MULTILINEAR MULTIPLIER THEOREMS AND APPLICATIONS

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Abstract. We obtain new multilinear multiplier theorems for symbols of restricted smoothness which lie locally in certain Sobolev spaces. We provide applications concerning the boundedness of the commutators of Calderón and Calderón-Coifman-Journé.

1. Introduction

The theory of multilinear multipliers has made significant advances in recent years. An n-dimensional m-linear multiplier is a bounded function σ on $(\mathbb{R}^n)^m$ associated with an m-linear operator T_σ on $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ in the following way:

$$T_\sigma(f_1, \dots, f_m)(x) = \int_{(\mathbb{R}^n)^m} \widehat{f_1}(\xi_1) \cdots \widehat{f_m}(\xi_m)\sigma(\xi_1, \dots, \xi_m) e^{2\pi i x\cdot(\xi_1+\cdots+\xi_m)} d\xi_1 \cdots d\xi_m, \tag{1.1}$$

where f_j, $j = 1, \dots, m$, are Schwartz functions in $\mathbb{R}^n$, and $\widehat{f_j}(\xi_j) = \int_{\mathbb{R}^n} f_j(x)e^{-2\pi i x\cdot\xi_j} dx$ is the Fourier transform of f_j. A classical result of Coifman and Meyer [9, 10] says that if for all sufficiently large multiindices $\alpha_1, \dots, \alpha_m \in (\mathbb{Z}^+ \cup \{0\})^n$ we have

$$\left|\partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_m}^{\alpha_m} \sigma(\xi_1, \dots, \xi_m)\right| \lesssim (|\xi_1| + \cdots + |\xi_m|)^{-(|\alpha_1|+\cdots+|\alpha_m|)} \tag{1.2}$$

for all $(\xi_1, \dots, \xi_m) \in (\mathbb{R}^n)^m \setminus \{(0, \dots, 0)\}$, then T_σ admits a bounded extension from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $1 < p_1, \dots, p_m \le \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$, and $1 \le p < \infty$. The extension of this theorem to indices $p > 1/m$ was simultaneously obtained by Kenig and Stein [29] (when $m = 2$) and Grafakos and Torres [22]. This theorem provides an m-linear extension of Mikhlin's classical linear multiplier result [30]. Hörmander [25] obtained an improvement of Mikhlin's theorem showing that when $m = 1$, T_σ maps $L^{p_1}(\mathbb{R}^n)$ to $L^{p_1}(\mathbb{R}^n)$, $1 < p_1 < \infty$ under the weaker condition

$$\sup_{j\in\mathbb{Z}} \left\|(I - \Delta)^{s/2}\big(\sigma(2^j\cdot)\widehat{\Psi}\big)\right\|_{L^2(\mathbb{R}^n)} < \infty,$$

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where $s > n/2$ and $\widehat{\Psi}$ is a smooth function supported in an annulus centered at the origin. Here Δ is the Laplacian and $(I-\Delta)^{s/2}$ is an operator given on the Fourier transform side by multiplication with $(1+4\pi^2|\xi|^2)^{s/2}$. Hörmander's theorem was extended to L^r-based Sobolev spaces and to indices $p_1 \le 1$, with L^{p_1} replaced by the Hardy space H^{p_1}, by Calderón and Torchinsky [5].

The adaptation of Hörmander's theorem to the multilinear setting was first obtained by Tomita [40]. This theorem was later extended by Grafakos and Si [20] to the range $p < 1$ by replacing L^2-based Sobolev spaces by L^r-based Sobolev spaces. The endpoint cases where some p_j are equal to infinity were treated by Grafakos, Miyachi, and Tomita [18]. Fujita and Tomita [13] provided weighted extensions of these results and also noticed that the operator $(I-\Delta)^{s/2}$ in $(\mathbb{R}^n)^m$ can be replaced by $(I-\Delta_{\xi_1})^{s_1/2}\cdots(I-\Delta_{\xi_m})^{s_m/2}$, where Δ_{ξ_j} is the Laplacian in the ξ_jth variable. The bilinear version of the Calderón and Torchinsky theorem was proved by Miyachi and Tomita [31], while the m-linear version (for general m) was proved by Grafakos and Nguyen [16] and Grafakos, Miyachi, Nguyen, and Tomita [17].

To study certain multilinear singular integrals, such as multicommutators, there is a need for a multilinear multiplier theorem that can handle symbols on $(\mathbb{R}^n)^m$ which, for instance, have one derivative in each variable but no two derivatives in a given variable. We notice that in the case where s_j are positive integers for all j, replacing $(I-\Delta)^{s/2}$ on $(\mathbb{R}^n)^m$ by $(I-\Delta_{\xi_1})^{s_1/2}\cdots(I-\Delta_{\xi_m})^{s_m/2}$, as in Fujita and Tomita [13], reflects the following decay condition for the derivatives of σ

$$\left|\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}\cdots\partial_{\xi_m}^{\beta_m}\sigma(\xi_1,\dots,\xi_m)\right| \lesssim (|\xi_1|+\cdots+|\xi_m|)^{-\sum_{j=1}^m|\beta_j|}, \tag{1.3}$$

where each multiindex β_j satisfies $|\beta_j| \le s_j$. In this case a given coordinate of ξ_j could be differentiated as many as s_j times. In this article we study multipliers that satisfy the following coordinate-wise version of (1.3)

$$\begin{aligned}\left|\partial_{\xi_{11}}^{\beta_{11}}\cdots\partial_{\xi_{1n}}^{\beta_{1n}}\partial_{\xi_{21}}^{\beta_{21}}\cdots\partial_{\xi_{2n}}^{\beta_{2n}}\cdots\partial_{\xi_{m1}}^{\beta_{m1}}\cdots\partial_{\xi_{mn}}^{\beta_{mn}}\sigma(\xi_1,\dots,\xi_m)\right|& \\ \lesssim (|\xi_1|+\cdots+|\xi_m|)^{-\sum_{j=1}^m\sum_{\ell=1}^n\beta_{j\ell}}&,\end{aligned} \tag{1.4}$$

where $\xi_j = (\xi_{j1},\dots,\xi_{jn})$ and each $\beta_{j\ell}$ is at most s_j/n. Condition (1.4) weakens the Coifman-Meyer hypothesis (1.2) and also (1.3) in the sense that it does not allow any one-dimensional variable to be differentiated more than an appropriate number of times.

We now state our first main result concerning the operator T_σ in (1.1). Here and throughout the ith coordinate of the vector ξ_j in $\mathbb{R}^n$ is denoted by ξ_{ji}. We denote partial derivatives in the ξ_{ji} variable by $\partial_{\xi_{ji}}$. Also the Laplacian Δ_{ξ_j} on $\mathbb{R}^n$ is given by $\partial_{\xi_{j1}}^2+\cdots+\partial_{\xi_{jn}}^2$. We have a result that extends condition (1.4) in the Sobolev space setting. We define $(I-\partial_{\xi_{i\ell}}^2)^{\frac{\gamma_{i\ell}}{2}} f(\xi)$ as the linear operator $((1+4\pi^2|\eta_{i\ell}|^2)^{\frac{\gamma_{i\ell}}{2}} \widehat{f}(\eta))^{\vee}(\xi)$ related to the multiplier $(1+4\pi^2|\eta_{i\ell}|^2)^{\frac{\gamma_{i\ell}}{2}}$.

Theorem 1.1. *Suppose that $1 \le r \le 2$ and $\gamma_{i\ell} > 1/r$ for all $1 \le i \le m$ and $1 \le \ell \le n$. Let σ be a bounded function on $\mathbb{R}^{mn}$ such that*

$$\sup_{j\in\mathbb{Z}} \Big\| \prod_{\substack{1\le i\le m\\ 1\le \ell\le n}} (I-\partial_{\xi_{i\ell}}^2)^{\frac{\gamma_{i\ell}}{2}} \big[\sigma(2^j\cdot)\widehat{\Psi}\big] \Big\|_{L^r(\mathbb{R}^{mn})} = A < \infty,$$

where $\widehat{\Psi}$ is a smooth function supported in the annulus $\frac{1}{2} \le |(\xi_1,\dots,\xi_m)| \le 2$ in $\mathbb{R}^{mn}$ that satisfies

$$\sum_{j\in\mathbb{Z}} \widehat{\Psi}(2^{-j}(\xi_1,\dots,\xi_m)) = 1, \qquad \text{for all } (\xi_1,\dots,\xi_m) \in (\mathbb{R}^n)^m \setminus \{0\}.$$

If $1 < p_i < \infty$, $i = 1,\dots,m$, satisfy $\max_{1\le i\le m} \max_{1\le \ell\le n} \frac{1}{\gamma_{i\ell}} < \min_{1\le i\le m} p_i$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, then we have

$$\|T_\sigma\|_{L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim A. \tag{1.5}$$

Taking $\gamma_{i\ell} = \gamma_i/n$ for all $\ell \in \{1,\dots,n\}$ and using simple embeddings between Sobolev spaces we deduce the following consequence of Theorem 1.1.

Corollary 1.2. *Let $1 \le r \le 2$ and suppose that $\gamma_i > n/r$ for all $i = 1,\dots,m$. Let σ be a bounded function on $\mathbb{R}^{mn}$ such that*

$$\sup_{j\in\mathbb{Z}} \Big\| (I-\Delta_{\xi_1})^{\frac{\gamma_1}{2}} \cdots (I-\Delta_{\xi_m})^{\frac{\gamma_m}{2}} \big[\sigma(2^j\cdot)\widehat{\Psi}\big] \Big\|_{L^r(\mathbb{R}^{mn})} = A < \infty, \tag{1.6}$$

where Ψ is as in Theorem 1.1. Then (1.5) holds for $1 < p_i < \infty, \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ satisfying $\max_{1\le i\le m} \frac{n}{\gamma_i} < \min_{1\le i\le m} p_i$.

We also provide an endpoint case of Corollary 1.2 when all $p_i = 1$. Let $H^1(\mathbb{R}^n)$ denote the classical Hardy space on $\mathbb{R}^n$. We note that when $m = 1$, boundedness for T_σ is known to hold from H^1 to L^1.

Theorem 1.3. *Let σ be a bounded function on $\mathbb{R}^{mn}$ which satisfies (1.6) with $r = 1$ and $\min_{1\le i\le m} \gamma_i > n$. Then we have*

$$\|T_\sigma\|_{H^1(\mathbb{R}^n)\times\cdots\times H^1(\mathbb{R}^n)\to L^{1/m,\infty}(\mathbb{R}^n)} \lesssim A. \tag{1.7}$$

Another extension of the Coifman-Meyer multiplier theorem is in the multiparameter setting. In this case (1.2) is relaxed to

$$\begin{aligned} &\big|\partial_{\xi_{11}}^{\alpha_{11}} \cdots \partial_{\xi_{1n}}^{\alpha_{1n}} \partial_{\xi_{21}}^{\alpha_{21}} \cdots \partial_{\xi_{2n}}^{\alpha_{2n}} \cdots \partial_{\xi_{m1}}^{\alpha_{m1}} \cdots \partial_{\xi_{mn}}^{\alpha_{mn}} \sigma(\xi_1,\dots,\xi_m)\big| \\ &\quad\lesssim\ (|\xi_{11}| + \cdots + |\xi_{m1}|)^{-(\alpha_{11}+\cdots+\alpha_{m1})} \cdots (|\xi_{1n}| + \cdots + |\xi_{mn}|)^{-(\alpha_{1n}+\cdots+\alpha_{mn})} \end{aligned} \tag{1.8}$$

for sufficiently large indices $\alpha_{i\ell}$. Such a condition was first considered by Muscalu, Pipher, Tao, and Thiele [35, 36], who obtained boundedness for the associated operator in the case $m = 2$, i.e., from $L^{p_1} \times L^{p_2}$ to L^p when $1/p_1 + 1/p_2 = 1/p$ and $1/2 < p < \infty$.

In this article we also prove a multilinear multiplier theorem that extends condition (1.8). Precisely, we study multilinear multipliers σ that satisfy (1.8) but α_{ji} are restricted. To handle the case of fractional derivatives we state our condition in terms of Sobolev spaces. We denote by $(I - \partial^2_{\xi_{j\ell}})^{\frac{\gamma_{j\ell}}{2}}$ the operator given on the Fourier transform side by multiplication by $(1 + 4\pi^2|y_{j\ell}|^2)^{\frac{\gamma_{j\ell}}{2}}$, where y_j is the dual variable of ξ_j. We now state our multiparameter version of Theorem 1.1, which extends the results in [35, 36] for Hörmander type multipliers with minimal smoothness. We do not need the time-frequency analysis used in [35, 36].

Theorem 1.4. *Let $1 \le r \le 2$ and $\gamma_{i\ell} > 1/r$ for all $1 \le i \le m$ and $1 \le \ell \le n$. Suppose that σ is a bounded function on $\mathbb{R}^{mn}$ such that*

$$\sup_{k_1,\dots,k_n \in \mathbb{Z}} \Big\| \prod_{j=1}^m (I - \partial^2_{\xi_{j1}})^{\frac{\gamma_{j1}}{2}} \cdots (I - \partial^2_{\xi_{jn}})^{\frac{\gamma_{jn}}{2}} \Big[\sigma\big(D_{k_1,\dots,k_n}\Xi\big) \prod_{\ell=1}^n \widehat{\Psi_\ell}(\xi_{1\ell},\dots,\xi_{m\ell}) \Big] \Big\|_{L^r(\mathbb{R}^{mn})} = A < \infty,$$

where

$$D_{k_1,\dots,k_n}\Xi = \begin{bmatrix} 2^{k_1}\xi_{11} & 2^{k_2}\xi_{12} & \dots & 2^{k_n}\xi_{1n} \\ 2^{k_1}\xi_{21} & 2^{k_2}\xi_{22} & \dots & 2^{k_n}\xi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 2^{k_1}\xi_{m1} & 2^{k_2}\xi_{m2} & \dots & 2^{k_n}\xi_{mn} \end{bmatrix},$$

for some $\widehat{\Psi_\ell}$ smooth functions on $\mathbb{R}^m$ supported in the annulus $\frac{1}{2} \le |\eta| \le 2$ satisfying

$$\sum_{k\in\mathbb{Z}} \widehat{\Psi_\ell}(2^{-k}\eta) = 1, \qquad \text{for all } \eta \in \mathbb{R}^m \setminus \{0\}. \tag{1.9}$$

If $1 < p_i < \infty$, $i = 1,\dots,m$, satisfy $\max_{1\le i\le m} \max_{1\le \ell\le n} \frac{1}{\gamma_{i\ell}} < \min_{1\le i\le m} p_i$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, then we have

$$\|T_\sigma\|_{L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim A.$$

A version of Theorem 1.4 was proved by Chen and Lu [6] when $r = m = 2$ and when the differential operator $(I - \partial^2_{\xi_{j1}})^{\frac{\gamma_{j1}}{2}} \cdots (I - \partial^2_{\xi_{jn}})^{\frac{\gamma_{jn}}{2}}$ is replaced by $(I - \Delta_{\xi_j})^{\frac{\gamma_j}{2}}$, where $\gamma_j = \gamma_{j1} + \cdots + \gamma_{jn}$; besides allowing r to be less than 2 and $m \ge 2$, Theorem 1.4 improves that of Chen and Lu [6] in the sense that only a restricted number of derivatives falls on each coordinate, while in [6] all derivatives could fall on a single coordinate ξ_j of the multiplier. An immediate consequence of Theorem 1.4 is the following:

Corollary 1.5. *Let $\sigma_\ell(\xi_{1\ell},\dots,\xi_{m\ell})$ be bounded functions on $\mathbb{R}^m$ for $1\le \ell\le n$. Let*

$$\sigma(\xi_1,\dots,\xi_m)=\prod_{\ell=1}^{n}\sigma_\ell(\xi_{1\ell},\dots,\xi_{m\ell}),$$

where $\xi_i=(\xi_{i1},\dots,\xi_{in})\in\mathbb{R}^n$, $1\le i\le m$. Suppose that for some $\gamma_{i\ell}$ and r as in Theorem 1.4 we have

$$\sup_{1\le\ell\le n}\sup_{k\in\mathbb{Z}}\Big\|(I-\partial^2_{\xi_{1\ell}})^{\frac{\gamma_{1\ell}}{2}}\cdots(I-\partial^2_{\xi_{m\ell}})^{\frac{\gamma_{m\ell}}{2}}\Big[\sigma_\ell(2^k\cdot)\widehat{\Psi_\ell}\Big]\Big\|_{L^r(\mathbb{R}^m)}=B<\infty$$

where $\widehat{\Psi_\ell}$ is a smooth function supported in an annulus in $\mathbb{R}^m$ that satisfies (1.9)*. Then for $1<p_i<\infty$, $1\le i\le m$, $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$ and $\max\limits_{1\le i\le m}\max\limits_{1\le\ell\le n}\frac{1}{\gamma_{i\ell}}<\min\limits_{1\le i\le m}p_i$ we have*

$$\|T_\sigma\|_{L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}\lesssim B^n.$$

As an application, we use this corollary to give a short proof of the boundedness of Calderón-Coifman-Journé commutators (Proposition 6.7).

Finally, we use arrows to denote elements of $\mathbb{R}^{nm}$, i.e., $\vec{\xi}=(\xi_1,\dots,\xi_m)$, where $\xi_j\in\mathbb{R}^n$.

2. Preliminaries

The following lemma will be useful in the sequel.

Lemma 2.1. *Let $\gamma_{i\ell},\gamma_j,\gamma>0$, $1\le i,j\le m$, $1\le\ell\le n$, $M>0$. Let D^Γ be a differential operator on $\mathbb{R}^{mn}$ of one of the following three types:*

$$\prod_{i=1}^{m}\prod_{\ell=1}^{n}(I-\partial^2_{\xi_{i\ell}})^{\frac{\gamma_{i\ell}}{2}};$$
$$(I-\Delta_{\xi_1})^{\frac{\gamma_1}{2}}\cdots(I-\Delta_{\xi_m})^{\frac{\gamma_m}{2}};$$
$$(I-\Delta_{\xi_1}-\cdots-\Delta_{\xi_m})^{\frac{\gamma}{2}}.$$

Let $1<r\le 2$, $1\le\rho\le r$ and let ϕ be a smooth function with compact support. Then there is a constant $C=C(\rho,r,\phi,n,\gamma_{i\ell},\gamma_j,\gamma,M)$ such that

$$\big\|\mathcal{F}\big(D^\Gamma(\phi f)\big)\big\|_{L^{\rho'}(\mathbb{R}^{mn})}\le C\big\|D^\Gamma(f)\big\|_{L^r(\mathbb{R}^{mn})}\tag{2.1}$$

is valid for all Schwartz functions f supported in the ball $\big\{\xi\in\mathbb{R}^m:\ |\xi|\le M\big\}$, where $\mathcal{F}$ is the Fourier transform.

Moreover, if D_δ is an operator of one of the following three types:

$$\prod_{i=1}^{m}\prod_{\ell=1}^{n}(I-\partial^2_{\xi_{i\ell}})^{\frac{\delta}{2}};$$
$$(I-\Delta_{\xi_1})^{\frac{\delta}{2}}\cdots(I-\Delta_{\xi_m})^{\frac{\delta}{2}};$$

$$(I-\Delta_{\xi_1}-\cdots-\Delta_{\xi_m})^{\frac{\delta}{2}}$$

then for D^Γ and D_δ of the same type and $\delta>0$ we have

$$\big\|\mathcal{F}(D^\Gamma D_{-\delta}(\phi f))\big\|_{L^\infty(\mathbb{R}^{mn})}\le C'\big\|D^\Gamma D_\delta(f)\big\|_{L^1(\mathbb{R}^{mn})} \tag{2.2}$$

for all Schwartz functions f supported in the ball $\big\{\xi\in\mathbb{R}^m:\ |\xi|\le M\big\}$.

Proof. We will focus on the case when $D^\Gamma=(I-\Delta_{\xi_1})^{\frac{\gamma_1}{2}}\cdots(I-\Delta_{\xi_m})^{\frac{\gamma_m}{2}}$ with $\xi_1,\dots,\xi_m\in\mathbb{R}$, while other cases can be handled in a similar way. We abbreviate $\|f\|_{L^r_{\vec\gamma}(\mathbb{R}^m)}=\big\|D^\Gamma(f)\big\|_{L^r(\mathbb{R}^m)}$.

Let $\omega_{\vec\gamma}(x)=(1+|x_1|^2)^{\gamma_1/2}\cdots(1+|x_m|^2)^{\gamma_m/2}$. We have the following lemma.

Lemma 2.2. *Let $1<r\le\infty$ and $\vec\gamma>0$ be a vector in $\mathbb{R}^m$. Assume that f is a function defined on $\mathbb{R}^m$, supported in the ball $\big\{\xi\in\mathbb{R}^m:\ |\xi|\le M\big\}$. Then there exists a constant $C_{m,M}$ such that*

$$\Big\|\widehat{f}\omega_{\vec\gamma}\Big\|_{L^\infty(\mathbb{R}^m)}\le C_{d,M}\Big\|\widehat{f}\omega_{\vec\gamma}\Big\|_{L^r(\mathbb{R}^m)}. \tag{2.3}$$

Proof. Let φ be a Schwartz function on $\mathbb{R}^m$ such that $\widehat{\varphi}(\xi)=1$ for all $\xi\in\mathbb{R}^d$, $|\xi|\le M$. Then we have $f(\xi)=f(\xi)\widehat{\varphi}(\xi)$. Taking the inverse Fourier transform yields

$$\widehat{f}(x)=\big(\widehat{f}*\varphi\big)(x)=\int_{\mathbb{R}^m}\widehat{f}(x-y)\varphi(y)\,dy.$$

Since $\omega_{\vec\gamma}(x)\lesssim\omega_{\vec\gamma}(x-y)\omega_{\vec\gamma}(y)$ for all $y\in\mathbb{R}^m$, we have for every $x\in\mathbb{R}^m$

$$\begin{aligned}
\Big|\widehat{f}(x)\Big|\omega_{\vec\gamma}(x) &= \omega_{\vec\gamma}(x)\Big|\int_{\mathbb{R}^m}\widehat{f}(x-y)\varphi(y)\,dy\Big|\\
&\lesssim \int_{\mathbb{R}^m}\omega_{\vec\gamma}(x-y)\Big|\widehat{f}(x-y)\Big|\omega_{\vec\gamma}(y)|\varphi(y)|\,dy\\
&\lesssim \Big\|\omega_{\vec\gamma}(x-\cdot)\widehat{f}(x-\cdot)\Big\|_{L^r(\mathbb{R}^m,dy)}\|\omega_{\vec\gamma}\,\varphi\|_{L^{r'}(\mathbb{R}^m,dy)}\\
&\lesssim \Big\|\widehat{f}\omega_{\vec\gamma}\Big\|_{L^r(\mathbb{R}^m,dy)},
\end{aligned}$$

where we used Hölder's inequality in the penultimate inequality and the implicit constant in the last inequality depends only on φ which relies only on the support of f, i.e., the constant M in the statement of the lemma and the dimension m. This proves (2.3). □

Let us continue the proof of Lemma 2.1. Interpolating between (2.3) with integration index r' and the trivial estimate

$$\Big\|\widehat{f}\omega_{\vec\gamma}\Big\|_{L^{r'}(\mathbb{R}^m)}\le\Big\|\widehat{f}\omega_{\vec\gamma}\Big\|_{L^{r'}(\mathbb{R}^m)}$$

yields

$$\Big\|\widehat{f}\omega_{\vec\gamma}\Big\|_{L^{\rho'}(\mathbb{R}^m)}\lesssim\Big\|\widehat{f}\omega_{\vec\gamma}\Big\|_{L^{r'}(\mathbb{R}^m)},$$

for all $1 < r' \le \rho' \le \infty$. In particular by the Hausdorff-Young inequality we have

$$\Big\|\widehat{f\phi}\omega_{\vec\gamma}\Big\|_{L^{\rho'}(\mathbb{R}^m)} \lesssim \Big\|\widehat{f\phi}\omega_{\vec\gamma}\Big\|_{L^{r'}(\mathbb{R}^m)} \lesssim \|f\phi\|_{L^r_{\vec\gamma}(\mathbb{R}^m)}, \tag{2.4}$$

for all $1 \le \rho \le r \le 2$.

For fixed $\xi \in \mathbb{R}^m$ we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^m} \omega_{\vec\gamma}(x)\big(\widehat{f} * \widehat{\phi}\big)(x)e^{2\pi i\xi\cdot x}dx\Big| \\
= &\Big|\int_{\mathbb{R}^m} \omega_{\vec\gamma}(x)\Big\{\int_{\mathbb{R}^n} \widehat{f}(x-y)\widehat{\phi}(y)dy\Big\}e^{2\pi\xi\cdot x}dx\Big| \\
= &\Big|\int_{\mathbb{R}^m} \widehat{\phi}(y)\Big\{\int_{\mathbb{R}^m} \omega_{\vec\gamma}(x)\widehat{f}(x-y)e^{2\pi i\xi\cdot x}dx\Big\}dy\Big| \\
= &\Big|\int_{\mathbb{R}^m} \widehat{\phi}(y)\Big\{\int_{\mathbb{R}^m} \omega_{\vec\gamma}(x+y)\widehat{f}(x)e^{2\pi i\xi\cdot x}dx\Big\}e^{2\pi i\xi\cdot y}dy\Big| \\
\le &\int_{\mathbb{R}^m} |\widehat{\phi}(y)|\Big|\int_{\mathbb{R}^m} F^y_{\vec\gamma}(x)\omega_{\vec\gamma}(x)\widehat{f}(x)e^{2\pi i\xi\cdot x}dx\Big|dy, \qquad (2.5)
\end{aligned}$$

where $F^y_{\vec\gamma}(x) = \frac{\omega_{\vec\gamma}(x+y)}{\omega_{\vec\gamma}(x)}$. It is routine to verify that $F^y_{\vec\gamma}(x)$ is a Mihlin multiplier, which is bounded on $L^r(\mathbb{R}^m)$ for any fixed $\vec\gamma > 0$ and $y \in \mathbb{R}^m$ with multiplier norm at most a multiple of $(1+|y|^2)^{c(\vec\gamma)}$, where $c(\vec\gamma) > 0$. More precisely, $F^y_{\vec\gamma}$ satisfies the estimate

$$\Big(\int_{\mathbb{R}^m}\Big|\int_{\mathbb{R}^m} F^y_{\vec\gamma}(x)\omega_{\vec\gamma}(x)\widehat{f}(x)e^{2\pi i\xi\cdot x}dx\Big|^r d\xi\Big)^{\frac{1}{r}} \le C(1+|y|^2)^{c(\vec\gamma)}\,\|f\|_{L^r_{\vec\gamma}(\mathbb{R}^m)} \tag{2.6}$$

for all $1 < r < \infty$. Then applying Minkowski's inequality in (2.5) and combining with (2.6) yields

$$\|f\phi\|_{L^r_{\vec\gamma}(\mathbb{R}^m)} \le C\|f\|_{L^r_{\vec\gamma}(\mathbb{R}^m)}.$$

Applying this inequality to the last term of (2.4), we obtain (2.1).

We next verify (2.2) for $D^\Gamma = (I-\Delta_{\xi_1})^{\frac{\gamma_1}{2}}\cdots(I-\Delta_{\xi_m})^{\frac{\gamma_m}{2}}$. Notice that

$$\|\mathcal{F}\big(D^\Gamma D_{-\delta}(\phi f)\big)\|_{L^\infty(\mathbb{R}^m)} = \|\omega_{\vec\gamma-\delta}\widehat{\phi f}\|_{L^\infty(\mathbb{R}^m)} \le \|\omega_{\vec\gamma+\delta}\widehat{\phi f}\|_{L^\infty(\mathbb{R}^m)}.$$

We claim that

$$\|\omega_{\vec\gamma}\widehat{\phi f}\|_{L^\infty(\mathbb{R}^m)} \le C_{\phi,\vec\gamma}\|\omega_{\vec\gamma}\widehat{f}\|_{L^\infty(\mathbb{R}^m)} \ \ \forall\,\vec\gamma,$$

which implies that

$$\|\omega_{\vec\gamma+\delta}\widehat{\phi f}\|_{L^\infty(\mathbb{R}^m)} \le C_{\phi,\vec\gamma}\|\omega_{\vec\gamma+\delta}\widehat{f}\|_{L^\infty(\mathbb{R}^m)} \le C\|D^\Gamma D_\delta(f)\|_{L^1(\mathbb{R}^m)}.$$

This gives (2.2).

It remains to prove the claim. It is easy to see that

$$\big|\omega_{\vec\gamma}(x)\widehat{\phi f}(x)\big| = \Big|\omega_{\vec\gamma}(x)\int\widehat{\phi}(x-y)\widehat{f}(y)dy\Big|$$

$$
\begin{aligned}
&= \Big|\omega_{\vec\gamma}(x)\int(\omega_{\vec\gamma})^{-1}(y)\widehat{\phi}(x-y)\omega_{\vec\gamma}(y)\widehat{f}(y)dy\Big| \\
&\le \|\omega_{\vec\gamma}\widehat{f}\|_{L^\infty}\int\big|\omega_{\vec\gamma}(x)(\omega_{\vec\gamma})^{-1}(y)\widehat{\phi}(x-y)\big|dy \\
&= \|\omega_{\vec\gamma}\widehat{f}\|_{L^\infty}\int\omega_{\vec\gamma}(x)(\omega_{\vec\gamma})^{-1}(x-y)\big|\widehat{\phi}(y)\big|dy.
\end{aligned}
$$

We notice that $\omega_{\vec\gamma}(x)\le\omega_{\vec\gamma}(x-y)\omega_{\vec\gamma}(y)$, so the last integral is bounded by

$$\int\omega_{\vec\gamma}(y)\big|\widehat{\phi}(y)\big|dy\le C_{\phi,\vec\gamma},$$

which verifies the claim.

This completes the proof of Lemma 2.1. □

We will also need a reverse square function inequality associated with Littlewood-Paley operators acting on each variable separately. We denote variables in $\mathbb{R}^{nl}$ by $(z_1,\dots,z_n)$, where each z_j lies in $\mathbb{R}^l$. Fix a smooth function $\widehat{\Psi}$ supported in an annulus in $\mathbb{R}^l$ satisfying $\sum_{j\in\mathbb{Z}}\widehat{\Psi}(2^{-j}z)=1$ for all $z\neq0$. For $j\in\mathbb{Z}$, define a Littlewood-Paley operator

$$\Delta_j^{(k)}(f)=\big(\widehat{f}(z_1,z_2,\dots,z_n)\widehat{\Psi}(2^{-j}z_k)\big)^\vee \tag{2.7}$$

acting on functions f on $\mathbb{R}^{nl}$. We need the following result.

Lemma 2.3. *For $f\in L^p(\mathbb{R}^{nl})$ with $1<p<\infty$ we have*

$$\Big\|\Big(\sum_{j_1}\cdots\sum_{j_n}|\Delta_{j_1}^{(1)}\cdots\Delta_{j_n}^{(n)}(f)|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^{nl})}\lesssim\|f\|_{L^p(\mathbb{R}^{nl})}. \tag{2.8}$$

Conversely, for $0<p<\infty$ there exists a constant C such that for any f in $L^2(\mathbb{R}^{nl})$ satisfying

$$\|(\sum_{j_1}\cdots\sum_{j_n}|\Delta_{j_1}^{(1)}\cdots\Delta_{j_n}^{(n)}(f)|^2)^{1/2}\|_{L^p(\mathbb{R}^{nl})}<\infty \tag{2.9}$$

we have

$$\|f\|_{L^p(\mathbb{R}^{nl})}\le C\Big\|\Big(\sum_{j_1}\cdots\sum_{j_n}|\Delta_{j_1}^{(1)}\cdots\Delta_{j_n}^{(n)}(f)|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^{nl})}. \tag{2.10}$$

Proof. The proof of (2.8) is well known and is omitted; see for instance [14, Theorem 6.1.6] when $l=1$ but the same idea works for all l. So we now focus on (2.10) which we prove inductively. The case $n=1$ is the reverse of the Littlewood-Paley inequality when $p>1$. When $n=1$ and $p\le1$, then by [15, Theorem 2.2.9] there is a polynomial Q on $\mathbb{R}^l$ such that

$$\|f-Q\|_{H^p(\mathbb{R}^l)}\lesssim\Big\|\Big(\sum_{j_1}|\Delta_{j_1}^{(1)}(f)|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^l)}<\infty\,.$$

Since f lies in $L^2(\mathbb{R}^l)$, it follows that $f-Q$ is a locally integrable function which lies in $H^p(\mathbb{R}^l)$ and thus $\|f-Q\|_{L^p} \lesssim \|f-Q\|_{H^p(\mathbb{R}^l)} < \infty$. Therefore $Q=0$ and (2.10) follows.

Assume that the assertion is valid for $n \geq 1$. We will prove that the assertion is also true for $n+1$. Fix $f \in L^2(\mathbb{R}^{(n+1)l})$ and for each $k \in \mathbb{Z}$, we denote $f_k = \Delta_k^{(n+1)}(f)$ as in (2.7). Let r_k be the Rademacher functions reindexed by $k \in \mathbb{Z}$. For fixed $x_{n+1} \in \mathbb{R}^l$, set $g_t(x_1,\dots,x_{n+1}) = \sum_k f_k(x_1,\dots,x_{n+1}) r_k(t)$. Then we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^l} \cdots \int_{\mathbb{R}^l} \Big(\sum_k |f_k(x_1,\dots,x_n,x_{n+1})|^2 \Big)^{p/2} dx_1 \cdots dx_n \\
&\quad \lesssim \int_{\mathbb{R}^l} \cdots \int_{\mathbb{R}^l} \int_0^1 \Big| \sum_k f_k(x_1,\dots,x_n,x_{n+1}) r_k(t) \Big|^p dt\, dx_1 \cdots dx_n \\
&\quad = C \int_0^1 \int_{\mathbb{R}^l} \cdots \int_{\mathbb{R}^l} |g_t(x_1,\dots,x_n,x_{n+1})|^p dx_1 \cdots dx_n\, dt,
\end{aligned}$$

where we used the property of Rademacher functions; see for instance [14, Appendix C]. By the induction hypothesis, the preceding expression is bounded by a multiple of

$$\begin{aligned}
&\int_0^1 \int_{(\mathbb{R}^l)^n} \Big(\sum_{j_1} \cdots \sum_{j_n} |\Delta_{j_1}^{(1)} \cdots \Delta_{j_n}^{(n)} g_t(x_1,\dots,x_n,x_{n+1})|^2 \Big)^{p/2} dx_1 \cdots dx_n\, dt \\
&\lesssim \int_0^1 \int_{(\mathbb{R}^l)^n} \int_{[0,1]^n} \Big| \sum_{j_1} \cdots \sum_{j_n} \Delta_{j_1}^{(1)} \cdots \Delta_{j_n}^{(n)} g_t(x_1,\dots,x_n,x_{n+1}) \prod_{i=1}^n r_{j_i}(t_i) \Big|^p dt_1 \cdots dt_n d\vec{x}\, dt \\
&= \int_{(\mathbb{R}^l)^n} \int_{[0,1]^{n+1}} \Big| \sum_{j_1,\dots,j_n,k} \Delta_{j_1}^{(1)} \cdots \Delta_{j_n}^{(n)} f_k(x_1,\dots,x_n,x_{n+1}) r_k(t) \prod_{i=1}^n r_{j_i}(t_i) \Big|^p dt_1 \cdots dt_n\, dt\, d\vec{x} \\
&\lesssim \int_{(\mathbb{R}^l)^n} \Big(\sum_{j_1} \cdots \sum_{j_n} \sum_k |\Delta_{j_1}^{(1)} \cdots \Delta_{j_n}^{(n)} f_k(x_1,\dots,x_n,x_{n+1})|^2 \Big)^{p/2} dx_1 \cdots dx_n,
\end{aligned}$$

once again the properties of Rademacher functions were used and $d\vec{x} = dx_1 \cdots dx_n$.

Let $(\varphi_t)_{t>0}$ be the approximate identity. Denote by $\varphi_t * f$ the convolution of φ with $f(x_1,\dots,x_n,\cdot)$ for fixed $(x_1,\dots,x_n)$. It follows that

$$\begin{aligned}
&\int_{(\mathbb{R}^l)^{n+1}} |f(x_1,\dots,x_{n+1})|^p dx_1 \cdots dx_{n+1} \\
&\lesssim \int_{(\mathbb{R}^l)^n} \int_{\mathbb{R}^l} \sup_{t>0} \Big| [\varphi_t * f(x_1,\dots,x_n,\cdot)](x_{n+1}) \Big|^p dx_{n+1}\, dx_1 \cdots dx_n \\
&\lesssim \int_{(\mathbb{R}^l)^n} \int_{\mathbb{R}^l} \Big(\sum_k |\Delta_k^{(n+1)} f(x_1,\dots,x_{n+1})|^2 \Big)^{p/2} dx_{n+1}\, dx_1 \cdots dx_n \\
&= \int_{\mathbb{R}^l} \int_{(\mathbb{R}^l)^n} \Big(\sum_k |f_k(x_1,\dots,x_{n+1})|^2 \Big)^{p/2} dx_1 \cdots dx_n\, dx_{n+1}
\end{aligned}$$

$$\lesssim \int_{(\mathbb{R}^l)^{n+1}} \Big(\sum_{j_1} \cdots \sum_{j_n} \sum_{k} |\Delta_{j_1}^{(1)} \cdots \Delta_{j_n}^{(n)} \Delta_k^{(n+1)} f(x_1, \dots, x_n, x_{n+1})|^2 \Big)^{p/2} dx_1 \cdots dx_n dx_{n+1},$$

where in the last step we use the inequality in the preceding alignment. To make this argument precise, we work with finitely many terms and then pass to limit using Fatou's lemma. □

Remark 2.4. *In both* (2.8) *and* (2.10) *we do not need the full set of variables. For example, we have*

$$\Big\| \Big(\sum_{j_1 \in \mathbb{Z}} \cdots \sum_{j_q \in \mathbb{Z}} |\Delta_{j_1}^{(1)} \cdots \Delta_{j_q}^{(q)} (f)|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^{nl})} \approx \|f\|_{L^p(\mathbb{R}^{nl})}$$

for any $1 \le q \le n$ *by applying Lemma 2.3 to* f *as a function of* $(x_1, \dots, x_q)$.

Remark 2.5. *The same proof as in Lemma 2.3 can be used to show that if* $1 \le q \le n$ *and* $f_{j_1,\dots,j_q} \in L^2(\mathbb{R}^{nl}), j_i \in \mathbb{Z}, 1 \le i \le q$, *and if there exists a constant* $B > 1$ *such that*

$$\operatorname{supp} \widehat{f_{j_1,\dots,j_q}} \subset \Big\{ (\xi_1, \dots, \xi_n) \in \mathbb{R}^{nl} : \frac{2^{j_i}}{B} \le |\xi_i| \le 2^{j_i} B, i = 1, \dots, q \Big\},$$

then

$$\Big\| \sum_{j_1,\dots,j_q} f_{j_1,\dots,j_q} \Big\|_{L^p(\mathbb{R}^{nl})} \le C \Big\| \Big(\sum_{j_1,\dots,j_q} |f_{j_1,\dots,j_q}|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R}^{nl})}$$

for $0 < p < \infty$.

Remark 2.6. *As a consequence of* (2.10) *one can derive the following inequality:*

$$\|f\|_{L^p(\mathbb{R}^{nl})} \le C \|f\|_{H^p(\mathbb{R}^l \times \cdots \times \mathbb{R}^l)} \qquad \text{for} \quad f \in L^2(\mathbb{R}^{nl}), \ 0 < p \le 1$$

where $H^p(\underbrace{\mathbb{R}^l \times \cdots \times \mathbb{R}^l}_{n \text{ times}})$ *denotes the multiparameter Hardy space; on this see* [23].

3. The proof of Theorem 1.1

We introduce the open cones

$$U_k = \Big\{ (\xi_1, \dots, \xi_m) \in (\mathbb{R}^n)^m \ : \ \max_{j \ne k} |\xi_j| < \frac{1}{5m} |\xi_k| \Big\}, \qquad 1 \le k \le m$$

and for $1 \le k \ne l \le m$,

$$W_{k,l} = \Big\{ (\xi_1, \dots, \xi_m) \in (\mathbb{R}^n)^m \ : \ \max_{j \ne k,l} |\xi_j| < \frac{11}{10} |\xi_k|, \ \frac{1}{10m} |\xi_\ell| < |\xi_k| \le 2|\xi_l| \Big\}.$$

We now construct smooth homogeneous functions of degree zero Φ_k and $\Psi_{k,l}$ supported in U_k and $W_{k,l}$, respectively, and such that

$$\sum_{1 \le k \le m} \Phi_k(\xi_1, \dots, \xi_m) + \sum_{1 \le k \ne l \le m} \Psi_{k,l}(\xi_1, \dots, \xi_m) = 1$$

for every $(\xi_1,\dots,\xi_m)$ in $\big(\mathbb{R}^n\big)^m\setminus\{(0,\dots,0)\}$; such functions can be constructed by following the hint of Exercise 7.5.4 in [15]. In the support of Φ_k the vector with the largest magnitude is ξ_l, while in the support of $\Psi_{k,l}$ the vector with the largest magnitude is ξ_l and the one with the second largest magnitude is ξ_k. The above partition of unity decomposes σ as:

$$\sigma=\sum_{k=1}^m \sigma\Phi_k+\sum_{\substack{k=1\\k\neq j}}^m\sum_{j=1}^m \sigma\,\Psi_{j,k}. \tag{3.1}$$

We will prove the required assertion for each piece of this decomposition, i.e., for the multipliers $\sigma\,\Phi_k$ for $1\le k\le m$ and $\sigma\,\Psi_{j,k}$ for each pair (j,k) in the previous sum. In view of the symmetry of the decomposition, it suffices to consider two cases: $\sigma\Phi_m$ and $\sigma\Psi_{m,m-1}$ in the sum in (3.1). Thus it is enough to prove boundedness for the m-linear operators whose symbols are $\sigma_1=\sigma\,\Phi_m$ and $\sigma_2=\sigma\,\Psi_{m,m-1}$. These correspond to the m-linear operators T_{σ_1} and T_{σ_2}, respectively. Note that σ_1 is supported in the set where

$$\max(|\xi_1|,\dots,|\xi_{m-1}|)<\tfrac{|\xi_m|}{5m}.$$

Also σ_2 is supported in the set where

$$\max(|\xi_1|,\dots,|\xi_{m-2}|)<\tfrac{11}{10}\,|\xi_{m-1}| \qquad\text{and}\qquad \tfrac{1}{10m}<\tfrac{|\xi_{m-1}|}{|\xi_m|}\le 2\,.$$

Fix a Schwartz function θ whose Fourier transform is supported in the annulus $\frac12\le|\xi|\le 2$ and $\sum_{j\in\mathbf{Z}}\widehat{\theta}(2^{-j}\xi)=1$ for $\xi\in\mathbb{R}^n\setminus\{0\}$. Associated with θ we define the Littlewood–Paley operator $\Delta_j^\theta(g)=g*\theta_{2^{-j}}$, where $\theta_t(x)=t^{-n}\theta(t^{-1}x)$ for $t>0$. The function θ can be extended to the function Θ defined on $\mathbb{R}^{nm}$ by setting $\widehat{\Theta}(\vec{\xi}\,)=\widehat{\Theta}(\xi_1,\dots,\xi_m)=\widehat{\theta}(\xi_1+\cdots+\xi_m)$. For given Schwartz functions f_j we have

$$\begin{aligned}
&\Delta_j^\theta\big(T_{\sigma_1}(f_1,\dots,f_m)\big)(x)\\
&\qquad=\int_{\mathbb{R}^{mn}}\widehat{\theta}(2^{-j}(\xi_1+\cdots+\xi_m))\sigma_1(\vec{\xi}\,)\widehat{f_1}(\xi_1)\cdots\widehat{f_m}(\xi_m)e^{2\pi i x\cdot(\xi_1+\cdots+\xi_m)}d\vec{\xi}\\
&\qquad=\int_{\mathbb{R}^{mn}}\widehat{\Theta}(2^{-j}\vec{\xi}\,)\sigma_1(\vec{\xi}\,)\widehat{f_1}(\xi_1)\cdots\widehat{f_m}(\xi_m)e^{2\pi i x\cdot(\xi_1+\cdots+\xi_m)}d\vec{\xi}.
\end{aligned}$$

Note that for all $\vec{\xi}=(\xi_1,\dots,\xi_m)$ in the support of the function $\widehat{\Theta}(2^{-j}\vec{\xi}\,)\sigma_1(\vec{\xi}\,)$, we always have $2^{j-2}\le|\xi_m|\le 2^{j+2}$. Therefore we can take a Schwartz function η whose Fourier transform is supported in $\frac18\le|\xi_m|\le 8$ and identical to 1 on $\frac14\le|\xi_m|\le 4$ and insert the factor $\widehat{\eta}(2^{-j}\xi_m)$ into the above integral without changing the outcome. More specifically

$$\begin{aligned}
&\Delta_j^\theta\big(T_{\sigma_1}(f_1,\dots,f_m)\big)(x)\\
&=\int_{\mathbb{R}^{mn}}\widehat{\Theta}(2^{-j}\vec{\xi}\,)\sigma_1(\vec{\xi}\,)\widehat{f_1}(\xi_1)\cdots\widehat{f_{m-1}}(\xi_{m-1})\widehat{\eta}(2^{-j}\xi_m)\widehat{f_m}(\xi_m)e^{2\pi i x\cdot(\xi_1+\cdots+\xi_m)}d\vec{\xi}.
\end{aligned}$$

Now define $\widehat{\Psi_*}(\vec{\xi}) = \sum_{|k|\le 4} \widehat{\Psi}(2^{-k}\vec{\xi})$ and note that $\widehat{\Psi_*}(2^{-j}\vec{\xi})$ is equal to 1 on the annulus $\{\vec{\xi} \in \mathbb{R}^{mn}: 2^{j-4} \le |\vec{\xi}| \le 2^{j+4}\}$ which contains the support of $\sigma_1\widehat{\Theta}(2^{-j}\cdot)$. Then we write

$$\begin{aligned}&\Delta_j^\theta\big(T_{\sigma_1}(f_1,\dots,f_m)\big)(x)\\ &= \int_{\mathbb{R}^{mn}} \widehat{\Psi_*}(2^{-j}\vec{\xi})\widehat{\Theta}(2^{-j}\vec{\xi})\sigma_1(\vec{\xi})\widehat{f_1}(\xi_1)\cdots\widehat{f_{m-1}}(\xi_{m-1})\widehat{\eta}(2^{-j}\xi_m)\widehat{f_m}(\xi_m)e^{2\pi i x\cdot(\xi_1+\cdots+\xi_m)}d\vec{\xi}.\end{aligned}$$

Taking the inverse Fourier transform, we obtain that $\Delta_j^\theta\big(T_{\sigma_1}(f_1,\dots,f_m)\big)(x)$ is equal to

$$\int_{(\mathbb{R}^n)^m} 2^{mnj}(\sigma_1^j\widehat{\Psi_*}\widehat{\Theta})^\vee\big(2^j(x-y_1),\dots,2^j(x-y_m)\big)\prod_{i=1}^{m-1} f_i(y_i)\,(\Delta_j^\eta f_m)(y_m)\,d\vec{y}, \tag{3.2}$$

where $d\vec{y} = dy_1\cdots dy_m$, and $\sigma_1^j(\xi_1,\xi_2,\dots,\xi_m) = \sigma_1(2^j\xi_1, 2^j\xi_2,\dots,2^j\xi_m)$.

Recall our assumptions that $\max_{1\le i\le m}\max_{1\le \ell\le n}\frac{1}{\gamma_{i\ell}} < r$ and $\max_{1\le i\le m}\max_{1\le \ell\le n}\frac{1}{\gamma_{i\ell}} < \min(p_1,\dots,p_m)$. If $r>1$ we pick ρ such that $1<\rho<2$ and $\max_{1\le i\le m}\max_{1\le \ell\le n}\frac{1}{\gamma_{i\ell}} < \rho < \min(p_1,\dots,p_m,r)$. If $r=1$, we set $\rho=1$.

Define a weight for $(y_1,\dots,y_m)\in(\mathbb{R}^n)^m$ by setting

$$w_{\vec{\gamma}}(y_1,\dots,y_m) = \prod_{1\le i\le m}\prod_{1\le \ell\le n}(1+4\pi^2|y_{i\ell}|^2)^{\frac{\gamma_{i\ell}}{2}}.$$

Let us first suppose that $\rho>1$. We have

$$\begin{aligned}&|\Delta_j^\theta\big(T_{\sigma_1}(f_1,\dots,f_{m-1},f_m)\big)(x)|\\
\le&\int_{(\mathbb{R}^n)^m} w_{\vec{\gamma}}\big(2^j(x-y_1),\dots,2^j(x-y_m)\big)\ |(\sigma_1^j\,\widehat{\Psi_*}\widehat{\Theta})^\vee(2^j(x-y_1),\dots,2^j(x-y_m))|\\
&\qquad\times\frac{2^{mnj}|f_1(y_1)\cdots f_{m-1}(y_{m-1})(\Delta_j^\eta f_m)(y_m)|}{w_{\vec{\gamma}}\big(2^j(x-y_1),\dots,2^j(x-y_m)\big)}\,d\vec{y}\\
\le&\Big[\int_{(\mathbb{R}^n)^m}\big|\big(w_{\vec{\gamma}}\,(\sigma_1^j\,\widehat{\Psi_*}\widehat{\Theta})^\vee\big)(2^j(x-y_1),\dots,2^j(x-y_m))\big|^{\rho'}d\vec{y}\Big]^{\frac{1}{\rho'}}\\
&\quad\times 2^{mnj}\left(\int_{(\mathbb{R}^n)^m}\frac{|f_1(y_1)\cdots f_{m-1}(y_{m-1})(\Delta_j^\eta f_m)(y_m)|^\rho}{w_{\rho\vec{\gamma}}\big(2^j(x-y_1),\dots,2^j(x-y_m)\big)}\,d\vec{y}\right)^{\frac{1}{\rho}}\\
\le&C\left(\int_{(\mathbb{R}^n)^m}\Big|w_{\vec{\gamma}}(y_1,\dots,y_m)(\sigma_1^j\,\widehat{\Psi_*}\widehat{\Theta})^\vee(y_1,\dots,y_m)\Big|^{\rho'}d\vec{y}\right)^{\frac{1}{\rho'}}\\
&\quad\times\left(\int_{(\mathbb{R}^n)^m}\frac{2^{mnj}|f_1(y_1)\cdots f_{m-1}(y_{m-1})(\Delta_j^\eta f_m)(y_m)|^\rho}{\big(\prod_{\ell=1}^n(1+2^j|x_\ell-y_{1\ell}|)^{\rho\gamma_{1\ell}}\big)\cdots\big(\prod_{\ell=1}^n(1+2^j|x_\ell-y_{m\ell}|)^{\rho\gamma_{m\ell}}\big)}\,d\vec{y}\right)^{\frac{1}{\rho}}\\
\le&C\Big\|\prod_{i=1}^m\prod_{\ell=1}^n(I-\partial_{\xi_{i\ell}}^2)^{\frac{\gamma_{i\ell}}{2}}(\sigma(2^j(\cdot))\,\widehat{\Psi_*})\Big\|_{L^r}\prod_{i=1}^{m-1}\left(\int_{\mathbb{R}^n}\frac{2^{jn}|f_i(y_i)|^\rho}{\prod_{\ell=1}^n(1+2^j|x_\ell-y_{i\ell}|)^{\rho\gamma_{i\ell}}}\,dy_i\right)^{\frac{1}{\rho}}\end{aligned}$$

$$\times \left(\int_{\mathbb{R}^n} \frac{2^{jn} |(\Delta_j^\eta f_m)(y_m)|^\rho}{\prod_{\ell=1}^n (1 + 2^j |x_\ell - y_{m\ell}|)^{\rho \gamma_{m\ell}}} \, dy_m \right)^{\frac{1}{\rho}}$$

$$\leq C' A \left[\prod_{i=1}^{m-1} \mathcal{M}(|f_i|^\rho)(x)^{\frac{1}{\rho}} \right] \mathcal{M}(|\Delta_j^\eta f_m|^\rho)(x)^{\frac{1}{\rho}}$$

where $\mathcal{M}$ is the strong maximal function given as $\mathcal{M} = M^{(1)} \circ \cdots \circ M^{(n)}$, where $M^{(j)}$ is the Hardy-Littlewood maximal operator acting in the jth variable. Here we made use of (2.1) since $\sigma_1^j(\xi) = \sigma(2^j(\xi))\Phi_m(2^j\xi) = \sigma(2^j(\xi))\Phi_m(\xi)$.

We now turn to the case where $r = 1$ in which case $\rho = 1$. We choose $\gamma'_{i\ell} < \gamma_{i\ell}$ and $\delta > 0$ such that

$$\frac{1}{\gamma_{i\ell}} = \frac{1}{\gamma'_{i\ell} + \delta} < \frac{1}{\gamma'_{i\ell}} < \frac{1}{\gamma'_{i\ell} - \delta} < r = 1$$

for all i, ℓ. The preceding argument with $\gamma'_{i\ell} - \delta$ in place of $\gamma_{i\ell}$ yields that

$$|\Delta_j^\theta \big(T_{\sigma_1}(f_1, \dots, f_m)\big)| \leq C' \Big\| w_{\vec{\gamma}} (\sigma_1^j \, \widehat{\Psi_*} \widehat{\Theta}) \Big\|_{L^\infty} \left[\prod_{i=1}^{m-1} \mathcal{M}(|f_i|) \right] \mathcal{M}(|\Delta_j^\eta f_m|) \, .$$

In view of (2.2) we obtain

$$\Big\| w_{\vec{\gamma}} (\sigma_1^j \, \widehat{\Psi_*} \widehat{\Theta}) \Big\|_{L^\infty} \lesssim \Big\| \prod_{i=1}^m \prod_{\ell=1}^n (I - \partial_{\xi_{i\ell}}^2)^{\frac{\gamma'_{i\ell} + \delta}{2}} (\sigma(2^j(\cdot)) \, \widehat{\Psi_*}) \Big\|_{L^1} \lesssim A \, .$$

Thus, we have obtained the estimate

$$|\Delta_j^\theta \big(T_{\sigma_1}(f_1, \dots, f_{m-1}, f_m)\big)(x)| \lesssim A \left[\prod_{i=1}^{m-1} \mathcal{M}(|f_i|^\rho)(x)^{\frac{1}{\rho}} \right] \mathcal{M}(|\Delta_j^\eta f_m|^\rho)(x)^{\frac{1}{\rho}}$$

from which it follows that

$$\left(\sum_{j \in \mathbb{Z}} |\Delta_j^\theta T_{\sigma_1}(f_1, \dots, f_{m-1}, f_m)|^2 \right)^{\frac{1}{2}} \lesssim A \left[\prod_{i=1}^{m-1} \mathcal{M}(|f_i|^\rho)^{\frac{1}{\rho}} \right] \left(\sum_{j \in \mathbb{Z}} \mathcal{M}(|\Delta_j^\eta f_m|^\rho)^{\frac{2}{\rho}} \right)^{\frac{1}{2}} .$$

The claimed bound follows by applying Hölder's inequality with exponents $p_1, \dots, p_m$ and using the boundedness of $\mathcal{M}$ on $L^{p_i/\rho}$, $i = 1, \dots, m$, and the Fefferman-Stein [12] vector-valued Hardy-Littlewood maximal function inequality on $L^{p_m/\rho}$. (Note $1 < 2/\rho \leq 2$.)

Next we deal with σ_2. Using the notation introduced earlier, we write

$$T_{\sigma_2}(f_1, \dots, f_{m-1}, f_m) = \sum_{j \in \mathbb{Z}} T_{\sigma_2}(f_1, \dots, f_{m-1}, \Delta_j^\theta f_m) \, .$$

We introduce another Littlewood–Paley operator Δ_j^ζ, which is given on the Fourier transform by multiplying with a bump $\widehat{\zeta}(2^{-j}\xi)$, where $\widehat{\zeta}$ is equal to one on the annulus $\{\xi \in \mathbb{R}^n : \frac{1}{2^k} \leq |\xi| \leq 4\}$ with $\frac{1}{2^k} \leq \frac{1}{20m}$, vanishes off the annulus $\frac{1}{2^{k+1}} \leq |\xi| \leq 8$, and $\sum_j \widehat{\zeta}(2^{-j}\xi) = k + 3$. The key observation in this case is that

$$T_{\sigma_2}(f_1,\dots,f_{m-1},\Delta_j^\theta f_m) = T_{\sigma_2}\big(f_1,\dots,f_{m-2},\Delta_j^\zeta f_{m-1},\Delta_j^\theta f_m\big)\,. \tag{3.3}$$

As in the previous case, we have

$$\begin{aligned} &T_{\sigma_2}\big(f_1,\dots,f_{m-2},\Delta_j^\zeta f_{m-1},\Delta_j^\theta f_m\big)(x) \\ &= \int_{(\mathbf{R}^n)^m} \sigma_2(\vec{\xi}\,)\prod_{i=1}^{m-2}\widehat{f}_i(\xi_i)\ \widehat{\Delta_j^\zeta f_{m-1}}(\xi_{m-1})\widehat{\Delta_j^\theta f_m}(\xi_m) e^{2\pi i x\cdot(\xi_1+\cdots+\xi_m)}\,d\vec{\xi}. \end{aligned} \tag{3.4}$$

The integrand in the right-hand side of (3.4) is supported in $\frac{1}{2}2^j \le |\xi_1|+\cdots+|\xi_m| \le \frac{11m}{5}2^j$. Thus one may insert the factor

$$\widehat{\Psi_*}(2^{-j}\xi_1,\dots,2^{-j}\xi_m) = \sum_{|k|\le m+1}\widehat{\Psi}(2^{-j-k}\xi_1,\dots,2^{-j-k}\xi_m)$$

in the integrand.

A similar calculation as in the case for σ_1 yields the estimate

$$|T_{\sigma_2}(f_1,\dots,f_{m-2},\Delta_j^\zeta f_{m-1},\Delta_j^\theta f_m)| \lesssim A\bigg(\prod_{i=1}^{m-2}\mathcal{M}(|f_i|^\rho)^{\frac{1}{\rho}}\bigg)\mathcal{M}(|\Delta_j^\zeta f_{m-1}|^\rho)^{\frac{1}{\rho}}\mathcal{M}(|\Delta_j^\theta f_m|^\rho)^{\frac{1}{\rho}}\,.$$

Summing over j and taking L^p norms yields

$$\begin{aligned} &\big\|T_{\sigma_2}(f_1,\dots,f_{m-1},f_m)\big\|_{L^p(\mathbb{R}^n)} \\ &\le C\,A\bigg\|\bigg[\prod_{i=1}^{m-2}\mathcal{M}(|f_i|^\rho)^{\frac{1}{\rho}}\bigg]\sum_{j\in\mathbf{Z}}\mathcal{M}(|\Delta_j^\theta f_{m-1}|^\rho)^{\frac{1}{\rho}}\mathcal{M}(|\Delta_j^\eta f_m|^\rho)^{\frac{1}{\rho}}\bigg\|_{L^p} \\ &\le C\,A\bigg\|\bigg[\prod_{i=1}^{m-2}\mathcal{M}(|f_i|^\rho)^{\frac{1}{\rho}}\bigg]\bigg(\sum_{j\in\mathbf{Z}}\mathcal{M}(|\Delta_j^\theta f_{m-1}|^\rho)^{\frac{2}{\rho}}\bigg)^{\frac{1}{2}}\bigg(\sum_{j\in\mathbf{Z}}\mathcal{M}(|\Delta_j^\eta f_m|^\rho)^{\frac{2}{\rho}}\bigg)^{\frac{1}{2}}\bigg\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Applying Hölder's inequality, the boundedness of $\mathcal{M}$ on $L^{p_i/\rho}$, $i=1,\dots,m-1$, and the Fefferman-Stein [12] vector-valued Hardy-Littlewood maximal function inequality on $L^{p_{m-1}/\rho}$ or on $L^{p_m/\rho}$ (noting that $1<2/\rho\le 2$) concludes the proof of Theorem 1.1.

Remark 3.1. *In case I we obtained the estimate*

$$|\Delta_j^\theta\big(T_{\sigma_1}(f_1,\dots,f_{m-1},f_m)\big)| \lesssim A\bigg[\prod_{i=1}^{m-1}\mathcal{M}(|f_i|^\rho)^{\frac{1}{\rho}}\bigg]\mathcal{M}(|\Delta_j^\eta f_m|^\rho)^{\frac{1}{\rho}}.$$

In case II we obtained the estimate

$$|T_{\sigma_2}(f_1,\dots,f_{m-1},\Delta_j^\theta f_m)| \lesssim A\bigg(\prod_{i=1}^{m-2}\mathcal{M}(|f_i|^\rho)^{\frac{1}{\rho}}\bigg)\mathcal{M}(|\Delta_j^\zeta f_{m-1}|^\rho)^{\frac{1}{\rho}}\mathcal{M}(|\Delta_j^\theta f_m|^\rho)^{\frac{1}{\rho}}.$$

By symmetry for any $1 \le j_0 \le m$ *we have*

$$|\Delta_j^\theta\big(T_{\sigma\Phi_{j_0}}(f_1,\dots,f_m)\big)| \lesssim A\Big[\prod_{\substack{1\le i\le m\\ i\neq j_0}} \mathcal{M}(|f_i|^\rho)^{\frac{1}{\rho}}\Big]\mathcal{M}(|\Delta_j^\eta f_{j_0}|^\rho)^{\frac{1}{\rho}}$$

and for $\sigma\Psi_{j_0,k_0}$, $k_0 \neq j_0$

$$|T_{\sigma\Psi_{j_0,k_0}}(f_1,\dots,\Delta_j^\theta f_{j_0},\dots,f_m)| \lesssim A\Big[\prod_{\substack{1\le i\le m\\ i\notin\{j_0,k_0\}}} \mathcal{M}(|f_i|^\rho)^{\frac{1}{\rho}}\Big]\mathcal{M}(|\Delta_j^\eta f_{j_0}|^\rho)^{\frac{1}{\rho}}\mathcal{M}(|\Delta_j^\zeta f_{k_0}|^\rho)^{\frac{1}{\rho}}.$$

4. The proof of Theorem 1.3

Recall the sets U_k and $W_{k,l}$ and the functions Φ_k and $\Psi_{k,l}$ in the proof of Theorem 1.1. Then we can decompose σ as in (3.1). As showed in the proof of Theorem 1.1, it suffices to establish the claimed estimate for T_{σ_1} and T_{σ_2} with $\sigma_1 = \sigma\Phi_m$ and $\sigma_2 = \sigma\Psi_{m-1,m}$.

We first consider $T_{\sigma_1}(f_1,\dots,f_m)$, where f_j are fixed Schwartz functions. We will prove

$$\Big\|\Big(\sum_j \Delta_j^\theta(T_{\sigma_1}(f_1,\dots,f_m))|^2\Big)^{1/2}\Big\|_{L^{1/m,\infty}(\mathbb{R}^n)} \lesssim A\,\|f_1\|_{H^1(\mathbb{R}^n)}\cdots\|f_m\|_{H^1(\mathbb{R}^n)}. \tag{4.1}$$

Let $H^{1/m,\infty}$ denote the weak Hardy space of all bounded tempered distributions whose smooth maximal function lies in weak $L^{1/m}$. Given $0 < p < \infty$, for F in $L^2(\mathbb{R}^n)$ there is a polynomial Q on $\mathbb{R}^n$ such that

$$\|F-Q\|_{L^{p,\infty}(\mathbb{R}^n)} \le C_{p,n}\,\|F-Q\|_{H^{p,\infty}(\mathbb{R}^n)} \approx \Big\|\Big(\sum_j |\Delta_j(F)|^2\Big)^{1/2}\Big\|_{L^{p,\infty}(\mathbb{R}^n)}, \tag{4.2}$$

by a result of He [24]. But the fact that F lies in L^2 implies that $Q=0$. Applying (4.2) with $F = T_{\sigma_1}(f_1,\dots,f_m)$, for which we observe that $\|T_{\sigma_1}(f_1,\dots,f_m)\|_{L^2(\mathbb{R}^n)} < \infty$ for Schwartz functions f_j, we conclude from (4.1) that (1.7) holds for σ_1.

To verify (4.1), we recall (3.2) and set $\omega_{\gamma_i}(y) = (1+4\pi^2|y|^2)^{\frac{\gamma_i}{2}}$ for $y\in\mathbb{R}^n$. Choose γ_j' and $\delta>0$ such that $n < \gamma_i'-\delta < \gamma_i' < \gamma_i'+\delta = \gamma_i$ for all $1\le i\le m$.

Now we rewrite

$$\begin{aligned}
&|\Delta_j^\theta\big(T_{\sigma_1}(f_1,\dots,f_m)\big)(x)| \\
&\le \int_{(\mathbb{R}^n)^m} \Big\{\prod_{i=1}^m \omega_{\gamma_i'-\delta}(2^j(x-y_i))\Big\}|(\sigma_1^j\,\widehat{\Psi_*}\widehat{\Theta})^\vee(2^j(x-y_1),\dots,2^j(x-y_m))| \\
&\qquad\qquad \times \frac{2^{mnj}|f_1(y_1)|\cdots|f_{m-1}(y_{m-1})||(\Delta_j^\eta f_m)(y_m)|}{\prod_{i=1}^m \omega_{\gamma_i'-\delta}(2^j(x-y_i))}\,d\vec{y} \\
&\lesssim \Big\|\Big(\prod_{i=1}^m \omega_{\gamma_i'-\delta}\Big)(\sigma_1^j\,\widehat{\Psi_*}\widehat{\Theta})^\vee\Big\|_{L^\infty}\Big(\prod_{i=1}^{m-1} M(f_i)(x)\Big)M(\Delta_j^\eta f_m)(x)
\end{aligned}$$

$$\lesssim \Big\| \prod_{i=1}^{m} \prod_{\ell=1}^{n} (I - \partial_{\xi_{i\ell}}^2)^{\frac{\gamma_{i\ell}}{2}} (\sigma(2^j(\cdot))\, \widehat{\Psi_*}) \Big\|_{L^1} \bigg(\prod_{i=1}^{m-1} M(f_i)(x) \bigg) M(\Delta_j^\eta f_m)(x)$$
$$\lesssim A \bigg(\prod_{i=1}^{m-1} M(f_i)(x) \bigg) M(\Delta_j^\eta f_m)(x)$$

as a consequence of the fact that $\gamma_i' - \delta > n$ for all $1 \le i \le m$ and (2.2). Here M is the uncentered Hardy-Littlewood maximal operator.

Thus, we proved

$$|\Delta_j^\theta \big(T_{\sigma_1}(f_1, \dots, f_m)\big)| \lesssim A \bigg(\prod_{i=1}^{m-1} M(f_i) \bigg) M(\Delta_j^\eta f_m).$$

Using the preceding inequality and Fefferman-Stein vector-valued inequality [12], we obtain

$$\big\| T_{\sigma_1}(f_1, \dots, f_{m-1}, f_m) \big\|_{H^{1/m,\infty}(\mathbb{R}^n)}$$
$$\lesssim \Big\| \Big\{ \sum_j |\Delta_j^\theta \big(T_{\sigma_1}(f_1, \dots, f_m)\big)|^2 \Big\}^{\frac{1}{2}} \Big\|_{L^{1/m,\infty}(\mathbb{R}^n)}$$
$$\lesssim A \Big\| \Big\{ \sum_j M(\Delta_j^\eta f_m)^2 \Big\}^{\frac{1}{2}} \Big\|_{L^{1,\infty}(\mathbb{R}^n)} \prod_{i=1}^{m-1} \big\| M(f_i) \big\|_{L^{1,\infty}(\mathbb{R}^n)}$$
$$\lesssim A \Big\| \Big\{ \sum_j |\Delta_j^\eta f_m|^2 \Big\}^{\frac{1}{2}} \Big\|_{L^1(\mathbb{R}^n)} \prod_{i=1}^{m-1} \|f_i\|_{L^1(\mathbb{R}^n)} \lesssim A \prod_{i=1}^{m} \|f_i\|_{H^1(\mathbb{R}^n)} .$$

This proves estimate (1.7) for σ_1.

Next we deal with σ_2. From (3.3), we have

$$T_{\sigma_2}(f_1, \dots, f_{m-1}, f_m) = \sum_{j \in \mathbb{Z}} T_{\sigma_2}(f_1, \dots, \Delta_j^\zeta f_{m-1}, \Delta_j^\theta f_m),$$

where $T_{\sigma_2}\big(f_1, \dots, f_{m-2}, \Delta_j^\zeta f_{m-1}, \Delta_j^\theta f_m\big)$ is defined in (3.4). A similar calculation as in the case for σ_1 yields the estimate

$$|T_{\sigma_2}(f_1, \dots, f_{m-2}, \Delta_j^\zeta f_{m-1}, \Delta_j^\theta f_m)| \lesssim A \bigg(\prod_{i=1}^{m-2} M(f_i) \bigg) M(\Delta_j^\zeta f_{m-1}) M(\Delta_j^\theta f_m) \,.$$

Summing over j, taking $L^{1/m,\infty}$ quasinorms, using Fefferman-Stein inequality [12], and applying the Littlewood-Paley characterization of H^1 we deduce

$$\big\| T_{\sigma_2}(f_1, \dots, f_{m-1}, f_m) \big\|_{L^{1/m,\infty}(\mathbb{R}^n)}$$
$$\lesssim A \Big\| \prod_{i=1}^{m-2} M(f_i) \sum_{j \in \mathbb{Z}} M\big(\Delta_j^\zeta f_{m-1}\big)\ M\left(\Delta_j^\theta f_m\right) \Big\|_{L^{1/m,\infty}(\mathbb{R}^n)}$$

$$
\begin{aligned}
&\lesssim A\Big\|\Big\{\prod_{i=1}^{m-2} M(f_i)\Big\}\Big\{\sum_{j\in\mathbb{Z}} M\big(\Delta_j^{\zeta} f_{m-1}\big)^2\Big\}^{\frac12}\Big\{\sum_{j\in\mathbb{Z}} M\left(\Delta_j^{\theta} f_m\right)^2\Big\}^{\frac12}\Big\|_{L^{1/m,\infty}(\mathbb{R}^n)}\\
&\lesssim A\Big(\prod_{i=1}^{m-2}\|M(f_i)\|_{L^{1,\infty}}\Big)\Big\|\Big\{\sum_{j\in\mathbb{Z}} M\big(\Delta_j^{\zeta} f_{m-1}\big)^2\Big\}^{\frac12}\Big\|_{L^{1,\infty}}\Big\|\Big\{\sum_{j\in\mathbb{Z}} M\big(\Delta_j^{\theta} f_m\big)^2\Big\}^{\frac12}\Big\|_{L^{1,\infty}}\\
&\lesssim A\Big(\prod_{i=1}^{m-2}\|f_i\|_{L^1(\mathbb{R}^n)}\Big)\Big\|\Big\{\sum_{j\in\mathbb{Z}} |\Delta_j^{\zeta} f_{m-1}|^2\Big\}^{\frac12}\Big\|_{L^1(\mathbb{R}^n)}\Big\|\Big\{\sum_{j\in\mathbb{Z}} |\Delta_j^{\theta} f_m|^2\Big\}^{\frac12}\Big\|_{L^1(\mathbb{R}^n)}\\
&\lesssim A\prod_{i=1}^{m}\|f_i\|_{H^1(\mathbb{R}^n)}.
\end{aligned}
$$

This concludes the proof of Theorem 1.3.

5. The proof of Theorem 1.4

We provide the proof of Theorem 1.4 next, which is similar to the proof of Theorem 1.1 but could be read independently.

Since the detailed proof of Theorem 1.4 is notationally cumbersome, we first present a proof in the case where $m = 4$ and $n = 3$, i.e., the case of 4 variables and 3 coordinates. This case captures all the ideas of the general case. Then we discuss the general case at the end.

Consider the following matrix of the coordinates of all variables:

$$
\begin{bmatrix}
\xi_{11} & \xi_{12} & \xi_{13}\\
\xi_{21} & \xi_{22} & \xi_{23}\\
\xi_{31} & \xi_{32} & \xi_{33}\\
\xi_{41} & \xi_{42} & \xi_{43}
\end{bmatrix}
=
\begin{bmatrix}
\xi_1\\ \xi_2\\ \xi_3\\ \xi_4
\end{bmatrix}.
$$

Along each column we encounter two cases: the case where the largest coordinate is larger than all the other ones (case I) and the other case where the largest coordinate is comparable to the second largest (case II). Such a splitting along all columns produces 8 cases. We only study a representative of these 8 cases, and in each one of those we make an arbitrary assumption about the largest variable. The case below illustrates the general one. Assume that:

- along column 1: case I (largest in modulus variable is ξ_{41});
- along column 2: case II (largest in modulus variable is ξ_{42} and second largest is ξ_{12});
- along column 3: case I (largest in modulus variable is ξ_{23}).

We denote the symbol associated with this case by

$$
\tau = \sigma_{I,II,I}^{41,(42,12),23}.
$$

This symbol is obtained by multiplying σ by a function of the form

$$\Phi\Big(\frac{|\xi_{11}|}{|\xi_{41}|},\frac{|\xi_{21}|}{|\xi_{41}|},\frac{|\xi_{31}|}{|\xi_{41}|}\Big)\Phi\Big(\frac{|\xi_{12}|}{|\xi_{42}|},\frac{|\xi_{22}|}{|\xi_{42}|},\frac{|\xi_{32}|}{|\xi_{42}|}\Big)\Psi\Big(\frac{|\xi_{12}|}{|\xi_{42}|}\Big)\Phi\Big(\frac{|\xi_{13}|}{|\xi_{23}|},\frac{|\xi_{33}|}{|\xi_{23}|},\frac{|\xi_{43}|}{|\xi_{23}|}\Big)$$

where $\Phi(u_1,u_2,u_3)$ is supported in $\big[0,\frac{11}{200}\big]\times[0,\frac{11}{200}\big]\times[0,\frac{1}{20}\big]$ while $\Psi(u)$ is supported in $\big[\frac{1}{40},2\big]$; see the proof of Theorem 1.1 or [15] (pages 570-571 or Exercise 7.5.4).

Fix a Schwartz function θ whose Fourier transform is supported in $[\frac12,2]\cup[-2,-\frac12]$ and satisfies $\sum_{j\in\mathbf{Z}}\widehat{\theta}(2^{-j}v)=1$ for $v\in\mathbb{R}\setminus\{0\}$. Associated with θ we define the Littlewood–Paley operator $\Delta_j^{(i)}(f)=f*_i\theta_{2^{-j}}$, where $\theta_t(u)=t^{-n}\theta(t^{-1}u)$ for $t>0$ and $*_i$ denotes the convolution in the ith variable. In a Littlewood-Paley operator $\Delta_j^{(k)}$ the upper letter inside the parenthesis indicates the coordinate on which it acts, so $1\le k\le 3$. We write

$$T_\tau(f_1,f_2,f_3,f_4)=\sum_{j_1}\sum_{j_2}\sum_{j_3}T_\tau\big(f_1,\Delta_{j_3}^{(3)}f_2,f_3,\Delta_{j_2}^{(2)}\Delta_{j_1}^{(1)}f_4\big)$$

and we have

$$T_\tau\big(f_1,\Delta_{j_3}^{(3)}f_2,f_3,\Delta_{j_2}^{(2)}\Delta_{j_1}^{(1)}f_4\big)(x)=$$
$$\int_{\mathbb{R}^{12}}\tau(\vec{\xi}\,)\widehat{f_1}(\xi_1)\widehat{\theta}(2^{-j_3}\xi_{23})\widehat{f_2}(\xi_2)\widehat{f_3}(\xi_3)\widehat{\theta}(2^{-j_2}\xi_{42})\widehat{\theta}(2^{-j_1}\xi_{41})\widehat{f_4}(\xi_4)e^{2\pi ix\cdot(\xi_1+\xi_2+\xi_3+\xi_4)}d\vec{\xi}.$$

Since ξ_{41} is the largest variable among $\xi_{11},\xi_{21},\xi_{31},\xi_{41}$, we have that

$$|\xi_{41}|\le|\xi_{11}|+|\xi_{21}|+|\xi_{31}|+|\xi_{41}|\le\frac{232}{200}|\xi_{41}|,\qquad |\xi_{11}+\xi_{21}+\xi_{31}+\xi_{41}|\approx|\xi_{41}|$$

and since ξ_{42} is the largest variable among $\xi_{12},\xi_{22},\xi_{32},\xi_{42}$, we have that

$$|\xi_{42}|\le|\xi_{12}|+|\xi_{22}|+|\xi_{32}|+|\xi_{42}|\le\frac{232}{200}|\xi_{42}|.$$

Likewise

$$|\xi_{23}|\le|\xi_{13}|+|\xi_{23}|+|\xi_{33}|+|\xi_{43}|\le\frac{232}{200}|\xi_{23}|,\qquad |\xi_{13}+\xi_{23}+\xi_{33}+\xi_{43}|\approx|\xi_{23}|.$$

We may therefore insert in the preceding integral the function

$$\widehat{\Omega}\big(D_{-j_1,-j_2,-j_3}(\xi_1,\xi_2,\xi_3,\xi_4)\big)=\widehat{\Theta}(2^{-j_1}(\xi_{11}+\xi_{21}+\xi_{31}+\xi_{41}))\widehat{\Theta}(2^{-j_3}(\xi_{13}+\xi_{23}+\xi_{33}+\xi_{43})),$$

where $\widehat{\Theta}(u)=\widehat{\theta}(u/2)+\widehat{\theta}(u)+\widehat{\theta}(2u)$; notice that $\widehat{\Theta}$ equals 1 on the support of $\widehat{\theta}$. We denote by $\widetilde{\Delta}_j$ the Littlewood-Paley operators associated to Θ. Let

$$D_{j_1,j_2,j_3}(\xi_1,\xi_2,\xi_3,\xi_4)=\left(\begin{bmatrix}\xi_{11}&\xi_{12}&\xi_{13}\\ \xi_{21}&\xi_{22}&\xi_{23}\\ \xi_{31}&\xi_{32}&\xi_{33}\\ \xi_{41}&\xi_{42}&\xi_{43}\end{bmatrix}\begin{bmatrix}2^{j_1}&0&0\\ 0&2^{j_2}&0\\ 0&0&2^{j_3}\end{bmatrix}\right)$$

and

$$\tau^{j_1,j_2,j_3}(\xi_1,\xi_2,\xi_3,\xi_4)=\tau\Big(D_{j_1,j_2,j_3}(\xi_1,\xi_2,\xi_3,\xi_4)\Big).$$

For the same reason we may also insert the function

$$\begin{aligned}&\widehat{\Psi^*}\big(D_{-j_1,-j_2,-j_3}(\xi_1,\xi_2,\xi_3,\xi_4)\big)\\&=\widehat{\Psi^*_1}(2^{-j_1}(\xi_{11},\xi_{21},\xi_{31},\xi_{41}))\widehat{\Psi^*_2}(2^{-j_2}(\xi_{12},\xi_{22},\xi_{32},\xi_{42}))\widehat{\Psi^*_3}(2^{-j_3}(\xi_{13},\xi_{23},\xi_{33},\xi_{43}))\end{aligned}$$

where

$$\widehat{\Psi^*_\ell}(u_1,u_2,u_3,u_4)=\sum_{|k|\le 1}\widehat{\Psi_\ell}(2^{-k}(u_1,u_2,u_3,u_4))\,,$$

and Ψ_ℓ is as in the hypotheses of the theorem.

Additionally, in case II there is the second largest variable which is comparable to the largest one. Therefore we can take a Schwartz function η whose Fourier transform is supported in $[\frac{1}{256},8]\cup[-8,-\frac{1}{256}]$ and identical to 1 on $[\frac{1}{128},4]\cup[-4,-\frac{1}{128}]$ and insert the factor $\widehat{\eta}(2^{-j_2}\xi_{12})$ into the above integral without changing the outcome. Let us denote the Littlewood-Paley operator associated with η by $\overline{\Delta}_j$.

We may therefore rewrite

$$\begin{aligned}T_\tau\big(f_1,\Delta^{(3)}_{j_3}f_2,f_3,\Delta^{(2)}_{j_2}\Delta^{(1)}_{j_1}f_4\big)&=T_\tau\big(\overline{\Delta}^{(2)}_{j_2}f_1,\Delta^{(3)}_{j_3}f_2,f_3,\Delta^{(2)}_{j_2}\Delta^{(1)}_{j_1}f_4\big)\\&=\widetilde{\Delta}^{(1)}_{j_1}\widetilde{\Delta}^{(3)}_{j_3}T_\tau\big(\overline{\Delta}^{(2)}_{j_2}f_1,\Delta^{(3)}_{j_3}f_2,f_3,\Delta^{(2)}_{j_2}\Delta^{(1)}_{j_1}f_4\big)\,.\end{aligned}$$

Manipulations with the Fourier transform give that the above can be expressed as

$$\begin{aligned}&\int_{\mathbb{R}^{12}}2^{4(j_1+j_2+j_3)}\Big(\tau^{j_1,j_2,j_3}\widehat{\Psi^*}\,\widehat{\Omega}\Big)^\vee\big(D_{j_1,j_2,j_3}(x-y_1,x-y_2,x-y_3,x-y_4)\big)\\&\qquad(\overline{\Delta}^{(2)}_{j_2}f_1)(y_1)(\Delta^{(3)}_{j_3}f_2)(y_2)f_3(y_3)(\Delta^{(2)}_{j_2}\Delta^{(1)}_{j_1}f_4)(y_4)dy_1dy_2dy_3dy_4\,.\end{aligned}$$

If $r=1$, set $\rho=1$. If $r>1$ pick ρ such that $1<\rho<2$ and that

$$\max_{1\le i\le m}\max_{1\le\ell\le n}\frac{1}{\gamma_{i\ell}}<\rho<\min(p_1,\dots,p_m,r).$$

Recall that $\mathcal{M}$ is the strong maximal function. Setting $\omega_\beta(y)=(1+4\pi^2|y|^2)^{\frac{\beta}{2}}$ for $y\in\mathbb{R}$, we write

$$\begin{aligned}&\Big|T_\tau\big(f_1,\Delta^{(3)}_{j_3}f_2,f_3,\Delta^{(2)}_{j_2}\Delta^{(1)}_{j_1}f_4\big)(x_1,x_2,x_3)\Big|\\
\le&\int_{\mathbb{R}^{12}}2^{\frac{4(j_1+j_2+j_3)}{\rho'}}\Big\{\prod_{i=1}^4\prod_{\ell=1}^3\omega_{\gamma_{i\ell}}(2^{j_\ell}(x_\ell-y_{i\ell}))\Big\}(\tau^{j_1,j_2,j_3}\widehat{\Psi^*}\widehat{\Omega})^\vee(D_{j_1,j_2,j_3}(x-y_1,x-y_2,x-y_3,x-y_4))\\
&\qquad\frac{2^{\frac{j_1+j_2+j_3}{\rho}}(\overline{\Delta}^{(2)}_{j_2}f_1)(y_1)}{\prod_{\ell=1}^3\omega_{\gamma_{1\ell}}(2^{j_\ell}(x_\ell-y_{1\ell}))}\quad\frac{2^{\frac{j_1+j_2+j_3}{\rho}}(\Delta^{(3)}_{j_3}f_2)(y_2)}{\prod_{\ell=1}^3\omega_{\gamma_{2\ell}}(2^{j_\ell}(x_\ell-y_{2\ell}))}\\
&\qquad\frac{2^{\frac{j_1+j_2+j_3}{\rho}}f_3(y_3)}{\prod_{\ell=1}^3\omega_{\gamma_{3\ell}}(2^{j_\ell}(x_\ell-y_{3\ell}))}\frac{2^{\frac{j_1+j_2+j_3}{\rho}}(\Delta^{(2)}_{j_2}\Delta^{(1)}_{j_1}f_4)(y_4)}{\prod_{\ell=1}^3\omega_{\gamma_{4\ell}}(2^{j_\ell}(x_\ell-y_{4\ell}))}dy_1dy_2dy_3dy_4\\
\le&CA\mathcal{M}(|\overline{\Delta}^{(2)}_{j_2}f_1|^\rho)^{\frac{1}{\rho}}\mathcal{M}(|\Delta^{(3)}_{j_3}f_2|^\rho)^{\frac{1}{\rho}}\mathcal{M}(|f_3|^\rho)^{\frac{1}{\rho}}\mathcal{M}(|\Delta^{(2)}_{j_2}\Delta^{(1)}_{j_1}f_4|^\rho)^{\frac{1}{\rho}}\end{aligned}$$

where we used that $\rho\gamma_{i\ell} > 1$ for all i, ℓ and also that

$$\Big(\int_{\mathbb{R}^{12}} \Big| \Big\{ \prod_{i=1}^{4}\prod_{\ell=1}^{3} \omega_{\gamma_{i\ell}}(y_{i\ell}) \Big\} (\tau^{j_1,j_2,j_3}\widehat{\Psi^*}\widehat{\Omega})^\vee(y_1,y_2,y_3,y_4)\Big|^{\rho'} dy_1dy_2dy_3dy_4\Big)^{1/\rho'}$$

$$\lesssim \Big\| \prod_{i=1}^{4}\prod_{\ell=1}^{3} (I-\partial^2_{\xi_{i\ell}})^{\frac{\gamma_{i\ell}}{2}} (\sigma\circ D_{j_1,j_2,j_3})\,\widehat{\Psi^*}\Big\|_{L^r} \lesssim A$$

which is a consequence of Lemma 2.1 and of the fact that Ψ^* is a finite sum of Ψ_ℓ's.

We now estimate our operator. We write

$$T_\tau(f_1,f_2,f_3,f_4) = \sum_{j_1}\sum_{j_2}\sum_{j_3} \widetilde{\Delta}^{(1)}_{j_1}\widetilde{\Delta}^{(3)}_{j_3} T_\tau\big(\overline{\Delta}^{(2)}_{j_2} f_1, \Delta^{(3)}_{j_3} f_2, f_3, \Delta^{(2)}_{j_2}\Delta^{(1)}_{j_1} f_4\big)\,.$$

For each j_1 and j_3 we have the pointwise estimate

$$\begin{aligned}
&\big|\widetilde{\Delta}^{(1)}_{j_1}\widetilde{\Delta}^{(3)}_{j_3} \sum_{j_2} T_\tau\big(\overline{\Delta}^{(2)}_{j_2} f_1, \Delta^{(3)}_{j_3} f_2, f_3, \Delta^{(2)}_{j_2}\Delta^{(1)}_{j_1} f_4\big)\big| \\
&\quad\le CA\sum_{j_2} \mathcal{M}(|\overline{\Delta}^{(2)}_{j_2} f_1|^\rho)^{\frac1\rho}\mathcal{M}(|\Delta^{(3)}_{j_3} f_2|^\rho)^{\frac1\rho}\mathcal{M}(|f_3|^\rho)^{\frac1\rho}\mathcal{M}(|\Delta^{(2)}_{j_2}\Delta^{(1)}_{j_1} f_4|^\rho)^{\frac1\rho} \\
&\quad\le CA\Big(\sum_{j_2} \mathcal{M}(|\overline{\Delta}^{(2)}_{j_2} f_1|^\rho)^{\frac2\rho}\Big)^{\frac12}\mathcal{M}(|\Delta^{(3)}_{j_3} f_2|^\rho)^{\frac1\rho}\mathcal{M}(|f_3|^\rho)^{\frac1\rho}\Big(\sum_{j_2}\mathcal{M}(|\Delta^{(2)}_{j_2}\Delta^{(1)}_{j_1} f_4|^\rho)^{\frac2\rho}\Big)^{\frac12}.
\end{aligned}$$

We now apply Lemma 2.3 (hypothesis (2.9) is easy to check), more precisely by Remark 2.4, to write

$$\big\|T_\tau(f_1,f_2,f_3,f_4)\big\|_{L^p} \lesssim \Big\|\Big(\sum_{j_1}\sum_{j_3}\Big|\widetilde{\Delta}^{(1)}_{j_1}\widetilde{\Delta}^{(3)}_{j_3}\sum_{j_2} T_\tau\big(\overline{\Delta}^{(2)}_{j_2} f_1, \Delta^{(3)}_{j_3} f_2, f_3, \Delta^{(2)}_{j_2}\Delta^{(1)}_{j_1} f_4\big)\Big|^2\Big)^{\frac12}\Big\|_{L^p}$$

and using the preceding estimate we control this expression by

$$A\Big\|\Big(\sum_{j_2}\mathcal{M}(|\overline{\Delta}^{(2)}_{j_2} f_1|^\rho)^{\frac2\rho}\Big)^{\frac12}\Big(\sum_{j_3}\mathcal{M}(|\Delta^{(3)}_{j_3} f_2|^\rho)^{\frac2\rho}\Big)^{\frac12}\mathcal{M}(|f_3|^\rho)^{\frac1\rho}\Big(\sum_{j_2}\sum_{j_1}\mathcal{M}(|\Delta^{(2)}_{j_2}\Delta^{(1)}_{j_1} f_4|^\rho)^{\frac2\rho}\Big)^{\frac12}\Big\|_{L^p}.$$

The required conclusion follows by applying Hölder's inequality, the Fefferman-Stein inequality [12], and Lemma 2.3 using the facts that $1 \le \rho < 2$ and $\rho < p_i$ for all i.

We now show how to modify the above proof to obtain the general case. We decompose σ as a finite sum of symbols of the form $\sigma\Theta_\nu$, with Θ_ν a smooth function homogeneous of degree 0 which is supported in an appropriate cone. We fix one of these $\sigma\Theta_\nu$ and we call it τ. Next, we introduce some notation. We define sets I and II that depend on the coordinates of the points in the support of Θ_ν (which all share the same size properties). We consider the set $\{1,2,\dots,n\}$ that indexes the columns of the $m\times n$ matrix $(\xi_{kl})_{\{1\le k\le m, 1\le l\le n\}}$. We split the set $\{1,2,\dots,n\}$ into two pieces I and II, by placing $l\in I$ if the lth column follows in the first case (where there the largest variable dominates all the other ones) and placing $l\in II$ if the

lth column follows in the second case (where there the largest variable and the second largest are comparable). To make the notation a bit simpler, without loss of generality we suppose that $I = \{1, \dots, q\}$ and $II = \{q+1, \dots, n\}$ for some q. Notice that one of these sets could be empty.

Recall the notation for the Littlewood-Paley operators $\Delta_j^{(l)}$ as in the case $m = 4$, $n = 3$. For the purposes of this theorem we introduce a slightly more refined notation using two upper indices in $\Delta_j^{(k,l)}$. The first index shows the function f_k on which $\Delta_j^{(k,l)}$ acts and the second one the coordinate ξ_{kl} of the variable ξ_k on which $\Delta_j^{(k,l)}$ acts.

Define a map

$$u : \{1, 2, \dots, n\} \to \{1, 2, \dots, m\}$$

such that for each l, $u(l)$ denotes the index such that $\xi_{u(l)l}$ is largest among ξ_{kl}. Also define a map

$$\bar{u} : \{q+1, \dots, n\} \to \{1, 2, \dots, m\}$$

such that $\xi_{\bar{u}(l)l}$ is second largest among ξ_{kl}. u and $\bar{u}$ are functions depending on $\xi = (\xi_1, \dots, \xi_m)$. But since all ξ in the support of Θ_ν share the same size properties, we see that they are fixed when Θ_ν is fixed.

We always have $\bar{u}(l) \neq u(l)$ for all l in $\{q+1, \dots, n\}$. We also define

$$\Delta_j^{(u(r),r)} \vec{f} = \Delta_j^{(u(r),r)}(f_1, \dots, f_m) = (f_1, \dots, \Delta_j^{(r)} f_{u(r)}, \dots, f_m)$$

and we extend this definition to the case where $\Delta_{j_{i_1}}^{(u(i_1),i_1)} \cdots \Delta_{j_{i_r}}^{(u(i_r),i_r)}$ acts on $(f_1, \dots, f_m)$. Additionally, we use the definitions of $\widetilde{\Delta}_j$ and $\overline{\Delta}_j$ as introduced in the special case $m = 4$, $n = 3$. In the last part of the proof, we will use the following convention of summation. Suppose

$$E_i = \{\lambda \in \mathbb{Z} : q+1 \le \lambda \le n, \lambda \in u^{-1}[i]\}$$

and F is an expression involving some indices $j_1, \dots, j_n$. If the above set E_i is not empty, say $E_i = \{\lambda_1, \dots, \lambda_\nu\}$, then

$$\sum_{\substack{j_\lambda \in \mathbb{Z} \\ \lambda \in u^{-1}[i] \\ q+1 \le \lambda \le n}} F = \sum_{j_{\lambda_1} \in \mathbb{Z}} \cdots \sum_{j_{\lambda_\nu} \in \mathbb{Z}} F.$$

If $E_i = \emptyset$, then

$$\sum_{\substack{j_\lambda \in \mathbb{Z} \\ \lambda \in u^{-1}[i] \\ q+1 \le \lambda \le n}} F = F.$$

Let τ be the multilinear multiplier associated with a given fixed mapping u. We write

$$T_\tau(f_1, \dots, f_m)$$

$$
\begin{aligned}
&= \sum_{j_1,\dots,j_n\in\mathbb{Z}} T_\tau\big[\Delta_{j_1}^{(u(1),1)}\cdots\Delta_{j_n}^{(u(n),n)}(f_1,\dots,f_m)\big]\\
&= \sum_{j_1,\dots,j_q\in\mathbb{Z}} \widetilde{\Delta}_{j_1}^{(u(1),1)}\cdots\widetilde{\Delta}_{j_q}^{(u(q),q)} \sum_{j_{q+1},\dots,j_n\in\mathbb{Z}} T_\tau\Big[\prod_{\kappa=1}^{q}\Delta_{j_\kappa}^{(u(\kappa),\kappa)}\prod_{\lambda=q+1}^{n}\Delta_{j_\lambda}^{(u(\lambda),\lambda)}\overline{\Delta}_{j_\lambda}^{(\overline{u}(\lambda),\lambda)}\vec{f}\,\Big].
\end{aligned}
$$

The estimates in the case $m=4$ and $n=3$ show that the term in the interior sum satisfies

$$
\begin{aligned}
&\Big|\widetilde{\Delta}_{j_1}^{(u(1),1)}\cdots\widetilde{\Delta}_{j_q}^{(u(q),q)} \sum_{j_{q+1},\dots,j_n\in\mathbb{Z}} T_\tau\Big[\prod_{\kappa=1}^{q}\Delta_{j_\kappa}^{(u(\kappa),\kappa)}\prod_{\lambda=q+1}^{n}\Delta_{j_\lambda}^{(u(\lambda),\lambda)}\overline{\Delta}_{j_\lambda}^{(\overline{u}(\lambda),\lambda)}(f_1,\dots,f_m)\Big]\Big|\\
&\qquad\lesssim A\sum_{j_{q+1},\dots,j_n\in\mathbb{Z}}\prod_{i=1}^{m}\mathcal{M}\Big(\Big|\prod_{\substack{1\le\kappa\le q\\ \kappa\in u^{-1}[i]}}\Delta_{j_\kappa}^{(i,\kappa)}\prod_{\substack{q+1\le\lambda\le n\\ \lambda\in u^{-1}[i]}}\Delta_{j_\lambda}^{(i,\lambda)}\prod_{\substack{q+1\le\mu\le n\\ \mu\in\overline{u}^{-1}[i]}}\overline{\Delta}_{j_\mu}^{(i,\mu)}f_i\Big|^\rho\Big)^{\frac{1}{\rho}},
\end{aligned}
$$

where $u^{-1}[i]=\{k\in\{1,\dots,n\}:\ u(k)=i\}$ and with the understanding that if any of the index sets is empty, then the corresponding Littlewood-Paley operators do not appear. Applying the Cauchy-Schwarz inequality $n-q$ times successively for the indices $j_{q+1},j_{q+2},\dots,j_n$ we estimate the last displayed expression by

$$
A\prod_{i=1}^{m}\Big[\sum_{\substack{j_\lambda\in\mathbb{Z}\\ \lambda\in u^{-1}[i]\\ q+1\le\lambda\le n}}\sum_{\substack{j_\mu\in\mathbb{Z}\\ \mu\in\overline{u}^{-1}[i]\\ q+1\le\mu\le n}}\mathcal{M}\Big(\Big|\prod_{\substack{1\le\kappa\le q\\ \kappa\in u^{-1}[i]}}\Delta_{j_\kappa}^{(i,\kappa)}\prod_{\substack{q+1\le\lambda\le n\\ \lambda\in u^{-1}[i]}}\Delta_{j_\lambda}^{(i,\lambda)}\prod_{\substack{q+1\le\mu\le n\\ \mu\in\overline{u}^{-1}[i]}}\overline{\Delta}_{j_\mu}^{(i,\mu)}f_i\Big|^\rho\Big)^{\frac{2}{\rho}}\Big]^{\frac12}. \tag{5.1}
$$

When $I\neq\emptyset$, we use Lemma 2.3 and (5.1) to obtain

$$
\begin{aligned}
&\big\|T_\tau(f_1,\dots,f_m)\big\|_{L^p}\\
&=\Big\|\sum_{j_1,\dots,j_q\in\mathbb{Z}}\widetilde{\Delta}_{j_1}^{(u(1),1)}\cdots\widetilde{\Delta}_{j_q}^{(u(q),q)}\sum_{j_{q+1},\dots,j_n\in\mathbb{Z}}T_\tau\Big[\prod_{\kappa=1}^{q}\Delta_{j_\kappa}^{(u(\kappa),\kappa)}\prod_{\lambda=q+1}^{n}\Delta_{j_\lambda}^{(u(\lambda),\lambda)}\overline{\Delta}_{j_\lambda}^{(\overline{u}(\lambda),\lambda)}\vec{f}\,\Big]\Big\|_{L^p}\\
&\lesssim\Big\|\Big[\sum_{j_1,\dots,j_q\in\mathbb{Z}}\Big|\widetilde{\Delta}_{j_1}^{(u(1),1)}\cdots\widetilde{\Delta}_{j_q}^{(u(q),q)}\sum_{j_{q+1},\dots,j_n\in\mathbb{Z}}T_\tau\Big[\prod_{\kappa=1}^{q}\Delta_{j_\kappa}^{(u(\kappa),\kappa)}\prod_{\lambda=q+1}^{n}\Delta_{j_\lambda}^{(u(\lambda),\lambda)}\overline{\Delta}_{j_\lambda}^{(\overline{u}(\lambda),\lambda)}\vec{f}\,\Big]\Big|^2\Big]^{\frac12}\Big\|_{L^p}\\
&\lesssim A\Big\|\Big[\sum_{j_1,\dots,j_q\in\mathbb{Z}}\prod_{i=1}^{m}\Big\{\sum_{\substack{j_\lambda\in\mathbb{Z}\\ \lambda\in u^{-1}[i]\\ q+1\le\lambda\le n}}\sum_{\substack{j_\mu\in\mathbb{Z}\\ \mu\in\overline{u}^{-1}[i]\\ q+1\le\mu\le n}}\mathcal{M}\Big(\Big|\prod_{\substack{1\le\kappa\le q\\ \kappa\in u^{-1}[i]}}\Delta_{j_\kappa}^{(i,\kappa)}\prod_{\substack{q+1\le\lambda\le n\\ \lambda\in u^{-1}[i]}}\Delta_{j_\lambda}^{(i,\lambda)}\prod_{\substack{q+1\le\mu\le n\\ \mu\in\overline{u}^{-1}[i]}}\overline{\Delta}_{j_\mu}^{(i,\mu)}f_i\Big|^\rho\Big)^{\frac{2}{\rho}}\Big\}\Big]^{\frac12}\Big\|_{L^p}\\
&=A\Big\|\Big(\prod_{i=1}^{m}\sum_{\substack{j_\kappa\in\mathbb{Z}\\ \kappa\in u^{-1}[i]\\ 1\le\kappa\le q}}\sum_{\substack{j_\lambda\in\mathbb{Z}\\ \lambda\in u^{-1}[i]\\ q+1\le\lambda\le n}}\sum_{\substack{j_\mu\in\mathbb{Z}\\ \mu\in\overline{u}^{-1}[i]\\ q+1\le\mu\le n}}\mathcal{M}\Big(\Big|\prod_{\substack{1\le\kappa\le q\\ \kappa\in u^{-1}[i]}}\Delta_{j_\kappa}^{(i,\kappa)}\prod_{\substack{q+1\le\lambda\le n\\ \lambda\in u^{-1}[i]}}\Delta_{j_\lambda}^{(i,\lambda)}\prod_{\substack{q+1\le\mu\le n\\ \mu\in\overline{u}^{-1}[i]}}\overline{\Delta}_{j_\mu}^{(i,\mu)}f_i\Big|^\rho\Big)^{\frac{2}{\rho}}\Big)^{\frac12}\Big\|_{L^p}.
\end{aligned}
$$

Otherwise, when $I = \emptyset$, from (5.1) we can see that $T_\tau(f_1, \ldots, f_m)$ is controlled by

$$A \prod_{i=1}^{m} \Bigg[\sum_{\substack{j_\lambda \in \mathbb{Z} \\ \lambda \in u^{-1}[i] \\ 1 \le \lambda \le n}} \sum_{\substack{j_\mu \in \mathbb{Z} \\ \mu \in \overline{u}^{-1}[i] \\ 1 \le \mu \le n}} \mathcal{M}\Big(\Big| \prod_{\substack{1 \le \lambda \le n \\ \lambda \in u^{-1}[i]}} \Delta_{j_\lambda}^{(i,\lambda)} \prod_{\substack{1 \le \mu \le n \\ \mu \in \overline{u}^{-1}[i]}} \overline{\Delta}_{j_\mu}^{(i,\mu)} f_i \Big|^\rho \Big)^{\frac{2}{\rho}} \Bigg]^{\frac{1}{2}}.$$

At this point we apply Hölder's inequality and the Fefferman-Stein inequality [12] using the facts that $1 \le \rho < 2$ and $\rho < p_i$ for all i. Then we control $\big\| T_\tau(f_1, \ldots, f_m) \big\|_{L^p}$ by a constant multiple of

$$A \prod_{i=1}^{m} \Bigg\| \Big(\sum_{\substack{j_\kappa \in \mathbb{Z} \\ \kappa \in u^{-1}[i] \\ 1 \le \kappa \le q}} \sum_{\substack{j_\lambda \in \mathbb{Z} \\ \lambda \in u^{-1}[i] \\ q+1 \le \lambda \le n}} \sum_{\substack{j_\mu \in \mathbb{Z} \\ \mu \in \overline{u}^{-1}[i] \\ q+1 \le \mu \le n}} \Big| \prod_{\substack{1 \le \kappa \le q \\ \kappa \in u^{-1}[i]}} \Delta_{j_\kappa}^{(i,\kappa)} \prod_{\substack{q+1 \le \lambda \le n \\ \lambda \in u^{-1}[i]}} \Delta_{j_\lambda}^{(i,\lambda)} \prod_{\substack{q+1 \le \mu \le n \\ \mu \in \overline{u}^{-1}[i]}} \overline{\Delta}_{j_\mu}^{(i,\mu)} f_i \Big|^2 \Big)^{\frac{1}{2}} \Bigg\|_{L^{p_i}}$$

or

$$A \prod_{i=1}^{m} \Bigg\| \Big(\sum_{\substack{j_\lambda \in \mathbb{Z} \\ \lambda \in u^{-1}[i] \\ 1 \le \lambda \le n}} \sum_{\substack{j_\mu \in \mathbb{Z} \\ \mu \in \overline{u}^{-1}[i] \\ 1 \le \mu \le n}} \Big| \prod_{\substack{1 \le \lambda \le n \\ \lambda \in u^{-1}[i]}} \Delta_{j_\lambda}^{(i,\lambda)} \prod_{\substack{1 \le \mu \le n \\ \mu \in \overline{u}^{-1}[i]}} \overline{\Delta}_{j_\mu}^{(i,\mu)} f_i \Big|^2 \Big)^{\frac{1}{2}} \Bigg\|_{L^{p_i}}$$

and by the Littlewood-Paley theorem the last expression is bounded by A times the product of the L^{p_i} norms of the f_i.

This concludes the proof of Theorem 1.4.

Remark 5.1. *We see from the proof that we do not use the property that $\xi_{kl} \in \mathbb{R}$, so the same argument generalizes our result to the case when each $f_k(\xi_{k1}, \ldots, \xi_{kn})$ is defined on $(\mathbb{R}^d)^n$ with $\xi_{kl} \in \mathbb{R}^d$. This covers* [6, Theorem 1.10]*, as we claimed in the introduction.*

6. Applications: Calderón-Coifman-Journé commutators

6.1. **Calderón commutator.** In 1965 Calderón [2] introduced the (first-order) commutator

$$\mathcal{C}_1(f; a)(x) = \text{p.v.} \int_{\mathbb{R}} \frac{A(x) - A(y)}{(x-y)^2} f(y) dy,$$

where a is the derivative of a Lipschitz function A and f is a test function on the real line. It is known that $\mathcal{C}_1$ is a bounded operator in $L^p(\mathbb{R}), 1 < p < \infty$, if A is a Lipchitz function on $\mathbb{R}$ and

$$\|\mathcal{C}_1(f; a)\|_{L^p(\mathbb{R})} \le C_p \|a\|_{L^\infty(\mathbb{R})} \|f\|_{L^p(\mathbb{R})}, \quad 1 < p < \infty.$$

See Calderón [2, 3] and Coifman-Meyer [8] for its history.

Viewed as a bilinear operator acting on the pair (f, a), then the operator $\mathcal{C}_1$ can be written as a bilinear multiplier operator

$$\mathcal{C}_1(f;a)(x) = -i\pi \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{f}(\xi)\, \widehat{a}(\eta)\ \big(\mathrm{sgn}\,(\eta)\Phi\big(\xi/\eta\big)\big)\ e^{2\pi i x(\xi+\eta)}\, d\xi d\eta\,,$$

where Φ is the following Lipschitz function on the real line:

$$\Phi(s) = \begin{cases} -1, & s \le -1; \\ 1+2s, & -1 < s \le 0; \\ 1, & s > 0. \end{cases} \tag{6.1}$$

The operator $\mathcal{C}_1$ is too singular to fall under the scope of multilinear Calderón-Zygmund theory [22]. However it was shown to be bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$ when $1 < p_1, p_2 < \infty$ and $(1/p_1 + 1/p_2)^{-1} = p > 1/2$; see C. Calderón [4]. See also Coifman-Meyer [8] and Duong-Grafakos-Yan [11]. The boundedness of $\mathcal{C}_1$ on L^p for $p \ge 1$ was also studied by Muscalu [33] via time-frequency analysis.

In this work we will apply Theorem 1.1 to obtain a direct proof of the boundedness of $\mathcal{C}_1$ from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$ in the full range of $p > 1/2$. Our proof is based on exploiting the (limited) smoothness of the function Φ, measured in terms of a Sobolev space norm. A partial result using a similar idea in this direction with the restriction $p > 2/3$ has been obtained by [32].

For $r \ge 1$ and $\gamma > 0$, we recall the Sobolev space $L^r_\gamma(\mathbb{R}^n)$, $\gamma > 0$ of all functions g with $\|(I-\Delta)^{\gamma/2} g\|_{L^p} < \infty$. For $\vec{\gamma} = (\gamma_1, \dots, \gamma_n)$, we denote by $L^r_{\vec{\gamma}}(\mathbb{R}^n)$ the class of distributions f such that

$$\Big\| \prod_{1\le \ell \le n} (I - \partial_\ell^2)^{\frac{\gamma_\ell}{2}} f \Big\|_{L^r(\mathbb{R}^n)} < \infty.$$

It is easy to verify using multiplier theorems that $L^r_\gamma(\mathbb{R}^n) \subset L^r_{\vec{\gamma}}(\mathbb{R}^n)$, where $\gamma = |\vec{\gamma}| = \gamma_1 + \cdots + \gamma_n$. The spaces $L^r_{\vec{\gamma}}(\mathbb{R}^n)$ are sometimes referred to as Sobolev spaces with dominating mixed smoothness in the literature, see [37] for more details and references.

To begin, we need the following characterizations of Sobolev norms, given by Stein [38], [39, Lemma 3, p. 136].

Lemma 6.1 (Stein)**.** *(i) Let* $0 < \alpha < 1$ *and* $\max(1, 2n/(n+2\alpha)) < p < \infty$*. Then* $f \in L^p_\alpha(\mathbb{R}^n)$ *if and only if* $f \in L^p(\mathbb{R}^n)$ *and* $I_\alpha(f) \in L^p(\mathbb{R}^n)$ *where*

$$I_\alpha(f)(x) = \left(\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\alpha}} dy \right)^{1/2}.$$

Moreover,

$$\|f\|_{L^p_\alpha(\mathbb{R}^n)} \simeq \|f\|_{L^p(\mathbb{R}^n)} + \|I_\alpha(f)\|_{L^p(\mathbb{R}^n)}.$$

(ii) Let $1 \leq \alpha < \infty$ and $1 < p < \infty$. Then $f \in L^p_\alpha(\mathbb{R}^n)$ if and only if $f \in L^p_{\alpha-1}(\mathbb{R}^n)$ and for $1 \leq j \leq n$, $\frac{\partial f}{\partial x_j} \in L^p_{\alpha-1}(\mathbb{R}^n)$. Furthermore, we have

$$\|f\|_{L^p_\alpha(\mathbb{R}^n)} \simeq \|f\|_{L^p_{\alpha-1}(\mathbb{R}^n)} + \sum_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \right\|_{L^p_{\alpha-1}(\mathbb{R}^n)}.$$

Throughout this section fix a nondecreasing smooth function h on $\mathbb{R}$ such that

$$h(t) = \begin{cases} 3, & \text{if } t \in [4, +\infty); \\ \text{smooth}, & \text{if } t \in [2, 4); \\ t, & \text{if } t \in [1/8, 2); \\ \text{smooth}, & \text{if } t \in [1/32, 1/8); \\ 1/16, & \text{otherwise}. \end{cases} \tag{6.2}$$

Lemma 6.2. *Let u be a function supported in the rectangle*

$$\{(y_1, y_2): \ |y_1| \leq 101/100, 1/4 \leq y_2 \leq 7/4\}$$

in $\mathbb{R}^2$ such that $\nabla u \in L^\infty(\mathbb{R}^2)$, and $u(x) \in L^r_\gamma(\mathbb{R}^2)$ with $1 < \gamma < 2$, $2/\gamma < r < \infty$. Define $U(y_1, y_2) = u(y_1/h(y_2), y_2)$. Then $U \in L^r_\gamma(\mathbb{R}^2)$ and

$$\|U\|_{L^r_\gamma(\mathbb{R}^2)} \leq C\big(\|\nabla u\|_{L^\infty(\mathbb{R}^2)} + \|u\|_{L^r_\gamma(\mathbb{R}^2)}\big).$$

Proof. Because of Lemma 6.1, it suffices to show for $\alpha = \gamma - 1$ and $2/(1+\alpha) < r < \infty$ that $U \in L^r_1(\mathbb{R}^2)$, $I_\alpha(U) \in L^r(\mathbb{R}^2)$ and $I_\alpha(\partial_j U) \in L^r(\mathbb{R}^2)$ with $j = 1, 2$. The first assertion follows trivially by checking the derivatives directly while the second one is verified in a way similar to the third one, where we adapt an argument found in Triebel [41, Section 4.3] with a suitable change of variables.

Next, we show that $I_\alpha(\partial_1 U) \in L^r(\mathbb{R}^2)$. We will estimate the following expression

$$\|I_\alpha(\partial_1 U)\|^r_{L^r(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \frac{|\partial_1 U(y) - \partial_1 U(y')|^2}{|y - y'|^{2+2\alpha}} \, dy \right)^{r/2} dy'.$$

Denote by B a finite ball centered at 0 containing the support of $\partial_1 U$. Then it is easy to check that, since $\partial_1 U \in L^\infty$, $r(1+\alpha) = r\gamma > 2$,

$$\|I_\alpha(\partial_1 U)\|^r_{L^r(\mathbb{R}^2)} \leq C \left(\|\nabla u\|^r_{L^\infty} + \int_{3B} \left(\int_{3B} \frac{|\partial_1 U(y) - \partial_1 U(y')|^2}{|y - y'|^{2+2\alpha}} \, dy \right)^{r/2} dy' \right),$$

where C is a constant depending on B.

Denote $x = (x_1, x_2)$, $y = (y_1, y_2)$. One writes $y = \varphi(x)$ and $x = \psi(y)$ in the form

$$\begin{cases} y_1 &= \varphi_1(x_1, x_2) = x_1 h(x_2), \\ y_2 &= \varphi_2(x_1, x_2) = x_2 \end{cases}$$

and

$$\begin{cases} x_1 &= \psi_1(y_1, y_2) = y_1/h(y_2), \\ x_2 &= \psi_2(y_1, y_2) = y_2, \end{cases}$$

where h is a function defined in (6.2). By the change of variables $y = \varphi(x)$ with $|\det\varphi'(x)| < C < \infty$, direct computations give

$$\partial_1 U(y) = \frac{\partial}{\partial y_1}[u(\psi(y))] = \partial_1 u(\psi(y)) \cdot \frac{1}{h(y_2)},$$
$$\partial_2 U(y) = \frac{\partial}{\partial y_2}[u(\psi(y))] = -\partial_1 u(\psi(y)) \cdot \frac{y_1 h'(y_2)}{h(y_2)^2} + \partial_2 u(\psi(y)),$$

and the fact that $|\psi(y) - \psi(y')| \le \max\{\|\nabla\psi_1\|_\infty, \|\nabla\psi_2\|_\infty\}|y - y'|$, we have

$$\begin{aligned}
&\|I_\alpha(\partial_1 U)\|^r_{L^r(\mathbb{R}^2)} \\
&\le C\|\nabla u\|^r_{L^\infty(\mathbb{R}^2)} + C\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{|\partial_1 U(y) - \partial_1 U(y')|^2}{|\psi(y)-\psi(y')|^{2+2\alpha}}dy\right)^{r/2} dy' \\
&\le C\|\nabla u\|^r_{L^\infty(\mathbb{R}^2)} + C\int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2}\frac{\left|\frac{\partial_1 u(\psi(y))}{h(y_2)} - \frac{\partial_1 u(\psi(y'))}{h(y_2')}\right|^2}{|\psi(y)-\psi(y')|^{2+2\alpha}}dy\right]^{r/2} dy' \\
&\le C\|\nabla u\|^r_{L^\infty(\mathbb{R}^2)} + C\int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2}\frac{\left|\frac{\partial_1 u(x)}{h(x_2)} - \frac{\partial_1 u(x')}{h(x_2')}\right|^2}{|x-x'|^{2+2\alpha}}|\det\varphi'(x)|dx\right]^{r/2} |\det\varphi'(x')|\,dx' \\
&\le C\|\nabla u\|^r_{L^\infty(\mathbb{R}^2)} + C\int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2}\frac{\left|\frac{\partial_1 u(x)}{h(x_2)} - \frac{\partial_1 u(x')}{h(x_2')}\right|^2}{|x-x'|^{2+2\alpha}}dx\right]^{r/2} dx'.
\end{aligned}$$

Now take $\eta(x_1, x_2) \in C_0^\infty(\mathbb{R}^2)$ assuming value 1 on the support of $\partial_1 u$ so that the support of η is just a bit larger than that of $\partial_1 u$, and $h(x_2) = x_2$ on the support of η. Define $\tilde{h}(x_1, x_2) = \eta(x_1, x_2)/h(x_2)$ and then write

$$\begin{aligned}
\frac{\partial_1 u(x)}{h(x_2)} - \frac{\partial_1 u(x')}{h(x_2')} &= \partial_1 u(x)\tilde{h}(x) - \partial_1 u(x')\tilde{h}(x') \\
&= [\partial_1 u(x) - \partial_1 u(x')]\tilde{h}(x') + \partial_1 u(x)[\tilde{h}(x) - \tilde{h}(x')],
\end{aligned}$$

which yields

$$\begin{aligned}
\|I_\alpha(\partial_1 U)\|^r_{L^r(\mathbb{R}^2)} &\le C\|\nabla u\|^r_{L^\infty(\mathbb{R}^2)} + C\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{|\partial_1 u(x) - \partial_1 u(x')|^2}{|x-x'|^{2+2\alpha}}dx\right)^{r/2} dx' \\
&\quad + C\|\nabla u\|^r_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{|\tilde{h}(x) - \tilde{h}(x')|^2}{|x-x'|^{2+2\alpha}}dx_1dx_2\right)^{r/2} dx_1'dx_2' \\
&\le C\|\nabla u\|^r_{L^\infty(\mathbb{R}^2)} + C\|\partial_1 u\|^r_{L^r_\alpha(\mathbb{R}^2)} + C\|\nabla u\|^r_{L^\infty}\|\tilde{h}\|^r_{L^r_\alpha(\mathbb{R}^2)} \\
&\le C\left(\|\nabla u\|_{L^\infty(\mathbb{R}^2)} + \|u\|_{L^r_\gamma(\mathbb{R}^2)}\right)^r.
\end{aligned}$$

A similar argument as the one above shows that

$$\begin{aligned}\|I_\alpha(\partial_2 U)\|_{L^r(\mathbb{R}^2)}^r &= \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{|\partial_2 U(y)-\partial_2 U(y')|^2}{|y-y'|^{2+2\alpha}}dy\right)^{r/2}dy' \\ &\le C\|\nabla u\|_{L^\infty(\mathbb{R}^2)}^r + C\|\partial_1 u\|_{L_\alpha^r(\mathbb{R}^2)}^r + C\|\partial_2 u\|_{L_\alpha^r(\mathbb{R}^2)}^r \\ &\le C\left(\|\nabla u\|_{L^\infty(\mathbb{R}^2)} + \|u\|_{L_\gamma^r(\mathbb{R}^2)}\right)^r.\end{aligned}$$

Also, by repeating the preceding argument we obtain,

$$\|I_\alpha(U)\|_{L^r(\mathbb{R}^2)} \le C\left(\|u\|_{L^\infty(\mathbb{R}^2)} + \|u\|_{L_\alpha^r(\mathbb{R}^2)}\right) \le C\,\|u\|_{L_\gamma^r(\mathbb{R}^2)},$$

where we used the Sobolev embedding theorem in the last inequality with $\gamma r > 2$. The proof of Lemma 6.2 is now complete. □

For g, h on $\mathbb{R}$ define a the tensor $g\otimes h$ as the following function on $\mathbb{R}^2$ by setting $(g\otimes h)(\xi,\eta) = g(\xi)h(\eta)$.

Lemma 6.3. *Let $1<\gamma<2$, $2/\gamma<r<\infty$. Let $f\in L_\gamma^r(\mathbb{R})$ supported in $[-1,1]$ with $f'\in L^\infty$, and $\widehat{\Theta}$ is a smooth function supported in an annulus centered at 0 with size comparable to 1, then we have*

$$\left\|f\otimes\widehat{\Theta}\right\|_{L_\gamma^r(\mathbb{R}^2)} \le C(f),$$

where $C(f)$ is a finite constant depending on f.

Proof. We use the same idea as in the proof of Lemma 6.2. It suffices to prove that $f\otimes\widehat{\Theta}\in L_1^r(\mathbb{R}^2)$ and that $I_\alpha(\partial^\beta(f\otimes\widehat{\Theta}))\in L^r(\mathbb{R}^2)$ with $|\beta|=1$. It is easy to check that $\|f\otimes\widehat{\Theta}\|_{L_1^r}\le C\|f\|_{L_1^r}$, so we only prove that $I_\alpha(\partial_\xi(f\otimes\widehat{\Theta}))\in L^r(\mathbb{R}^2)$.

Note that $f\otimes\widehat{\Theta}$ is compactly supported and we can choose a function $\varphi(\xi,\eta)\in C_0^\infty(\mathbb{R}^2)$ assuming 1 on the support of $f\otimes\widehat{\Theta}$ and therefore $f\otimes\widehat{\Theta} = f(\xi)\varphi(\xi,\eta)\widehat{\Theta}(\eta)\varphi(\xi,\eta)$. Then $\int_{\mathbb{R}^2}|I_\alpha(\partial_\xi(f\otimes\widehat{\Theta}))|^r d\xi d\eta$ is split into the parts

$$\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{|[f'(\xi)\varphi(\xi,\eta)-f'(\xi')\varphi(\xi',\eta')]\widehat{\Theta}(\eta')\varphi(\xi',\eta')|^2}{|(\xi,\eta)-(\xi',\eta')|^{2+2\alpha}}d\xi'd\eta'\right)^{r/2}d\xi d\eta$$

and

$$\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{|f'(\xi)\varphi(\xi,\eta)[\widehat{\Theta}(\eta)\varphi(\xi,\eta)-\widehat{\Theta}(\eta')\varphi(\xi',\eta')]|^2}{|(\xi,\eta)-(\xi',\eta')|^{2+2\alpha}}d\xi'd\eta'\right)^{r/2}d\xi d\eta.$$

We prove only that the first one is finite since the latter can be proved similarly.

To prove the boundedness of the first one, we split it further via the identity

$$f'(\xi)\varphi(\xi,\eta)-f'(\xi')\varphi(\xi',\eta') = (f'(\xi)-f'(\xi'))\varphi(\xi,\eta)+f'(\xi')(\varphi(\xi,\eta)-\varphi(\xi',\eta')).$$

The integral containing the second part is finite because f' is bounded and $\varphi \in L^r_\gamma(\mathbb{R}^2)$. For the other part, the simple change of variables $\eta' \to (\eta-\eta')/(\xi-\xi')$ shows that it is dominated by

$$C\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}}\frac{|f'(\xi)-f'(\xi')|^2}{|\xi-\xi'|^{1+2\alpha}}d\xi'\right)^{r/2}|\varphi(\xi,\eta)|\,d\xi d\eta,$$

which, by Lemma 6.1, is bounded by $\|f\|^r_{L^r_\gamma(\mathbb{R})}$ since $\varphi \in C_0^\infty(\mathbb{R}^2)$. □

Lemma 6.4. *Let $\gamma \in (1,2)$ and $1 < r < \frac{1}{\gamma-1}$. Then $\|\Phi\varphi\|_{L^r_\gamma(\mathbb{R})} < \infty$, where φ is a smooth function with compact support, and Φ is the function in* (6.1)*.*

Proof. To obtain the claim, we need to show that $D^\gamma(\varphi\Phi) = \big((1+|\xi|^2)^{\gamma/2}\widehat{\varphi\Phi}\big)^\vee \in L^r(\mathbb{R})$. Since

$$\|D^\gamma(\varphi\Phi)\|_{L^r(\mathbb{R})} \approx \|\varphi\Phi\|_{L^r(\mathbb{R})} + \left\|\big(|\xi|^\gamma\widehat{\varphi\Phi}\big)^\vee\right\|_{L^r(\mathbb{R})},$$

and trivially $\varphi\Phi \in L^r(\mathbb{R})$, we reduce the proof to establishing $\big\|\big(|\xi|^\gamma\widehat{\varphi\Phi}\big)^\vee\big\|_{L^r(\mathbb{R})} < \infty$. By the Kato-Ponce inequality for homogeneous type [7], [35], [19], it suffices to show that $\big(|\xi|^\gamma\widehat{\Phi}\big)^\vee$ lies in $L^r(\mathbb{R})$. Indeed, for $\gamma \in (1,2)$ we write

$$\begin{aligned}\widehat{\Phi}(\xi)|\xi|^\gamma &= \frac{1}{\xi}\,\xi\,\widehat{\Phi}(\xi)\,|\xi|^\gamma = \frac{1}{2\pi i}\frac{1}{\xi}\,\widehat{\Phi}'(\xi)\,|\xi|^\gamma\\ &= -\,i\frac{1}{\pi\xi}\,\widehat{\chi_{[-1,0]}}(\xi)\,|\xi|^\gamma = -i\frac{1}{\pi\xi}\,\frac{e^{2\pi i\xi}-1}{2\pi i\xi}\,|\xi|^\gamma\\ &= -\,i\frac{1}{\pi}\,\frac{e^{2\pi i\xi}-1}{2\pi i}\,|\xi|^{\gamma-2} = -\frac{1}{2\pi^2}(e^{2\pi i\xi}-1)\,|\xi|^{\gamma-2}.\end{aligned}$$

Taking inverse Fourier transforms we obtain that

$$\big(\widehat{\Phi}(\xi)|\xi|^\gamma\big)^\vee(x) = c_\gamma(|x+1|^{1-\gamma} - |x|^{1-\gamma})$$

and this function lies in $L^r(\mathbb{R})$ when $1 < r < \frac{1}{\gamma-1}$ and γ is very close to 2. □

The preceding result can be lifted to $\mathbb{R}^2$ as follows.

Lemma 6.5. *Let $\gamma \in (1,2)$ and $\frac{2}{\gamma} < r < \frac{1}{\gamma-1}$, and let θ be a smooth function supported in $\frac{1}{2} \le |\xi| \le 2$ on the real line. Define $U(\xi,\eta) = \Phi(\frac{\xi}{\eta})\theta(\frac{\xi}{\eta})\widehat{\psi}(\xi,\eta)$, where $\widehat{\psi}$ is a smooth function supported in an annulus centered at zero. Then $\|U\|_{L^r_\gamma(\mathbb{R}^2)} < \infty$.*

Proof. Set

$$u(\xi,\eta) = \Phi(\xi)\theta(\xi)\widehat{\Psi}(\xi\eta,\eta)$$

and

$$U(\xi,\eta) = \Phi(\xi/\eta)\theta(\xi/\eta)\widehat{\Psi}(\xi,\eta).$$

Since $h(\eta)=\eta$ on the support of the function U. We now apply Lemma 6.2 to obtain

$$\|U\|_{L^r_\gamma(\mathbb{R}^2)} \le C\big(\|\nabla u\|_{L^\infty(\mathbb{R}^2)} + \|u\|_{L^r_\gamma(\mathbb{R}^2)}\big).$$

Thus, it is enough to show that $\|u\|_{L^r_\gamma(\mathbb{R}^2)} < \infty$. We introduce a compactly supported smooth function $\widehat{\Theta}(\eta)$ which is equal to 1 on the support of $\eta \mapsto \theta(\xi)\widehat{\Psi}(\xi\eta,\eta)$ for any ξ. the Kato-Ponce inequality ([28] [19]) allows us to estimate the Sobolev norm of u as follows:

$$\begin{aligned}\|u\|_{L^r_\gamma(\mathbb{R}^2)} &= \big\|\Phi(\xi)\theta(\xi)\widehat{\Theta}(\eta)\widehat{\Psi}(\xi\eta,\eta)\big\|_{L^r_\gamma(\mathbb{R}^2)} \\ &\lesssim \big\|\Phi(\xi)\theta(\xi)\widehat{\Theta}(\eta)\big\|_{L^r_\gamma(\mathbb{R}^2)}\big\|\widehat{\Psi}(\xi\eta,\eta)\big\|_{L^\infty(\mathbb{R}^2)} \\ &\quad + \big\|\widehat{\Psi}(\xi\eta,\eta)\big\|_{L^r_\gamma(\mathbb{R}^2)}\big\|\Phi(\xi)\theta(\xi)\widehat{\Theta}(\eta)\big\|_{L^\infty(\mathbb{R}^2)}.\end{aligned}$$

We are left with establishing $\|\Phi(\xi)\theta(\xi)\widehat{\Theta}(\eta)\|_{L^r_\gamma(\mathbb{R}^2)} < \infty$, since all other terms on the right of the above inequality are finite. This is achieved via Lemmas 6.4 and 6.3. Thus the proof of Lemma 6.5 is complete. □

Using these ideas we are able to deduce the following result concerning $\mathcal{C}_1$.

Proposition 6.6. *The Calderón commutator $\mathcal{C}_1$ maps $L^{p_1}(\mathbb{R})\times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$ when $1/p_1 + 1/p_2 = 1/p$, $1<p_1,p_2<\infty$, and $1/2<p<\infty$.*

Proof. Note that $\sigma(\xi,\eta)=\operatorname{sgn}(\eta)\Phi(\xi/\eta)$ has an obvious modification which is continuous on $\mathbb{R}^2\backslash\{0\}$. We denote the latter by $\operatorname{sgn}(\eta)\Phi(\xi/\eta)$ as well since there is no chance to introduce any confusion.

We introduce a smooth function with compact support θ on the real line which is supported in two small intervals, say, of length $1/100$ centered at the points -1 and 0. Then we write

$$1 = \theta(\xi/\eta) + 1 - \theta(\xi/\eta)$$

and we decompose the function $\operatorname{sgn}(\eta)\Phi(\xi/\eta) = \sigma_1(\xi,\eta)+\sigma_2(\xi,\eta)$, where

$$\sigma_1(\xi,\eta) = \operatorname{sgn}(\eta)\Phi(\xi/\eta)\theta(\xi/\eta) \quad \text{and} \quad \sigma_2(\xi,\eta) = \operatorname{sgn}(\eta)\Phi(\xi/\eta)(1-\theta(\xi/\eta)).$$

Let $\widehat{\Psi}$ be a smooth bump supported in the annulus $1/2<|(\xi,\eta)|<3/2$ in $\mathbb{R}^2$. The function σ_2 is smooth away from zero and $\sigma_2\widehat{\Psi}$ lies in $L^r_\gamma(\mathbb{R}^2)$ for any $r,\gamma>1$. For σ_1, we take arbitrarily $\gamma\in(1,2)$ and $\frac{2}{\gamma}<r<\frac{1}{\gamma-1}$. Then Lemma 6.5 gives

$$\sigma_1\widehat{\Psi}\in L^r_\gamma(\mathbb{R}^2)\subset L^r_{(\gamma/2,\gamma/2)}(\mathbb{R}^2).$$

Thus $\sigma\widehat{\Psi}\in L^r_{(\gamma/2,\gamma/2)}(\mathbb{R}^2)$ for $\gamma\in(1,2)$ and $\frac{2}{\gamma}<r<\frac{1}{\gamma-1}$. Hence, Corollary 1.2 with $m=2, n=1$, and $\gamma_i=\gamma/2$ implies that T_σ is bounded from $L^{p_1}(\mathbb{R})\times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$, where $1<p_1,p_2<\infty$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ satisfying $\min\{p_1,p_2\}>2/\gamma$. Taking γ sufficiently near to 2, we can cover the whole range $1<p_1,p_2<\infty$. □

6.2. **Commutators of Calderón-Coifman-Journé.** Now we focus on the boundedness properties of the following n-dimensional version of $\mathcal{C}_1$:

$$\begin{aligned}&\mathcal{C}_1^{(n)}(f,a)(x)\\&=\text{p.v.}\int_{\mathbb{R}^n} f(y)\left(\prod_{l=1}^{n}\frac{1}{(y_l-x_l)^2}\right)\int_{x_1}^{y_1}\cdots\int_{x_n}^{y_n} a(u_1,\dots,u_n)\,du_1\cdots du_n\,dy,\end{aligned}$$

where f is a function on $\mathbb{R}^n$, and $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, $y=(y_1,\dots,y_n)\in\mathbb{R}^n$. The operator $\mathcal{C}_1^{(n)}$ was introduced by a suggestion of Coifman when $n=2$. The $L^2\times L^\infty\to L^2$ bound for $\mathcal{C}_1^{(2)}$ was studied by Aguirre [1] and Journé [26, 27], namely,

$$\|\mathcal{C}_1^{(2)}(f,a)\|_{L^2(\mathbb{R}^2)}\le C\|a\|_{L^\infty(\mathbb{R}^2)}\|f\|_{L^2(\mathbb{R}^2)}.$$

For general $n\ge 2$, boundedness for $\mathcal{C}_1^{(n)}$ from $L^{p_1}\times L^{p_2}$ to L^p for $p>1/2$, could be derived by Muscalu's work on Calderón commutators on polydiscs [34, Theorem 6.1] via time-frequency analysis. In this section we will apply Corollary 1.5 to obtain a direct proof of the boundedness of $\mathcal{C}_1^{(n)}$ from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ in the full range of $p>1/2$.

Proposition 6.7. *Let* $1<p_1,p_2<\infty$, $1/2<p<\infty$ *and* $1/p=1/p_1+1/p_2$. *Then the operator* $\mathcal{C}_1^{(n)}(f,a)$ *is bounded from* $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ *to* $L^p(\mathbb{R}^n)$, *i.e.,*

$$\|\mathcal{C}_1^{(n)}(f,a)\|_{L^p(\mathbb{R}^n)}\le C_p\|a\|_{L^{p_1}(\mathbb{R}^n)}\|f\|_{L^{p_2}(\mathbb{R}^n)}.$$

Proof. The operator $\mathcal{C}_1^{(n)}(f,a)$ is a bilinear operator which can also be expressed in bilinear Fourier multiplier form as

$$\mathcal{C}_1^{(n)}(f,a)(x)=(i\pi)^n\iint_{\mathbb{R}^n\times\mathbb{R}^n}\widehat{f}(\xi_1,\dots,\xi_n)\,\widehat{a}(\eta_1,\dots,\eta_n)\,e^{2\pi i x\cdot(\xi+\eta)}m(\xi,\eta)\,d\xi d\eta\,,$$

where the symbol m is given by

$$m(\xi,\eta)=\prod_{i=1}^{n}\left[\operatorname{sgn}(\eta_i)\,\Phi\Big(\frac{\xi_i}{\eta_i}\Big)\right],$$

and $\xi=(\xi_1,\dots,\xi_n)$ and $\eta=(\eta_1,\dots,\eta_n)$. Let r and γ be as in Lemma 6.5. Since $m(\xi,\eta)=\prod_{i=1}^{n}\sigma(\xi_i,\eta_i)$ is a product of n equal pieces, by Corollary 1.5, it suffices to verify that

$$\sup_{k\in\mathbb{Z}}\|\sigma(2^k\cdot)\widehat{\Psi}\|_{L^r_{\gamma/2,\gamma/2}(\mathbb{R}^2)}=B<\infty.$$

Note that $\sigma(2^k\cdot)\widehat{\Psi}\in L^r_\gamma(\mathbb{R}^2)$ uniformly in k by Proposition 6.6, so they are also in $L^r_{\gamma/2,\gamma/2}(\mathbb{R}^2)$ uniformly due to that $L^r_\gamma(\mathbb{R}^2)\subset L^r_{\gamma/2,\gamma/2}(\mathbb{R}^2)$. This completes the proof of Proposition 6.7. □

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