambda\setminus\{k\}})L^2(x)} dy_k\\
&= \left\| \prod_{i\in\Lambda} \langle z_i\rangle^{s_i} \cdot K_0(z_1,\dots,z_m)\right\|_{L^2(z_\Lambda)} = \|\sigma_0\|_{W^{(s_1,\dots,s_m)}}.
\end{aligned}$$

Thus, by the assumption (3.10),

$$\left\| h^{(k,0)}\right\|_{L^2(\mathbb{R}^n)} \le 1. \tag{4.20}$$

On the other hand, using the vanishing moment condition of a_k and Taylor's formula, we can write $g_0(x)$ as

$$\begin{aligned} g_0(x) = \sum_{|\alpha|=N_k} C_\alpha \int_{\mathbb{R}^{mn}} \Big\{ \int_0^1 (1-t)^{N_k-1} \\ \times \partial_k^\alpha K_0\Big(x-y_1,\dots,x^t_{c_k,y_k},\dots,x-y_m\Big) \\ \times (y_k-c_k)^\alpha a_1(y_1)\cdots a_l(y_l) f_{l+1}(y_{l+1})\cdots f_m(y_m)\,dt\Big\}\,dy_1\cdots dy_m, \end{aligned}$$

where $\partial_k^\alpha K_0(z_1,\dots,z_m) = \partial_{z_k}^\alpha K_0(z_1,\dots,z_m)$ and $x^t_{c_k,y_k} = x-c_k-t(y_k-c_k)$. Hence the following inequality holds for $x\in E_J$:

$$\begin{aligned} \prod_{i\in J}\langle x-c_i\rangle^{s_i}\cdot|g_0(x)| \lesssim \sum_{|\alpha|=N_k}\int_{\mathbb{R}^{mn}} \Big\{ \int_0^1 \langle x^t_{c_k,y_k}\rangle^{s_k} \prod_{i\in J\setminus\{k\}} \langle x-y_i\rangle^{s_i} \\ \times \Big|\partial_k^\alpha K_0\Big(x-y_1,\dots,x^t_{c_k,y_k},\dots,x-y_m\Big)\Big| \\ \times \ell(Q_k)^{N_k}\prod_{i\in\mathrm{I}}|Q_i|^{-1/p_i}\,\chi_{Q_i}(y_i)\cdot\prod_{i\in\mathrm{II}}|f_i(y_i)|\;dt\Big\}\,dy_1\cdots dy_m. \end{aligned}$$

Using this inequality and arguing in the same way as before, we obtain the following estimate for $x\in E_J$:

$$\prod_{i\in J}\langle x-c_i\rangle^{s_i}\cdot|g_0(x)| \lesssim h^{(k,1)}(x)\prod_{i\in J}|Q_i|^{-1/p_i+1}\cdot\prod_{i\in\mathrm{I}\setminus J}|Q_i|^{-1/p_i}\cdot\prod_{i\in\mathrm{II}}M_q(f_i)(x), \tag{4.21}$$

where

$$\begin{aligned} h^{(k,1)}(x) = |Q_k|^{-1+N_k/n}\sum_{|\alpha|=N_k}\int_{\substack{0<t<1\\ y_k\in Q_k}} \\ \Big\|\langle x^t_{c_k,y_k}\rangle^{s_k}\prod_{i\in\Lambda\setminus\{k\}}\langle z_i\rangle^{s_i}\cdot\partial_k^\alpha K_0(z_1,\dots,x^t_{c_k,y_k},\dots,z_m)\Big\|_{L^2(z_{\Lambda\setminus\{k\}})} dt dy_k. \end{aligned}$$

Using Lemma 2.2, we obtain

$$\Big\|h^{(k,1)}\Big\|_{L^2(\mathbb{R}^n)} \lesssim |Q_k|^{N_k/n}\,. \tag{4.22}$$

From two estimates (4.19) and (4.21), we obtain

$$|g_0(x)| \lesssim \prod_{i\in J}\langle x-c_i\rangle^{-s_i}\,|Q_i|^{-1/p_i+1}\cdot\prod_{i\in\mathrm{I}\setminus J}|Q_i|^{-1/p_i}\cdot\prod_{i\in\mathrm{II}}M_q(f_i)(x)\times\min\{h^{(k,0)}(x),\,h^{(k,1)}(x)\}$$

for all $x\in E_J$ and for each $k\in J$. We take positive numbers $(\beta_k)_{k\in J}$ satisfying $\sum_{k\in J}\beta_k = 1$ and take a geometric mean of the above estimates to obtain

$$|g_0(x)|\,\chi_{E_J}(x) \lesssim \prod_{k\in J}u_k^J(x)\cdot\prod_{i\in\mathrm{I}\setminus J}|Q_i|^{-1/p_i}\,\chi_{Q_i^*}(x)\cdot\prod_{i\in\mathrm{II}}M_q(f_i)(x),$$

where

$$u_k^J(x) = \langle x-c_k\rangle^{-s_k}\,|Q_k|^{-1/p_k+1}\,\chi_{(Q_k^*)^c}(x)\,\Big(\min\{h^{(k,0)}(x),\,h^{(k,1)}(x)\}\Big)^{\beta_k}.$$

We choose β_k, $k\in J$, so that we have

$$\beta_k>0,\quad \frac{s_k}{n}>\frac{1}{p_k}-\frac{\beta_k}{2},\quad \sum_{k\in J}\beta_k = 1.$$

This is possible since $1/2 > \sum_{k\in J} \max\{0, 1/p_k - s_k/n\}$ by virtue of our condition (1.4). If we write $1/p_k - \beta_k/2 = 1/r_k$, then $r_k > 0$ and Hölder's inequality gives

$$\|u_k^J\|_{L^{p_k}} \le \left\| \langle x-c_k\rangle^{-s_k} |Q_k|^{-1/p_k+1} \chi_{(Q_k^*)^c}(x) \right\|_{L^{r_k}} \times \left\| \left(\min\{h^{(k,0)}(x),\, h^{(k,1)}(x)\}\right)^{\beta_k} \right\|_{L^{2/\beta_k}}.$$

Since $s_k r_k > n$, we have

$$\left\| \langle x-c_k\rangle^{-s_k} |Q_k|^{-1/p_k+1} \chi_{(Q_k^*)^c}(x) \right\|_{L^{r_k}} \approx \begin{cases} |Q_k|^{-1/p_k+1} & \text{if } |Q_k| \le 1 \\ |Q_k|^{-1/p_k+1-s_k/n+1/r_k} & \text{if } |Q_k| > 1. \end{cases}$$

By (4.20) and (4.22), we have

$$\begin{aligned} \left\| \left(\min\{h^{(k,0)}(x),\, h^{(k,1)}(x)\}\right)^{\beta_k} \right\|_{L^{2/\beta_k}} &\le \min\left\{ \left\|h^{(k,0)}\right\|_{L^2}^{\beta_k},\, \left\|h^{(k,1)}\right\|_{L^2}^{\beta_k} \right\} \\ &\lesssim \begin{cases} |Q_k|^{N_k\beta_k/n} & \text{if } |Q_k| \le 1 \\ 1 & \text{if } |Q_k| > 1. \end{cases} \end{aligned}$$

Thus

$$\|u_k^J\|_{L^{p_k}} \lesssim \begin{cases} |Q_k|^{N_k\beta_k/n-1/p_k+1} & \text{if } |Q_k| \le 1 \\ |Q_k|^{-1/p_k+1-s_k/n+1/r_k} & \text{if } |Q_k| > 1, \end{cases}$$

which implies (4.15) for $j = 0$ with $\gamma_k = N_k\beta_k - n/p_k + n$ and $\delta_k = n/p_k - n + s_k - n/r_k$. We have $\gamma_k > 0$ since N_k is sufficiently large and $\delta_k > 0$ since $\delta_k = n\beta_k/2 - n + s_k \ge n\beta_k/2 - n/p_k + s_k > 0$ by our choice of β_k. This completes the proof of Lemma 3.5. □

Proof of Lemma 3.6. Since the proof is similar to that of Lemma 3.5, we shall briefly indicate only the key points. We use the same notation as in the proof of Lemma 3.5. We also write

$$G(x) = GT_\sigma(a_1, \dots, a_l, f_{l+1}, \dots, f_m)(x) = \left(\sum_{j\in\mathbb{Z}} |g_j(x)|^2 \right)^{1/2}.$$

It is sufficient to prove the estimate

$$G(x)\chi_{E_J}(x) \lesssim \prod_{i\in \mathrm{I}} b_i^J(x) \cdot \prod_{i\in \mathrm{II}} \widetilde{f}_i(x) \tag{4.23}$$

for each subset J of I, where b_i^J and $\widetilde{f}_i$ have the same properties as in (4.5).

First we consider the case $J = \emptyset$, $E_\emptyset = Q_1^* \cap \dots \cap Q_l^*$. We divide the proof into the following three cases, (1)–(3), depending on the index k_1 involved in the assumption (3.9) with $k_2 = m+1$.

(1) $k_1 \in$ I. Without loss of generality, we assume $k_1 = 1$. We can write

$$g_j = T_{\sigma_j}(\Delta_j a_1, a_2, \dots, a_l, f_{l+1}, \dots, f_\rho, \dots, f_m).$$

By Lemma 2.7, we have

$$|g_j| \lesssim M_q(\Delta_j a_1) M_q(a_2) \cdots M_q(a_l) M_q(f_{l+1}) \cdots M_q(f_\rho).$$

Hence

$$G \lesssim \left(\sum_{j\in\mathbb{Z}} \{M_q(\Delta_j a_1)\}^2 \right)^{1/2} M_q(a_2) \cdots M_q(a_l) M_q(f_{l+1}) \cdots M_q(f_\rho).$$

Thus we obtain (4.23) for $J = \emptyset$ with

$$b_1^\emptyset = \left(\sum_{j\in\mathbb{Z}} \{M_q(\Delta_j a_1)\}^2 \right)^{1/2} \chi_{Q_1^*},$$

$$
\begin{aligned}
b_i^{\emptyset} &= M_q(a_i)\chi_{Q_i^*} \quad \text{for} \quad 2 \le i \le l, \\
\widetilde{f}_i^{\emptyset} &= M_q(f_i) \quad \text{for} \quad l+1 \le i \le \rho.
\end{aligned}
$$

(2) $k_1 \in \mathrm{II}$. Without loss of generality, we assume $k_1 = l+1$. We can write

$$
g_j = T_{\sigma_j}(a_1, \dots, a_l, \Delta_j f_{l+1}, f_{l+2}, \dots, f_\rho, \dots, f_m).
$$

By Lemma 2.7, we have

$$
|g_j| \lesssim M_q(a_1) \cdots M_q(a_l) M_q(\Delta_j f_{l+1}) M_q(f_{l+2}) \cdots M_q(f_\rho).
$$

Hence

$$
G \lesssim M_q(a_1) \cdots M_q(a_l) \bigg(\sum_{j \in \mathbb{Z}} \{M_q(\Delta_j f_{l+1})\}^2 \bigg)^{1/2} M_q(f_{l+2}) \cdots M_q(f_\rho).
$$

Thus we obtain (4.23) for $J = \emptyset$ with

$$
\begin{aligned}
b_i^{\emptyset} &= M_q(a_i)\chi_{Q_i^*} \quad \text{for} \quad 1 \le i \le l, \\
\widetilde{f}_{l+1}^{\emptyset} &= \bigg(\sum_{j \in \mathbb{Z}} \{M_q(\Delta_j f_{l+1})\}^2 \bigg)^{1/2}, \\
\widetilde{f}_i^{\emptyset} &= M_q(f_i) \quad \text{for} \quad l+2 \le i \le \rho.
\end{aligned}
$$

(3) $k_1 \in \mathrm{III}$. Without loss of generality, we assume $k_1 = \rho+1$. We can write

$$
g_j = T_{\sigma_j}(a_1, \dots, a_l, f_{l+1}, \dots, f_\rho, \Delta_j f_{\rho+1}, \dots, f_m).
$$

Lemma 2.7 yields

$$
|g_j| \lesssim (\zeta_j * |a_1|^q)^{1/q} M_q(a_2) \cdots M_q(a_l) M_q(f_{l+1}) \cdots M_q(f_\rho) (\zeta_j * |\Delta_j f_{\rho+1}|^q)^{1/q}.
$$

Hence

$$
G \lesssim \bigg(\sum_{j \in \mathbb{Z}} (\zeta_j * |a_1|^q)^{2/q} (\zeta_j * |\Delta_j f_{\rho+1}|^q)^{2/q} \bigg)^{1/2} M_q(a_2) \cdots M_q(a_l) M_q(f_{l+1}) \cdots M_q(f_\rho).
$$

Thus we obtain (4.23) for $J = \emptyset$ with

$$
\begin{aligned}
b_1^{\emptyset} &= \bigg(\sum_{j \in \mathbb{Z}} (\zeta_j * |a_1|^q)^{2/q} (\zeta_j * |\Delta_j f_{\rho+1}|^q)^{2/q} \bigg)^{1/2} \chi_{Q_1^*}, \\
b_i^{\emptyset} &= M_q(a_i)\chi_{Q_i^*} \quad \text{for} \quad 2 \le i \le l, \\
\widetilde{f}_i^{\emptyset} &= M_q(f_i) \quad \text{for} \quad l+1 \le i \le \rho.
\end{aligned}
$$

Finally we prove (4.23) for $J \ne \emptyset$. The proof is immediate. Observe that the estimate of $g_j(x)$ on E_J, $J \ne \emptyset$, given in the latter half of the proof of Lemma 3.5 holds in the present case as well, since we did not use the assumption (3.9) in that argument. Also observe that there we have actually proved the estimate

$$
\sum_{j \in \mathbb{Z}} |g_j(x)| \, \chi_{E_J}(x) \lesssim b_1^J(x) \cdots b_l^J(x) \widetilde{f}_{l+1}^J(x) \cdots \widetilde{f}_\rho^J(x)
$$

for $J \ne \emptyset$. Thus the estimate (4.23) for $J \ne \emptyset$ also holds since

$$
G(x) = \bigg(\sum_{j \in \mathbb{Z}} |g_j(x)|^2 \bigg)^{1/2} \le \sum_{j \in \mathbb{Z}} |g_j(x)| \, .
$$

This completes the proof of Lemma 3.6. □

5. The space L^1 and weak type estimates

In this section, we prove that if we replace H^1 by L^1, then we obtain the weak type estimate for T_σ under the same regularity assumption on the multipliers. Precisely, we prove the following theorem.

Theorem 5.1. *Let $s_1, \dots, s_m$, $p_1, \dots, p_m$, and p satisfy the same assumptions as in Theorem 1.1. Define X_i, $i = 1, \dots, m$, by $X_i = H^{p_i}$ if $p_i \neq 1$ and $X_i = L^1$ if $p_i = 1$. Then*

$$\|T_\sigma\|_{X_1 \times \cdots \times X_m \longrightarrow L^{(p,\infty)}} \lesssim \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\psi} \right\|_{W^{(s_1, \dots, s_m)}}. \tag{5.1}$$

The conditions given above are optimal in the sense that if (5.1) holds then we must have $s_1, \dots, s_m \geq n/2$ and (1.7) for every nonempty subset J of $\{1, 2, \dots, m\}$.

The proof depends on the following lemma, which is a slight generalization of the remark given in Stein [21, 5.24].

Lemma 5.2. *Let p_0, p_1, q_0, q_1, r satisfy $n/(n+1) < p_0 < 1 < p_1 < \infty$, $0 < q_0 < r < q_1 < \infty$, and $1/p_0 - 1/q_0 = 1/p_1 - 1/q_1 = 1 - 1/r$. Let T be a linear operators that maps $L^1(\mathbb{R}^n)$ to $\mathcal{M}(\mathbb{R})^n$, the space of all measurable functions on $\mathbb{R}^n$. Assume that there are M_0 and M_1 positive constants such that for all $f \in L^1(\mathbb{R}^n)$ we have*

$$\|T(f)\|_{L^{(q_0,\infty)}} \leq M_0 \|f\|_{H^{p_0}}, \tag{5.2}$$

$$\|T(f)\|_{L^{(q_1,\infty)}} \leq M_1 \|f\|_{L^{p_1}} \tag{5.3}$$

whenever the the right hand sides are finite. Then

$$\|T(f)\|_{L^{(r,\infty)}} \leq C M_0^{1-\theta} M_1^{\theta} \|f\|_{L^1}$$

for all $f \in L^1(\mathbb{R}^n)$, where C is a constant depending only on p_0, p_1, q_0, q_1, r, and n, and θ is given by $1 = (1-\theta)/p_0 + \theta/p_1$.

Proof. Let $f \in L^1(\mathbb{R}^n)$ and we assume $\|f\|_{L^1} = 1$. Let $0 < \lambda < \infty$ be given. We apply the Calderón-Zygmund decomposition to f at height $\delta\lambda^r$, where δ is a positive constant to be determined later. Thus we obtain a family of disjoint cubes $\{Q_j\}$ such that

$$\delta\lambda^r < \frac{1}{|Q_j|} \int_{Q_j} |f(x)|\, dx \leq 2^n \delta\lambda^r,$$

$$|f(x)| \leq \delta\lambda^r \quad \text{for a.e.} \quad x \notin \bigcup_j Q_j,$$

$$\sum_j |Q_j| \leq (\delta\lambda^r)^{-1},$$

and we write $f = g + b$, $b = \sum_j b_j$ with

$$b_j(x) = \big(f(x) - f_{Q_j}\big) \chi_{Q_j}(x), \quad f_{Q_j} = \frac{1}{|Q_j|} \int_{Q_j} f(x)\, dx.$$

For g, we have

$$\|g\|_{L^{p_1}}^{p_1} \leq \|g\|_{L^\infty}^{p_1 - 1} \|g\|_{L^1} \lesssim (\delta\lambda^r)^{p_1 - 1}.$$

Thus (5.3) gives

$$|\{x : |T(g)(x)| > \lambda\}| \leq \big(M_1 \|g\|_{L^{p_1}} \lambda^{-1}\big)^{q_1}$$
$$\lesssim \Big(M_1 (\delta\lambda^r)^{1-1/p_1} \lambda^{-1}\Big)^{q_1} = \Big(M_1 \delta^{1-1/p_1}\Big)^{q_1} \lambda^{-r}.$$

Each b_j satisfies

$$\operatorname{supp} b_j \subset Q_j, \quad \int b_j(x)\, dx = 0, \quad \frac{1}{|Q_j|} \int_{Q_j} |b_j(x)|\, dx \lesssim \delta\lambda^r,$$

and thus $|Q_j|^{-1/p_0}(\delta\lambda^r)^{-1} b_j$ is a constant multiple of an L^1-atom for H^{p_0} since $n/(n+1) < p_0 < 1$. Hence we have

$$\|b\|_{H^{p_0}}^{p_0} \lesssim \sum_j \left(|Q_j|^{1/p_0}\,\delta\lambda^r\right)^{p_0} \le (\delta\lambda^r)^{p_0}(\delta\lambda^r)^{-1} = (\delta\lambda^r)^{p_0-1}.$$

Thus (5.2) gives

$$\begin{aligned}|\{x:\ |Tb(x)|>\lambda\}| &\le \left(M_0\,\|b\|_{H^{p_0}}\,\lambda^{-1}\right)^{q_0}\\ &\lesssim \left(M_0\,(\delta\lambda^r)^{1-1/p_0}\,\lambda^{-1}\right)^{q_0} = \left(M_0\delta^{1-1/p_0}\right)^{q_0}\lambda^{-r}.\end{aligned}$$

Combining the above estimates with the fact that $T(f)=T(g)+T(b)$, we obtain

$$|\{x:\ |T(f)(x)|>2\lambda\}| \lesssim \left\{\left(M_0\delta^{1-1/p_0}\right)^{q_0} + \left(M_1\delta^{1-1/p_1}\right)^{q_1}\right\}\lambda^{-r}.$$

Choosing δ so that it minimizes the last expression, we obtain

$$|\{x:\ |T(f)(x)|>2\lambda\}| \lesssim \left(M_0^{1-\theta}M_1^{\theta}\lambda^{-1}\right)^r.$$

This completes the proof of Lemma 5.2. □

Proof of Theorem 5.1. Suppose $s_1,\dots,s_m$ and $p_1,\dots,p_m$ satisfy the assumptions of the theorem and suppose for example $p_1=1$. If we take $\epsilon>0$ sufficiently small, then $s_1,\dots,s_m$ also satisfy the assumptions of the theorem with $p_1=1$ replaced by $1\pm\epsilon$. Thus Theorem 1.1 yields two estimates

$$\begin{aligned}\|T_\sigma(f_1,f_2,\dots,f_m)\|_{L^{(p_-,\infty)}} &\lesssim A\,\|f_1\|_{H^{1-\epsilon}}\,\|f_2\|_{H^{p_2}}\cdots\|f_m\|_{H^{p_m}},\\ \|T_\sigma(f_1,f_2,\dots,f_m)\|_{L^{(p_+,\infty)}} &\lesssim A\,\|f_1\|_{L^{1+\epsilon}}\,\|f_2\|_{H^{p_2}}\cdots\|f_m\|_{H^{p_m}},\end{aligned}$$

where $A=\sup_{j\in\mathbb{Z}}\left\|\sigma(2^j\cdot)\widehat{\psi}\right\|_{W^{(s_1,\dots,s_m)}}$ and $p_\pm$ is given by $1/(1\pm\epsilon)+1/p_2+\dots+1/p_m = 1/p_\pm$. We freeze the functions $f_2,\dots,f_m$ and apply Lemma 5.2 to the linear operator $f_1\mapsto T_\sigma(f_1,f_2,\dots,f_m)$ to obtain

$$\|T_\sigma(f_1,f_2,\dots,f_m)\|_{L^{(p,\infty)}} \lesssim A\,\|f_1\|_{L^1}\,\|f_2\|_{H^{p_2}}\cdots\|f_m\|_{H^{p_m}}.$$

Repeated application of the same argument gives the desired weak type estimate.

The necessity of the conditions $s_i \ge n/2$ and (1.7) can be shown by the same method as in [13, Theorem 5.1]. This completes the proof of Theorem 5.1. □

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