THE HÖRMANDER MULTIPLIER THEOREM, III: THE COMPLETE BILINEAR CASE VIA INTERPOLATION

LOUKAS GRAFAKOS AND HANH VAN NGUYEN

Abstract. We develop a special multilinear complex interpolation theorem that allows us to prove an optimal version of the bilinear Hörmander multiplier theorem concerning symbols that lie in the Sobolev space $L^r_s(\mathbb{R}^{2n})$, $2 \le r < \infty$, $rs > 2n$, uniformly over all annuli. More precisely, given a smoothness index s, we find the largest open set of indices $(1/p_1, 1/p_2)$ for which we have boundedness for the associated bilinear multiplier operator from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $1/p = 1/p_1 + 1/p_2$, $1 < p_1, p_2 < \infty$.

1. Introduction

Multipliers are linear operators of the form

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)\sigma(\xi)e^{2\pi i x\cdot\xi}d\xi\,,$$

where f is a Schwartz function on $\mathbb{R}^n$ and $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx$ is its Fourier transform.

Let Ψ be a Schwartz function whose Fourier transform is supported in the annulus of the form $\{\xi : 1/2 < |\xi| < 2\}$ which satisfies $\sum_{j\in\mathbb{Z}} \widehat{\Psi}(2^{-j}\xi) = 1$ for all $\xi \neq 0$. We denote by Δ the Laplacian and by $(I-\Delta)^{s/2}$ the operator given on the Fourier transform by multiplication by $(1+4\pi^2|\xi|^2)^{s/2}$; also for $s > 0$, and we denote by L^r_s the Sobolev space of all functions h on $\mathbb{R}^n$ with norm $\|h\|_{L^r_s} := \|(I-\Delta)^{s/2}h\|_{L^r} < \infty$. Extending an earlier result of Mikhlin [15], the optimal version of the Hörmander multiplier theorem says that if

$$\sup_{k\in\mathbb{Z}} \big\|\widehat{\Psi}\sigma(2^k\cdot)\big\|_{L^r_s} < \infty \tag{1}$$

and

$$\Big|\frac{1}{p} - \frac{1}{2}\Big| < \frac{s}{n}\,, \tag{2}$$

then T_σ is bounded from $L^p(\mathbb{R}^n)$ to itself for $1 < p < \infty$. Hörmander's [13] original version of this theorem stated boundedness in the entire interval $1 < p < \infty$ provided $s > n/2$. A restriction on the indices first appeared in Calderón and Torchinsky [1], while condition (2) appeared in [5]; this condition is sharp as examples are given in [5] indicating that the theorem fails in general when $\big|\frac{1}{p} - \frac{1}{2}\big| > \frac{s}{n}$. Moreover, recently Slavíková [19] provided an example showing that boundedness may also fail even on the critical line $\big|\frac{1}{p} - \frac{1}{2}\big| = \frac{s}{n}$.

In this paper we provide bilinear analogues of these results. The study of the Hörmander multiplier theorem in the multilinear setting was initiated by Tomita [21] and was further

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studied by Fujita, Grafakos, Miyachi, Nguyen, Si, Tomita (see [2], [7] [11], [8], [17], [18]) among others. For a given function σ on $\mathbb{R}^{2n}$ we define a bilinear operator

$$T_\sigma(f_1, f_2)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\sigma(\xi_1,\xi_2)e^{2\pi i x\cdot(\xi_1+\xi_2)}d\xi_1 d\xi_2$$

originally defined on pairs of $\mathcal{C}_0^\infty$ functions f_1, f_2 on $\mathbb{R}^n$. We fix a Schwartz function Ψ on $\mathbb{R}^{2n}$ whose Fourier transform is supported in the annulus $1/2 \le |(\xi_1,\xi_2)| \le 2$ and satisfies

$$\sum_{j\in\mathbb{Z}} \widehat{\Psi}(2^{-j}(\xi_1,\xi_2)) = 1, \qquad (\xi_1,\xi_2) \neq 0.$$

The following theorem is the main result of this paper:

Theorem 1.1. *Let $2 \le r < \infty$, $s > \frac{2n}{r}$, $1 < p_1, p_2 \le \infty$ and let $1/p = 1/p_1 + 1/p_2 > 0$.*
(a) Let $n/2 < s \le n$. Suppose that

$$\frac{1}{p_1} < \frac{s}{n}, \frac{1}{p_2} < \frac{s}{n}, 1 - \frac{s}{n} < \frac{1}{p} < \frac{s}{n} + \frac{1}{2}. \tag{3}$$

Then for all $\mathcal{C}_0^\infty(\mathbb{R}^n)$ functions f_1, f_2 we have

$$\|T_\sigma(f_1,f_2)\|_{L^p(\mathbb{R}^n)} \le C \sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\Psi}\|_{L^r_s(\mathbb{R}^{2n})}\|f_1\|_{L^{p_1}(\mathbb{R}^n)}\|f_2\|_{L^{p_2}(\mathbb{R}^n)}. \tag{4}$$

Moreover, if (4) holds for all $f_1, f_2 \in \mathcal{C}_0^\infty$ and all σ satisfying (1), then we must necessarily have

$$\frac{1}{p_1} \le \frac{s}{n}, \frac{1}{p_2} \le \frac{s}{n}, 1 - \frac{s}{n} \le \frac{1}{p} \le \frac{s}{n} + \frac{1}{2}. \tag{5}$$

(b) Let $n < s \le 3n/2$ and satisfy

$$\frac{1}{p} < \frac{s}{n} + \frac{1}{2}\,. \tag{6}$$

Then (4) holds. Moreover, if (4) holds for all $f_1, f_2 \in \mathcal{C}_0^\infty$ and all σ satisfying (1), then we must necessarily have

$$\frac{1}{p} \le \frac{s}{n} + \frac{1}{2}\,. \tag{7}$$

(c) If $s > \frac{3n}{2}$ then (4) holds for all $1 < p_1, p_2 < \infty$ and $\frac{1}{2} < p < \infty$.

This theorem uses two main tools: First, the optimal $n/2$-derivative result in the local L^2-case contained in [6] and a special type of multilinear interpolation suitable for the purposes of this problem (see Theorem 3.1 below). Figure 1 (Section 4), plotted on a slanted $(1/p_1, 1/p_2)$ plane, shows the regions of boundedness for T_σ in the two cases $n/2 < s \le n$ and $n < s \le 3n/2$. Note also that in the former case, the condition $1 - \frac{s}{n} < \frac{1}{p}$ is only needed when $p > 2$.

Finally, we mention that the necessity of conditions (3), (5), and (7) in Theorem 1.1 are consequences of Theorems 2 and 3 in [6]; these say that if boundedness holds, then we must necessarily have

$$\frac{1}{p_1} \le \frac{s}{n}, \quad \frac{1}{p_2} \le \frac{s}{n}, \quad \frac{1}{p} \le \frac{s}{n} + \frac{1}{2}.$$

Also, if T_σ maps $L^{p_1}\times L^{p_2}$ to L^p and $p>2$, then duality implies that T_σ maps $L^{p'}\times L^{p_2}$ to $L^{p_1'}$. Now p' plays the role of p_1 and so constraint $\frac{1}{p_1}\le\frac{s}{n}$ becomes $1-\frac{s}{n}\le\frac{1}{p}$. This proves (5). So the main contribution of this work is the sufficiency of the conditions in (3) and (6).

2. Preliminary material for interpolation

In this section we briefly discuss three lemmas needed in our interpolation.

Lemma 2.1. *Let $0<p_0<p<p_1\le\infty$ be related as in $1/p=(1-\theta)/p_0+\theta/p_1$ for some $\theta\in(0,1)$. Given $f\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ and $\varepsilon>0$, there exist smooth functions h_j^ε, $j=1,\dots,N_\varepsilon$, supported in cubes with pairwise disjoint interiors, and nonzero complex constants c_j^ε such that the functions*

$$f^{z,\varepsilon}=\sum_{j=1}^{N_\varepsilon}|c_j^\varepsilon|^{\frac{p}{p_0}(1-z)+\frac{p}{p_1}z}\,h_j^\varepsilon \tag{8}$$

satisfy

$$\big\|f^{\theta,\varepsilon}-f\big\|_{L^{p_0}}<\varepsilon\quad\text{and}\quad\begin{cases}\big\|f^{\theta,\varepsilon}-f\big\|_{L^{p_1}}<\varepsilon & \text{if } p_1<\infty\\[2ex] \big\|f^{\theta,\varepsilon}\big\|_{L^\infty}\le\big\|f\big\|_{L^\infty}+\varepsilon & \text{if } p_1=\infty\end{cases} \tag{9}$$

and

$$\|f^{it,\varepsilon}\|_{L^{p_0}}^{p_0}\le\|f\|_{L^p}^p+\varepsilon',\quad \|f^{1+it,\varepsilon}\|_{L^{p_1}}\le\big(\|f\|_{L^p}^p+\varepsilon'\big)^{\frac{1}{p_1}},$$

where ε' depends on $\varepsilon,p_0,p_1,p,\|f\|_{L^p}$ and tends to zero as $\varepsilon\to0$.

Proof. Given $f\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ and $\varepsilon>0$, by uniform continuity there are N_ε cubes Q_j^ε (with disjoint interiors) and nonzero complex constants c_j^ε such that

$$\Big\|f-\sum_{j=1}^{N_\varepsilon}c_j^\varepsilon\chi_{Q_j^\varepsilon}\Big\|_{L^{p_0}}^{\min(1,p_0)}<\frac{\varepsilon^{\min(1,p_0)}}{2},\qquad \Big\|f-\sum_{j=1}^{N_\varepsilon}c_j^\varepsilon\chi_{Q_j^\varepsilon}\Big\|_{L^{p_1}}^{\min(1,p_1)}<\frac{\varepsilon^{\min(1,p_1)}}{2},$$

and

$$\Big\|f-\sum_{j=1}^{N_\varepsilon}c_j^\varepsilon\chi_{Q_j^\varepsilon}\Big\|_{L^p}<\varepsilon. \tag{10}$$

Find smooth functions g_j^ε satisfying $0\le g_j^\varepsilon\le\chi_{Q_j^\varepsilon}$ such that

$$\Big\|f-\sum_{j=1}^{N_\varepsilon}c_j^\varepsilon g_j^\varepsilon\Big\|_{L^{p_0}}^{\min(1,p_0)}<\frac{\varepsilon^{\min(1,p_0)}}{2}\qquad\text{and}\qquad \Big\|f-\sum_{j=1}^{N_\varepsilon}c_j^\varepsilon g_j^\varepsilon\Big\|_{L^{p_1}}^{\min(1,p_1)}<\frac{\varepsilon^{\min(1,p_1)}}{2},$$

where the last estimate is required only when $p_1<\infty$. We set $h_j^\varepsilon=e^{i\phi_j^\varepsilon}g_j^\varepsilon$, where ϕ_j^ε is the argument of the complex number c_j^ε. Then h_j^ε is that function claimed in (8). Observe that

$$f^{\theta,\varepsilon}=\sum_{j=1}^{N_\varepsilon}|c_j^\varepsilon|h_j^\varepsilon=\sum_{j=1}^{N_\varepsilon}c_j^\varepsilon g_j^\varepsilon$$

satisfies (9) when $p_1 < \infty$; in the case $p_1 = \infty$ we have

$$|f^{\theta,\varepsilon}| \le \sum_{j=1}^{N_\varepsilon} |c_j^\varepsilon| \chi_{Q_j^\varepsilon} = \Big| \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \chi_{Q_j^\varepsilon} \Big| \le \Big| \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \chi_{Q_j^\varepsilon} - f \Big| + |f| \le \frac{\varepsilon}{2} + |f| \le \varepsilon + \|f\|_{L^\infty}.$$

Now we have

$$\|f^{it,\varepsilon}\|_{L^{p_0}}^{p_0} \le \sum_{j=1}^{N_\varepsilon} |c_j^\varepsilon|^p |Q_j^\varepsilon| = \Big\| \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \chi_{Q_j^\varepsilon} \Big\|_{L^p}^p \le \Big(\varepsilon^{\min(1,p)} + \|f\|_{L^p}^{\min(1,p)} \Big)^{\frac{p}{\min(1,p)}},$$

having made use of (10).

Given $a, c > 0$ and $\varepsilon > 0$ set $\varepsilon' = \varepsilon'(\varepsilon, a, c) = (\varepsilon^a + c^a)^{1/a} - c$. Then $(\varepsilon^a + c^a)^{1/a} \le \varepsilon' + c$ and $\varepsilon' \to 0$ as $\varepsilon \to 0$. Then for a suitable ε' that only depends on $\varepsilon, p, p_0, p_1, \|f\|_{L^p}$, the preceding estimate gives: $\|f^{it,\varepsilon}\|_{L^{p_0}}^{p_0} \le \|f\|_{L^p}^p + \varepsilon'$ and analogously $\|f^{1+it,\varepsilon}\|_{L^{p_1}} \le \big(\|f\|_{L^p}^p + \varepsilon'\big)^{1/p_1}$ when $p_1 < \infty$; notice that if $p_1 = \infty$ then $\|f^{1+it,\varepsilon}\|_{L^\infty} \le 1$ and the right hand side of the inequality is equal to 1, thus the inequality is still valid. □

Lemma 2.2. *Given a domain Ω on the complex plane and (M, μ) a measure space, let $V : \Omega \times M \to \mathbb{C}$ be a function such that $V(\cdot, x)$ is analytic on Ω for almost every $x \in M$. If the function*

$$V^*(z, x) = \sup_{w:|w-z|<\frac{1}{2}\mathrm{dist}(z,\partial\Omega)} \Big| \frac{dV}{dw}(w, x) \Big|, \quad x \in M \tag{11}$$

is integrable over M for each $z \in \Omega$, then the mapping $z \longmapsto V(z, \cdot)$ is an analytic function from Ω to the Banach space $L^1(M, d\mu)$.

Proof. Fix $z \in \Omega$ and denote $r_z = \frac{1}{2}\mathrm{dist}(z, \partial\Omega)$. It is enough to show that

$$\lim_{h\to 0} \Big\| \frac{V(z+h, \cdot) - V(z, \cdot)}{h} - \frac{dV}{dz}(z, \cdot) \Big\|_{L^1(M,d\mu)} = 0. \tag{12}$$

The assumption yields that for some set M_0 with $\mu(M \setminus M_0) = 0$, we have

$$\lim_{h\to 0} \frac{V(z+h, x) - V(z, x)}{h} = \frac{dV}{dz}(z, x)$$

for all $x \in M_0$. Thus for each $x \in M_0$ and $h \in \mathbb{C}$ with $|h| < r_z$ we can write

$$\begin{aligned} \Big| \frac{V(z+h, x) - V(z, x)}{h} - \frac{dV}{dz}(z, x) \Big| &= \Big| \frac{1}{h} \int_0^h \frac{dV}{dw}(w, x) dw - \frac{dV}{dz}(z, x) \Big| \\ &\le 2 \sup_{w:|w-z|<r_z} \Big| \frac{dV}{dw}(w, x) \Big| \\ &= 2V^*(z, x). \end{aligned}$$

Since $V^*(z, \cdot)$ is integrable on M_0, the Lebesgue dominated convergence theorem yields

$$\begin{aligned} &\lim_{h\to 0} \int_{M_0} \Big| \frac{V(z+h, x) - V(z, x)}{h} - \frac{dV}{dz}(z, x) \Big| d\mu(x) \\ &= \int_{M_0} \lim_{h\to 0} \Big| \frac{V(z+h, x) - V(z, x)}{h} - \frac{dV}{dz}(z, x) \Big| d\mu(x) = 0. \end{aligned}$$

This yields (12) and completes the proof, as the last integral is over the entire space M. ☐

Lemma 2.3. *Given $0 < a < b < \infty$, $\Omega = \{z \in \mathbb{C} \ : \ a < \Re(z) < b\}$, and a measure space (M, μ) of finite measure, let $H : \Omega \times \mathbb{R}^d \times M \to \mathbb{C}$ be a measurable function so that $H(\cdot, \xi, x)$ be analytic on Ω and continuous on $\overline{\Omega}$ for each $(\xi, x) \in \mathbb{R}^d \times M$. Suppose that*

$$\sup_{w\in\overline{\Omega}} \Big| H(w, \xi, x) \Big| + \sup_{w\in\Omega} \Big| \frac{dH}{dw}(w, \xi, x) \Big| \le C(1 + |\xi|)^{-d-1} \tag{13}$$

for all $(\xi, x) \in \mathbb{R}^d \times M$. If φ be a bounded measurable function on $\mathbb{R}^d$, then the mapping $z \longmapsto V(z, \cdot)$, defined by

$$V(z, x) = \int_{\mathbb{R}^d} |\varphi(\xi)|^z e^{iArg(\varphi(\xi))} H(z, \xi, x) d\xi,$$

is an analytic function from Ω to the Banach space $L^1(M, d\mu)$ and is continuous on $\overline{\Omega}$.

Proof. Let $K = \{\xi \in \mathbb{R}^d : \ \varphi(\xi) \neq 0\}$. By assumption, for each $x \in M$ we have

$$\frac{dV}{dz}(z, x) = \int_K |\varphi(\xi)|^z \ln(|\varphi(\xi)|) e^{iArg(\varphi(\xi))} H(z, \xi, x) d\xi + \int_K |\varphi(\xi)|^z e^{iArg(\varphi(\xi))} \frac{dH}{dz}(z, \xi, x) d\xi.$$

As for each $z \in \Omega$ we have

$$\big| \, |\varphi(\xi)|^z \ln(|\varphi(\xi)|) \big| \le \sup_{|t|\le 1} |t|^a \log \frac{1}{|t|} + (1 + \|\varphi\|_{L^\infty})^b \log(1 + \|\varphi\|_{L^\infty}) = c < \infty$$

and H satisfies assumption (13), the associated function $V^*(z, \cdot)$ defined in (11) is bounded and thus integrable over M. Therefore, using Lemma 2.2 we deduce that $z \longmapsto V(z, \cdot)$ is analytic from Ω to $L^1(M, d\mu)$.

Using Lebesgue's dominated convergence theorem and the fist part of assumption (13) we easily deduce that $V(z, \cdot)$ is continuous up to the boundary of Ω. ☐

Lemma 2.4 ([3]). *Let F be analytic on the open strip $S = \{z \in \mathbb{C} \ : \ 0 < \Re(z) < 1\}$ and continuous on its closure. Assume that for all $0 \le \tau \le 1$ there exist functions A_τ on the real line such that*

$$|F(\tau + it)| \le A_\tau(t) \qquad \text{for all } t \in \mathbb{R},$$

and suppose that there exist constants $A > 0$ and $0 < a < \pi$ such that for all $t \in \mathbb{R}$ we have

$$0 < A_\tau(t) \le \exp\left\{ A e^{a|t|} \right\}.$$

Then for $0 < \theta < 1$ we have

$$|F(\theta)| \le \exp\left\{ \frac{\sin(\pi\theta)}{2} \int_{-\infty}^{\infty} \left[\frac{\log |A_0(t)|}{\cosh(\pi t) - \cos(\pi\theta)} + \frac{\log |A_1(t)|}{\cosh(\pi t) + \cos(\pi\theta)} \right] dt \right\}.$$

In calculations it is crucial to note that

$$\frac{\sin(\pi\theta)}{2} \int_{-\infty}^{\infty} \frac{dt}{\cosh(\pi t) - \cos(\pi\theta)} = 1 - \theta \, , \quad \frac{\sin(\pi\theta)}{2} \int_{-\infty}^{\infty} \frac{dt}{\cosh(\pi t) + \cos(\pi\theta)} = \theta.$$

3. Multilinear interpolation

In this section we prove the main tool needed to derive Theorem 1.1 by interpolation. We denote by $\vec{\xi} = (\xi_1, \dots, \xi_m)$ elements of $\mathbb{R}^{mn}$, where $\xi_j \in \mathbb{R}^n$. We fix a Schwartz function Ψ on $\mathbb{R}^{mn}$ whose Fourier transform is supported in the annulus $1/2 \le |\vec{\xi}\,| \le 2$ and satisfies

$$\sum_j \widehat{\Psi}(2^{-j}\vec{\xi}\,) = 1, \qquad 0 \neq \vec{\xi} \in \mathbb{R}^{mn}.$$

Theorem 3.1. *Let $0 < p_1^0, \dots, p_m^0 \le \infty$, $0 < p_1^1, \dots, p_m^1 \le \infty$, $0 < q_0, q_1 \le \infty$, $0 \le s_0, s_1 < \infty$, $1 < r_0, r_1 < \infty$, $0 < \theta < 1$, and let*

$$\frac{1}{p_l} = \frac{1-\theta}{p_l^0} + \frac{\theta}{p_l^1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \quad s = (1-\theta)s_0 + \theta s_1$$

for $l = 1, \dots, m$. Assume $r_0 s_0 > mn$, and $r_1 s_1 > mn$ and that for all $f_l \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $l = 1, \dots, m$, we have

$$\|T_\sigma(f_1, \dots, f_m)\|_{L^{q_k}(\mathbb{R}^n)} \le K_k \sup_{j\in\mathbb{Z}} \left\|\sigma(2^j\cdot)\widehat{\Psi}\right\|_{L^{r_k}_{s_k}(\mathbb{R}^{mn})} \prod_{l=1}^m \|f_l\|_{L^{p_l^k}(\mathbb{R}^n)}$$

for $k = 0, 1$ where K_0, K_1 are positive constants. Then the intermediate estimate holds:

$$\|T_\sigma(f_1, \dots, f_m)\|_{L^q(\mathbb{R}^n)} \le C_*\, K_0^{1-\theta} K_1^\theta \sup_{j\in\mathbb{Z}} \left\|\sigma(2^j\cdot)\widehat{\Psi}\right\|_{L^r_s(\mathbb{R}^{mn})} \prod_{l=1}^m \|f_l\|_{L^{p_l}(\mathbb{R}^n)} \tag{14}$$

for all $f_l \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, where C_* depends on all the indices, on θ, and on the dimension.

Consequently, if $p_l < \infty$ for all $l \in \{1, \dots, m\}$, then T_σ admits a bounded extension from $L^{p_1} \times \cdots \times L^{p_m}$ to L^q that satisfies (14).

Proof. Fix a smooth function $\widehat{\Phi}$ on $\mathbb{R}^{mn}$ such that $\mathrm{supp}(\Phi) \subset \left\{\frac14 \le |\vec{\xi}\,| \le 4\right\}$ and $\widehat{\Phi} \equiv 1$ on the support of the function $\widehat{\Psi}$. Denote $\varphi_j = (I-\Delta)^{\frac{s}{2}}[\sigma(2^j\cdot)\widehat{\Psi}]$ and define

$$\sigma_z(\vec{\xi}\,) = \sum_{j\in\mathbb{Z}} (I-\Delta)^{-\frac{s_0(1-z)+s_1 z}{2}} \left[|\varphi_j|^{r(\frac{1-z}{r_0}+\frac{z}{r_1})} e^{i\mathrm{Arg}\,(\varphi_j)}\right](2^{-j}\vec{\xi}\,)\widehat{\Phi}(2^{-j}\vec{\xi}\,). \tag{15}$$

This sum has only finitely many terms and we now estimate its L^∞ norm.

Fix $\vec{\xi} \in \mathbb{R}^{mn}$. Then there is a j_0 such that $|\vec{\xi}\,| \approx 2^{j_0}$ and there are only two terms in the sum in (15). For these terms we estimate the L^∞ norm of $(I-\Delta)^{-\frac{s_0(1-z)+s_1 z}{2}}\left[|\varphi_j|^{r(\frac{1-z}{r_0}+\frac{z}{r_1})} e^{i\mathrm{Arg}\,(\varphi_j)}\right]$. For $z = \tau + it$ with $0 \le \tau \le 1$, let $s_\tau = (1-\tau)s_0 + \tau s_1$ and $1/r_\tau = (1-\tau)/r_0 + \tau/r_1$. By the Sobolev embedding theorem we have

$$\begin{aligned}
&\left\|(I-\Delta)^{-\frac{s_0(1-z)+s_1 z}{2}}\left[|\varphi_j|^{r(\frac{1-z}{r_0}+\frac{z}{r_1})} e^{i\mathrm{Arg}\,(\varphi_j)}\right]\right\|_{L^\infty(\mathbb{R}^{mn})} \\
&\quad\le C(r_\tau, s_\tau, mn)\left\|(I-\Delta)^{-\frac{s_0(1-z)+s_1 z}{2}}\left[|\varphi_j|^{r(\frac{1-z}{r_0}+\frac{z}{r_1})} e^{i\mathrm{Arg}\,(\varphi_j)}\right]\right\|_{L^{r_\tau}_{s_\tau}(\mathbb{R}^{mn})} \\
&\quad\le C(r_\tau, s_\tau, n)\left\|(I-\Delta)^{it\frac{s_0-s_1}{2}}\left[|\varphi_j|^{r(\frac{1-z}{r_0}+\frac{z}{r_1})} e^{i\mathrm{Arg}\,(\varphi_j)}\right]\right\|_{L^{r_\tau}(\mathbb{R}^{mn})} \\
&\quad\le C'(r_\tau, s_\tau, mn)(1+|s_0-s_1|\,|t|)^{mn/2+1}\left\||\varphi_j|^{r(\frac{1-z}{r_0}+\frac{z}{r_1})} e^{i\mathrm{Arg}\,(\varphi_j)}\right\|_{L^{r_\tau}(\mathbb{R}^{mn})}
\end{aligned}$$

$$\le C''(r_0,r_1,s_0,s_1,\tau,mn)(1+\,|t|)^{mn/2+1}\Big\| |\varphi_j|^{r(\frac{1-\tau}{r_0}+\frac{\tau}{r_1})}\Big\|_{L^{r_\tau}(\mathbb{R}^{mn})}$$
$$= C''(r_0,r_1,s_0,s_1,\tau,mn)(1+\,|t|)^{mn/2+1}\big\|\varphi_j\big\|^{r/r_\tau}_{L^r(\mathbb{R}^{mn})}.$$

It follows from this that

$$\|\sigma_{\tau+it}\|_{L^\infty(\mathbb{R}^{mn})} \le C''(r_0,r_1,s_0,s_1,\tau,mn)(1+\,|t|)^{mn/2+1}\Big(\sup_{j\in\mathbb{Z}}\big\|\sigma(2^j\cdot)\widehat{\Psi}\big\|_{L^r_s(\mathbb{R}^{mn})}\Big)^{r/r_\tau}. \quad (16)$$

Let T_{σ_z} be the family of operators associated to the multipliers σ_z. Let ε be given.

Suppose first that $\min(p_l^0,p_l^1)<\infty$ for all $l\in\{1,\dots,m\}$. This forces $p_l<\infty$ for all l.

Case I: $\min(q_0,q_1)>1$. This assumption implies that $q>1$, hence $q',q_0',q_1'<\infty$. Fix $f_l,g\in\mathcal{C}_0^\infty(\mathbb{R}^n)$. For given $\varepsilon>0$, for every $l\in\{1,\dots,m\}$, by Lemma 2.1 there exist functions $f_l^{z,\varepsilon}$ and $g^{z,\varepsilon}$ of the form (8) such that

$$\|f_l^{\theta,\varepsilon}-f_l\|_{L^{p_l^1}}<\varepsilon,\quad \|f_l^{\theta,\varepsilon}-f_l\|_{L^{p_l^0}}<\varepsilon,\quad \|g^{\theta,\varepsilon}-g\|_{L^{q_0'}}<\varepsilon,\quad \|g^{\theta,\varepsilon}-g\|_{L^{q_1'}}<\varepsilon, \quad (17)$$

when $\max(p_l^0,p_l^1)<\infty$, while one of the first two inequalities is replaced by $\|f_l^{\theta,\varepsilon}\|_{L^\infty}\le \|f_l\|_{L^{p_l^k}}+\varepsilon=\|f_l\|_{L^\infty}+\varepsilon$ when $p_l^k=\max(p_l^0,p_l^1)=\infty$, and that

$$\|f_l^{it,\varepsilon}\|_{L^{p_l^0}}\le\big(\|f_l\|^{p_l}_{L^{p_l}}+\varepsilon'\big)^{\frac{1}{p_l^0}},\quad \|f_l^{1+it\varepsilon}\|_{L^{p_l^1}}\le\big(\|f_l\|^{p_l}_{L^{p_l}}+\varepsilon'\big)^{\frac{1}{p_l^1}},$$
$$\|g^{it,\varepsilon}\|_{L^{q_0'}}\le\big(\|g\|^{q'}_{L^{q'}}+\varepsilon'\big)^{\frac{1}{q_0'}},\quad \|g^{1+it,\varepsilon}\|_{L^{q_1'}}\le\big(\|g\|^{q'}_{L^{q'}}+\varepsilon'\big)^{\frac{1}{q_1'}}.$$

Define

$$\begin{aligned}
F(z) &= \int_{\mathbb{R}^n} T_{\sigma_z}(f_1^{z,\varepsilon},\dots,f_m^{z,\varepsilon})g^{z,\varepsilon}\,dx\\
&= \int_{\mathbb{R}^{mn}} \sigma_z(\vec{\xi})\widehat{f_1^{z,\varepsilon}}(\xi_1)\cdots\widehat{f_m^{z,\varepsilon}}(\xi_m)\widehat{g^{z,\varepsilon}}(-(\xi_1+\cdots+\xi_m))\,d\vec{\xi}\\
&= \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^{mn}}(I-\Delta)^{-\frac{s_0(1-z)+s_1z}{2}}\Big[|\varphi_j|^{r(\frac{1-z}{r_0}+\frac{z}{r_1})}e^{i\mathrm{Arg}\,(\varphi_j)}\Big](2^{-j}\xi)\widehat{\Phi}(2^{-j}\vec{\xi})\\
&\quad\times\Big(\prod_{l=1}^m\widehat{f_l^{z,\varepsilon}}(\xi_l)\Big)\widehat{g^{z,\varepsilon}}(-(\xi_1+\cdots+\xi_m))\,d\vec{\xi}\\
&= \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^{mn}}\Big[|\varphi_j|^{r(\frac{1-z}{r_0}+\frac{z}{r_1})}e^{i\mathrm{Arg}\,(\varphi_j)}\Big](2^{-j}\vec{\xi})\\
&\quad\times(I-\Delta)^{-\frac{s_0(1-z)+s_1z}{2}}\Big[\widehat{\Phi}(2^{-j}\vec{\xi})\Big(\prod_{l=1}^m\widehat{f_l^{z,\varepsilon}}(\xi_l)\Big)\widehat{g^{z,\varepsilon}}(-(\xi_1+\cdots+\xi_m))\Big](\vec{\xi})\,d\vec{\xi}.
\end{aligned}$$

Notice that

$$(I-\Delta)^{-\frac{s_0(1-z)+s_1z}{2}}\Big[\widehat{\Phi}(2^{-j}\vec{\xi})\Big(\prod_{l=1}^m\widehat{f_l^{z,\varepsilon}}(\xi_l)\Big)\widehat{g^{z,\varepsilon}}(-(\xi_1+\cdots+\xi_m))\Big](\vec{\xi})$$

is equal to a finite sum (over $k_1, \dots, k_m, l$) of terms of the form

$$|c^{\varepsilon}_{k_1}|^{\frac{p_1}{p^0_1}(1-z)+\frac{p_1}{p^1_1}z} \cdots |c^{\varepsilon}_{k_m}|^{\frac{p_m}{p^0_m}(1-z)+\frac{p_m}{p^1_m}z} |d^{\varepsilon}_l|^{\frac{q'}{q'_0}(1-z)+\frac{q'}{q'_1}z} (I-\Delta)^{-\frac{s_0(1-z)+s_1 z}{2}} \left[\widehat{\Phi}(2^{-j}\cdot)\zeta_{k_1,\dots,k_m,l}\right](\vec{\xi}),$$

which we call $H(z,\vec{\xi})$, where $\zeta_{k_1,\dots,k_m,l}$ are Schwartz functions. Thus $H(z,\vec{\xi})$ is an analytic function in z. Moreover $H(z,\vec{\xi})$ can be thought of as a function of three variables $H(z,\vec{\xi},x_0)$, being constant in the variable x_0, where $\{x_0\}$ is a measure space of one element equipped with counting measure. With this interpretation, it is not hard to verify that $H(z,\vec{\xi},x_0)$ satisfies (13).

Lemma 2.3 guarantees that $F(z)$ is analytic on the strip $0 < \Re(z) < 1$ and continuous up to the boundary. Furthermore, by Hölder's inequality,

$$|F(it)| \le \left\| T_{\sigma_{it}}(f^{it,\varepsilon}_1, \dots, f^{it,\varepsilon}_m) \right\|_{L^{q_0}} \|g^{\varepsilon}_{it}\|_{L^{q'_0}},$$

and noting that only the terms with $j = k-1, k, k+1$ survive in the sum in (15) for $\sigma_{it}(2^k\cdot)\widehat{\Psi}$, the Kato-Ponce inequality [10, 14] applied as $\|(I-\Delta)^{s/2}(F\widehat{\Phi})\|_{L^{r_0}} \le C\|(I-\Delta)^{s/2}(F)\|_{L^{r_0}}$ yields

$$\begin{aligned}
&\|T_{\sigma_{it}}(f^{it,\varepsilon}_1, \dots, f^{it,\varepsilon}_m)\|_{L^{q_0}} \\
&\le K_0 \sup_{k\in\mathbb{Z}} \left\| \sigma_{it}(2^k\cdot)\widehat{\Psi} \right\|_{L^{r_0}_{s_0}} \prod_{l=1}^m \|f^{it,\varepsilon}_l\|_{L^{p^0_l}} \\
&\le C_{n,r_0,s_0} K_0 \sup_{k\in\mathbb{Z}} \left\| (I-\Delta)^{\frac{s_0}{2}} (I-\Delta)^{-\frac{s_0(1-it)+s_1 it}{2}} \left[|\varphi_k|^{r(\frac{1-it}{r_0}+\frac{it}{r_1})} e^{i\mathrm{Arg}\,(\varphi_k)} \right] \right\|_{L^{r_0}} \prod_{l=1}^m \|f^{it,\varepsilon}_l\|_{L^{p^0_l}} \\
&\le C(m,n,r_0,s_0)(1+|s_1-s_0|\,|t|)^{\frac{mn}{2}+1} K_0 \sup_{j\in\mathbb{Z}} \|\varphi_j\|^{\frac{r}{r_0}}_{L^r} \prod_{l=1}^m \big(\|f_l\|^{p_l}_{L^{p_l}} + \varepsilon' \big)^{\frac{1}{p^0_l}} \\
&= C(m,n,r_0,s_0,s_1)(1+|t|)^{\frac{mn}{2}+1} K_0 \sup_{j\in\mathbb{Z}} \left\| (I-\Delta)^{\frac{s}{2}} [\sigma(2^j\cdot)\widehat{\Psi}] \right\|^{\frac{r}{r_0}}_{L^r} \prod_{l=1}^m \big(\|f_l\|^{p_l}_{L^{p_l}} + \varepsilon' \big)^{\frac{1}{p^0_l}}.
\end{aligned}$$

Thus, for some constant $C = C(m,n,r_0,s_0,s_1)$ we have

$$|F(it)| \le C(1+|t|)^{\frac{mn}{2}+1} K_0 \sup_{j\in\mathbb{Z}} \left\| (I-\Delta)^{\frac{s}{2}} [\sigma(2^j\cdot)\widehat{\Psi}] \right\|^{\frac{r}{r_0}}_{L^r} \big(\|g\|^{q'}_{L^{q'}} + \varepsilon' \big)^{\frac{1}{q'_0}} \prod_{l=1}^m \big(\|f_l\|^{p_l}_{L^{p_l}} + \varepsilon' \big)^{\frac{p_l}{p^0_l}}.$$

Similarly, we can choose the constant $C = C(m,n,r_1,s_0,s_1)$ above large enough so that

$$|F(1+it)| \le C(1+|t|)^{\frac{mn}{2}+1} K_1 \sup_{j\in\mathbb{Z}} \left\| (I-\Delta)^{\frac{s}{2}} [\sigma(2^j\cdot)\widehat{\Psi}] \right\|^{\frac{r}{r_1}}_{L^r} \big(\|g\|^{q'}_{L^{q'}} + \varepsilon' \big)^{\frac{1}{q'_1}} \prod_{l=1}^m \big(\|f_l\|^{p_l}_{L^{p_l}} + \varepsilon' \big)^{\frac{1}{p^1_l}}.$$

Note that $F(z)$ is a combination of finite terms of the form

$$\Lambda_{k_1,\dots,k_m,l}(z) \int_{\mathbb{R}^{mn}} \sigma_z(\vec{\xi}) \widehat{h^{1,\varepsilon}_{j_1}}(\xi_1) \cdots \widehat{h^{m,\varepsilon}_{j_m}}(\xi_m) \widehat{g^{\varepsilon}_j}(-(\xi_1+\cdots+\xi_m))\, d\vec{\xi},$$

where

$$\Lambda_{k_1,\dots,k_m,l}(z) = |c^{\varepsilon}_{k_1}|^{\frac{p_1}{p^0_1}(1-z)+\frac{p_1}{p^1_1}z} \cdots |c^{\varepsilon}_{k_m}|^{\frac{p_m}{p^0_m}(1-z)+\frac{p_m}{p^1_m}z} |d^{\varepsilon}_l|^{\frac{q'}{q'_0}(1-z)+\frac{q'}{q'_1}z},$$

and $h_{j_1}^{1,\varepsilon}$, g_j^{ε} are smooth functions with compact support. Thus for $z=\tau+it$, $t\in\mathbb{R}$ and $0\le\tau\le 1$ it follows from (16) and from the definition of $F(z)$ that

$$|F(z)|\le C(\tau,\epsilon,f_1,\dots,f_m,g,r_l,p_l,q_0,q_1)(1+\,|t|)^{\frac{mn}{2}+1}\Big(\sup_{j\in\mathbb{Z}}\Big\|\sigma(2^j\cdot)\widehat{\Psi}\Big\|_{L^r_s}\Big)^{\frac{r}{r_\tau}}=A_\tau(t).$$

As $A_\tau(t)\le\exp(Ae^{a|t|})$, the admissible growth hypothesis of Lemma 2.4 is satisfied. Applying Lemma 2.4 we obtain

$$|F(\theta)|\le C\,K_0^{1-\theta}K_1^{\theta}\sup_{j\in\mathbb{Z}}\Big\|(I-\Delta)^{\frac{s}{2}}[\sigma(2^j\cdot)\widehat{\psi}]\Big\|_{L^r}\big(\,\|g\|_{L^{q'}}^{q'}+\varepsilon'\big)^{\frac{1}{q'}}\prod_{l=1}^{m}\big(\,\|f_l\|_{L^{p_l}}^{p_l}+\varepsilon'\big)^{\frac{1}{p_l}}. \tag{18}$$

But

$$F(\theta)=\int_{\mathbb{R}^n}T_\sigma(f_1^{\theta,\varepsilon},\dots,f_m^{\theta,\varepsilon})\,g^{\theta,\varepsilon}\,dx$$

and then we have

$$\begin{aligned}\int_{\mathbb{R}^n}T_\sigma(f_1,\dots,f_m)\,g\,dx=F(\theta)&+\int_{\mathbb{R}^n}\big[T_\sigma(f_1,\dots,f_m)-T_\sigma(f_1^{\theta,\varepsilon},\dots,f_m^{\theta,\varepsilon})\big]\,g\,dx\\&+\int_{\mathbb{R}^n}T_\sigma(f_1^{\theta,\varepsilon},\dots,f_m^{\theta,\varepsilon})\big(g-g^{\theta,\varepsilon}\big)\,dx.\end{aligned}\tag{19}$$

A telescoping identity yields

$$|T_\sigma(f_1,\dots,f_m)-T_\sigma(f_1^{\theta,\varepsilon},\dots,f_m^{\theta,\varepsilon})|\le\sum_{l=1}^{m}|T_\sigma(f_1,\dots,f_{l-1},f_l-f_l^{\theta,\varepsilon},f_{l+1}^{\theta,\varepsilon},\dots,f_m^{\theta,\varepsilon})|.$$

For every fixed l, applying the hypothesis that T_σ is bounded from $L^{p_1^k}\times\cdots\times L^{p_m^k}$ to L^{q_k} for $k=0,1$ we obtain

$$\big\|T_\sigma(f_1,\dots,f_{l-1},f_l-f_l^{\theta,\varepsilon},f_{l+1}^{\theta,\varepsilon},\dots,f_m^{\theta,\varepsilon})\big\|_{L^{q_k}}\lesssim\big\|f_l-f_l^{\theta,\varepsilon}\big\|_{L^{p_l^k}}\prod_{j\neq l}\big(\|f_j\|_{L^{p_j^k}}^{p_j}+\varepsilon'\big)^{\frac{1}{p_j}}.$$

In view of the inequality $\|h\|_{L^q}\le\|h\|_{L^{q_0}}^{1-\theta}\|h\|_{L^{q_1}}^{\theta}$ these estimates yield

$$\big\|T_\sigma(f_1,\dots,f_{l-1},f_l-f_l^{\theta,\varepsilon},f_{l+1}^{\theta,\varepsilon},\dots,f_m^{\theta,\varepsilon})\big\|_{L^q}\lesssim\big\|f_l-f_l^{\theta,\varepsilon}\big\|_{L^{p_l^0}}^{1-\theta}\big\|f_l-f_l^{\theta,\varepsilon}\big\|_{L^{p_l^1}}^{\theta}\prod_{j\neq l}\big(\|f_j\|_{L^{p_j^k}}^{p_j}+\varepsilon'\big)^{\frac{1}{p_j}}.$$

As $0<\theta<1$ and one of p_l^0 or p_l^1 is strictly less than infinity, the expression on the right above is bounded by a constant multiple of $\varepsilon^{\min(\theta,1-\theta)}$ and hence it tends to zero as $\varepsilon\to 0$ because of (9). This proves that (in fact for all $0<q<\infty$)

$$\big\|T_\sigma(f_1,\dots,f_m)-T_\sigma(f_1^{\theta,\varepsilon},\dots,f_m^{\theta,\varepsilon})\big\|_{L^q}\le E_\varepsilon, \tag{20}$$

where $E_\varepsilon\to 0$ as $\varepsilon\to 0$. Returning to (19) and using (18) and Hölder's inequality we write

$$\begin{aligned}&\Big|\int T_\sigma(f_1,\dots,f_m)(x)\,g(x)\,dx\Big|\\&\qquad\le CK_0^{1-\theta}K_1^{\theta}\sup_{j\in\mathbb{Z}}\Big\|(I-\Delta)^{\frac{s}{2}}[\sigma(2^j\cdot)\widehat{\psi}]\Big\|_{L^r}\big(\,\|g\|_{L^{q'}}^{q'}+\varepsilon'\big)^{\frac{1}{q'}}\prod_{l=1}^{m}\big(\,\|f_l\|_{L^{p_l}}^{p_l}+\varepsilon'\big)^{\frac{1}{p_l}}\end{aligned}$$

$$+ E_\varepsilon \|g\|_{L^{q'}} + C\|g - g^{\theta,\varepsilon}\|_{L^{q_0'}} \prod_{l=1}^m \|f_l^{\theta,\varepsilon}\|_{L^{p_l^0}}$$

Recalling (17) and using that each $\|f_l^{\theta,\varepsilon}\|_{L^{p_l^0}}$ remains bounded as $\varepsilon \to 0$ we obtain

$$\left|\int T_\sigma(f_1,\dots,f_m)\, g\, dx\right| \le C K_0^{1-\theta} K_1^\theta \sup_{j\in\mathbb{Z}} \left\|(I-\Delta)^{\frac{s}{2}}[\sigma(2^j\cdot)\widehat{\psi}]\right\|_{L^r} \|g\|_{L^{q'}} \prod_{l=1}^m \|f_l\|_{L^{p_l}}$$

by letting $\varepsilon \to 0$. Taking the supremum over all functions $g \in L^{q'}$ with $\|g\|_{L^{q'}} = 1$ yields the sought estimate (14) in Case I.

Case II: $\min(q_0, q_1) \le 1$.

Here we will make use of two following lemmas proved by Stein and Weiss [20].

Lemma 3.2 ([20]). *Let $U : \overline{S} \longrightarrow \mathbb{R}$ be an upper semi-continuous function of admissible growth and subharmonic in the unit strip S. Then for $z_0 = x_0 + iy_0 \in S$ we have*

$$U(z_0) \le \int_{-\infty}^{+\infty} U\big(i(y_0+t)\big)\omega(1-x_0,t)dt + \int_{-\infty}^{+\infty} U\big(1+i(y_0+t)\big)\omega(x_0,t)dt,$$

where

$$\omega(x,y) = \frac{1}{2}\frac{\sin \pi x}{\cos \pi x + \cosh \pi y}.$$

Lemma 3.3 ([20]). *Let $0 < c \le 1$ and let (M,μ) be a measure space with finite measure. If a function $V(z,\cdot)$ is analytic from the unit strip S to $L^1(M,\mu)$, then $\log \int_M |V(z,x)|^c\, d\mu$ is subharmonic on S.*

We now continue the proof of the second case. We fix functions f_l as in the previous case. Choose an integer $\rho > 1$ such that $\rho \ge \rho \min(q_0,q_1) > q$. Take an arbitrary positive simple function g with $\|g\|_{L^{\rho'}} = 1$. Assume that $g = \sum_{k=1}^N c_k \chi_{E_k}$, where $c_k > 0$ and E_k are pairwise disjoint measurable sets of finite measure and compact support. For $z \in \mathbb{C}$, set

$$g^z = \sum_{k=1}^N c_k^{\lambda(z)} \chi_{E_k}, \quad \text{where } \lambda(z) = \rho'\left[1 - \frac{q}{\rho}\left(\frac{1-z}{q_0} + \frac{z}{q_1}\right)\right].$$

Now consider

$$G(z) = \int_{\mathbb{R}^n} |T_{\sigma_z}(f_1^{z,\varepsilon},\dots,f_m^{z,\varepsilon})(x)|^{\frac{q}{\rho}} |g^z(x)|\, dx = \sum_{k=1}^N \int_{E_k} \left|c_k^{\frac{\rho}{q}\lambda(z)} T_{\sigma_z}(f_1^{z,\varepsilon},\dots,f_m^{z,\varepsilon})(x)\right|^{\frac{q}{\rho}} dx.$$

Let $V(z,x) = T_{\sigma_z}(f_1^{z,\varepsilon},\dots,f_m^{z,\varepsilon})(x)$. Then $V(z,x)$ can be represented as a finite sum of terms of the form

$$\int_{\mathbb{R}^{mn}} e^{P(z)} |\varphi_j(\vec{\xi})|^{\frac{r}{r_0}(1-z)+\frac{r}{r_1}z} e^{i\mathrm{Arg}\,(\varphi_j)} (I-\Delta)^{-\frac{s_0(1-z)+s_1 z}{2}} \left[e^{2\pi i x 2^j\cdot(\sum_{\kappa=1}^m \xi_\kappa)} \widehat{\Phi}(\vec{\xi}) \prod_{\kappa=1}^m \widehat{h_\kappa^\varepsilon}(2^j\xi_\kappa)\right](\vec{\xi})d\vec{\xi},$$

where h_κ^ε are the smooth functions with compact support in (8) and P is a polynomial. Setting

$$H(z,\vec{\xi},x) = (I-\Delta)^{-\frac{s_0}{2}(1-z)-\frac{s_1}{2}z}\Big[e^{2\pi i 2^j x\cdot(\xi_1+\cdots+\xi_n)}\widehat{\Phi}(\vec{\xi})\prod_{\kappa=1}^m \widehat{h_\kappa^\varepsilon}(2^j\xi_\kappa)\Big],$$

we note that $H(z,\vec{\xi},x)$ is analytic in z, smooth in ξ and bounded in x, as long as x remains in a compact set. Moreover H satisfies (13). Applying Lemma 2.3 we obtain that for all $(\vec{\xi},x)$ the mapping $H(\cdot,\vec{\xi},x)$ is analytic from S to $L^1(E_k,dx)$ Then Lemma 3.3 applies and yields that $\log G$ is subharmonic on S. Using Hölder's inequality with indices $\frac{\rho q_0}{q}$ and $\left(\frac{\rho q_0}{q}\right)'$ and the fact that the $L^{\rho'}$-norm of g is equal to 1, we have

$$\begin{aligned} G(it) &\le \left\{\int_{\mathbb{R}^n}\left|T_{\sigma_{it}}(f_1^{it,\varepsilon},\dots,f_m^{it,\varepsilon})(x)\right|^{q_0}dx\right\}^{\frac{q}{\rho q_0}}\left\|g^{it}\right\|_{L^{(\frac{\rho q_0}{q})'}} \\ &\le C\left((1+|t|)^{\frac{mn}{2}+1}\right)^{\frac{q}{\rho}}\left(K_0\sup_{j\in Z}\left\|\sigma(2^j\cdot)\widehat{\psi}\right\|_{L_s^r}^{\frac{r}{r_0}}\prod_{l=1}^m\left(\|f_l\|_{L^{p_l}}^{p_l}+\varepsilon'\right)^{\frac{1}{p_l}}\right)^{\frac{q}{\rho}}. \end{aligned}$$

Similarly, we can estimate

$$\begin{aligned} G(1+it) &\le \left\{\int_{\mathbb{R}^n}\left|T_{\sigma_{it}}(f_1^{1+it,\varepsilon},\dots,f_m^{1+it,\varepsilon})(x)\right|^{q_1}dx\right\}^{\frac{q}{\rho q_1}}\left\|g^{1+it}\right\|_{L^{(\frac{\rho q_1}{q})'}} \\ &\le C\left((1+|t|)^{\frac{mn}{2}+1}\right)^{\frac{q}{\rho}}\left(K_1\sup_{j\in Z}\left\|\sigma(2^j\cdot)\widehat{\psi}\right\|_{L_s^r}^{\frac{r}{r_1}}\prod_{l=1}^m\left(\|f_l\|_{L^{p_l}}^{p_l}+\varepsilon'\right)^{\frac{1}{p_l}}\right)^{\frac{q}{\rho}}. \end{aligned}$$

Applying Lemma 3.2 to $U=\log G$ (with $y_0=0$ and $x_0=\theta$) and using that for $0<\theta<1$ we have

$$\begin{aligned} \frac{\sin(\pi(1-\theta))}{2}\int_{-\infty}^{+\infty}\frac{1}{\cosh(\pi t)+\cos(\pi(1-\theta))}\,dt &= 1-\theta\,, \\ \frac{\sin(\pi\theta)}{2}\int_{-\infty}^{+\infty}\frac{1}{\cosh(\pi t)+\cos(\pi\theta)}\,dt &= \theta\,, \end{aligned}$$

(see [3, Page 48]) we obtain

$$G(\theta)\le C_*'\left(K_0^{1-\theta}K_1^\theta\sup_{j\in Z}\left\|\sigma(2^j\cdot)\widehat{\psi}\right\|_{L_s^r}\prod_{l=1}^m\left(\|f_l\|_{L^{p_l}}^{p_l}+\varepsilon'\right)^{\frac{1}{p_l}}\right)^{\frac{q}{\rho}}. \tag{21}$$

Notice that as

$$G(\theta)=\int_{\mathbb{R}^n}\left|T_\sigma(f_1^{\theta,\varepsilon},\dots,f_m^{\theta,\varepsilon})(x)\right|^{\frac{q}{\rho}}g(x)\,dx,$$

inequality (21) implies that

$$\begin{aligned} \left\|T_\sigma(f_1^{\theta,\varepsilon},\dots,f_m^{\theta,\varepsilon})\right\|_{L^q} &= \left\|\left|T_\sigma(f_1^{\theta,\varepsilon},\dots,f_m^{\theta,\varepsilon})\right|^{\frac{q}{\rho}}\right\|_{L^\rho}^{\frac{\rho}{q}} \\ &= \sup\left\{\int\left|T_\sigma(f_1^{\theta,\varepsilon},\dots,f_m^{\theta,\varepsilon})(x)\right|^{\frac{q}{\rho}}g(x)\,dx:\ g\ge 0, g\text{ simple}, \|g\|_{L^{\rho'}}=1\right\}^{\frac{\rho}{q}} \end{aligned}$$

$$\leq (C'_*)^{\frac{\varrho}{q}} K_0^{1-\theta} K_1^{\theta} \sup_{j\in Z} \left\|\sigma(2^j\cdot)\widehat{\psi}\right\|_{L^r_s} \prod_{l=1}^m \big(\|f_l\|_{L^{p_l}}^{p_l} + \varepsilon'\big)^{\frac{1}{p_l}}. \tag{22}$$

Finally, we use

$$\|T_\sigma(f_1,\dots,f_m)\|_{L^q} \leq (1+2^{\frac{1}{q}-1})\big[\|T_\sigma(f_1,\dots,f_m)-T_\sigma(f_1^{\theta,\varepsilon},\dots,f_m^{\theta,\varepsilon})\|_{L^q}+\|T_\sigma(f_1^{\theta,\varepsilon},\dots,f_m^{\theta,\varepsilon})\|_{L^q}\big]$$

and we note that for the second term we use (22), while the first term tends to zero, in view of (20). Letting $\varepsilon \to 0$, we deduce (14).

We now turn to the case where $\min(p_l^0, p_l^1) = \infty$ for some (but not all) l in $\{1,\dots,m\}$. Then we must have $p_l = \infty$ for these l, and for these l we set $f_l^{z,\varepsilon} = f$, while for the remaining l the functions $f_l^{z,\varepsilon}$ are defined as before; we notice that the preceding argument works with only minor modifications.

Finally we consider the case where $p_l^0 = p_l^1 = \infty$ for all $1 \leq l \leq m$. Here we also take $f_l^{z,\varepsilon} = f_l$ for all l in $\{1,\dots,m\}$. Now (19) becomes

$$\int_{\mathbb{R}^n} T_\sigma(f_1,\dots,f_m)\, g\, dx = F(\theta) + \int_{\mathbb{R}^n} T_\sigma(f_1,\dots,f_m)\big(g - g^{\theta,\varepsilon}\big)\, dx. \tag{23}$$

Hence, in Case I, when $\min(q_0, q_1) > 1$, we have

$$\begin{aligned} &\left| \int T_\sigma(f_1,\dots,f_m)(x)\, g(x)\, dx \right| \\ &\qquad \leq C K_0^{1-\theta} K_1^{\theta} \sup_{j\in\mathbb{Z}} \left\|(I-\Delta)^{\frac{s}{2}}[\sigma(2^j\cdot)\widehat{\psi}]\right\|_{L^r} \big(\|g\|_{L^{q'}}^{q'} + \varepsilon'\big)^{\frac{1}{q'}} \prod_{l=1}^m \|f_l\|_{L^\infty} \\ &\qquad\quad + C\big\|g - g^{\theta,\varepsilon}\big\|_{L^{q_0'}} \prod_{l=1}^m \big\|f_l\big\|_{L^\infty}. \end{aligned}$$

Passing the limit as $\varepsilon \to 0$ to obtain

$$\left|\int T_\sigma(f_1,\dots,f_m)\, g\, dx\right| \leq C K_0^{1-\theta} K_1^{\theta} \sup_{j\in\mathbb{Z}} \left\|(I-\Delta)^{\frac{s}{2}}[\sigma(2^j\cdot)\widehat{\psi}]\right\|_{L^r} \|g\|_{L^{q'}} \prod_{l=1}^m \|f_l\|_{L^\infty}.$$

The result in Case II, which is when $\min(q_0, q_1) \leq 1$, can be obtained from that in Case I by choosing $\rho > 1$ such that $\rho \min(q_0, q_1) > q$ and by arguing as before, replacing each term $\big(\|f_l\|_{L^{p_l}}^{p_l} + \varepsilon'\big)^{\frac{1}{p_l}}$ by $\|f_l\|_{L^\infty}$. This concludes the proof of the theorem in all cases. □

Note that the proof of Theorem 3.1 is much simpler in the case $r_0 = r_1 = 2$, and this was proved earlier in [8, Theorem 6.1, Step 1]; see also [9, Theorem 2.3]. In this case, the domains can be arbitrary Hardy spaces. We state the theorem in this case (without providing a proof):

Theorem 3.4 ([8]). *Let $p_l^0, p_l^1, p_l, q_0, q_1, q, s_0, s_1, s$ and $\theta \in (0,1)$ be as in Theorem 3.1 for $l = 1,\dots,m$. Assume that $s_0, s_1 > \frac{mn}{2}$, $p_l^0, p_l^1 < \infty$ for all l, and that*

$$\|T_\sigma(f_1,\dots,f_m)\|_{L^{q_k}(\mathbb{R}^n)} \leq K_k \sup_{j\in\mathbb{Z}} \left\|\sigma(2^j\cdot)\widehat{\Psi}\right\|_{L^2_{s_k}(\mathbb{R}^{mn})} \prod_{l=1}^m \|f_l\|_{H^{p_l^k}(\mathbb{R}^n)}$$

for $k = 0, 1$ where K_0, K_1 are positive constants. Then we have the intermediate estimate:

$$\|T_\sigma(f_1, \dots, f_m)\|_{L^q(\mathbb{R}^n)} \le C_* \, K_0^{1-\theta} K_1^{\theta} \sup_{j\in\mathbb{Z}} \Big\| \sigma(2^j \cdot)\widehat{\Psi} \Big\|_{L^2_s(\mathbb{R}^{mn})} \prod_{l=1}^{m} \|f_l\|_{H^{p_l}(\mathbb{R}^n)}$$

for all Schwartz functions f_l with vanishing moments of all orders, where C_* depends on all the indices, θ, and the dimension.

4. The proof of the main result via interpolation

We now turn to the proof of Theorem 1.1.

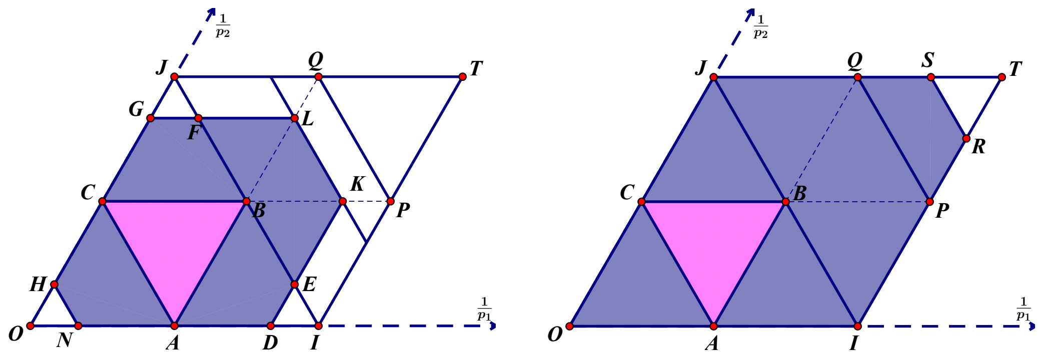
Proof. (a) Assume $n/2 < s \le n$ and let

$$\Gamma_1 = \Big\{ \Big(\frac{1}{p_1}, \frac{1}{p_2}\Big) \; : \; \frac{1}{p_1} < \frac{s}{n}, \frac{1}{p_2} < \frac{s}{n}, 1 - \frac{s}{n} < \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < \frac{s}{n} + \frac{1}{2} \Big\}.$$

We will prove that

$$\|T_\sigma(f_1, f_2)\|_{L^p(\mathbb{R}^n)} \le C \sup_{j\in\mathbb{Z}} \|\sigma(2^j \cdot)\widehat{\Psi}\|_{L^r_s(\mathbb{R}^{2n})} \|f_1\|_{L^{p_1}(\mathbb{R}^n)} \|f_2\|_{L^{p_2}(\mathbb{R}^n)} \tag{24}$$

for every $(\frac{1}{p_1}, \frac{1}{p_2}) \in \Gamma_1$, which is a convex set with vertices D, K, L, G, H and N (see Figure 1A below). By multilinear real interpolation [4, Corollary 7.2.4], we only need to verify the boundedness of T_σ at points in Γ_1 near its vertices D, K, L, G, H, N which do not lie in Γ_1.



(A) $\frac{n}{2} < s \le n$ (B) $n < s \le \frac{3n}{2}$

Figure 1. Boundedness holds in the shaded regions and unboundedness in the white regions. The local L^2 region is shaded in a lighter color.

As showed in [4, 11], the Hörmander condition $\sup_{j\in\mathbb{Z}} \|\sigma(2^j \cdot)\widehat{\Psi}\|_{L^r_s(\mathbb{R}^{2n})}$ is invariant under duality. For $1 \le p < \infty$, by duality, if T_σ maps $L^{p_1} \times L^{p_2} \to L^p$, then it also maps $L^{p'} \times L^{p_2} \to L^{p_1'}$. Therefore, if T_σ is bounded near D, then T_σ is also bounded near N by duality. By symmetry, if T_σ is bounded near N, D and K then it is bounded near H, G and L as well. From these reductions, it remains to prove (24) at points in Γ_1 near D and K.

With $s_1 > \frac{n}{2}$ and $r_1 s_1 > 2n$, we recall the following [6, Theorem 1]:

$$\|T_\sigma(f_1, f_2)\|_{L^1(\mathbb{R}^n)} \le C \sup_{j\in\mathbb{Z}} \|\sigma(2^j \cdot)\widehat{\Psi}\|_{L^{r_1}_{s_1}(\mathbb{R}^{2n})} \|f_1\|_{L^2(\mathbb{R}^n)} \|f_2\|_{L^2(\mathbb{R}^n)}. \tag{25}$$

By duality it follows from (25) that when $s_1 > \frac{n}{2}$ and $r_1 s_1 > 2n$ we have

$$\|T_\sigma(f_1,f_2)\|_{L^2(\mathbb{R}^n)} \le C \sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\Psi}\|_{L^{r_1}_{s_1}(\mathbb{R}^{2n})} \|f_1\|_{L^2(\mathbb{R}^n)} \|f_2\|_{L^\infty(\mathbb{R}^n)}. \tag{26}$$

Theorem 1.1 in [17] (with $s_1 = s_2$ in [17] being γ below) implies that

$$\|T_\sigma(f_1,f_2)\|_{L^q(\mathbb{R}^n)} \le C \sup_{j\in\mathbb{Z}} \|(I-\Delta_{\xi_1})^{\frac{\gamma}{2}}(I-\Delta_{\xi_2})^{\frac{\gamma}{2}}\left[\sigma(2^j\cdot)\widehat{\Psi}\right]\|_{L^2(\mathbb{R}^{2n})} \|f_1\|_{L^{q_1}(\mathbb{R}^n)} \|f_2\|_{L^{q_2}(\mathbb{R}^n)}$$

for $\gamma > \frac{n}{2}$, where $1 < q_1, q_2 \le \infty$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} < \frac{2\gamma}{n} + \frac{1}{2}$. Given $s_2 > n$, choose $\gamma = \frac{s_2}{2} > \frac{n}{2}$ and observing the trivial estimate

$$\sup_{j\in\mathbb{Z}} \|(I-\Delta_{\xi_1})^{\frac{\gamma}{2}}(I-\Delta_{\xi_2})^{\frac{\gamma}{2}}\left[\sigma(2^j\cdot)\widehat{\Psi}\right]\|_{L^2(\mathbb{R}^{2n})} \le C \sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\Psi}\|_{L^2_{s_2}(\mathbb{R}^{2n})},$$

we obtain

$$\|T_\sigma(f_1,f_2)\|_{L^q(\mathbb{R}^n)} \le C \sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\Psi}\|_{L^2_{s_2}(\mathbb{R}^{2n})} \|f_1\|_{L^{q_1}(\mathbb{R}^n)} \|f_2\|_{L^{q_2}(\mathbb{R}^n)} \tag{27}$$

for all $1 < q_1, q_2 \le \infty$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} < \frac{s_2}{n} + \frac{1}{2}$.

We now use Theorem 3.1 to interpolate between (26) and (27) (for $q_1 = q$ near 1 and $q_2 = \infty$). We obtain (24) at points $D_1(\frac{1}{p_1}, 0)$ with $\frac{1}{p_1} < \frac{s}{n}$ which are near the point $D(\frac{s}{n}, 0)$. Similarly, interpolating between (25) and (27) (q_1 near 1, $q_2 = 2$) yields (24) at points $K_1(\frac{1}{p_1}, \frac{1}{2})$ with $\frac{1}{p_1} < \frac{s}{n}$ near $K(\frac{s}{n}, \frac{1}{2})$. This yields (24) on Γ_1 and completes part (a).

(b) Assume $n < s \le \frac{3n}{2}$. Since $r \ge 2$, the Kato-Poince inequality [10] implies that

$$\sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\Psi}\|_{L^2_s(\mathbb{R}^{2n})} \lesssim \sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\Psi}\|_{L^r_s(\mathbb{R}^{2n})}. \tag{28}$$

Combining estimates (28) and (27) yields (24) in the open pentagon $OIRSJ$ union the open segments OI and OJ. This completes the second part of Theorem 1.1.

(c) In the last case when $s > \frac{3n}{2}$, notice that condition (7) reduces to $p > \frac{1}{2}$ and since

$$\sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\Psi}\|_{L^r_{\frac{3n}{2}}(\mathbb{R}^{2n})} \le \sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\Psi}\|_{L^r_s(\mathbb{R}^{2n})},$$

the case in part (b) applies and yields (24) for every point in the entire rhombus $OITJ$ union the open segments OI and OJ. The proof of Theorem 1.1 is now complete. □

5. An application

We consider the following multiplier on $\mathbb{R}^{2n}$: $m_{a,b}(\xi_1,\xi_2) = \psi(\xi_1,\xi_2)|(\xi_1,\xi_2)|^{-b}e^{i|(\xi_1,\xi_2)|^a}$ where $a > 0$, $a \ne 1$, $b > 0$, and ψ is a smooth function on $\mathbb{R}^{2n}$ which vanishes in a neighborhood of the origin and is equal to 1 in a neighborhood of infinity. One can verify that $m_{a,b}$ satisfies (1) on $\mathbb{R}^{2n}$ with $s = b/a$ and any $r > 2n/s$.

The range of p's for which $m_{a,b}$ is a bounded bilinear multiplier on $L^p(\mathbb{R}^{2n})$ can be completely described by the equation $|\frac{1}{p} - \frac{1}{2}| \le \frac{b/a}{2n}$ (see Hirschman [12, comments after Theorem 3c], Wainger [22, Part II], and Miyachi [16, Theorem 3]); similar examples of multipliers of limited boundedness are contained in Miyachi and Tomita [17, Section 7].

As a consequence of Theorem 1.1 we obtain that the bilinear multiplier operator associated with $m_{a,b}$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ in the following cases:

(i) when $n \geq b/a > n/2$ and

$$\frac{1}{p_1} < \frac{b}{an}, \frac{1}{p_2} < \frac{b}{an}, 1 - \frac{b}{an} < \frac{1}{p} < \frac{b}{an} + \frac{1}{2}.$$

(ii) when $3n/2 \geq b/a > n$ and

$$\frac{1}{p} < \frac{b}{an} + \frac{1}{2};$$

(iii) when $b/a > 3n/2$ in the entire range of exponents $1 < p_1, p_2 \leq \infty$, $\frac{1}{2} < p < \infty$.

The boundedness of this specific bilinear multiplier is unknown to us outside the above range of indices.

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Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: grafakosl@missouri.edu

Department of Mathematics, The University of Alabama, Tuscaloosa, AL 35487, USA
E-mail address: hvnguyen@ua.edu