A SHARP VARIANT OF THE MARCINKIEWICZ THEOREM WITH MULTIPLIERS IN SOBOLEV SPACES OF LORENTZ TYPE

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ABSTRACT. Given a bounded measurable function σ on $\mathbb{R}^n$, we let T_σ be the operator obtained by multiplication on the Fourier transform by σ. Let $0 < s_1 \leq s_2 \leq \cdots \leq s_n < 1$ and ψ be a Schwartz function on the real line whose Fourier transform $\widehat{\psi}$ is supported in $[-2, -1/2] \cup [1/2, 2]$ and which satisfies $\sum_{j \in \mathbb{Z}} \widehat{\psi}\left(2^{-j}\xi\right) = 1$ for all $\xi \neq 0$. In this work we provide a sharp form of the Marcinkiewicz multiplier theorem on L^p by finding an almost optimal function space with the property that, if the function

$$(\xi_1, \ldots, \xi_n) \mapsto \prod_{i=1}^{n}(I - \partial_i^2)^{\frac{s_i}{2}}\Big[\prod_{i=1}^{n}\widehat{\psi}(\xi_i)\sigma(2^{j_1}\xi_1, \ldots, 2^{j_n}\xi_n)\Big]$$

belongs to it uniformly in $j_1, \ldots, j_n \in \mathbb{Z}$, then T_σ is bounded on $L^p(\mathbb{R}^n)$ when $|\frac{1}{p} - \frac{1}{2}| < s_1$ and $1 < p < \infty$. In the case where $s_i \neq s_{i+1}$ for all i, it was proved in [13] that the Lorentz space $L^{\frac{1}{s_1}, 1}(\mathbb{R}^n)$ is the function space sought. Here we address the significantly more difficult general case when for certain indices i we might have $s_i = s_{i+1}$. We obtain a version of the Marcinkiewicz multiplier theorem in which the space $L^{\frac{1}{s_1}, 1}$ is replaced by an appropriate Lorentz space associated with a certain concave function related to the number of terms among $s_2, \ldots, s_n$ that equal s_1. Our result is optimal up to an arbitrarily small power of the logarithm in the defining concave function of the Lorentz space.

1. INTRODUCTION

Let $\mathscr{C}_0^\infty(\mathbb{R}^n)$ be the space of smooth functions with compact support on $\mathbb{R}^n$. Given any function σ in $L^\infty(\mathbb{R}^n)$, we consider the multiplier operator T_σ defined for all $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ by

$$T_\sigma f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)\sigma(\xi)e^{2\pi i x \cdot \xi}d\xi, \quad x \in \mathbb{R}^n.$$

As usual, here and in the sequel, $\widehat{f}$ denotes the Fourier transform of f given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x \cdot \xi}\,dx, \quad \xi \in \mathbb{R}^n.$$

The theory of multipliers is vast and extensive but basic material about them can be found in [18], [11] and [23].

A classical problem in harmonic analysis is to find good sufficient conditions on functions σ guaranteeing that T_σ extends to a bounded operator on $L^p(\mathbb{R}^n)$ for some

$1 < p < \infty$. If this is the case, then σ is called an L^p Fourier multiplier. This problem has a long history going back to Bernstein, Hardy, Weyl, Marcinkiewicz, Mikhlin and was studied in the sixties by several mathematicians including Calderón [4], Hirschman [17], Hörmander [18], de Leeuw [24], Carleson and Sjölin [8].

The significance of the multiplier problem lies in the fact that many classical L^p boundedness problems in analysis can be described in terms of Fourier multipliers. Several conditions on σ are known to imply boundedness for T_σ on $L^p(\mathbb{R}^n)$. We are not going into a complete historical overview of multiplier theory, but we focus on versions of the Marcinkiewicz multiplier theorem. We start with the classical result of Marcinkiewicz [26], first proved in the context of two-dimensional Fourier series, which basically says (in n dimensions) that if for all $\alpha_j \in \{0, 1\}$

$$(1.1) \qquad \left| \partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_n}^{\alpha_n} m(\xi) \right| \le C_\alpha |\xi_1|^{-\alpha_1} \cdots |\xi_n|^{-\alpha_n}, \qquad \xi_j \ne 0 \text{ when } \alpha_j = 1,$$

then T_m is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$.

In order to fine-tune this theorem we discuss a version of it where the derivatives α_j could be fractional. To describe this we introduce a Schwartz function ψ on $\mathbb{R}$ whose Fourier transform is supported in $[-2, -1/2] \cup [1/2, 2]$ and which satisfies $\sum_{j \in \mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1$ for all $\xi \ne 0$. We then define a function Ψ on $\mathbb{R}^n$ such that

$$(1.2) \qquad \widehat{\Psi} = \underbrace{\widehat{\psi} \otimes \cdots \otimes \widehat{\psi}}_{n \text{ times}} .$$

Here,

$$(g_1 \otimes \cdots \otimes g_n)(x_1, \ldots, x_n) := g_1(x_1) \cdots g_n(x_n), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n ,$$

stands for the tensor product of functions $g_j \colon \mathbb{R} \to \mathbb{C}$, $1 \le j \le n$. We use the following notation for the differential operator

$$\Gamma(s_1, \ldots, s_n) := (I - \partial_1^2)^{s_1/2} \cdots (I - \partial_n^2)^{s_n/2},$$

where ∂_j denotes differentiation in the jth variable. We also introduce the multi-dilation operator

$$D_{j_1, \ldots, j_n} g(\xi_1, \ldots, \xi_n) := g(2^{j_1}\xi_1, \ldots, 2^{j_n}\xi_n), \quad (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n ,$$

where g is a function on $\mathbb{R}^n$ and $j_1, \ldots, j_n \in \mathbb{Z}$.

When $0 < 1/r < s_1 \le \cdots \le s_n < 1$, it was shown in [15] that if

$$(1.3) \qquad \sup_{j_1, \ldots, j_n \in \mathbb{Z}} \left\| \Gamma(s_1, \ldots, s_n) \left[\widehat{\Psi} D_{j_1, \ldots, j_n} \sigma \right] \right\|_{L^r(\mathbb{R}^n)} < \infty ,$$

then T_σ maps $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $|\frac{1}{p} - \frac{1}{2}| < s_1$. Earlier versions of this result were provided by Carbery [5], who considered the case in which the multiplier lies in a product-type L^2-based Sobolev space, and Carbery and Seeger [6, Remark after Prop. 6.1], who considered the case $s_1 = \cdots = s_n > |\frac{1}{p} - \frac{1}{2}| = \frac{1}{r}$. The positive direction of Carbery and Seeger's result in the range $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{r}$ also appeared in [7, Condition (1.4)]; note that in these cases the range is expressed in terms of the integrability of the multiplier and not in terms of its smoothness. An alternative L^p version of the Marcinkiewcz multiplier theorem was proved by Coifman, Rubio de Francia and Semmes [9]. For the endpoint cases, we mention the work of Kislyakov [21] and Tao and Wright [32]; related to the latter see Seeger and Tao [29]. An extension of the Marcinkiewicz multiplier theorem to general Banach spaces was obtained by Hytönen [19].

A weakening of the condition in (1.3) was provided in [13], where the L^r space was replaced by the locally larger Lorentz space $L^{1/s_1,1}(\mathbb{R}^n)$. But this was achieved under the additional hypothesis that $s_1 < s_2 < \cdots < s_n$; the case $n = 2$ was first proved in [12]. In this paper we deal with the more complicated case when a streak of s_j's could be identical. In this case, the Lorentz-space estimate from [13] fails (see Example 4.4 below for the proof of this assertion). Nevertheless, we show that a limiting version of the Marcinkiewicz multiplier theorem can still be obtained. We achieve this goal shrinking the original Lorentz space $L^{1/s_1,1}(\mathbb{R}^n)$ by inserting in the defining function a certain power of the logarithm. The class of function spaces that is suitable for solving this problem is described below. Given a concave function $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ that is positive on $(0, \infty)$ and satisfies $\varphi(0) = 0$, we define the *Lorentz space* $\Lambda_\varphi = \Lambda_\varphi(\mathbb{R}^n)$ to be the space of all Lebesgue measurable functions f on $\mathbb{R}^n$ for which

$$\|f\|_{\Lambda_\varphi} := \int_0^\infty f^*(t)\, d\varphi(t) < \infty\,,$$

where f^* denotes the non-increasing rearrangement of f with respect to Lebesgue measure. If $1 \le p < \infty$ and $\varphi(t) = t^{1/p}$ for all $t \ge 0$, then we recover the classical Lorentz space $L^{p,1}$. It is straightforward to verify that for two concave functions φ and ψ the continuous inclusion $\Lambda_\varphi \hookrightarrow \Lambda_\psi$ holds if and only if there exists a positive constant γ such that $\psi(t) \le \gamma \varphi(t)$ for all $t > 0$.

In what follows, for given $s \in (0, 1)$ and $\beta \in \mathbb{R}$, we consider a concave function $\phi_{s,\beta}$ such that

$$(1.4) \qquad\qquad \phi_{s,\beta}(t) \approx t^s \log^\beta\left(e + \tfrac{1}{t}\right), \quad t > 0\,.$$

In view of the aforementioned fact, for all $s \in (0, 1)$ and $\beta > 0$, one has $\Lambda_{\phi_{s,\beta}} \hookrightarrow L^{\frac{1}{s},1}$.

The main result of this paper is the following theorem.

Theorem 1.1. *Let $1 < p < \infty$ and Ψ be as in (1.2). Let $0 < s_1 \le s_2 \le \cdots \le s_n < 1$ and assume that there are exactly d numbers among $s_2, \ldots, s_n$ that equal s_1. In addition, assume that $s_1 > |1/p - 1/2|$. If a function $\sigma \in L^\infty(\mathbb{R}^n)$ satisfies*

$$(1.5) \qquad K := \sup_{j_1,\ldots,j_n \in \mathbb{Z}} \left\| \Gamma\left(s_1, \ldots, s_n\right) \left[\widehat{\Psi}\, D_{j_1,\ldots,j_n} \sigma \right] \right\|_{\Lambda_{\phi_{s_1,d}}(\mathbb{R}^n)} < \infty\,,$$

then there is constant $C = C(s_1, \ldots, s_n, p, n, d, \psi)$ such that, for every $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$, we have

$$(1.6) \qquad\qquad \|T_\sigma f\|_{L^p(\mathbb{R}^n)} \le CK \|f\|_{L^p(\mathbb{R}^n)}\,.$$

Thus, T_σ admits a bounded extension from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with the same bound.

Naturally, the theorem remains invariant under any permutation of the variables. It was only stated in the case where the index s_j corresponds to variable ξ_j for simplicity. We also point out that the power d of the logarithm in condition (1.5) can be slightly lowered if we allow it to depend on s_1; on this improvement see Section 7.

Throughout the paper we use standard notation. Given two nonnegative functions f and g defined on the same set A, we write $f \lesssim g$, if there is a constant $c > 0$ such that $f(x) \le cg(x)$ for all $x \in A$, while $f \approx g$ means that both $f \lesssim g$ and $g \lesssim f$ hold. If X and Y are Banach spaces, then $X \hookrightarrow Y$ means that $X \subset Y$ and the inclusion map is continuous. If X and Y are Banach spaces, then we write $X = Y$ if $X \hookrightarrow Y$ and $Y \hookrightarrow X$. The measure space of all Lebesgue's measurable subsets

of $\mathbb{R}^n$ equipped with Lebesgue measure λ_n is denoted by $(\mathbb{R}^n, \lambda_n)$. For simplicity of notation, λ denotes the Lebesgue measure restricted to Lebesgue's measurable subset of $\mathbb{R}_+ := [0, \infty)$. We use C to describe an inessential constant that may vary from occurrence to occurrence.

2. Background material

Let $(\Omega, \mu) := (\Omega, \Sigma, \mu)$ be a σ-finite measure space and let $L^0(\mu)$ denote the space of all (equivalence classes) of scalar valued (real or complex) Σ-measurable functions on (Ω, μ) (on Ω for short) that are finite μ-a.e. A Banach space $X \subset L^0(\mu)$ is said to be a Banach function space over Ω if for all $f, g \in L^0(\mu)$ with $|g| \leq |f|$ μ-a.e. and $f \in X$, one has $g \in X$ and $\|g\|_X \leq \|f\|_X$. The Köthe dual space X' of a Banach function space X on Ω is a Banach function space of those $f \in L^0(\mu)$ for which $\|f\|_{X'} := \sup \left\{ \int_\Omega |fg| \, d\mu : \|g\|_X \leq 1 \right\}$ is finite.

Given $f \in L^0(\mu)$, its *distribution function* is defined by $\mu_f(\tau) = \mu(\{x \in \Omega : |f(x)| > \tau\})$, $\tau > 0$, and its *nonincreasing rearrangement* by $f^*(t) = \inf\{\tau \geq 0 : \mu_f(\tau) \leq t\}$, $t \geq 0$. A Banach function space E is called a *rearrangement-invariant* (r.i.) space if $\|f\|_E = \|g\|_E$ whenever $\mu_f = \mu_g$ and $f \in E$.

Let E be an r.i. space on $\mathbb{R}_+$ and let (Ω, μ) be a measure space. Then we define the r.i. space $E(\Omega)$ on Ω to be the space of all $f \in L^0(\mu)$ such that $f^* \in E$ with $\|f\|_{E(\Omega)} = \|f^*\|_E$. Many properties of r.i. spaces can be expressed in terms of conditions on their Boyd indices. Recall that for any r.i. space E on $\mathbb{R}_+$, we define the dilation operators σ_s for $0 < s < \infty$ by

$$\sigma_s f(t) = f(t/s), \quad f \in E, \, t \geq 0 \, .$$

Since $s \mapsto \|\sigma_s\|_E = \sup_{\|f\|_E \leq 1} \|\sigma_s f\|_E$ is a finite submultiplicative function on $(0, \infty)$, the Boyd indices given by

$$\alpha_E := \lim_{s \to 0+} \frac{\log \|\sigma_s\|_E}{\log s}, \quad \beta_E := \lim_{s \to \infty} \frac{\log \|\sigma_s\|_E}{\log s}$$

are well defined and satisfy $0 \leq \alpha_E \leq \beta_E \leq 1$ (see [22, p. 99]).

In the theory of operators on r.i. spaces the Lorentz and the Marcinkiewicz space play a fundamental role. Let $\mathcal{P}$ be the set of functions $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ that are concave, positive on $(0, \infty)$ and $\varphi(0) = 0$.

Given $\varphi \in \mathcal{P}$, the *Lorentz space* Λ_φ on $\mathbb{R}_+$ consists of all $f \in L^0(\lambda)$ such that

$$\|f\|_{\Lambda_\varphi} := \int_0^\infty f^*(t) d\varphi(t) = \varphi(0+)f^*(0+) + \int_0^\infty f^*(t)\varphi'(t) \, dt \, ,$$

where φ' is the derivative of φ, which exists except at a countable set. We note that the functional $\|\cdot\|_{\Lambda_\varphi}$ induced by an increasing function $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ is a norm if and only if φ is concave and $\varphi(0) = 0$ [25]. We note that Λ_φ is a separable space if and only if $\varphi(0+) = 0$ and $\varphi(+\infty) := \lim_{t \to \infty} \varphi(t) = \infty$.

Let $\mathcal{Q}$ be the set of all quasi-concave functions $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$, that is, of all positive functions φ on $(0, \infty)$ such that $\varphi(s) \leq \max\{1, s/t\}\varphi(t)$ for all $s, t > 0$. Note that, for any $\varphi \in \mathcal{Q}$, the function φ_* given by $\varphi_*(t) := t/\varphi(t)$ for all $t > 0$ is also a quasi-concave function. We also note that for every quasi-concave function there exists a *concave majorant* defined by

$$\widetilde{\varphi}(t) = \inf_{s > 0} \left(1 + \frac{t}{s}\right) \varphi(s) \, ,$$

which satisfies $\varphi(t) \leq \widetilde{\varphi}(t) \leq 2\varphi(t)$ for all $t > 0$.

For each given $\varphi \in \mathcal{Q}$, the *Marcinkiewicz space* M_φ on $\mathbb{R}_+$ is the r.i. space of all $f \in L^0(\lambda)$ equipped with the norm

$$\|f\|_{M_\varphi} := \sup_{t>0} \varphi(t) f^{**}(t) \,,$$

where $f^{**}(t) := \frac{1}{t} \int_0^t f^*(s) \, ds$ for all $t > 0$.

We will consider the Lorentz space $\Lambda_\varphi(\mathbb{R}^n)$ and the Marcinkiewicz space $M_\varphi(\mathbb{R}^n)$ over the measure space $(\mathbb{R}^n, \lambda_n)$. We will use the Köthe duality between Lorentz and Marcinkiewicz spaces, which states that for any $\varphi \in \mathcal{P}$ with $\varphi(0+) = 0$, we have

$$\Lambda_\varphi(\mathbb{R}^n)' = M_{\varphi_*}(\mathbb{R}^n)$$

with equality of norms. As a consequence, we have the following variant of Hölder's inequality (see, e.g., [22, Theorem 5.2] or [2, Chapter 1, Theorem 2.4]):

$$(2.1) \qquad \int_{\mathbb{R}^n} |fg| \, d\lambda_n \leq \|f\|_{\Lambda_\varphi(\mathbb{R}^n)} \|g\|_{M_{\varphi_*}(\mathbb{R}^n)} \,.$$

In what follows, for simplicity of notation, we often write Λ_φ and M_φ for short instead of $\Lambda_\varphi(\mathbb{R}^n)$ and $M_\varphi(\mathbb{R}^n)$.

We will also consider a class $\mathcal{B}$ of all measurable functions $\psi \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that the function m_ψ is finite and measurable, where

$$m_\psi(t) := \sup_{s>0} \frac{\psi(st)}{\psi(s)}, \quad t > 0 \,.$$

The lower and the upper index of a function $\psi \in \mathcal{B}$ are defined by

$$\gamma_\psi = \lim_{t \to 0+} \frac{\log m_\psi(t)}{\log t}, \quad \delta_\psi = \lim_{t \to \infty} \frac{\log m_\psi(t)}{\log t} \,.$$

We have $-\infty < \gamma_\psi \leq \delta_\psi < \infty$ (see [22, Section 2, p. 53]). Note that $\varphi, \psi \in \mathcal{B}$ with $\varphi \approx \psi$ implies $\gamma_\varphi = \gamma_\psi$ and $\delta_\varphi = \delta_\psi$.

In the sequel we will use the following properties without any references:

(i) Every function $\psi \in \mathcal{B}$ with $0 < \gamma_\psi \leq \delta_\psi < 1$ is equivalent to its *concave majorant* (see [22, Corollary 2, p. 55]).

(ii) If $\varphi \in \mathcal{P}$ with $\gamma_\varphi > 0$, then it follows from [22, Lemma 2.1.4] that

$$(2.2) \qquad \varphi(t) \approx \int_0^t \frac{\varphi(s)}{s} \, ds, \quad t > 0 \,.$$

In particular this implies that,

$$\|f\|_{\Lambda_\varphi} \approx \int_0^\infty f^*(t) \frac{\varphi(t)}{t} \, dt, \quad f \in \Lambda_\varphi(\mathbb{R}^n) \,,$$

up to multiplicative constants depending only on φ.

(iii) If $\varphi \in \mathcal{Q}$ with $\delta_\varphi < 1$, then $\gamma_{\varphi_*} = 1 - \delta_\varphi > 0$ and so by applying (2.2), we conclude that

$$\|f\|_{M_\varphi} \approx \sup_{t>0} \varphi(t) f^*(t), \quad f \in M_\varphi(\mathbb{R}^n) \,.$$

(iv) If $\varphi \in \mathcal{P}$ with $0 < \gamma_\varphi \leq \delta_\varphi < 1$, then the Lorentz space $\Lambda_\varphi(\mathbb{R}^n)$ is separable (by $\varphi(0+) = 0$ and $\varphi(+\infty) = \infty$). In particular, it follows that the space $\mathscr{C}_0^\infty(\mathbb{R}^n)$ is

dense in $\Lambda_\varphi(\mathbb{R}^n)$.

Throughout the paper we consider two families of special concave functions associated with indices $s \in (0,1)$ and $\beta \in \mathbb{R}$. The function $\phi_{s,\beta}$ was defined in (1.4). In addition, we let $\omega_{s,\beta}$ be a concave function satisfying

$$\omega_{s,\beta}(t) \approx t^s \log^\beta(e+t), \quad t > 0.$$

Basic properties of the functions $\phi_{s,\beta}$ and $\omega_{s,\beta}$ are summarized in the following proposition.

Proposition 2.1. *Given $s > 0$, $\beta \in \mathbb{R}$, consider the functions ϕ and ω defined by $\phi(t) := t^s \log^\beta \left(e + \frac{1}{t}\right)$ and $\omega(t) := t^s \log^\beta(e+t)$ for all $t > 0$. Then $\phi, \omega \in \mathcal{B}$ with $\gamma_\phi = \delta_\phi = s$ and $\gamma_\omega = \delta_\omega = s$. If, in addition, $s \in (0,1)$, then ϕ and ω are equivalent to their concave majorant denoted by $\phi_{s,\beta}$ and $\omega_{s,\beta}$, respectively.*

Proof. We first focus on the case when $\beta \geq 0$. We observe that, for all $a, b \geq 0$ we have

$$\log(e + ab) < \log[(e+a)(e+b)] = \log(e+a) + \log(e+b)$$
$$\leq 2[\log(e+a)] \left[\log(e+b)\right].$$

This shows that, for $C = 1/\log^\beta(e+1)$, we have

$$C\phi(t) \leq m_\phi(t) = \sup_{r>0} \frac{\phi(rt)}{\phi(r)} \leq 2^\beta \phi(t), \quad t > 0,$$

and so $m_\phi \approx \phi$. Since ϕ is continuous, it follows that $\phi \in \mathcal{B}$ and we have

$$\gamma_\phi = \lim_{t\to 0+} \frac{\log m_\phi(t)}{\log t} = \lim_{t\to 0+} \frac{\log \phi(t)}{\log t} = s + \beta \lim_{t\to 0+} \frac{\log(\log(e + t^{-1}))}{\log t} = s$$

and

$$\delta_\phi = s + \beta \lim_{t\to\infty} \frac{\log(\log(e + t^{-1}))}{\log t} = s.$$

Similarly, we deduce that $\omega \in \mathcal{B}$ with $m_\omega \approx \omega$ and $\gamma_\omega = \delta_\omega = s$.

Notice that $m_{\phi_\beta}(t) = m_{\omega_{-\beta}}(t)$ and $m_{\omega_\beta}(t) = m_{\phi_{-\beta}}(t)$ for $t > 0$ and $\beta \in \mathbb{R}$, where we set $\phi_\beta := \phi$ and $\omega_\beta := \omega$. These equalities yield the conclusion for $\beta < 0$ using the case $-\beta > 0$. If $s \in (0,1)$, then the required statements about concave majorants follow from the preceding results combined with property (i). □

We will need the following lemma. For completeness we include a proof.

Lemma 2.2. *Let $\psi \in \mathcal{B}$ with $\delta_\psi < p$ for some $0 < p \leq 1$. Then there exists a constant $C = C(\psi, p) > 0$ such that, for any $h \in L^0(\lambda_n)$, we have*

$$\int_0^\infty h^{**}(t)^p \frac{\psi(t)}{t} dt \leq C \int_0^\infty h^*(t)^p \frac{\psi(t)}{t} dt.$$

Proof. Observe that for any $t > 0$, we have an obvious estimate

$$h^{**}(t) = \int_0^1 h^*(ts) ds = \sum_{k=1}^\infty \int_{2^{-k}}^{2^{-k+1}} h^*(ts) ds \leq \sum_{k=1}^\infty 2^{-k} h^*(t/2^k).$$

Combining with subadditivity of the function $t \mapsto t^p$, defined on $\mathbb{R}_+$, yields

$$\int_0^\infty h^{**}(t)^p \frac{\psi(t)}{t} dt \leq \sum_{k=1}^\infty 2^{-kp} \int_0^\infty h^*(t/2^k)^p \frac{\psi(t)}{t} dt$$

$$= \sum_{k=1}^{\infty} 2^{-kp} \int_0^{\infty} h^*(t)^p \frac{\psi(2^k t)}{t} \, dt$$

$$\leq C(p, \psi) \int_0^{\infty} h^*(t)^p \frac{\psi(t)}{t} \, dt \,,$$

where $C(p, \psi) := \sum_{k=1}^{\infty} 2^{-kp} m_{\psi}(2^k)$.

We complete the proof by showing that $C(p, \psi) < \infty$. To see this observe that by $\delta_{\psi} < p$, we can find $\varepsilon > 0$ so that $\alpha := p - \delta_{\psi} - \varepsilon > 0$. It follows from the definition of δ_{ψ} that there is an integer $k_0 = k_0(\varepsilon) > 0$ such that $m_{\psi}(2^k) \leq 2^{k(\delta_{\psi} + \varepsilon)}$ for each $k \geq k_0$ and hence

$$\sum_{k=k_0}^{\infty} 2^{-kp} m_{\psi}(2^k) \leq \sum_{k=k_0}^{\infty} 2^{-k\alpha} < \infty \,.$$

This concludes the proof. $\qquad\qquad\square$

We now prove the following result on boundedness of the Fourier transform between corresponding Lorentz spaces on $\mathbb{R}^n$. Before doing so, we point out that various variants of the Hausdorff-Young inequality in the setting of Lorentz spaces are available in the literature (see, e.g., [1], [28], [30]) but the result below appears to be new.

Lemma 2.3. *Let $\varphi \in \mathcal{P}$ such that $1/2 < \gamma_{\varphi} \leq \delta_{\varphi} < 1$. Then there exists a constant $C > 0$ such that, for any $f \in \Lambda_{\varphi}$, we have*

$$\|\widehat{f}\|_{\Lambda_{\psi}} \leq C\|f\|_{\Lambda_{\varphi}} \,,$$

where $\psi(t) := t\varphi(1/t)$ for all $t > 0$.

Proof. Since φ is concave, it is easy to check that ψ is a concave function on $(0, \infty)$. Clearly, $m_{\psi}(t) = t \, m_{\varphi}(1/t)$ for all $t > 0$ and so

$$\gamma_{\psi} = 1 - \delta_{\varphi} \quad \text{and} \quad \delta_{\psi} = 1 - \gamma_{\varphi} \,.$$

Hence it follows by assumption on indices of φ that $0 < \gamma_{\psi} \leq \delta_{\psi} < 1/2$.

We use the pointwise estimate for the Fourier transform due to Jodeit and Torchinsky [20, Theorem 4.6], which states that there exists a constant $D > 0$ such that, for any $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$, we have

$$\int_0^t (\widehat{f})^*(s)^2 \, ds \leq D \int_0^t \left(\int_0^{\frac{1}{s}} f^*(\tau) \, d\tau \right)^2 ds \,.$$

Combining this estimate with

$$(\widehat{f})^*(t) \leq \frac{1}{t^{1/2}} \left(\int_0^t ((\widehat{f})^*(s))^2 \, ds \right)^{\frac{1}{2}}, \quad t > 0 \,,$$

we obtain that for any simple function $f \in \Lambda_{\varphi}$,

$$\|\widehat{f}\|_{\Lambda_{\psi}} \leq D \int_0^{\infty} \left(\frac{1}{t} \int_0^t \left(\int_0^{\frac{1}{s}} f^*(\tau) \, d\tau \right)^2 ds \right)^{\frac{1}{2}} \frac{\psi(t)}{t} \, dt = \int_0^{\infty} g^{**}(t)^{\frac{1}{2}} \frac{\psi(t)}{t} \, dt \,,$$

where g is given by

$$g(t) := \left(\int_0^{\frac{1}{t}} f^*(\tau) \, d\tau \right)^2, \quad t > 0 \,.$$

Clearly, g is non-negative and nonincreasing and so it follows from Lemma 2.2 (by $\delta_\psi < 1/2$) that there exists $C > 0$ such that

$$\int_0^\infty g^{**}(t)^{\frac{1}{2}} \frac{\psi(t)}{t} \, dt \leq C \int_0^\infty g(t)^{\frac{1}{2}} \frac{\psi(t)}{t} \, dt \, .$$

In consequence, we obtain

$$\|\widehat{f}\|_{\Lambda_\psi} \lesssim \int_0^\infty \left(\int_0^{\frac{1}{t}} f^*(s) \, ds \right) \frac{\psi(t)}{t} \, dt = \int_0^\infty f^{**}(t^{-1}) \frac{\psi(t)}{t^2} \, dt$$

$$= \int_0^\infty f^{**}(t)\psi(t^{-1}) \, dt = \int_0^\infty f^{**}(t) \frac{\varphi(t)}{t} \, dt$$

$$\lesssim \int_0^\infty f^*(t) \frac{\varphi(t)}{t} \, dt \, ,$$

where the last estimate follows by Lemma 2.2 with $p = 1$ (by $\delta_\varphi < 1$).

Recall that $\gamma_\varphi > 0$ implies that

$$\|f\|_{\Lambda_\varphi} \approx \int_0^\infty f^*(t) \, \frac{\varphi(t)}{t} \, dt \, .$$

Thus the required estimate follows by density of simple functions in the Lorentz space Λ_φ. □

3. Preliminary results

In this section we prove various auxiliary results that will be crucial in the proof of Theorem 1.1. We start with an estimate for an integral.

Lemma 3.1. *If $k \geq 2$, $\alpha_1, \ldots, \alpha_k > 1$, $r_1, \ldots, r_k > 0$ are such that $(\alpha_1 - 1)/r_1 \leq (\alpha_2 - 1)/r_2 \leq \cdots \leq (\alpha_k - 1)/r_k$ and $a > 1$, then*

$$(3.1) \qquad \int \cdots \int_{\substack{u_1, \ldots, u_k \geq 1 \\ u_1^{r_1} \cdots u_k^{r_k} > a}} u_1^{-\alpha_1} \cdots u_k^{-\alpha_k} \, du_1 \cdots du_k \approx a^{\frac{1-\alpha_1}{r_1}} [\log(e+a)]^{d'} \, ,$$

up to multiplicative constants independent of a. Here, d' is the number of elements in $\{(\alpha_2 - 1)/r_2, (\alpha_3 - 1)/r_3, \ldots, (\alpha_k - 1)/r_k\}$ that are equal to $(\alpha_1 - 1)/r_1$.

Proof. To prove (3.1) we proceed by induction. First we verify the case $k = 2$. In this case the u_1 integral is over the region $u_1 \geq \max\{1, (au_2^{-r_2})^{1/r_1}\}$ and so evaluating the u_1 integral gives

$$(3.2) \qquad \iint_{\substack{u_1, u_2 \geq 1 \\ u_1^{r_1} u_2^{r_2} > a}} u_1^{-\alpha_1} u_2^{-\alpha_2} \, du_1 du_2 = C \int_{u_2=1}^\infty u_2^{-\alpha_2} \max\{1, (au_2^{-r_2})^{1/r_1}\}^{1-\alpha_1} \, du_2 \, .$$

If the maximum equals 1, then the integral is over the region $a^{1/r_2} \leq u_2 < \infty$ and the u_2 integration produces $Ca^{1/r_2(1-\alpha_2)} \leq Ca^{1/r_1(1-\alpha_1)}$. Thus, the corresponding part of (3.2) is bounded from above by the right-hand side of (3.1), and if $a \in (1,2)$ then one has the lower bound as well.

If the maximum equals $(au_2^{-r_2})^{1/r_1}$, then the integral is over the region $1 \leq u_2 \leq a^{1/r_2}$ and so the corresponding part of (3.2) becomes

$$\int_{u_2=1}^{a^{\frac{1}{r_2}}} u_2^{-\alpha_2} (au_2^{-r_2})^{(1-\alpha_1)/r_1} \, du_2 = a^{\frac{1-\alpha_1}{r_1}} \int_{u_2=1}^{a^{\frac{1}{r_2}}} u_2^{-\alpha_2 - r_2/r_1(1-\alpha_1)} \, du_2 \, .$$

Now if $(\alpha_2 - 1)/r_2 > (\alpha_1 - 1)/r_1$ then this is bounded from above by the right-hand side of (3.1) with $d' = 0$ and if $(\alpha_2 - 1)/r_2 = (\alpha_1 - 1)/r_1$ then the same estimate holds with $d' = 1$. In addition, one has the corresponding lower bound if $a > 2$. This concludes the proof of estimate (3.1) when $k = 2$.

Assume by induction that (3.1) holds for an integer $k - 1$ (in place of k). Then

$$\int \cdots \int_{\substack{u_1,\ldots,u_k \geq 1 \\ u_1^{r_1} \cdots u_k^{r_k} > a}} u_1^{-\alpha_1} \cdots u_k^{-\alpha_k} \, du_1 \cdots du_k = \int_{u_2=1}^{\infty} \cdots \int_{u_k=1}^{\infty} \frac{\int_{u_1=L}^{\infty} u_1^{-\alpha_1} \, du_1}{u_k^{\alpha_k} \cdots u_2^{\alpha_2}} \, du_k \cdots du_2 \,,$$

where $L = \max\{1, (au_2^{-r_2} \cdots u_k^{-r_k})^{1/r_1}\}$. As $\alpha_1 > 1$, the u_1 integral is convergent and the preceding expression equals

$$(3.3) \qquad c \int_{u_2=1}^{\infty} \cdots \int_{u_k=1}^{\infty} \max\left\{1, (au_2^{-r_2} \cdots u_k^{-r_k})^{\frac{1}{r_1}}\right\}^{1-\alpha_1} \frac{du_k}{u_k^{\alpha_k}} \cdots \frac{du_2}{u_2^{\alpha_2}} \,.$$

The part of the integral in (3.3) over the set where the maximum equals 1 is

$$(3.4) \qquad c \int \cdots \int_{\substack{u_2,\ldots,u_k \geq 1 \\ u_2^{r_2} \cdots u_k^{r_k} > a}} u_k^{-\alpha_k} \cdots u_2^{-\alpha_2} \, du_k \cdots du_2 \approx a^{\frac{1-\alpha_2}{r_2}} \log^{d''}(e+a) \,,$$

where the equivalence holds by the induction hypothesis and d'' is the number of elements in $\{(\alpha_3 - 1)/r_3, \ldots, (\alpha_k - 1)/r_k\}$ that are equal to $(\alpha_2 - 1)/r_2$. Note that if $(\alpha_1 - 1)/r_1 < (\alpha_2 - 1)/r_2$, then the expression on the right in (3.4) is bounded from above by

$$(3.5) \qquad\qquad\qquad Ca^{\frac{1-\alpha_1}{r_1}} \log^{d'}(e+a) \,.$$

Now if $(\alpha_1 - 1)/r_1 = (\alpha_2 - 1)/r_2$, then we have $d' = d'' + 1$ and then the expression on the right in (3.4) is also bounded by (3.5). In addition, we also have the corresponding lower bound in both cases within the range $a \in (1, 2)$. We now turn to the part of the integral in (3.3) over the set where the maximum equals $(au_2^{-r_2} \cdots u_k^{-r_k})^{\frac{1}{r_1}}$. It can be expressed as

$$(3.6) \qquad ca^{\frac{1-\alpha_1}{r_1}} \int \cdots \int_{\substack{u_2,\ldots,u_k \geq 1 \\ u_2^{r_2} \cdots u_k^{r_k} \leq a}} u_k^{\frac{r_k}{r_1}(\alpha_1-1)-\alpha_k} \cdots u_2^{\frac{r_2}{r_1}(\alpha_1-1)-\alpha_2} \, du_k \cdots du_2 \,.$$

First, we observe that we have the following upper bound for (3.6):

$$ca^{\frac{1-\alpha_1}{r_1}} \int_1^{a^{\frac{1}{r_2}}} \cdots \int_1^{a^{\frac{1}{r_k}}} u_k^{\frac{r_k}{r_1}(\alpha_1-1)-\alpha_k} \cdots u_2^{\frac{r_2}{r_1}(\alpha_1-1)-\alpha_2} \, du_k \cdots du_2$$

$$\leq Ca^{\frac{1-\alpha_1}{r_1}} \log^{d'}(e+a) \,,$$

where the logarithm appears exactly when $\frac{r_k}{r_1} = \frac{\alpha_k-1}{\alpha_1-1}$ (d' times) and the remaining integrals produce a constant. Conversely, if $a > 2$ then we have an analogous lower bound for (3.6) as well:

$$ca^{\frac{1-\alpha_1}{r_1}} \int_1^{a^{\frac{1}{(k-1)r_2}}} \cdots \int_1^{a^{\frac{1}{(k-1)r_k}}} u_k^{\frac{r_k}{r_1}(\alpha_1-1)-\alpha_k} \cdots u_2^{\frac{r_2}{r_1}(\alpha_1-1)-\alpha_2} \, du_k \cdots du_2$$

$$\approx Ca^{\frac{1-\alpha_1}{r_1}} \log^{d'}(e+a) \,.$$

The claim follows. $\square$

We denote by $\mathcal{M}$ the strong maximal operator defined at point as the supremum of the averages of a given function over all rectangles with sides parallel to the axes that contain the point. Then we define

$$\mathcal{M}_{L^q}(g)(x_1,\ldots,x_n) = \mathcal{M}(|g|^q)(x_1,\ldots,x_n)^{\frac{1}{q}},$$

a version of the strong maximal function with respect to an exponent $q \in (1,\infty)$.

Lemma 3.2. *Let $0 < 1/q < s_1 \le s_2 \le \cdots \le s_n < 1$. Suppose that exactly d of the numbers $s_2,\ldots,s_n$ are equal to s_1, where $1 \le d \le n-1$. Then for g in $L^1_{loc}(\mathbb{R}^n)$ with $\mathcal{M}_{L^q}(g)(0) = 1$ and $a > 0$ we have*

$$(3.7) \qquad \left| \left\{ y \in \mathbb{R}^n \setminus [-1,1]^n : \frac{|g(y)|}{\prod_{i=1}^n (1+|y_i|)^{s_i}} > a \right\} \right| \le C a^{-\frac{1}{s_1}} \log^d \left(e + \frac{1}{a} \right).$$

Proof. For $j_1,\ldots,j_n$ nonnegative integers define

$$R_{j_1,\ldots,j_n} = \left\{ (y_1,\ldots,y_n) \in \mathbb{R}^n : \begin{cases} 2^{j_i} < |y_i| \le 2^{j_i+1} & \text{if } j_i \ge 1 \\ |y_i| \le 1 & \text{if } j_i = 0, \end{cases} \quad 1 \le i \le n \right\}$$

and notice that the family of rectangles $R_{j_1,\ldots,j_n}$ is a tiling of $\mathbb{R}^n$ when $j_1,\ldots,j_n$ run over all nonnegative integers.

In the sequel we denote by y the vector $(y_1,\ldots,y_n)$. For $a > 0$ and $j_1,\ldots,j_n$ nonnegative integers, we have

$$\left| \{ y \in R_{j_1,\ldots,j_n} : |g(y)| > a \} \right| \le \frac{1}{a^q} \int_{R_{j_1,\ldots,j_n}} |g(y)|^q dy \le a^{-q} 2^{j_1+\cdots+j_n+2n}$$

since we are assuming that $\mathcal{M}_{L^q}(g)(0) = 1$. Thus, in view of the trivial estimate $|R_{j_1,\ldots,j_n}| \le 2^{j_1+\cdots+j_n+2n}$, we obtain

$$(3.8) \qquad \left| \{ y \in R_{j_1,\ldots,j_n} : |g(y)| > a \} \right| \le 2^{2n} 2^{j_1+\cdots+j_n} \min\{1, a^{-q}\}.$$

It follows from (3.8) that, for all $j_1,\ldots,j_n \ge 0$, we have

$$(3.9) \qquad \left| \left\{ y \in R_{j_1,\ldots,j_n} : \frac{|g(y)|}{(1+|y_1|)^{s_1} \cdots (1+|y_n|)^{s_n}} > a \right\} \right|$$
$$\le 2^{2n} 2^{j_1+\cdots+j_n} \min\left\{ 1, (a 2^{j_1 s_1 + \cdots j_n s_n})^{-q} \right\}.$$

We let $g_1 = g \chi_{\mathbb{R}^n \setminus R_{0,\ldots,0}}$. Using (3.9), we get that

$$\left| \left\{ y \in \mathbb{R}^n : \frac{|g_1(y)|}{(1+|y_1|)^{s_1} \cdots (1+|y_n|)^{s_n}} > a \right\} \right|$$
$$\le \sum_{\substack{j_1,\ldots,j_n=0 \\ j_1+\cdots+j_n>0}}^{\infty} \left| \left\{ y \in R_{j_1,\ldots,j_n} : \frac{|g(y)|}{(1+|y_1|)^{s_1} \cdots (1+|y_n|)^{s_n}} > a \right\} \right|$$
$$\le \sum_{\substack{j_1,\ldots,j_n=0 \\ j_1+\cdots+j_n>0}}^{\infty} 2^{j_1+\cdots+j_n+2n} \min\left\{ 1, a^{-q} (2^{j_1 s_1 + \cdots + j_n s_n})^{-q} \right\}$$
$$\le 2^{2n+s_1 q + \cdots + s_n q} \int\cdots\int_{[0,\infty)^n \setminus [0,1]^n} \min\{1, a^{-q} \prod_{\rho=1}^n \max\{1, t_\rho^{s_\rho}\}^{-q} \} \, dt_1 \cdots dt_n$$
$$(3.10) \qquad =: 2^{2n+s_1 q + \cdots + s_n q} I(a,n),$$

where the last inequality follows from the monotonicity of the integrand. Let S be the set of all $(t_1, \ldots t_n) \in [0, \infty)^n \setminus [0, 1]^n$ such that

$$(3.11) \qquad a \max\{1, t_1^{s_1}\} \cdots \max\{1, t_n^{s_n}\} \leq 1.$$

If S is nonempty, then we must have $a \leq 1$. Let us fix a two-set partition $I = \{i_1, \ldots, i_m\}$ and $J = \{j_1, \ldots, j_k\}$ of $\{1, 2, \ldots, n\}$. We split S as a union of sets $S_{I,J}$ (ranging over all such pairs of partitions) for which

$$(3.12) \qquad (t_1, \ldots, t_n) \in S_{I,J} \iff t_i \leq 1 \text{ for all } i \in I \text{ and } t_j > 1 \text{ for all } j \in J.$$

Then the n-dimensional measure $|S_{I,J}|$ of $S_{I,J}$ is at most the k-th dimensional measure of

$$S_{I,J}^k = \left\{ (t_{j_1}, \ldots, t_{j_k}) : \ t_{j_1}^{s_{j_1}} \cdots t_{j_k}^{s_{j_k}} \leq \frac{1}{a} \right\} \cap [1, \infty)^k,$$

as the vector of the remaining m coordinates is contained in the cube $[0, 1]^m$ which has m-th dimensional measure equal to 1. Let us assume, without loss of generality, that $s_{j_1} \leq s_{j_2} \leq \cdots \leq s_{j_k}$ (i.e., $j_1 < j_2 < \cdots < j_k$).

We make the following observation: if $(t_{j_1}, \ldots, t_{j_k}) \in S_{I,J}^k$, then

$$1 \leq t_{j_i} \leq a^{-\frac{1}{s_1}}, \quad 1 \leq i \leq k.$$

Indeed, as all $t_{j_i} \geq 1$, we have $1 \leq t_{j_i}^{s_{j_i}} \leq t_{j_1}^{s_{j_1}} \cdots t_{j_k}^{s_{j_k}} \leq a^{-1}$, which implies that $1 \leq t_{j_i} \leq a^{-1/s_{j_i}} \leq a^{-1/s_1}$. Thus we conclude that

$$|S_{I,J}^k| \leq \int_{t_{j_2}=1}^{a^{-\frac{1}{s_1}}} \cdots \int_{t_{j_k}=1}^{a^{-\frac{1}{s_1}}} \left| \left\{ t_{j_1} : \ 1 \leq t_{j_1} \leq a^{-\frac{1}{s_{j_1}}} t_{j_2}^{-\frac{s_{j_2}}{s_{j_1}}} \cdots t_{j_k}^{-\frac{s_{j_k}}{s_{j_1}}} \right\} \right| dt_{j_k} \cdots dt_{j_2}$$

$$\leq a^{-\frac{1}{s_{j_1}}} \chi_{a \leq 1} \int_{t_{j_2}=1}^{a^{-\frac{1}{s_1}}} \cdots \int_{t_{j_k}=1}^{a^{-\frac{1}{s_1}}} t_{j_2}^{-\frac{s_{j_2}}{s_{j_1}}} \cdots t_{j_k}^{-\frac{s_{j_k}}{s_{j_1}}} dt_{j_k} \cdots dt_{j_2}$$

$$\leq C a^{-\frac{1}{s_{j_1}}} \chi_{a \leq 1} \log^{d'} \left(\frac{1}{a} \right)^{\frac{1}{s_1}} \leq C' a^{-\frac{1}{s_1}} \chi_{a \leq 1} \log^d \left(e + \frac{1}{a} \right),$$

where d' is the number of elements of the set $\{s_{j_2}, \ldots, s_{j_k}\}$ that are equal to s_{j_1}. The integrals associated with these variables produce a logarithm, while all other integrals are convergent on $[1, \infty)$. The last inequality holds independently of the relationship between d and d' if $s_{j_1} > s_1$, while if $s_{j_1} = s_1$ then it is satisfied since $d' \leq d$. Summing over all partitions (I, J) of $\{1, 2, \ldots, n\}$ yields the required estimate for $I(a, n)$, defined in (3.10), whenever (3.11) holds.

Now let S' be the set of all $(t_1, \ldots t_n) \in [0, \infty)^n \setminus [0, 1]^n$ such that

$$(3.13) \qquad a \max\{1, t_1^{s_1}\} \cdots \max\{1, t_n^{s_n}\} > 1.$$

Then S' is complementary to S in $[0, \infty)^n \setminus [0, 1]^n$. Writing S' as a union of sets $S'_{I,J}$ over all partitions (I, J) of $\{1, 2, \ldots, n\}$ as in (3.12), matters reduce to estimating the integral

$$(3.14) \qquad \frac{1}{a^q} \int \cdots \int_{\substack{t_{j_1}, \ldots, t_{j_k} \geq 1 \\ t_{j_1}^{s_{j_1}} \cdots t_{j_k}^{s_{j_k}} > \frac{1}{a}}} (t_{j_1}^{s_{j_1}} \cdots t_{j_k}^{s_{j_k}})^{-q} \, dt_{j_1} \cdots dt_{j_k}$$

for each subset $\{j_1, \ldots, j_k\}$ of $\{1, \ldots, n\}$. Now if $a > 1$, the integral in (3.14) is over the set $[1, \infty)^k$ and, as $s_{j_1} q > 1, \ldots, s_{j_k} q > 1$, the expression in (3.14) is bounded by

$$C a^{-q} \chi_{a > 1} \leq C a^{-\frac{1}{s_1}},$$

since $q > 1/s_1$. So we focus attention to the case $a \leq 1$ in (3.14). Let us again assume, without loss of generality, that $s_{j_1} \leq s_{j_2} \leq \cdots \leq s_{j_k}$. To estimate (3.14) we use Lemma 3.1. Inequality (3.1) in this lemma implies that, if d' is the number of terms in $\{s_{j_2}, \ldots, s_{j_k}\}$ that are equal to s_{j_1}, then (3.14) is bounded by

$$Ca^{-\frac{1}{s_{j_1}}} \log^{d'} \left(e + a^{-1}\right) \leq Ca^{-\frac{1}{s_1}} \log^d \left(e + a^{-1}\right),$$

where the last inequality is due to the fact that either $s_{j_1} = s_1$ and $d' \leq d$, or $s_{j_1} < s_1$ (as $a < 1$).

Summing over all partitions (I, J) of $\{1, 2, \ldots, n\}$ yields the required estimate for $I(a, n)$, defined in (3.10), whenever (3.13) holds. This completes the proof of (3.7). □

Corollary 3.3. *Let* $0 < 1/q < s_1 \leq s_2 \leq \cdots \leq s_n < 1$ *with exactly d numbers among* $s_2, \ldots, s_n$ *being equal to s_1. For g in $L^1_{loc}(\mathbb{R}^n)$ with $\mathcal{M}_{L^q}(g)(0) = 1$ we have*

$$(3.15) \qquad \left(\frac{|g(y_1, \ldots, y_n)|\chi_{\mathbb{R}^n \setminus [-1,1]^n}}{\prod_{i=1}^n (1 + |y_i|)^{s_i}}\right)^* (t) \leq \frac{C}{\omega_{s_1, -s_1 d}(t)}.$$

Proof. Note that the inverse function to

$$(0, \infty) \ni a \mapsto a^{-\frac{1}{s_1}} \log^d \left(e + \frac{1}{a}\right),$$

that appears in Lemma 3.2, is equivalent to

$$t \mapsto t^{-s_1} \log^{s_1 d}(e + t) = \left(\omega_{s_1, -s_1 d}(t)\right)^{-1}.$$

This proves (3.15). □

Proposition 3.4. *Assume that h is supported in the cube $[-1, 1]^n = Q_0$ and that $s_1 > 1/r > 1/q > 0$. Then*

$$(3.16) \qquad \|h\|_{M_{\omega_{s_1, -s_1 d}}} \lesssim \|h\|_{L^{r,\infty}} \lesssim \mathcal{M}_{L^q}(h)(0).$$

Proof. We first notice that the function h is supported in a set of measure 2^n, and therefore $h^*(t) = 0$ if $t > 2^n$. Since the function $\omega_{s_1, -s_1 d}(t)/t$ is non-increasing, we have

$$\|h\|_{M_{\omega_{s_1, -s_1 d}}} = \sup_{t > 0} \frac{\omega_{s_1, -s_1 d}(t)}{t} \int_0^t h^*(s) \, ds = \sup_{t \in (0, 2^n)} \frac{\omega_{s_1, -s_1 d}(t)}{t} \int_0^t h^*(s) \, ds$$

$$\lesssim \sup_{t \in (0, 2^n)} \frac{t^{\frac{1}{r}}}{t} \int_0^t h^*(s) \, ds \lesssim \|h\|_{L^{r,\infty}}.$$

Notice that the first inequality above makes use of the fact that $\omega_{s_1, -s_1 d}(t) \lesssim t^{1/r}$ for $t \in (0, 2^n)$ as $1/r < s_1$. This proves the first inequality in (3.16). The second inequality in (3.16) follows from the natural embedding of $L^q([-1, 1]^n)$ in $L^{r,\infty}([-1, 1]^n)$, as $r < q$. □

Combining the results of Corollary 3.3 and Proposition 3.4 we obtain the following.

Corollary 3.5. *Let* $0 < 1/q < s_1 \leq s_2 \leq \cdots \leq s_n < 1$ *with exactly d numbers among* $s_2, \ldots, s_n$ *being equal to s_1. For g in $L^1_{loc}(\mathbb{R}^n)$ with $\mathcal{M}_{L^q}(g)(0) = 1$ we have*

$$(3.17) \qquad \left(\frac{g(y_1, \ldots, y_n)}{\prod_{i=1}^n (1 + |y_i|)^{s_i}}\right)^* (t) \leq \frac{C}{\omega_{s_1, -s_1 d}(t)}.$$

Consequently, for any $g \in L^1_{loc}(\mathbb{R}^n)$ and any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we have

$$(3.18) \qquad \left\| \frac{g(x_1 + 2^{-j_1} y_1, \ldots, x_n + 2^{-j_n} y_n)}{\prod_{i=1}^n (1 + |y_i|)^{s_i}} \right\|_{M_{\omega_{s_1}, -s_1 d}(dy_1 \cdots dy_n)} \leq C \mathcal{M}_{L^q}(g)(x),$$

for any $j_1, \ldots, j_n \in \mathbb{Z}$.

Proof. To prove (3.17) we split $g = g_0 + g_1$, where $g_0 = g\chi_{[-1,1]^n}$ and $g_1 = g\chi_{\mathbb{R}^n \setminus [-1,1]^n}$ and we apply Corollary 3.3 to g_1 and Proposition 3.4 to g_0. Now (3.17) applied to $g/\mathcal{M}_{L^q}(g)(0)$ yields (3.18) when $x = 0$ and $j_1 = \cdots = j_n = 0$. The general case of (3.18) can be obtained by a translation and a dilation. $\qquad \square$

4. A LIMITING CASE SOBOLEV EMBEDDING

The following embedding of a Lorentz-Sobolev space into the space of essentially bounded functions is an important ingredient for the proof of Theorem 1.1.

Proposition 4.1. *Let $0 < s_1 \leq s_2 \leq \cdots \leq s_n < 1$, where exactly d of the numbers $s_2, \ldots, s_n$ are equal to s_1. Then*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|\Gamma(s_1, \ldots, s_n)f\|_{\Lambda_{\phi_{s_1}, (1-s_1)d}(\mathbb{R}^n)}.$$

Proof. For a given function f, we denote

$$g = \Gamma(s_1, \ldots, s_n)f.$$

We write $f = \Gamma(-s_1, \ldots, -s_n)g = (G_{s_1} \otimes \cdots \otimes G_{s_n}) * g$, where G_s is the one-dimensional kernel of $(I - \partial^2)^{-s/2}$. We recall the estimates

$$G_s(x) \lesssim |x|^{s-1} \quad \text{as } x \to 0$$

$$G_s(x) \lesssim e^{-c|x|} \quad \text{as } |x| \to \infty.$$

The Hölder inequality (2.1) yields

$$\|f\|_{L^\infty(\mathbb{R}^n)} = \|(G_{s_1} \otimes \cdots \otimes G_{s_n}) * g\|_{L^\infty(\mathbb{R}^n)}$$

$$\leq \sup_{(\tilde{x}_1, \ldots, \tilde{x}_n) \in \mathbb{R}^n} \int_{\mathbb{R}^n} G_{s_1}(x_1) \cdots G_{s_n}(x_n) |g(\tilde{x}_1 - x_1, \ldots, \tilde{x}_n - x_n)| \, dx_1 \cdots dx_n$$

$$\leq \|G_{s_1} \otimes \cdots \otimes G_{s_n}\|_{M_{\phi_{1-s_1}, d(s_1-1)}(\mathbb{R}^n)} \|g\|_{\Lambda_{\phi_{s_1}, (1-s_1)d}(\mathbb{R}^n)},$$

as for $t > 0$ we have

$$\frac{t}{\phi_{s_1, (1-s_1)d}(t)} = t^{1-s_1} \log^{d(s_1-1)}(e + \tfrac{1}{t}) = \phi_{1-s_1, d(s_1-1)}(t).$$

It remains to verify that

$$(4.1) \qquad G_{s_1} \otimes \cdots \otimes G_{s_n} \in M_{\phi_{1-s_1}, d(s_1-1)}(\mathbb{R}^n).$$

Given a subset I of $\{1, 2, \ldots, n\}$, we set $J = \{1, 2, \ldots, n\} \setminus I$ and write $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_{n-k})$ (there is a slight abuse of notation as one of the sets may be empty). Now observe that

$$|\{(x_1, \ldots, x_n) \in \mathbb{R}^n : G_{s_1}(x_1) \ldots G_{s_n}(x_n) > \lambda\}|$$

$$\leq \sum_{I \subseteq \{1,2,\ldots,n\}} |\{(x_{i_1}, \ldots, x_{i_k}) \in (-1,1)^k, \ (x_{j_1}, \ldots, x_{j_{n-k}}) \in (\mathbb{R} \setminus (-1,1))^{n-k} :$$

$$|x_{i_1}|^{s_{i_1}-1} \cdots |x_{i_k}|^{s_{i_k}-1} e^{-c(|x_{j_1}| + \cdots + |x_{j_{n-k}}|)} > \lambda\}|$$

$$\lesssim \sum_{I \subseteq \{1,2,\ldots,n\}} |\{(x_{i_1}, \ldots, x_{i_k}) \in (0,1)^k, \ (x_{j_1}, \ldots, x_{j_{n-k}}) \in (1,\infty)^{n-k} :$$

$$x_{i_1}^{s_{i_1}-1} \cdots x_{i_k}^{s_{i_k}-1} e^{-c(x_{j_1}+\cdots+x_{j_{n-k}})} > \lambda\}|.$$

We denote

$$S_{I,\lambda} = \{(x_{i_1}, \ldots, x_{i_k}) \in (0,1)^k, \ (x_{j_1}, \ldots, x_{j_{n-k}}) \in (1,\infty)^{n-k} :$$

$$x_{i_1}^{s_{i_1}-1} \cdots x_{i_k}^{s_{i_k}-1} e^{-c(x_{j_1}+\cdots+x_{j_{n-k}})} > \lambda\}.$$

We want to estimate $|S_{I,\lambda}|$. To this end, we fix $I \subseteq \{1,2,\ldots,n\}$ and $(x_{j_1}, \ldots, x_{j_{n-k}}) \in (1,\infty)^{n-k}$. Further, for a fixed $\lambda > 0$ we set $a = \lambda e^{c(x_{j_1}+\cdots+x_{j_{n-k}})}$. If $a > 1$ then we estimate

(4.2)
$$|\{(x_{i_1}, \ldots, x_{i_k}) \in (0,1)^k : \ x_{i_1}^{s_{i_1}-1} \cdots x_{i_k}^{s_{i_k}-1} > a\}|$$

$$= \int \cdots \int_{\substack{x_{i_1},\ldots,x_{i_k} \in (0,1) \\ x_{i_1}^{s_{i_1}-1} \cdots x_{i_k}^{s_{i_k}-1} > a}} dx_{i_1} \cdots dx_{i_k} = \int \cdots \int_{\substack{u_1,\ldots,u_k > 1 \\ u_1^{1-s_{i_1}} \cdots u_k^{1-s_{i_k}} > a}} u_1^{-2} \ldots u_k^{-2} \, du_1 \cdots du_k$$

$$\lesssim a^{-\frac{1}{1-s_{i_1}}} \log^{d'}(e+a),$$

where d' is the number of elements from the set $\{s_{i_2}, \ldots, s_{i_k}\}$ that are equal to s_{i_1}. We recall that the last inequality follows from Lemma 3.1. Notice that the estimate (4.2) is true also if $a \leq 1$ as the measure of the set on the left-hand side is at most 1, which is trivially bounded by the right-hand side. We also observe that

$$a^{-\frac{1}{1-s_{i_1}}} \log^{d'}(e+a) \lesssim \lambda^{-\frac{1}{1-s_{i_1}}} \log^{d'}(e+\lambda) e^{-\frac{c}{1-s_{i_1}}(x_{j_1}+\cdots+x_{j_{n-k}})}(x_{j_1}+\cdots+x_{j_{n-k}})^{d'}$$

$$\lesssim \lambda^{-\frac{1}{1-s_{i_1}}} \log^{d'}(e+\lambda) e^{-c'(x_{j_1}+\cdots+x_{j_{n-k}})},$$

where $c' < \frac{c}{1-s_{i_1}}$.

Thus, if $\lambda > 1$ then we have

$$|S_{I,\lambda}| \lesssim \int \cdots \int_{x_{j_1},\ldots,x_{j_{n-k}} > 1} \lambda^{-\frac{1}{1-s_{i_1}}} \log^{d'}(e+\lambda) e^{-c'(x_{j_1}+\cdots+x_{j_{n-k}})} \, dx_{j_1} \ldots dx_{j_{n-k}}$$

$$\lesssim \lambda^{-\frac{1}{1-s_{i_1}}} \log^{d'}(e+\lambda) \lesssim \lambda^{-\frac{1}{1-s_1}} \log^{d}(e+\lambda).$$

On the other hand, if $\lambda \leq 1$ then

(4.3)
$$|\{(x_{i_1}, \ldots, x_{i_k}) \in (0,1)^k : \ x_{i_1}^{s_{i_1}-1} \ldots x_{i_k}^{s_{i_k}-1} e^{-c(x_{j_1}+\cdots+x_{j_{n-k}})} > \lambda\}|$$

$$\lesssim \min\{1, \lambda^{-\frac{1}{1-s_{i_1}}} \log^{d'}(e+\lambda) e^{-c'(x_{j_1}+\cdots+x_{j_{n-k}})}\}.$$

If the minimum is equal to 1 then $e^{c'(x_{j_1}+\cdots+x_{j_{n-k}})} \leq \lambda^{-\frac{1}{1-s_{i_1}}} \log^{d'}(e+\lambda)$, and so $x_{j_1} + \cdots + x_{j_{n-k}} \lesssim \log(e\lambda^{-1})$. Then the measure of the corresponding part of the set $S_{I,\lambda}$ is bounded by constant times

$$\log^{n-k}(e\lambda^{-1}) \lesssim \lambda^{-\frac{1}{1-s_1}} \log^{d}(e+\lambda).$$

Finally, if the minimum in (4.3) is equal to $\lambda^{-\frac{1}{1-s_{i_1}}} \log^{d'}(e+\lambda) e^{-c'(x_{j_1}+\cdots+x_{j_{n-k}})}$, then

$$x_{j_1} + \cdots + x_{j_{n-k}} \geq \frac{1}{c'} \log(\lambda^{-\frac{1}{1-s_{i_1}}} \log^{d'}(e+\lambda),$$

and the measure of the corresponding part of the set $S_{I,\lambda}$ is bounded by constant times

$$\int \cdots \int_{\substack{x_{j_1},\ldots,x_{j_{n-k}}>1 \\ x_{j_1}+\cdots+x_{j_{n-k}} \geq \frac{1}{c'}\log(\lambda^{-\frac{1}{1-s_{i_1}}}\log^{d'}(e+\lambda))}} \lambda^{-\frac{1}{1-s_{i_1}}}\log^{d'}(e+\lambda)e^{-c'(x_{j_1}+\cdots+x_{j_{n-k}})}\,dx_{j_1}\ldots dx_{j_{n-k}}$$

$$\lesssim \int_{\frac{1}{c'}\log(\lambda^{-\frac{1}{1-s_{i_1}}}\log^{d'}(e+\lambda))}^{\infty} \lambda^{-\frac{1}{1-s_{i_1}}}\log^{d'}(e+\lambda)e^{-c'r}r^{n-k-1}\,dr$$

$$\lesssim \lambda^{-\frac{1}{1-s_{i_1}}}\log^{d'}(e+\lambda)\int_{\frac{1}{c'}\log(\lambda^{-\frac{1}{1-s_{i_1}}}\log^{d'}(e+\lambda))}^{\infty} e^{-c''r}\,dr$$

$$\lesssim \lambda^{-(1-\frac{c''}{c'})\frac{1}{1-s_{i_1}}}\log^{(1-\frac{c''}{c'})d'}(e+\lambda) \lesssim \lambda^{-\frac{1}{1-s_1}}\log^{d}(e+\lambda)\,.$$

The last inequality holds since c'' can be chosen to be any number less than c', and thus $1 - c''/c'$ can be arbitrarily small.

Altogether, we proved

$$|\{(x_1,\ldots,x_n) \in \mathbb{R}^n:\ G_{s_1}(x_1)\cdots G_{s_n}(x_n) > \lambda\}| \lesssim \lambda^{-\frac{1}{1-s_1}}\log^{d}(e+\lambda)\,.$$

This yields

$$(G_{s_1}\otimes\cdots\otimes G_{s_n})^*(t) \lesssim t^{s_1-1}(\log(e+1/t))^{d(1-s_1)},\quad t>0\,,$$

which in turn implies (4.1). $\qquad\square$

Next we show that the previous result is sharp, in the sense that the space $\Lambda_{\phi_{s_1,(1-s_1)d}}$ is locally the largest rearrangement-invariant space for which Proposition 4.1 holds.

Proposition 4.2. *Let $0 < s_1 \leq s_2 \leq \cdots \leq s_n < 1$, where exactly d of the numbers $s_2,\ldots,s_n$ are equal to s_1. Assume that E is a rearrangement-invariant space such that*

$$(4.4) \qquad \|f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|\Gamma(s_1,\ldots,s_n)f\|_{E(\mathbb{R}^n)}.$$

Then $E(\Omega) \hookrightarrow \Lambda_{\phi_{s_1,(1-s_1)d}}(\Omega)$ for all sets $\Omega \subseteq \mathbb{R}^n$ of finite measure.

Proof. To prove this claim, we set $g = \Gamma(s_1,\ldots,s_n)f$ and rewrite inequality (4.4) as

$$(4.5) \qquad \|(G_{s_1}\otimes\cdots\otimes G_{s_n})*g\|_{L^\infty(\mathbb{R}^n)} \lesssim \|g\|_{E(\mathbb{R}^n)}\,,$$

where G_s is the one-dimensional kernel of $(I-\partial^2)^{-s/2}$. For a given $(\tilde{x}_1,\ldots,\tilde{x}_n) \in \mathbb{R}^n$, we have

$$(4.6) \qquad \sup_{\|g\|_{E(\mathbb{R}^n)}\leq 1} |(G_{s_1}\otimes\cdots\otimes G_{s_n})*g(\tilde{x}_1,\ldots,\tilde{x}_n)|$$

$$= \sup_{\|g\|_{E(\mathbb{R}^n)}\leq 1} \left|\int_{\mathbb{R}^n} G_{s_1}(x_1)\cdots G_{s_n}(x_n)g(\tilde{x}_1-x_1,\ldots,\tilde{x}_n-x_n)\,dx_1\ldots dx_n\right|$$

$$= \|G_{s_1}\otimes\cdots\otimes G_{s_n}\|_{E'(\mathbb{R}^n)}\,,$$

where E' is the Köthe dual space of E. Thus, (4.5) implies that the function $G_{s_1}\otimes\cdots\otimes G_{s_n}$ belongs to $E'(\mathbb{R}^n)$. We next find a lower bound for the distribution function of $G_{s_1}\otimes\cdots\otimes G_{s_n}$. Since $G_{s_i}(x_i) \approx x_i^{s_i-1}$ if $0 < x_i < 1$ and $i \in \{1,2,\ldots,n\}$, we obtain for $\lambda > 1$,

$$|\{(x_1,\ldots,x_n) \in \mathbb{R}^n:\ G_{s_1}(x_1)\cdots G_{s_n}(x_n) > \lambda\}|$$

$$\gtrsim |\{(x_1, \ldots, x_n) \in (0,1)^n : x_1^{s_1-1} \cdots x_n^{s_n-1} > \lambda\}|$$
$$\approx \lambda^{-\frac{1}{1-s_1}} \log^d(e + \lambda),$$

where d is the number of elements from the set $\{s_2, \ldots, s_n\}$ that are equal to s_1. Note that the last equivalence follows by the calculation in (4.2) and by Lemma 3.1. This shows that

$$(G_{s_1} \otimes \cdots \otimes G_{s_n})^*(t) \gtrsim t^{s_1-1} \log^{(1-s_1)d}\left(e + \frac{1}{t}\right), \quad t \in (0, t_0)$$

for some $t_0 > 0$. This shows that if

$$g(t) := t^{s_1-1} \log^{(1-s_1)d}\left(e + \frac{1}{t}\right), \quad t > 0,$$

then the function $g\chi_{(0,t_0)} \in E' := E'(0,\infty)$. To reach the conclusion we observe that the embedding $E(\Omega) \hookrightarrow \Lambda_{\phi_{s_1,(1-s_1)d}}(\Omega)$ is, by duality, equivalent to $M_{\phi_{1-s_1,(s_1-1)d}}(\Omega) \hookrightarrow E'(\Omega)$. Now, if f is a function satisfying $\|f\|_{M_{\phi_{1-s_1,(s_1-1)d}}(\Omega)} \leq 1$, then

$$f^*(t) \leq f^{**}(t) \lesssim g(t), \quad t \in (0, |\Omega|).$$

Hence

$$\|f\|_{E'(\Omega)} = \|f^*\chi_{(0,|\Omega|)}\|_{E'} \lesssim \|g\chi_{(0,|\Omega|)}\|_{E'}$$
$$\lesssim \|g\chi_{(0,t_0)}\|_{E'} + \|\chi_{(t_0,|\Omega|)}\|_{E'}$$
$$\lesssim \|g\chi_{(0,t_0)}\|_{E'} \leq C.$$

In this chain of inequalities we used the monotonicity and rearrangement-invariance of the norm in the space E', the fact that the interval $(t_0, |\Omega|)$ can be split into a finite number of intervals of length at most t_0 and that the constant function on the interval $(0, t_0)$ is bounded from above by a multiple of the function $t^{s_1-1} \log^{d(1-s_1)}(e + \frac{1}{t})$. This completes the proof. $\square$

Corollary 4.3. *Let $0 < s_1 \leq s_2 \leq \cdots \leq s_n < 1$, where exactly d of the numbers $s_2, \ldots, s_n$ are equal to s_1. Assume that E is a rearrangement-invariant space such that $0 < \alpha_E \leq \beta_E < 1$ and*

$$(4.7) \qquad \|T_\sigma f\|_{L^p(\mathbb{R}^n)} \lesssim \sup_{j_1,\ldots,j_n \in \mathbb{Z}} \left\|\Gamma(s_1, \ldots, s_n)\left[\widehat{\Psi}\, D_{j_1,\ldots,j_n}\sigma\right]\right\|_{E(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.$$

Then $E(\Omega) \hookrightarrow \Lambda_{\phi_{s_1,(1-s_1)d}}(\Omega)$ for all sets $\Omega \subseteq \mathbb{R}^n$ of finite Lebesgue measure.

Proof. Assume that Φ is a smooth function on $\mathbb{R}^n$ with compactly supported Fourier transform and $a_1, \ldots, a_n$ are fixed integers. We recall the estimates

$$(4.8) \qquad \|\Gamma(s_1, \ldots, s_n)[\widehat{\Phi}F]\|_{E(\mathbb{R}^n)} \lesssim \|\Gamma(s_1, \ldots, s_n)F\|_{E(\mathbb{R}^n)}$$

and

$$(4.9) \qquad \|\Gamma(s_1, \ldots, s_n)[D_{a_1,\ldots,a_n}F]\|_{E(\mathbb{R}^n)} \lesssim \|\Gamma(s_1, \ldots, s_n)F\|_{E(\mathbb{R}^n)}$$

for any function F on $\mathbb{R}^n$. To verify (4.8) and (4.9) we first observe that they hold in the special case when $E = L^q$, $1 < q < \infty$. Then we choose $1 < q_1, q_2 < \infty$ such that $1/q_1 < \alpha_E \leq \beta_E < 1/q_2$, and the conclusion follows by interpolating between the L^{q_1} and L^{q_2} endpoints via Boyd's interpolation theorem [3, Theorem 1] (see also the beginning of Section 6 for the statement of this theorem).

Let us consider testing functions σ of the form

$$(4.10) \qquad \sigma = [(G_{s_1} \otimes \cdots \otimes G_{s_n}) * g]\widehat{\eta},$$

where η is a smooth function on $\mathbb{R}^n$ satisfying $\widehat{\eta} = 1$ on the cube $[7/8, 9/8]^n$ and such that the support of $\widehat{\eta}$ is contained in $[3/4, 5/4]^n$. Taking into account the support properties of $\widehat{\Psi}$ we deduce that $\widehat{\Psi} D_{j_1, \dots, j_n} \sigma = 0$ unless $j_i \in \{-1, 0, 1\}$ for each $i = 1, 2, \dots, n$. Inequality (4.7) combined with the fact that $\|T_\sigma\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} \gtrsim \|\sigma\|_{L^\infty(\mathbb{R}^n)}$ yields

$$(4.11) \qquad \|\sigma\|_{L^\infty(\mathbb{R}^n)} \lesssim \sup_{j_1, \dots, j_n \in \{-1, 0, 1\}} \left\| \Gamma(s_1, \dots, s_n) \left[\widehat{\Psi} D_{j_1, \dots, j_n} \sigma \right] \right\|_{E(\mathbb{R}^n)}.$$

Using (4.8) and (4.9), this implies

$$\|\sigma\|_{L^\infty(\mathbb{R}^n)} \lesssim \| \Gamma(s_1, \dots, s_n) \sigma \|_{E(\mathbb{R}^n)}.$$

An application of (4.10) and (4.8) then gives

$$\| [(G_{s_1} \otimes \cdots \otimes G_{s_n}) * g] \widehat{\eta} \|_{L^\infty(\mathbb{R}^n)} \lesssim \|g\|_{E(\mathbb{R}^n)}.$$

Since $\widehat{\eta} = 1$ on $[7/8, 9/8]^n$, the proof of Proposition 4.2 applied with $(\tilde{x}_1, \dots, \tilde{x}_n) \in [7/8, 9/8]^n$ yields the conclusion. $\square$

Example 4.4. We apply Corollary 4.3 with the Lorentz space $E = \Lambda_{\phi_{s_1, \beta}}$, where $\beta \in \mathbb{R}$ (note that $\alpha_{\Lambda_{\phi_{s, \beta}}} = \beta_{\Lambda_{\phi_{s, \beta}}} = s_1$). Thus, a necessary condition for inequality

$$(4.12) \qquad \|T_\sigma f\|_{L^p(\mathbb{R}^n)} \lesssim \sup_{j_1, \dots, j_n \in \mathbb{Z}} \left\| \Gamma(s_1, \dots, s_n) \left[\widehat{\Psi} D_{j_1, \dots, j_n} \sigma \right] \right\|_{\Lambda_{\phi_{s_1, \beta}}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}$$

to be satisfied is the validity of embedding $\Lambda_{\phi_{s_1, \beta}}(\Omega) \hookrightarrow \Lambda_{\phi_{s_1, (1-s_1)d}}(\Omega)$ for all sets $\Omega \subseteq \mathbb{R}^n$ of finite measure. This is equivalent to the pointwise estimate $\phi_{s_1, (1-s_1)d}(t) \lesssim \phi_{s_1, \beta}(t)$ for t near 0 (see, e.g., [27, Theorem 10.3.8]), which in turn yields the explicit necessary condition $\beta \geq (1 - s_1)d$. In particular, $\beta = 0$ is not allowed unless $d = 0$, and estimate

$$\|T_\sigma f\|_{L^p(\mathbb{R}^n)} \lesssim \sup_{j_1, \dots, j_n \in \mathbb{Z}} \left\| \Gamma(s_1, \dots, s_n) \left[\widehat{\Psi} D_{j_1, \dots, j_n} \sigma \right] \right\|_{L^{\frac{1}{s_1}, 1}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}$$

thus fails whenever at least one of the indices $s_2, \dots, s_n$ equals s_1. On the other hand, if $s_1 \leq 1/2$ then we will prove that condition (4.12) is satisfied whenever $\beta > (1 - s_1)d$, see Section 7 below.

5. THE CORE OF THE PROOF

In this section we prove Theorem 1.1 in the special case when $1/2 < s_1 < 1$. The general case then follows by interpolation; the details can be found in Section 6. We point out that in fact we prove a slightly stronger variant of Theorem 1.1 in this particular case; namely, we replace the constant K in (1.5) by the smaller constant

$$\widetilde{K} = \sup_{j_1, \dots, j_n \in \mathbb{Z}} \left\| \Gamma(s_1, \dots, s_n) \left[\widehat{\Psi} D_{j_1, \dots, j_n} \sigma \right] \right\|_{\Lambda_{\phi_{s_1, s_1 d}}(\mathbb{R}^n)}.$$

Proof of Theorem 1.1 : *case* $1/2 < s_1 < 1$. Given a Schwartz function ψ as in the statement of the theorem, we define a new Schwartz function ψ_b (ψ big) on $\mathbb{R}$ as follows:

$$(5.1) \qquad \widehat{\psi_b}(\xi) = \widehat{\psi}(\xi/2) + \widehat{\psi}(\xi) + \widehat{\psi}(2\xi).$$

Then $\widehat{\psi_b}$ is supported in the annulus $1/4 < |\xi| < 4$ and $\widehat{\psi_b} = 1$ on the support of $\widehat{\psi}$. Recalling the definition of Ψ given in (1.2), we introduce a Schwartz function Ψ_b on $\mathbb{R}^n$ by setting

$$\widehat{\Psi_b} = \overbrace{\widehat{\psi_b} \otimes \cdots \otimes \widehat{\psi_b}}^{n \text{ times}} .$$

For $j \in \mathbb{Z}$ we can define the Littlewood-Paley operators corresponding to ψ and ψ_b in the kth variable as the operators whose action on a function f on $\mathbb{R}^n$ is as follows:

$$\Delta_j^{\psi,k}(f)(x_1, \ldots, x_n) = \int_{\mathbb{R}} f(\ldots, x_k - y, \ldots) \, 2^j \psi(2^j y) \, dy$$

and

$$\Delta_j^{\psi_b,k}(f)(x_1, \ldots, x_n) = \int_{\mathbb{R}} f(\ldots, x_k - y, \ldots) \, 2^j \psi_b(2^j y) \, dy.$$

Since $\widehat{\psi_b} = 1$ on the support of $\widehat{\psi}$, $\widehat{\Psi_b}(2^{-j_1}\xi_1, \ldots, 2^{-j_n}\xi_n) = 1$ on the support of $\widehat{\Psi}(2^{-j_1}\xi_1, \ldots, 2^{-j_n}\xi_n)$ for each $j_1, \ldots, j_n \in \mathbb{Z}$ and so

$$\Delta_{j_1}^{\psi,1} \cdots \Delta_{j_n}^{\psi,n} T_\sigma(f)(x_1, \ldots, x_n)$$

$$= \int_{\mathbb{R}^n} \widehat{f}(\xi_1, \ldots \xi_n) \widehat{\Psi}(2^{-j_1}\xi_1, \ldots, 2^{-j_n}\xi_n) \, \sigma(\xi_1, \ldots, \xi_n) e^{2\pi i(x_1\xi_1 + \cdots + x_n\xi_n)} d\xi_1 \cdots d\xi_n$$

$$= \int_{\mathbb{R}^n} \widehat{f}(\xi_1, \ldots \xi_n) \widehat{\Psi_b}(2^{-j_1}\xi_1, \ldots, 2^{-j_n}\xi_n) \, \widehat{\Psi}(2^{-j_1}\xi_1, \ldots, 2^{-j_n}\xi_n)$$

$$\sigma(\xi_1, \ldots, \xi_n) e^{2\pi i(x_1\xi_1 + \cdots + x_n\xi_n)} d\xi_1 \cdots d\xi_n$$

$$= \int_{\mathbb{R}^n} (\Delta_{j_1}^{\psi_b,1} \cdots \Delta_{j_n}^{\psi_b,n} f)^\wedge(\xi_1, \ldots, \xi_n) \widehat{\Psi}(2^{-j_1}\xi_1, \ldots, 2^{-j_n}\xi_n)$$

$$\sigma(\xi_1, \ldots, \xi_n) e^{2\pi i(x_1\xi_1 + \cdots + x_n\xi_n)} d\xi_1 \cdots d\xi_n$$

$$= \int_{\mathbb{R}^n} 2^{j_1 + \cdots + j_n} (\Delta_{j_1}^{\psi_b,1} \cdots \Delta_{j_n}^{\psi_b,n} f)^\wedge(2^{j_1}\xi_1', \ldots, 2^{j_n}\xi_n')$$

$$\widehat{\Psi}(\xi_1', \ldots, \xi_n') \, \sigma(2^{j_1}\xi_1', \ldots, 2^{j_n}\xi_2') e^{2\pi i(2^{j_1}x_1\xi_1' + \cdots + 2^{j_n}x_n\xi_n')} d\xi_1' \cdots d\xi_n'$$

$$= \int_{\mathbb{R}^n} (\Delta_{j_1}^{\psi_b,1} \cdots \Delta_{j_n}^{\psi_b,n} f)(2^{-j_1}y_1', \ldots, 2^{-j_n}y_n')$$

$$\left[\widehat{\Psi} D_{j_1, \ldots, j_n} \sigma\right]^\wedge(y_1' - 2^{j_1}x_1, \ldots, y_n' - 2^{j_n}x_n) dy_1' \cdots dy_n'$$

$$= \int_{\mathbb{R}^n} (\Delta_{j_1}^{\psi_b,1} \cdots \Delta_{j_n}^{\psi_b,n} f)(2^{-j_1}y_1 + x_1, \ldots, 2^{-j_n}y_n + x_n)$$

$$\left[\widehat{\Psi} D_{j_1, \ldots, j_n} \sigma\right]^\wedge(y_1, \ldots, y_n) dy_1 \cdots dy_n$$

$$= \int_{\mathbb{R}^n} \frac{(\Delta_{j_1}^{\psi_b,1} \cdots \Delta_{j_n}^{\psi_b,n} f)(2^{-j_1}y_1 + x_1, \ldots, 2^{-j_n}y_n + x_n)}{(1 + |y_1|)^{s_1} \cdots (1 + |y_n|)^{s_n}}$$

$$(1 + |y_1|)^{s_1} \cdots (1 + |y_n|)^{s_n} \left[\widehat{\Psi} D_{j_1, \ldots, j_n} \sigma\right]^\wedge(y_1, \ldots, y_n) \, dy_1 \cdots dy_n.$$

Applying Hölder's inequality in the Lorentz-Marcinkiewicz setting (2.1), we obtain that $|\Delta_{j_1}^{\psi,1} \cdots \Delta_{j_n}^{\psi,n} T_\sigma(f)(x_1,\ldots,x_n)|$ is bounded by

$$\left\| \frac{\left(\Delta_{j_1}^{\psi_b,1} \cdots \Delta_{j_n}^{\psi_b,n} f\right)(2^{-j_1}y_1 + x_1, \ldots, 2^{-j_n}y_n + x_n)}{(1+|y_1|)^{s_1} \cdots (1+|y_n|)^{s_n}} \right\|_{M_{\omega_{s_1,-s_1 d}}(\mathbb{R}^n, dy_1 \cdots dy_n)}$$

$$\cdot \left\| (1+|y_1|)^{s_1} \cdots (1+|y_n|)^{s_n} [\widehat{\Psi} D_{j_1,\ldots,j_n}\sigma]^{\widehat{}}(y_1,\ldots,y_n) \right\|_{\Lambda_{\omega_{1-s_1,s_1 d}}(\mathbb{R}^n)}.$$

The first term in this product is estimated by Corollary 3.5 as follows: Since we are assuming $1 > s_1 > 1/2$, there is a q such that $1 < 1/s_1 < q < 2$. Then for this q we get

$$\left\| \frac{\left(\Delta_{j_1}^{\psi_b,1} \cdots \Delta_{j_n}^{\psi_b,n} f\right)(2^{-j_1}y_1 + x_1, \ldots, 2^{-j_n}y_n + x_n)}{(1+|y_1|)^{s_1} \cdots (1+|y_n|)^{s_n}} \right\|_{M_{\omega_{s_1,-s_1 d}}(\mathbb{R}^n)}$$

$$\leq C\mathcal{M}_{L^q}\left(|\Delta_{j_1}^{\psi_b,1} \cdots \Delta_{j_n}^{\psi_b,n} f| \right)(x_1,\ldots,x_n).$$

We estimate the second term in the product using Proposition 2.1 and Lemma 2.3, i.e., the Hausdorff-Young inequality adapted to these Lorentz spaces. We obtain

$$\left\| \prod_{i=1}^{n}(1+|y_i|)^{s_i} [\widehat{\Psi} D_{j_1,\ldots,j_n}\sigma]^{\widehat{}}(y_1,\ldots,y_n) \right\|_{\Lambda_{\omega_{1-s_1,s_1 d}}(\mathbb{R}^n)}$$

$$\leq C \left\| \prod_{i=1}^{n}(1+|y_i|^2)^{\frac{s_i}{2}} [\widehat{\Psi} D_{j_1,\ldots,j_n}\sigma]^{\widehat{}}(y_1,\ldots,y_n) \right\|_{\Lambda_{\omega_{1-s_1,s_1 d}}(\mathbb{R}^n)}$$

$$\leq C \left\| \Gamma(s_1,\ldots,s_n) [\widehat{\Psi} D_{j_1,\ldots,j_n}\sigma] \right\|_{\Lambda_{\phi_{s_1,s_1 d}}(\mathbb{R}^n)} \leq C\widetilde{K}.$$

where we used the fact that $1 < 1/s_1 < 2$, which is a hypothesis of Lemma 2.3.

We have now obtained the pointwise estimate

$$(5.2) \qquad |\Delta_{j_1}^{\psi,1} \cdots \Delta_{j_n}^{\psi,n} T_\sigma(f)| \leq C\widetilde{K} \mathcal{M}_{L^q}\left(|\Delta_{j_1}^{\psi_b,1} \cdots \Delta_{j_n}^{\psi_b,n} f| \right).$$

Now let $p \geq 2$. Applying the product type Littlewood-Paley theorem, the Fefferman-Stein inequality, and estimate (5.2) we obtain

$$\|T_\sigma(f)\|_{L^p(\mathbb{R}^n)} \leq C \left\| \left(\sum_{j_1,\ldots,j_n \in \mathbb{Z}} \left| \Delta_{j_1}^{\psi,1} \cdots \Delta_{j_n}^{\psi,n} T_\sigma(f) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}$$

$$\leq C\widetilde{K} \left\| \left(\sum_{j_1,\ldots,j_n \in \mathbb{Z}} \left| \mathcal{M}_{L^q}\left(|\Delta_{j_1}^{\psi_b,1} \cdots \Delta_{j_n}^{\psi_b,n} f| \right) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}$$

$$\leq C\widetilde{K} \left\| \left(\sum_{j_1,\ldots,j_n \in \mathbb{Z}} \left(\mathcal{M}\left(|\Delta_{j_1}^{\psi_b,1} \cdots \Delta_{j_n}^{\psi_b,n} f|^q \right) \right)^{\frac{2}{q}} \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}$$

$$\leq C\widetilde{K} \left\| \left(\sum_{j_1,\ldots,j_n \in \mathbb{Z}} \left(\mathcal{M}\left(|\Delta_{j_1}^{\psi_b,1} \cdots \Delta_{j_n}^{\psi_b,n} f|^q \right) \right)^{\frac{2}{q}} \right)^{\frac{q}{2}} \right\|_{L^{\frac{p}{q}}(\mathbb{R}^n)}^{\frac{1}{q}}$$

$$\leq C' \widetilde{K} \left\| \left(\sum_{j_1,\ldots,j_n \in \mathbb{Z}} |\Delta_{j_1}^{\psi_b,1} \cdots \Delta_{j_n}^{\psi_b,n} f|^{q \cdot \frac{2}{q}} \right)^{\frac{q}{2}} \right\|_{L^{\frac{p}{q}}(\mathbb{R}^n)}^{\frac{1}{q}}$$

$$\leq C'' \widetilde{K} \left\| \left(\sum_{j_1,\ldots,j_n \in \mathbb{Z}} |\Delta_{j_1}^{\psi_b,1} \cdots \Delta_{j_n}^{\psi_b,n} f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}$$

$$\leq C''' \widetilde{K} \|f\|_{L^p(\mathbb{R}^n)} .$$

The case $1 < p < 2$ follows by duality. $\qquad\square$

6. Interpolation

In the previous section we have proved the main theorem under the extra assumption that $1/2 < s_1 < 1$. This estimate will be useful for p near 1 or near ∞ while for $p = 2$ we can use the trivial L^∞ estimate for the multiplier. The final conclusion will be a consequence of an interpolation result (Theorem 6.6) discussed in this section.

We start with a few lemmas. In the proof we will use Boyd's interpolation theorem (see [3, Theorem 1]) which states: If E is an r.i. space on $\mathbb{R}_+$ such that $1/p_1 < \alpha_E \leq \beta_E < 1/p_0$ for some $1 < p_0 < p_1 < \infty$, then the r.i. space $E(\Omega)$ on (Ω, μ) is interpolation between $L^{p_0}(\mu)$ and $L^{p_1}(\mu)$, i.e., $L^{p_0}(\mu) \cap L^{p_1}(\mu) \hookrightarrow E(\Omega) \hookrightarrow L^{p_0}(\mu) + L^{p_1}(\mu)$ and for any linear operator T on $L^{p_0}(\mu) + L^{p_1}(\mu)$ such that T is bounded on $L^{p_j}(\mu)$ for $j = 0$ and $j = 1$, it follows that T is a bounded operator on $E(\Omega)$.

Lemma 6.1. *Let Φ be a smooth function on $\mathbb{R}^n$ with compactly supported Fourier transform. Then, for any $0 < s, s_2 \ldots, s_n < 1$, $\gamma > 0$ and any function F on $\mathbb{R}^n$, we have*

$$\left\| \Gamma\left(s, s_2, \ldots, s_n\right) \left[\widehat{\Phi} \, F \right] \right\|_{\Lambda_{\phi_s,\gamma}(\mathbb{R}^n)} \leq \left\| \Gamma\left(s, s_2, \ldots, s_n\right) F \right\|_{\Lambda_{\phi_s,\gamma}(\mathbb{R}^n)} .$$

Lemma 6.2. *Let $0 < s < 1$ and $\gamma > 0$. Then, for any $t_1, \ldots, t_n \in \mathbb{R}$, we have*

$$\left\| \Gamma\left(it_1, \ldots, it_n\right) f \right\|_{\Lambda_{\phi_s,\gamma}(\mathbb{R}^n)} \leq C(p,n)(1 + |t_1|) \cdots (1 + |t_n|) \|f\|_{\Lambda_{\phi_s,\gamma}(\mathbb{R}^n)} .$$

Lemma 6.3. *Let $0 < s < 1$. Let m be a function satisfying (1.1). Then, for any $\gamma > 0$, we have*

$$\|T_m(f)\|_{\Lambda_{\phi_s,\gamma}(\mathbb{R}^n)} \leq C(p,n) \|f\|_{\Lambda_{\phi_s,\gamma}(\mathbb{R}^n)} .$$

All these lemmas can be proved in the following way.

Proof. It is easy to check that if $\varphi \in \mathcal{P}$ and Λ_φ is the Lorentz space on $\mathbb{R}_+$, then $\|\sigma_t\|_E = m_\varphi(t)$ for all $t > 0$. This implies that $\alpha_{\Lambda_\varphi} = \gamma_\varphi$ and $\beta_{\Lambda_\varphi} = \delta_\varphi$.

Now we choose p_0 and $p_1 \in (1, \infty)$ such that $1/p_1 < s < 1/p_0$ and let $E := \Lambda_{\phi_s,\gamma}$. Combining the above fact with Proposition 2.1, we conclude that

$$1/p_1 < \alpha_E = \beta_E = s < 1/p_0 .$$

Since the estimates hold for L^{p_0} and L^{p_1} in place of the Lorentz space, Boyd's interpolation theorem completes the proof. $\qquad\square$

We will use the following lemma (see [14, Lemma 2.1]).

Lemma 6.4. *Let $0 < p_0 \leq p_1 < \infty$ and define p via $1/p = (1 - \theta)/p_0 + \theta/p_1$, where $0 < \theta < 1$. Given $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ and $\varepsilon > 0$, there exist smooth functions h_j^ε,*

$j = 1, \ldots, N_\varepsilon$, *supported in cubes with disjoint interiors, and there exist nonzero complex constants c_j^ε such that the functions*

$$(6.1) \qquad f_z^\varepsilon = \sum_{j=1}^{N_\varepsilon} |c_j^\varepsilon|^{\frac{p}{p_0}(1-z) + \frac{p}{p_1} z} h_j^\varepsilon$$

satisfy

$$(6.2) \qquad \left\| f_\theta^\varepsilon - f \right\|_{L^{p_0}} + \left\| f_\theta^\varepsilon - f \right\|_{L^{p_1}} + \left\| f_\theta^\varepsilon - f \right\|_{L^2} < \varepsilon$$

and

$$(6.3) \qquad \| f_{it}^\varepsilon \|_{L^{p_0}}^{p_0} \le \| f \|_{L^p}^p + \varepsilon', \quad \| f_{1+it}^\varepsilon \|_{L^{p_1}}^{p_1} \le \| f \|_{L^p}^p + \varepsilon',$$

where ε' depends on $\varepsilon, p, \| f \|_{L^p}$ and tends to zero as $\varepsilon \to 0$.

The next lemma is a variant of Lemma 3.7 from [16].

Lemma 6.5. *Let $0 < \alpha, \beta < 1$, $\gamma > 0$. Then for some constant $C(\alpha, \beta, \gamma)$ we have*

$$(6.4) \qquad \int_0^\infty (f^*(r) r^{\beta-\alpha})^*(y) \phi_{\alpha,\gamma}(y) \, \frac{dy}{y} \le C(\alpha, \beta, \gamma) \int_0^\infty f^*(r) \phi_{\beta,\gamma}(r) \, \frac{dr}{r} \,.$$

Proof. Recall that for given $s \in (0, 1)$ and $\gamma > 0$, we have the equivalence $\phi_{s,\gamma}(t) \approx t^s \log^\gamma \left(e + \frac{1}{t} \right)$ on $(0, \infty)$. Now observe that the estimate (6.4) is trivial when $\beta \le \alpha$ as $(f^*(r) r^{\beta-\alpha})^* = f^*(r) r^{\beta-\alpha}$. Thus we may assume that $\beta > \alpha$ in the proof below. We may also assume that

$$\int_0^\infty f^*(r) r^{\beta-1} \log^\gamma \left(e + \frac{1}{r} \right) dr < \infty \,,$$

otherwise the right-hand side of (6.4) is infinite. Then

$$\sup_{r>0} f^*(r) r^\beta \log^\gamma \left(e + \frac{1}{r} \right) \le C,$$

and thus $\lim_{r \to \infty} f^*(r) r^{\beta-\alpha} = 0$. Since the set of discontinuity points of f^* is at most countable (f^* is right continuous), we may assume without loss of generality that function f^* is continuous. Then $\sup_{y \le r < \infty} f^*(r) r^{\beta-\alpha}$ is attained for any $y > 0$ and so the set

$$M = \{ y \in (0, \infty) : \sup_{y \le r < \infty} f^*(r) r^{\beta-\alpha} > f^*(y) y^{\beta-\alpha} \}$$

is open. Hence, M is a countable union of open intervals, namely, $M = \bigcup_{k \in S} (a_k, b_k)$, where S is a countable set of positive integers. Also, observe that if $y \in (a_k, b_k)$, then

$$\sup_{y \le r < \infty} f^*(r) r^{\beta-\alpha} = f^*(b_k) b_k^{\beta-\alpha} \,.$$

We have

$$\int_0^\infty (f^*(r) r^{\beta-\alpha})^*(y) y^{\alpha-1} \log^\gamma \left(e + \frac{1}{y} \right) dy$$

$$\le \int_0^\infty \sup_{y \le r < \infty} f^*(r) r^{\beta-\alpha} y^{\alpha-1} \log^\gamma \left(e + \frac{1}{y} \right) dy$$

$$\le \int_{(0,\infty) \setminus M} f^*(y) y^{\beta-1} \log^\gamma \left(e + \frac{1}{y} \right) dy$$

$$+ \sum_{k \in S} f^*(b_k) b_k^{\beta-\alpha} \int_{a_k}^{b_k} y^{\alpha-1} \log^\gamma \left(e + \frac{1}{y}\right) dy \,.$$

Furthermore, for every $k \in S$,

$$f^*(b_k) b_k^{\beta-\alpha} \int_{a_k}^{b_k} y^{\alpha-1} \log^\gamma \left(e + \frac{1}{y}\right) dy$$

$$\leq f^*(b_k) b_k^{\beta-\alpha} \int_{\max(a_k, \frac{b_k}{2})}^{b_k} y^{\alpha-1} \log^\gamma \left(e + \frac{1}{y}\right) dy \cdot \frac{\int_0^{b_k} y^{\alpha-1} \log^\gamma(e + \frac{1}{y}) dy}{\int_{\frac{b_k}{2}}^{b_k} y^{\alpha-1} \log^\gamma(e + \frac{1}{y}) dy}$$

$$= C_{\alpha,\gamma} f^*(b_k) b_k^{\beta-\alpha} \int_{\max(a_k, \frac{b_k}{2})}^{b_k} y^{\alpha-1} \log^\gamma \left(e + \frac{1}{y}\right) dy$$

$$\leq C_{\alpha,\beta,\gamma} \int_{a_k}^{b_k} f^*(y) y^{\beta-1} \log^\gamma \left(e + \frac{1}{y}\right) dy \,.$$

Therefore,

$$\int_0^\infty (f^*(r) r^{\beta-\alpha})^*(y) y^{\alpha-1} \log^\gamma \left(e + \frac{1}{y}\right) dy$$

$$\leq \int_0^\infty f^*(y) y^{\beta-1} \log^\gamma \left(e + \frac{1}{y}\right) dy + C_{\alpha,\beta,\gamma} \sum_{k \in S} \int_{a_k}^{b_k} f^*(y) y^{\beta-1} \log^\gamma \left(e + \frac{1}{y}\right) dy$$

$$\leq (C_{\alpha,\beta,\gamma} + 1) \int_0^\infty f^*(y) y^{\beta-1} \log^\gamma \left(e + \frac{1}{y}\right) dy \,.$$

This proves (6.4). □

The main interpolation tool in this work is the following.

Theorem 6.6. *Let $1 < p_0 < \infty$ and suppose that $\frac{1}{2} < s_1^0 \leq s_2^0 \leq \cdots \leq s_n^0 < 1$ and that $0 < s_1^1 \leq s_2^1 \leq \cdots \leq s_n^1 < 1$. Assume that exactly d of the numbers $s_2^0, \ldots, s_n^0$ are equal to s_1^0, and exactly d of the numbers $s_2^1, \ldots, s_n^1$ are equal to s_1^1. Let Ψ be as in (1.2). Suppose that for all nonzero $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ we have*

$$(6.5) \quad \left\|T_\sigma f\right\|_{L^{p_0}(\mathbb{R}^n)} \leq K_0 \sup_{j_1,\ldots,j_n \in \mathbb{Z}} \left\|\Gamma\left(s_1^0, \ldots, s_n^0\right)[\widehat{\Psi} D_{j_1,\ldots,j_n} \sigma]\right\|_{\Lambda_{\phi_{s_1^0, s_1^0 d}}(\mathbb{R}^n)} \|f\|_{L^{p_0}(\mathbb{R}^n)}$$

and

$$(6.6) \quad \left\|T_\sigma f\right\|_{L^{p_1}(\mathbb{R}^n)} \leq K_1 \sup_{j_1,\ldots,j_n \in \mathbb{Z}} \left\|\Gamma\left(s_1^1, \ldots, s_n^1\right)[\widehat{\Psi} D_{j_1,\ldots,j_n} \sigma]\right\|_{\Lambda_{\phi_{s_1^1, (1-s_1^1) d}}(\mathbb{R}^n)} \|f\|_{L^{p_1}(\mathbb{R}^n)} \,.$$

Let $0 < \theta < 1$ and suppose

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{2}, \quad s_j = (1-\theta) s_j^0 + \theta s_j^1, \quad j = 1, \ldots, n.$$

Then there is a constant $C_* = C_*(p_0, \theta, n, d, \psi, s_j^0, s_j^1)$ such that for all $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$

$$\left\|T_\sigma f\right\|_{L^p(\mathbb{R}^n)} \leq C_* K_0^{1-\theta} K_1^\theta \sup_{j_1,\ldots,j_n \in \mathbb{Z}} \left\|\Gamma\left(s_1, \ldots, s_n\right)[\widehat{\Psi} D_{j_1,\ldots,j_n} \sigma]\right\|_{\Lambda_{\phi_{s_1, d}}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} \,.$$

Proof. Let us fix a function σ such that

$$(6.7) \quad \sup_{j_1,\ldots,j_n \in \mathbb{Z}} \left\|\Gamma(s_1, \ldots, s_n)[\widehat{\Psi} D_{j_1,\ldots,j_n} \sigma]\right\|_{\Lambda_{\phi_{s_1, d}}(\mathbb{R}^n)} < \infty$$

and for $j_1, \ldots, j_n \in \mathbb{Z}$ define

$$\varphi_{j_1,\ldots,j_n} = \Gamma(s_1,\ldots,s_n)\big[\widehat{\Psi}\, D_{j_1,\ldots,j_n}\sigma\big].$$

Since $\varphi_{j_1,\ldots,j_n} \in \Lambda_{\phi_{s_1,d}}(\mathbb{R}^n)$, we have $\sup_{\lambda>0}\phi_{s_1,d}(\lambda)\varphi^*_{j_1,\ldots,j_n}(\lambda) < \infty$ and so $\varphi^*_{j_1,\ldots,j_n}(\lambda)$ converges to 0 as $\lambda \to \infty$. Now by [2, Corollary 7.6 in Chapter 2], there is a measure preserving transformation $h_{j_1,\ldots,j_n} : \mathbb{R}^n \to (0,\infty)$ such that

$$(6.8) \qquad |\varphi_{j_1,\ldots,j_n}| = \varphi^*_{j_1,\ldots,j_n} \circ h_{j_1,\ldots,j_n}.$$

Recall that $s_1^0 \leq \cdots \leq s_n^0$ and $s_1^1 \leq \cdots \leq s_n^1$. For $z \in \mathbb{C}$ with $0 \leq \mathrm{Re}(z) \leq 1$, we define complex polynomials

$$P_\rho(z) = s_\rho^0(1-z) + s_\rho^1 z$$

for $\rho = 1, 2, \ldots, n$. Let $\widehat{\Psi_b} = \widehat{\psi_b} \otimes \cdots \otimes \widehat{\psi_b}$ where ψ_b is defined in (5.1). We define the family of multipliers

$$(6.9) \quad \sigma_z = \sum_{k_1,\ldots,k_n\in\mathbb{Z}} D_{-k_1,\ldots,-k_n}\left[\widehat{\Psi_b}\,\Gamma\big(-P_1(z),\ldots,-P_n(z)\big)\Big[\varphi_{k_1,\ldots,k_n}h_{k_1,\ldots,k_n}^{s_1-P_1(z)}\Big]\right].$$

As $P_j(\theta) = s_j$ for $1 \leq j \leq n$ and $\sum_{k_1,\ldots,k_n\in\mathbb{Z}}\widehat{\Psi}\big(2^{-k_1}\xi_1,\ldots,2^{-k_n}\xi_n\big) = 1$ when all $\xi_k \neq 0$, it follows that $\sigma_\theta = \sigma$ a.e.

Fix $f, g \in \mathscr{C}_0^\infty(\mathbb{R}^n)$. Given $\epsilon > 0$ find f_z^ϵ and g_z^ϵ as in Lemma 6.4. Thus we have $\|f_\theta^\epsilon - f\|_{L^p} + \|f_\theta^\epsilon - f\|_{L^2} < \epsilon$, $\|g_\theta^\epsilon - g\|_{L^{p'}} + \|g_\theta^\epsilon - g\|_{L^2} \leq \epsilon$,

$$\|f_{it}^\epsilon\|_{L^{p_0}(\mathbb{R}^n)}^{p_0} \leq \|f\|_{L^p(\mathbb{R}^n)}^p + \epsilon', \quad \|f_{1+it}^\epsilon\|_{L^2(\mathbb{R}^n)}^2 \leq \|f\|_{L^p(\mathbb{R}^n)}^p + \epsilon',$$

$$\|g_{it}^\epsilon\|_{L^{p_0'}(\mathbb{R}^n)}^{p_0'} \leq \|g\|_{L^{p'}(\mathbb{R}^n)}^{p'} + \epsilon', \quad \|g_{1+it}^\epsilon\|_{L^2(\mathbb{R}^n)}^2 \leq \|g\|_{L^{p'}(\mathbb{R}^n)}^{p'} + \epsilon'.$$

Now define on the unit strip $\{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq 1\}$ the following function

$$(6.10) \qquad F(z) = \int_{\mathbb{R}^n} \sigma_z(\xi)\widehat{f_z^\epsilon}(\xi)\widehat{g_z^\epsilon}(\xi)d\xi = \int_{\mathbb{R}^n} T_{\sigma_z}(f_z^\epsilon)(x)\,g_z^\epsilon(-x)\,dx$$

which is analytic in the interior of this strip and is continuous on its closure. Hölder's inequality and one hypothesis of the theorem give

$$|F(it)| \leq \|T_{\sigma_{it}}(f_{it}^\epsilon)\|_{L^{p_0}}\|g_{it}^\epsilon\|_{L^{p_0'}}$$

$$(6.11) \qquad \leq K_0 \sup_{j_1,\ldots,j_n\in\mathbb{Z}}\left\|\Gamma\big(s_1^0,\ldots,s_n^0\big)\big[\widehat{\Psi}\,D_{j_1,\ldots,j_n}\sigma_{it}\big]\right\|_{\Lambda_{\phi_{s_1^0,s_1^0 d}}}\|f_{it}^\epsilon\|_{L^{p_0}}\|g_{it}^\epsilon\|_{L^{p_0'}}.$$

Using the definition of σ_z with $z = it$, we have

$$\widehat{\Psi}\,D_{j_1,\ldots,j_n}\sigma_{it}$$

$$= \sum_{k_1,\ldots,k_n\in\mathbb{Z}}\widehat{\Psi}\,D_{j_1-k_1,\ldots,j_n-k_n}\left[\widehat{\Psi_b}\,\Gamma\big(-P_1(it),\ldots,-P_n(it)\big)\Big[\varphi_{k_1,\ldots,k_n}h_{k_1,\ldots,k_n}^{s_1-P_1(it)}\Big]\right].$$

In view of the support properties of the bumps $\widehat{\Psi}$ and $\widehat{\Psi_b}$, all terms in the sum above are zero if $k_i \notin \{j_i - 2, j_i - 1, j_i, j_i + 1, j_i + 2\}$ for some $i \in \{1, \ldots, n\}$. Using this observation and Lemma 6.1 with $\widehat{\Phi} = \widehat{\Psi}\,D_{a_1,\ldots,a_n}\widehat{\Psi_b}$, we write

$$\left\|\Gamma\big(s_1^0,\ldots,s_n^0\big)\big[\widehat{\Psi}\,D_{j_1,\ldots,j_n}\sigma_{it}\big]\right\|_{\Lambda_{\phi_{s_1^0,s_1^0 d}}}$$

$$\leq \sum_{\substack{1\leq i\leq n \\ -2\leq a_i\leq 2}} \left\| \Gamma\left(s_1^0,\ldots,s_n^0\right) \left[\widehat{\Psi}\left(D_{a_1,\ldots,a_n}\widehat{\Psi_b}\right) \right. \right.$$

$$\left. \left. D_{a_1,\ldots,a_n}\left\{ \Gamma\left(-P_1(it),\ldots,-P_n(it)\right)\left(\varphi_{j_1+a_1,\ldots,j_n+a_n}h_{j_1+a_1,\ldots,j_n+a_n}^{s_1-P_1(it)}\right)\right\}\right]\right\|_{\Lambda_{\phi_{s_1^0,s_1^0 d}}}$$

$$\leq \sum_{\substack{1\leq i\leq n \\ -2\leq a_i\leq 2}} \left\| \Gamma\left(s_1^0,\ldots,s_n^0\right) D_{a_1,\ldots,a_n}\left\{ \Gamma\left(-s_1^0,\ldots,-s_n^0\right) \right. \right.$$

$$\left. \left. \Gamma\left(it(s_1^0-s_1^1),\ldots,it(s_n^0-s_n^1)\right)\left(\varphi_{j_1+a_1,\ldots,j_n+a_n}h_{j_1+a_1,\ldots,j_n+a_n}^{s_1-P_1(it)}\right)\right\}\right\|_{\Lambda_{\phi_{s_1^0,s_1^0 d}}}$$

$$\leq C \sum_{\substack{1\leq i\leq n \\ -2\leq a_i\leq 2}} \left\| \Gamma\left(it(s_1^0-s_1^1),\ldots,it(s_n^0-s_n^1)\right)\left[\varphi_{j_1+a_1,\ldots,j_n+a_n}h_{j_1+a_1,\ldots,j_n+a_n}^{s_1-P_1(it)}\right]\right\|_{\Lambda_{\phi_{s_1^0,s_1^0 d}}},$$

as $-P_j(it) = -s_j^0 + it(s_j^0 - s_j^1)$. In the last inequality we made use of the fact that the function

$$\prod_{i=1}^n \left(\frac{1+4\pi^2|\xi_i|^2}{1+4\pi^2|\xi_i/2^{a_i}|^2} \right)^{s_i^0/2}$$

satisfies (1.1) and thus Lemma 6.3 applies. We continue estimating as follows:

$$C \sum_{\substack{1\leq i\leq n \\ -2\leq a_i\leq 2}} \left\| \Gamma\left(it(s_1^0-s_1^1),\ldots,it(s_n^0-s_n^1)\right)\left[\varphi_{j_1+a_1,\ldots,j_n+a_n}h_{j_1+a_1,\ldots,j_n+a_n}^{s_1-P_1(it)}\right]\right\|_{\Lambda_{\phi_{s_1^0,s_1^0 d}}(\mathbb{R}^n)}$$

$$\leq C \sum_{\substack{1\leq i\leq n \\ -2\leq a_i\leq 2}} (1+|t|)^n \left\| \varphi_{j_1+a_1,\ldots,j_n+a_n}h_{j_1+a_1,\ldots,j_n+a_n}^{s_1-s_1^0}\right\|_{\Lambda_{\phi_{s_1^0,s_1^0 d}}(\mathbb{R}^n)}$$

$$= C(1+|t|)^n \sum_{\substack{1\leq i\leq n \\ -2\leq a_i\leq 2}} \left\| \varphi_{j_1+a_1,\ldots,j_n+a_n}^*(r)r^{s_1-s_1^0}\right\|_{\Lambda_{\phi_{s_1^0,s_1^0 d}}((0,\infty),dr)}$$

$$\leq C(1+|t|)^n \sum_{\substack{1\leq i\leq n \\ -2\leq a_i\leq 2}} \left\| \varphi_{j_1+a_1,\ldots,j_n+a_n}^*\right\|_{\Lambda_{\phi_{s_1,s_1^0 d}}(0,\infty)}$$

$$= C(1+|t|)^n \sum_{\substack{1\leq i\leq n \\ -2\leq a_i\leq 2}} \left\| \varphi_{j_1+a_1,\ldots,j_n+a_n}\right\|_{\Lambda_{\phi_{s_1,s_1^0 d}}(\mathbb{R}^n)}$$

$$\leq C(1+|t|)^n \sup_{j_1,\ldots,j_n\in\mathbb{Z}} \left\| \varphi_{j_1,\ldots,j_n}\right\|_{\Lambda_{\phi_{s_1,d}}(\mathbb{R}^n)},$$

where we used successively Lemma 6.2, the fact that $\mathrm{Re}\, P_1(it) = s_1^0$, identity (6.8) together with the fact that $h_{j_1,\ldots,j_n}$ is measure-preserving, and Lemma 6.5. Inserting this estimate in (6.11) and using Lemma 6.4 we obtain

$$|F(it)| \leq CK_0(1+|t|)^n \sup_{j_1,\ldots,j_n\in\mathbb{Z}} \left\| \varphi_{j_1,\ldots,j_n}\right\|_{\Lambda_{\phi_{s_1,d}}(\mathbb{R}^n)} \left(\|f\|_{L^p}^p + \epsilon\right)^{\frac{1}{p_0}} \left(\|g\|_{L^{p'}}^{p'} + \epsilon\right)^{\frac{1}{p_0'}}.$$

A similar argument using the inequality

$$\left\| \varphi_{j_1,\ldots,j_n}\right\|_{\Lambda_{\phi_{s_1,(1-s_1^1)d}}(\mathbb{R}^n)} \leq \left\| \varphi_{j_1,\ldots,j_n}\right\|_{\Lambda_{\phi_{s_1,d}}(\mathbb{R}^n)}$$

yields

$$|F(1+it)| \leq C K_1 (1+|t|)^n \sup_{j_1,\ldots,j_n \in \mathbb{Z}} \|\varphi_{j_1,\ldots,j_n}\|_{\Lambda_{\phi_{s_1},d}(\mathbb{R}^n)} \left(\|f\|_{L^p}^p + \epsilon'\right)^{\frac{1}{2}} \left(\|g\|_{L^{p'}}^{p'} + \epsilon'\right)^{\frac{1}{2}}.$$

Moreover, for $\tau \in [0,1]$, we claim that $|F(\tau + it)| \leq A_\tau(t)$ where $A_\tau(t)$ has at most polynomial growth as $|t| \to \infty$; we prove this assertion at the end. Thus we can apply Hirschman's lemma ([11, Lemma 1.3.8]). Using the estimates for $|F(it)|$ and $|F(1+it)|$, for $\theta \in (0,1)$, we obtain

$$|F(\theta)| \leq C_* K_0^{1-\theta} K_1^\theta \sup_{j_1,\ldots,j_n \in \mathbb{Z}} \|\varphi_{j_1,\ldots,j_n}\|_{\Lambda_{\phi_{s_1},d}(\mathbb{R}^n)} \left(\|f\|_{L^p}^p + \epsilon'\right)^{\frac{1}{p}} \left(\|g\|_{L^{p'}}^{p'} + \epsilon'\right)^{\frac{1}{p'}}.$$

We write

$$\left| F(\theta) - \int_{\mathbb{R}^n} \widehat{T_\sigma(f)}\, \widehat{g}(\xi)\, d\xi \right| = \left| \int_{\mathbb{R}^n} \sigma(\xi) \widehat{f_\theta^\epsilon}(\xi) \widehat{g_\theta^\epsilon}(\xi)\, d\xi - \int_{\mathbb{R}^n} \sigma(\xi) \widehat{f}(\xi) \widehat{g}(\xi)\, d\xi \right|$$
$$\leq \|\sigma\|_{L^\infty} \left[\|f_\theta^\epsilon - f\|_{L^2} \|g\|_{L^2} + \|g_\theta^\epsilon - g\|_{L^2} \|f\|_{L^2} \right],$$

which tends to zero as $\epsilon \to 0$ (which implies $\epsilon' \to 0$). Thus

$$\left| \int_{\mathbb{R}^n} \widehat{T_\sigma(f)}\, \widehat{g}(\xi)\, d\xi \right| \leq C_* K_0^{1-\theta} K_1^\theta \sup_{j_1,\ldots,j_n \in \mathbb{Z}} \|\varphi_{j_1,\ldots,j_n}\|_{\Lambda_{\phi_{s_1},d}(\mathbb{R}^n)} \|f\|_{L^p} \|g\|_{L^{p'}}.$$

But the integral on the left is equal to $\int_{\mathbb{R}^n} T_\sigma(f)(x) g(-x) dx$. Taking the supremum over all functions $g \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ with $\|g\|_{L^{p'}} \leq 1$ we deduce for $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$:

$$\|T_\sigma(f)\|_{L^p(\mathbb{R}^n)} \leq C_* K_0^{1-\theta} K_1^\theta \sup_{j_1,\ldots,j_n \in \mathbb{Z}} \|\varphi_{j_1,\ldots,j_n}\|_{\Lambda_{\phi_{s_1},d}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.$$

Notice that the constant C_* depends on the parameters indicated in the statement.

We now return to the assertion that $|F(\tau + it)| \leq A_\tau(t)$, where $A_\tau(t)$ has at most polynomial growth in $|t|$, which was one of the hypotheses in Hirschman's lemma. Let $z = \tau + it$ where $t \in \mathbb{R}$ and $0 \leq \tau \leq 1$. We use that

$$|F(\tau + it)| \leq \|\sigma_{\tau+it}\|_{L^\infty} \|f_{\tau+it}\|_{L^2} \|g_{\tau+it}\|_{L^2},$$

and we notice that in view of (6.1), the L^2 norms of $f_{\tau+it}$ and $g_{\tau+it}$ are bounded by constants independent of t. We now estimate $\|\sigma_z\|_{L^\infty}$. Let E be the set of all $(\xi_1,\ldots,\xi_n) \in \mathbb{R}^n$ with some $\xi_i = 0$. Then for all $(\xi_1,\ldots,\xi_n) \in \mathbb{R}^n \setminus E$ there are only finitely many indices k_i in the summation defining $\sigma_z(\xi_1,\ldots,\xi_n)$ that produce a nonzero term, in fact the indices with $|\xi_i|/4 \leq 2^{k_i} \leq 4|\xi_i|$ for all $i \in \{1,\ldots,n\}$. Also, $P_\rho(\tau + it) = P_\rho(\tau) + (s_1^1 - s_1^0)(it)$, which implies that

$$(6.12) \quad \begin{aligned} &\Gamma\big(-P_1(\tau+it),\ldots,-P_n(\tau+it)\big) \\ &\quad = \Gamma\big(-P_1(\tau),\ldots,-P_n(\tau)\big)\, \Gamma\big(it(s_1^0 - s_1^1),\ldots,it(s_n^0 - s_n^1)\big). \end{aligned}$$

Applying identity (6.12), and using successively Proposition 4.1, Lemma 6.2, the fact that $\operatorname{Re} P_1(\tau+it) = P_1(\tau)$, identity (6.8) together with the fact that $h_{k_1,\ldots,k_n}$ is measure-preserving, and Lemma 6.5, we estimate $\|\sigma_{\tau+it}\|_{L^\infty}$ by

$$\sup_{\substack{\xi \in \mathbb{R}^n \setminus E \\ 1 \leq i \leq n \\ \frac{|\xi_i|}{4} \leq 2^{k_i} \leq 4|\xi_i|}} \sum \left\| \Gamma\big(-P_1(\tau+it),\ldots,-P_n(\tau+it)\big) \left[\varphi_{k_1,\ldots,k_n} h_{k_1,\ldots,k_n}^{s_1 - P_1(\tau+it)} \right] \right\|_{L^\infty}$$

$$\leq C\, (1+|t|)^n \sup_{\substack{\xi \in \mathbb{R}^n \setminus E \\ 1 \leq i \leq n \\ \frac{|\xi_i|}{4} \leq 2^{k_i} \leq 4|\xi_i|}} \sum \left\| \varphi_{k_1,\ldots,k_n} h_{k_1,\ldots,k_n}^{s_1 - P_1(\tau)} \right\|_{\Lambda_{\phi_{P_1(\tau),(1-P_1(\tau))d}}(\mathbb{R}^n)}$$

$$\leq C\left(1+|t|\right)^n \sup_{\substack{\xi\in\mathbb{R}^n\backslash E \\ 1\leq i\leq n \\ \frac{|\xi_i|}{4}\leq 2^{k_i}\leq 4|\xi_i|}} \sum \left\|\varphi^*_{k_1,\ldots,k_n}(r)r^{s_1-P_1(\tau)}\right\|_{\Lambda_{\phi_{P_1(\tau),(1-P_1(\tau))d}}(0,\infty)}$$

$$\leq C(1+|t|)^n \sup_{\substack{\xi\in\mathbb{R}^n\backslash E \\ 1\leq i\leq n \\ \frac{|\xi_i|}{4}\leq 2^{k_i}\leq 4|\xi_i|}} \sum \left\|\varphi^*_{k_1,\ldots,k_n}\right\|_{\Lambda_{\phi_{s_1,(1-P_1(\tau))d}}(0,\infty)}$$

$$\leq C\left(1+|t|\right)^n 5^n \sup_{k_1,\ldots,k_n\in\mathbb{Z}} \left\|\varphi_{k_1,\ldots,k_n}\right\|_{\Lambda_{\phi_{s_1,d}}(\mathbb{R}^n)}\,,$$

and the last expression is finite in view of assumption (6.7). This proves that $|F(\tau+it)|\leq A_\tau(t)$, where $A_\tau(t)\leq C'\left(1+|t|\right)^n$. $\qquad\square$

To prove Theorem 1.1 we apply Theorem 6.6 as follows: For the given p with $1<p<2$ we set $p_0=1+\epsilon$ for some small number ϵ, and we define θ in terms of $(1-\theta)/p_0+\theta/2=1/p$.

Given $0<s_1\leq\cdots\leq s_n$ with exactly d numbers among $s_2,\ldots,s_n$ equal to s_1, pick $\frac{1}{2}<s_1^0\leq\cdots\leq s_n^0$ and $0<s_1^1\leq\cdots\leq s_n^1\leq 1/2$ such that $s_j=(1-\theta)s_j^0+\theta s_j^1$. This relationship maintains proportions, and as the sequences are all increasing, it must be the case that the first $d+1$ terms in each sequence are equal. We pick these sequences so that $s_1^0=\cdots=s_{d+1}^0=\frac{1}{2}+\varepsilon$ and $s_1^1=\cdots=s_{d+1}^1$. We note that s_1^1 can be found thanks to the assumption $s_1>1/p-1/2$. Inequality (6.5) follows from the special case $1/2<s_1<1$ of Theorem 1.1 proved in Section 5, while inequality (6.6) follows from Proposition 4.1.

7. FINAL REMARKS

We conclude with some comments on Theorem 1.1. Let p, Ψ, $s_1,\ldots,s_n$ be as in that theorem. If a function σ in $L^\infty(\mathbb{R}^n)$ satisfies

$$(7.1)\qquad K':=\sup_{j_1,\ldots,j_n\in\mathbb{Z}}\left\|\Gamma\left(s_1,\ldots,s_n\right)\left[\widehat{\Psi}\,D_{j_1,\ldots,j_n}\sigma\right]\right\|_{\Lambda_{\phi_{s_1,(1-s_1)d}}(\mathbb{R}^n)}<\infty\,,$$

then we conjecture that there is constant $C=C(s_1,\ldots,s_n,p,n,d,\psi)$ such that, for every $f\in\mathscr{C}_0^\infty(\mathbb{R}^n)$, we have

$$(7.2)\qquad\qquad\|T_\sigma(f)\|_{L^p(\mathbb{R}^n)}\leq CK'\,\|f\|_{L^p(\mathbb{R}^n)}\,.$$

We recall that $\|T_\sigma\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}\gtrsim\|\sigma\|_{L^\infty(\mathbb{R}^n)}$, and the appearance of the space $\Lambda_{\phi_{s_1,(1-s_1)d}}(\mathbb{R}^n)$ in (7.1) is motivated by the fact that among all rearrangement-invariant spaces $E(\mathbb{R}^n)$ satisfying the Sobolev-type embedding

$$\|\sigma\|_{L^\infty(\mathbb{R}^n)}\lesssim\|\Gamma(s_1,\ldots,s_n)\sigma\|_{E(\mathbb{R}^n)},$$

$\Lambda_{\phi_{s_1,(1-s_1)d}}(\mathbb{R}^n)$ is locally the largest one, see Proposition 4.2. Our emphasis on the local behavior of the function spaces involved when investigating optimality questions is then justified by the local nature of condition (1.5).

At present, the validity of the aforementioned conjecture remains an open problem. We point out, however, that if $s_1\leq 1/2$ then the conjecture has to be true up to an arbitrarily small power of the logarithm. Namely, we claim that if $\delta>0$ then inequality (1.6) holds with the constant

$$K=\sup_{j_1,\ldots,j_n\in\mathbb{Z}}\left\|\Gamma\left(s_1,\ldots,s_n\right)\left[\widehat{\Psi}\,D_{j_1,\ldots,j_n}\sigma\right]\right\|_{\Lambda_{\phi_{s_1,(1-s_1+\delta)d}}(\mathbb{R}^n)}\,.$$

To see this it is enough to modify the proof of Theorem 6.6. Namely, we replace equation (6.9) by

$$\sigma_z = \sum_{k_1,\ldots,k_n \in \mathbb{Z}} D_{-k_1,\ldots,-k_n}$$

$$\left[\widehat{\Psi_b}\, \Gamma\big(-P_1(z),\ldots,-P_n(z)\big) \Big[\varphi_{k_1,\ldots,k_n} h_{k_1,\ldots,k_n}^{s_1-P_1(z)} (\log(e + h_{k_1,\ldots,k_n}^{-1}))^{(P_1(z)-s_1)d} \Big] \right]$$

and define the function F by (6.10). Then one can show that

$$|F(it)| \leq C K_0 (1+|t|)^n \sup_{j_1,\ldots,j_n \in \mathbb{Z}} \|\varphi_{j_1,\ldots,j_n}\|_{\Lambda_{\phi_{s_1,d(2s_1^0-s_1)}}(\mathbb{R}^n)} \left(\|f\|_{L^p}^p + \epsilon' \right)^{\frac{1}{p_0}} \left(\|g\|_{L^{p'}}^{p'} + \epsilon' \right)^{\frac{1}{p_0'}}$$

and

$$|F(1+it)| \leq C K_1 (1+|t|)^n \sup_{j_1,\ldots,j_n \in \mathbb{Z}} \|\varphi_{j_1,\ldots,j_n}\|_{\Lambda_{\phi_{s_1,d(1-s_1)}}(\mathbb{R}^n)} \left(\|f\|_{L^p}^p + \epsilon' \right)^{\frac{1}{2}} \left(\|g\|_{L^{p'}}^{p'} + \epsilon' \right)^{\frac{1}{2}}.$$

This then implies

$$\|T_\sigma(f)\|_{L^p(\mathbb{R}^n)} \leq C_* K_0^{1-\theta} K_1^\theta \sup_{j_1,\ldots,j_n \in \mathbb{Z}} \|\varphi_{j_1,\ldots,j_n}\|_{\Lambda_{\phi_{s_1,d(2s_1^0-s_1)}}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.$$

Choosing all parameters as in the proof of Theorem 1.1 with $\epsilon < \delta/2$ yields the conclusion.

In order to achieve a similar conclusion in the case when $s_1 > 1/2$, it would be desirable to have a good understanding of embeddings between Lorentz-Sobolev spaces of mixed smoothness. Namely, we expect that the desired result for $s_1 > 1/2$ might follow by combining the above result for $s_1 \leq 1/2$ with suitable Sobolev-type embeddings.

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