MULTILINEAR ROUGH SINGULAR INTEGRAL OPERATORS

LOUKAS GRAFAKOS, DANQING HE, PETR HONZÍK, AND BAE JUN PARK

Abstract. We study m-linear homogeneous rough singular integral operators $\mathcal{L}_\Omega$ associated with integrable functions Ω on $\mathbb{S}^{mn-1}$ with mean value zero. We prove boundedness for $\mathcal{L}_\Omega$ from $L^{p_1}\times\cdots\times L^{p_m}$ to L^p when $1<p_1,\dots,p_m<\infty$ and $1/p=1/p_1+\cdots+1/p_m$ in the largest possible open set of exponents when $\Omega\in L^q(\mathbb{S}^{mn-1})$ and $q\ge 2$. This set can be described by a convex polyhedron in $\mathbb{R}^m$.

1. Introduction

Let Ω be an integrable function on the unit sphere $\mathbb{S}^{n-1}$ with mean value 0. The rough singular integral operator associated with Ω is defined by

$$\mathcal{L}_\Omega f(x) := \text{p.v.}\int_{\mathbb{R}^n}\frac{\Omega(y/|y|)}{|y|^n}f(x-y)dy$$

initially for f in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$.

Calderón and Zygmund [2] proved that if $\Omega\in L\log L(\mathbb{S}^{n-1})$, then $\mathcal{L}_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ for all $1<p<\infty$. This result was improved by Coifman and Weiss [9] who replaced the condition $\Omega\in L\log L(\mathbb{S}^{n-1})$ by the less restrictive condition $\Omega\in H^1(\mathbb{S}^{n-1})$. The same conclusion was also obtained independently by Connett [11]. In the two dimensional case $n=2$, the weak type $(1,1)$ of $\mathcal{L}_\Omega$ was established by Christ [5] and independently by Hofmann [26] for $\Omega\in L^q(\mathbb{S}^1)$, $1<q\le\infty$, and by Christ and Rubio de Francia [6] for $\Omega\in L\log L(\mathbb{S}^1)$. These results were extended to all dimensions by Seeger [29].

In this paper we focus on analogous questions for m-linear singular integral operators. Throughout this paper we fix m to be an integer greater or equal to 2. Let Ω be an integrable function on $\mathbb{S}^{mn-1}$ with mean value zero, and we introduce a kernel K by setting

$$K(\vec{\boldsymbol{y}}) := \frac{\Omega(\vec{\boldsymbol{y}}\,')}{|\vec{\boldsymbol{y}}|^{mn}},\qquad \vec{\boldsymbol{y}}\neq 0,$$

where $\vec{\boldsymbol{y}}\,' := \vec{\boldsymbol{y}}/|\vec{\boldsymbol{y}}\,|\in\mathbb{S}^{mn-1}$ and $\vec{\boldsymbol{y}} := (y_1,\dots,y_m)\in(\mathbb{R}^n)^m$. Then the multilinear singular integral operator associated with Ω is defined as follows:

$$\mathcal{L}_\Omega\big(f_1,\dots,f_m\big)(x) := \text{p.v.}\int_{(\mathbb{R}^n)^m}K(\vec{\boldsymbol{y}})\prod_{j=1}^m f_j(x-y_j)\;d\,\vec{\boldsymbol{y}}$$

for Schwartz functions $f_1,\dots,f_m$ on $\mathbb{R}^n$, where $x\in\mathbb{R}^n$.

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The first important result concerning bilinear ($m = 2$) rough singular integrals appeared in the work of Coifman and Meyer [7] who obtained an estimate for $\mathcal{L}_\Omega$ when Ω possesses some smoothness. These authors actually showed that if Ω is a function of bounded variation on the circle $\mathbb{S}^1$, then the corresponding bilinear operator $\mathcal{L}_\Omega$ is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$ when $1 < p_1, p_2, p < \infty$ and $1/p_1 + 1/p_2 = 1/p$. Later, for general dimensions $n \geq 1$ and all $m \geq 2$, Grafakos and Torres [23] established the $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ boundedness of $\mathcal{L}_\Omega$ for all $1 < p_1, \dots, p_m < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$ when Ω is a Lipschitz function on $\mathbb{S}^{mn-1}$. The case of rough Ω was not really addressed until the work of Grafakos, He, and Honzík [17] who proved bilinear estimates in the full range $1 < p_1, p_2 < \infty$ under the condition $\Omega \in L^\infty(\mathbb{S}^{2n-1})$. These authors also showed that $\mathcal{L}_\Omega$ maps $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ if Ω is merely an L^2 function on $\mathbb{S}^{2n-1}$. This initial $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ estimate was refined by Grafakos, He, and Slavíková [20] replacing $\Omega \in L^2(\mathbb{S}^{2n-1})$ by $\Omega \in L^q(\mathbb{S}^{2n-1})$ for $q > 4/3$. Recently, He and Park [25] proved more points of boundedness for the bilinear rough singular integral operators in the range $1 < p_1, p_2 \leq \infty$ except the endpoint $p_1 = p_2 = \infty$.

Theorem A. [25] *Let $1 < p_1, p_2 \leq \infty$ and $1/2 < p < \infty$ with $1/p = 1/p_1 + 1/p_2$. Suppose that*

$$\max\Big(\frac{4}{3}, \frac{p}{2p-1}\Big) < q \leq \infty \tag{1.1}$$

and $\Omega \in L^q(\mathbb{S}^{2n-1})$ with $\int_{\mathbb{S}^{2n-1}} \Omega \, d\sigma = 0$, where $d\sigma$ denotes surface measure on $\mathbb{S}^{2n-1}$. Then the estimate

$$\big\|\mathcal{L}_\Omega\big\|_{L^{p_1}\times L^{p_2}\to L^p} \lesssim \|\Omega\|_{L^q(\mathbb{S}^{2n-1})} \tag{1.2}$$

is valid.

In this paper, we will study a multilinear analogue of Theorem A. In order to present our main results, we first introduce some notation. Let $J_m := \{1, \dots, m\}$. For $0 < s < 1$ and any subsets $J \subseteq J_m$, we let

$$\mathcal{H}_J^m(s) := \Big\{(t_1, \dots, t_m) \in (0,1)^m : \sum_{j\in J}(s - t_j) > -(1-s)\Big\},$$

$$\mathcal{O}_J^m(s) := \Big\{(t_1, \dots, t_m) \in (0,1)^m : \sum_{j\in J}(s - t_j) < -(1-s)\Big\}$$

and we define

$$\mathcal{H}^m(s) := \bigcap_{J\subseteq J_m} \mathcal{H}_J^m(s). \tag{1.3}$$

See Figure 1 for the shape of $\mathcal{H}^3(s)$ in the trilinear setting. We observe that

$$\mathcal{H}^m(s_1) \subset \mathcal{H}^m(s_2) \subset (0,1)^m \quad \text{ for } \ s_1 < s_2$$

and $\lim_{s\nearrow 1} \mathcal{H}^m(s) = (0,1)^m$. Another useful geometric object is the rectangle

$$\mathcal{V}_l^m(s) := \{(t_1, \dots, t_m) : 0 < t_l < 1 \quad \text{and} \quad 0 < t_j < s \ \text{ for } \ j \neq l\}, \tag{1.4}$$

where $l \in J_m$. As we will see from Lemma 5.4 below, $\mathcal{H}^m(s)$ is the convex hull of the rectangles $\mathcal{V}_l^m(s)$, $l = 1, \dots, m$, which reduces the geometric complexity in establishing the boundedness of $\mathcal{L}_\Omega$.

We reserve the letter s to denote $s = 1 - \frac{1}{q} = \frac{1}{q'}$, where q' the index conjugate to q. It is easy to check that

$$\mathcal{H}^2(s) = \{(t_1, t_2) \in (0,1)^2 : \ t_1 + t_2 < 1 + s\}.$$

In particular, $(\frac{1}{p_1}, \frac{1}{p_2}) \in \mathcal{H}^2(\frac{1}{q'})$ means that $\frac{p}{2p-1} < q$, which is exactly (1.1) when $q \geq 2$. In this work we generalize Theorem A to the multilinear setting when $q \geq 2$.

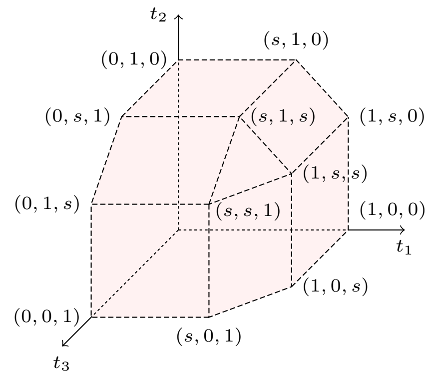

FIGURE 1. The region $\mathcal{H}^3(s)$

From this point on, let $d\sigma$ denote surface measure on $\mathbb{S}^{mn-1}$. The first main result of this paper as follows:

Theorem 1.1. *Let $2 \leq q \leq \infty$, $m \geq 2$, and $\Omega \in L^q(\mathbb{S}^{mn-1})$ with $\int_{\mathbb{S}^{mn-1}} \Omega \, d\sigma = 0$. Suppose that $1 < p_1, \dots, p_m < \infty$ and $1/m < p < \infty$ satisfy $1/p = 1/p_1 + \cdots + 1/p_m$ and*

$$(1/p_1, \dots, 1/p_m) \in \mathcal{H}^m(s), \tag{1.5}$$

where $s = 1 - 1/q \geq 1/2$. Then we have

$$\big\|\mathcal{L}_\Omega\big\|_{L^{p_1}\times\cdots\times L^{p_m}\to L^p} \lesssim \|\Omega\|_{L^q(\mathbb{S}^{mn-1})}. \tag{1.6}$$

We point out that Theorem 1.1 does not improve Theorem A when $m = 2$ as Theorem 1.1 needs a stronger condition that $\Omega \in L^q$ for $q \geq 2$ while $q > 4/3$ is assumed in Theorem A. Indeed, we mainly focus on extension of Theorem A to general $m \geq 3$ for which the arguments in the proof of Theorem A cannot be directly applied. Such an extension is naturally more complicated combinatorially, but also presents additional difficulties as $L^2 \times \cdots \times L^2$ maps into the nonlocally convex space $L^{2/m}$ when $m \geq 3$. To be specific, in the bilinear case, the initial estimate $L^2 \times L^2 \to L^1$ could be extended to two different end-point estimates $L^2 \times L^\infty \to L^2$ and $L^\infty \times L^2 \to L^2$ by duality, and accordingly $L^{p_1} \times L^{p_2} \to L^p$ for $2 \leq p_1, p_2 \leq \infty$ and $1 \leq p \leq 2$ by interpolation. This approach is difficult to extend to general m-linear cases for $m \geq 3$ as the space $L^{2/m}$, which is the target space in the initial estimate in m-linear cases, is no longer a Banach space for $m \geq 3$ and this prevents the duality argument used in the bilinear case. To overcome this obstacle, we will provide a new $L^{p_1} \times \cdots \times L^{p_m} \to L^p$ estimates for $p_1, \dots, p_m \geq 2$ in Proposition 6.1 below, but this new one requires the condition $q \geq 2$. Then a new remarkably powerful induction technique of Proposition 6.2 extends the estimate in Proposition 6.1 to arbitrary p_j.

Remark 1. It is proved in [18] that if $\Omega \in L^q(\mathbb{S}^{mn-1})$ for $q > \frac{2m}{m+1}$ and $\int_{\mathbb{S}^{mn-1}} \Omega d\sigma = 0$, then

$$\|\mathcal{L}_\Omega\|_{L^2\times\cdots\times L^2\to L^{2/m}} \lesssim \|\Omega\|_{L^q(\mathbb{S}^{mn-1})}.$$

In view of this, one might think that Theorem 1.1 also holds for certain $s < \frac{1}{2}$, and possibly for all $s > \frac{m-1}{2m}$ or even for all $s > 0$. To summarize, although the case $\Omega \in L^q(\mathbb{S}^{mn-1})$ with $q \geq 2$ is resolved in this paper, only partial results are currently known in the case $q < 2$ as it presents several challenges. Upon completion of this work we were informed of the related work of Dosidis and Slavíková [13] which considers the case $q < 2$ based on the induction technique (Proposition 6.2) introduced by us.

It is surprising that the condition (1.5), which originated from the extension in Proposition 6.2, is optimal. Our second main result (Theorem 1.2) provides the necessity of the condition (1.5). We note that the intersection in (1.3) can be actually taken over $J \subseteq J_m$ with $|J| \geq 2$ as the inequality $\sum_{j\in J}(s-t_j) > -(1-s)$ is trivial for $0 < t_j < 1$, $j = 1, \dots, m$, if $|J| \leq 1$.

Theorem 1.2. *Let $1 < q < \infty$ and $m \geq 2$. Suppose that $1 < p_1, \dots, p_m < \infty$ and $1/m < p < \infty$ satisfy $1/p = 1/p_1 + \cdots + 1/p_m$ and*

$$(1/p_1, \dots, 1/p_m) \in \bigcup_{J\subset J_m : |J|\geq 2} \mathcal{O}_J^m(1/q'). \tag{1.7}$$

Then there exists $\Omega \in L^q(\mathbb{S}^{mn-1})$ with $\int_{\mathbb{S}^{mn-1}} \Omega \, d\sigma = 0$ such that estimate (1.6) does not hold. In particular, for $q < \frac{2(m-1)}{m}$, there exists a function $\Omega \in L^q(\mathbb{S}^{mn-1})$ with $\int_{\mathbb{S}^{mn-1}} \Omega \, d\sigma = 0$ such that $\mathcal{L}_\Omega$ is unbounded from $L^2 \times \cdots \times L^2$ to $L^{2/m}$.

Thus, combining Theorems 1.1 and 1.2 we obtain that $\mathcal{H}^m(1/q')$ is the largest open set of indices $(1/p_1, \dots, 1/p_m)$ for which boundedness holds for $\mathcal{L}_\Omega$ when $\Omega \in L^q(\mathbb{S}^{mn-1})$ and $q \geq 2$. (Here q' is the dual index of q).

Remark 2. When $m = 2$, condition (1.7) is equivalent to $1/q' + 1 < 1/p$ and this implies that if $\|\mathcal{L}_\Omega\|_{L^{p_1}\times L^{p_2}\to L^p} \lesssim \|\Omega\|_{L^q(\mathbb{S}^{2n-1})}$ holds for $1 < p_1, p_2 < \infty$ with $1/p_1 + 1/p_2 = 1/p$, then we must have $\frac{p}{2p-1} \leq q$. This clearly indicates the necessity of one part of the condition (1.1) in Theorem A.

Remark 3. It is still unknown whether the bilinear estimate (1.2) holds when $q = \frac{p}{2p-1}$ in Theorem A. In general, we have no conclusion in Theorem 1.1 when

$$\sum_{j\in J_0} (s - 1/p_j) = -(1-s) \quad \text{for some} \quad J_0 \subseteq J_m$$

and

$$(1/p_1, \dots, 1/p_m) \in \Big(\bigcup_{J\subset J_m : |J|\geq 2} \mathcal{O}_J(s) \Big)^c.$$

The proof of Theorem 1.1 is based on the dyadic decomposition of Duoandikoetxea and Rubio de Francia [14] and on its m-linear adaptation contained in some of the aforementioned references. The main idea is as follows: We express the operator $\mathcal{L}_\Omega$ as $\sum_{\mu\in\mathbb{Z}} \mathcal{L}_\mu$ where $\|\mathcal{L}_\mu\|_{L^{p_1}\times\cdots\times L^{p_m}\to L^p} \lesssim 2^{\delta_0\mu}\|\Omega\|_{L^q}$ for all $1 < q < \infty$ and some $\delta_0 > 0$, depending on q. As the series is summable when $\mu < 0$, we focus on obtaining a good decay when $\mu \to +\infty$. Such an estimate is stated in (3.4) below. In order to obtain this estimate, we apply multilinear interpolation between an initial $L^2 \times \cdots \times L^2 \to L^{2/m}$ estimate with exponential decay $2^{-\widetilde{\delta}\mu}$ for some fixed number $\widetilde{\delta} > 0$ and general $L^{p_1} \times \cdots \times L^{p_m} \to L^p$ estimates with arbitrarily slow growth in Proposition 3.1.

The arbitrarily slow growth estimate obtained in Proposition 3.1 is actually the main contribution of this paper. Let us explain our strategy in more details. As already mentioned, unlike in the bilinear case, we are not able to obtain estimates for the local L^2 cases (namely $p_1, p_2, p' \in [2,\infty)$) from the initial estimates by duality. To overcome this obstacle, when $q = 2$, we refine the column-argument developed in [18] to obtain the estimate in the upper L^2 case (i.e., $p_1, p_2 \in [2,\infty)$). This combined with a modified Calderón-Zygmund argument developed in [25] yields the desired range for $q = 2$. For $q = \frac{2m}{m-1}$, based on the estimate for $q = 2$, a simple geometric observation about the range of indices leads to the estimates in the upper $L^{\frac{2m}{m+1}}$ case, and hence the full desired range by the modified Calderón-Zygmund argument. Repeating this process gives Proposition 3.1 for all $q \in [2,\infty)$. We remark that this induction argument still holds when $q < 2$, but the initial case $q = 2$ stops us from obtaining Theorem 1.1 for this range of q. As far as the proof of Theorem 1.2 is concerned, we adapt an idea appearing in [19], whose primordial form can be found in [12].

The sparse domination and weighted inequalities of linear and multilinear rough singular integrals are natural questions that arise upon establishing their L^p boundedness in (1.6). These problems in the linear and bilinear settings actually have been studied by [1, 3, 4, 10, 24, 27]. Our goal in this work is not to establish a comprehensive study of these operators, but to provide boundedness tools that will play a crucial role in their further investigation, including that of their sparse domination properties and their weighted estimates.

Organization. This paper is organized as follows. We first give the proof of Theorem 1.2 by constructing counterexamples in Section 2. We reduce Theorem 1.1 to Proposition 3.1 in Section 3. Section 4 contains some preliminaries and Section 5 is devoted to providing several key lemmas which are essential in the proof of Proposition 3.1. In the last section, we provide a detailed proof of Proposition 3.1.

Notation. Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of all natural numbers and all integers, respectively. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We use the symbol $A \lesssim B$ to indicate that $A \le CB$ for some constant $C > 0$ independent of the variable quantities A and B, and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$ hold simultaneously. We adopt the notation $\vec{\boldsymbol{\xi}} := (\xi_1, \dots, \xi_m) \in (\mathbb{R}^n)^m$ to denote m-tuples of elements of $\mathbb{R}^n$. We denote by χ_U the characteristic function of a set U. $\mathcal{C}^N(\mathbb{R})$ consists of functions on $\mathbb{R}$ of continuous derivatives up to order N. $\mathcal{S}(\mathbb{R}^n)$ is the class of Schwartz functions on $\mathbb{R}^n$.

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2. Proof of Theorem 1.2

Suppose that there exists a subset $J \subseteq \{1, \dots, m\}$ with $|J| \ge 2$ such that

$$\sum_{j\in J} \Big(\frac{1}{q'} - \frac{1}{p_j}\Big) < -\Big(1 - \frac{1}{q'}\Big),$$

which is equivalent to

$$\frac{1}{q} + \frac{|J|}{q'} < \sum_{j\in J} \frac{1}{p_j} =: \frac{1}{p_J}. \tag{2.1}$$

Here, we notice that $0 < p_J < 1$ as $|J| \ge 2$ and $\frac{1}{p_J} \ge 1 + \frac{|J|-1}{q'}$, and $1/p_J < |J|$ since $1 < p_j$ for all $j \in J$. Then we will show that there exists an $\Omega \in L^q(\mathbb{S}^{mn-1})$ with mean value zero such that

$$\|\mathcal{L}_\Omega\|_{L^{p_1}\times\cdots\times L^{p_m}\to L^p} = \infty.$$

Let $v_n = |B(0,1)|$ be the volume of the unit ball in $\mathbb{R}^n$. For any natural numbers N greater than 2, we define

$$f_j^N(y) := \begin{cases} v_n^{-1/p_j} 2^{Nn/p_j} \chi_{B(0,2^{-N})}(y), & j \in J \\ v_n^{-1/p_j} 2^{2n/p_j} \chi_{B(0,2^{-2})}(y), & j \in J_m \setminus J \end{cases}$$

so that the L^{p_j} norms of f_j are equal to 1 for all $j \in J_m$.

For $k \ge 2$, we define V_k^J to be a tubular neighborhood of radius comparable to 2^{-k} of the subspace $\{(x_1,\dots,x_m) \in (\mathbb{R}^n)^m : x_{\mathrm{i}} = x_j \text{ for i}, j \in J\}$. Precisely, we define

$$V_k^J := \bigcup_{x_0\in\mathbb{R}^n} \Big\{(y_1,\dots,y_m) \in (\mathbb{R}^n)^m : \ |x_0 - y_j| < \frac{4}{3\sqrt{|J|}} 2^{-k} \quad \text{for} \ \ j \in J\Big\}.$$

Then we define the function

$$\omega_k(\vec{z}\,) := 2^{kn(|J|-1/p_J)} \chi_{V_k^J \cap \mathbb{S}^{mn-1}}(\vec{z}\,)$$

on the sphere. We observe that the spherical measure of $V_k^J \cap \mathbb{S}^{mn-1}$ is proportional to $2^{-kn(|J|-1)}$ as we have freedom on the variables y_j for $j \in J_m \setminus J$ and on only one variable among y_j for $j \in J$. Therefore

$$\int_{\mathbb{S}^{mn-1}} \omega_k \, d\sigma \sim 2^{kn((|J|-1/p_J)-(|J|-1))} = 2^{-kn(1/p_J-1)}.$$

As $p_J < 1$, this expression tends to 0 like a power of 2^{-k} as $k \to \infty$. We set

$$\Omega_k := \omega_k - \alpha_k \chi_{(V_2^J)^c \cap \mathbb{S}^{mn-1}},$$

where α_k is a positive constant chosen so that Ω_k has vanishing integral. Note that $\alpha_k \sim 2^{-kn(1/p_J-1)}$ and

$$\begin{aligned} \big\|\Omega_k\big\|_{L^q(\mathbb{S}^{mn-1})}^q &\le 2^{knq(|J|-1/p_J)} \sigma(V_k^J \cap \mathbb{S}^{mn-1}) + \alpha_k^q \\ &\lesssim 2^{-kn(|J|-1-q(|J|-1/p_J))} + 2^{-knq(1/p_J-1)} \lesssim 2^{-\epsilon' kn} \end{aligned} \tag{2.2}$$

where $\epsilon' := \min\big\{|J| - 1 - q(|J| - 1/p_J), q(1/p_J - 1)\big\} > 0$, in view of (2.1), which is equivalent to $|J| - 1 - q(|J| - 1/p_J) > 0$. We now set

$$\Omega := \sum_{k=2}^{\infty} k \, \Omega_k$$

and then the estimate (2.2) clearly yields $\Omega \in L^q(\mathbb{S}^{mn-1})$.

We now see that for $x \in \mathbb{R}^n$ satisfying $1 < |x| < 2$ we have

$$\begin{aligned} \mathcal{L}_\Omega\big(f_1^N,\dots,f_m^N\big)(x) &= \sum_{k=2}^{\infty} k\mathcal{L}_{\Omega_k}\big(f_1^N,\dots,f_m^N\big)(x) \\ &= \sum_{k=2}^{\infty} k\mathcal{L}_{\omega_k}\big(f_1^N,\dots,f_m^N\big)(x) - \sum_{k=2}^{\infty} k \, \alpha_k \, \mathcal{L}_{\chi_{(V_2^J)^c\cap\mathbb{S}^{mn-1}}}\big(f_1^N,\dots,f_m^N\big)(x) \\ &=: \Xi_1(x) - \Xi_2(x) \end{aligned}$$

The term Ξ_2 is an error term for sufficiently large N. Indeed, if

$$1<|x|<2 \quad \text{and} \quad |x-y_j| \le \begin{cases} 2^{-N}, & j\in J \\ 2^{-2}, & j\in J_m\setminus J \end{cases}, \tag{2.3}$$

then

$$\frac{3}{4}\sqrt{m} < (|x|-2^{-2})\sqrt{m} \le |\vec{\boldsymbol{y}}| \le (|x|+2^{-2})\sqrt{m} < \frac{9}{4}\sqrt{m} \tag{2.4}$$

and thus,

$$\mathcal{L}_{\chi_{(V_2^J)^c\cap\mathbb{S}^{mn-1}}}\big(f_1^N,\dots,f_m^N\big)(x) \le \mathcal{L}_{\chi_{\mathbb{S}^{mn-1}}}\big(f_1^N,\dots,f_m^N\big)(x) \lesssim 2^{-Nn(|J|-1/p_J)},$$

which yields that

$$\Xi_2(x) \lesssim 2^{-Nn(|J|-1/p_J)} \sum_{k=2}^{\infty} k\, 2^{-kn(1/p_J-1)} \lesssim 2^{-Nn(|J|-1/p_J)} \le 1 \tag{2.5}$$

when N is large.

Moreover, (2.3) and (2.4) imply that

$$\left|\frac{x}{|\vec{\boldsymbol{y}}|}-\frac{y_j}{|\vec{\boldsymbol{y}}|}\right| \le 2^{-N}|\vec{\boldsymbol{y}}|^{-1} < \frac{4}{3\sqrt{m}}2^{-N} \le \frac{4}{3\sqrt{|J|}}2^{-N} \quad \text{for all } \ j\in J$$

and thus $\vec{\boldsymbol{y}}/|\vec{\boldsymbol{y}}|\in V_N^J$. In other words, $\omega_N(\vec{\boldsymbol{y}}/|\vec{\boldsymbol{y}}|)=2^{Nn(|J|-1/p_J)}$ for $\vec{\boldsymbol{y}}$ satisfying (2.3). This combined with the fact that $\mathcal{L}_{\omega_k}\big(f_1^N,\dots,f_m^N\big)\ge 0$ shows that

$$\begin{aligned}\Xi_1(x) &\ge N\mathcal{L}_{\omega_N}\big(f_1^N,\dots,f_m^N\big)(x)\\ &\gtrsim N2^{Nn(|J|-1/p_J)}\int_{(\mathbb{R}^n)^m} f_1^N(y_1)\cdots f_m^N(y_m)\,d\vec{\boldsymbol{y}} \sim N.\end{aligned}$$

This, together with (2.5), proves that for sufficiently large N

$$\mathcal{L}_\Omega\big(f_1^N,\dots,f_m^N\big)(x) \gtrsim N \quad \text{when } \ 1<|x|<2$$

and thus

$$\big\|\mathcal{L}_\Omega(f_1^N,\dots,f_m^N)\big\|_{L^p(\mathbb{R}^n)} \ge \big\|\mathcal{L}_\Omega(f_1^N,\dots,f_m^N)\big\|_{L^p(\{x\in\mathbb{R}^n:1<|x|<2\})} \gtrsim N.$$

Since N can be taken arbitrary large, we conclude the proof.

3. Proof of Theorem 1.1

We choose a Schwartz function $\Phi^{(m)}$ on $(\mathbb{R}^n)^m$ such that its Fourier transform $\widehat{\Phi^{(m)}}$ is supported in the annulus $\{\vec{\boldsymbol{\xi}}\in(\mathbb{R}^n)^m : 1/2\le|\vec{\boldsymbol{\xi}}|\le 2\}$ and satisfies $\sum_{j\in\mathbb{Z}}\widehat{\Phi_j^{(m)}}(\vec{\boldsymbol{\xi}})=1$ for $\vec{\boldsymbol{\xi}}\neq\vec{\boldsymbol{0}}$ where $\widehat{\Phi_j^{(m)}}(\vec{\boldsymbol{\xi}}):=\widehat{\Phi^{(m)}}(\vec{\boldsymbol{\xi}}/2^j)$. Recall that $K(\vec{\boldsymbol{y}}):=\frac{\Omega(\vec{\boldsymbol{y}}')}{|\vec{\boldsymbol{y}}|^{mn}}$ for $\vec{\boldsymbol{y}}\neq 0$. For $\gamma\in\mathbb{Z}$ let

$$K^\gamma(\vec{\boldsymbol{y}}):=\widehat{\Phi^{(m)}}(2^\gamma\vec{\boldsymbol{y}})K(\vec{\boldsymbol{y}}), \quad \vec{\boldsymbol{y}}\in(\mathbb{R}^n)^m$$

and then we observe that $K^\gamma(\vec{\boldsymbol{y}})=2^{\gamma mn}K^0(2^\gamma\vec{\boldsymbol{y}})$. For $\mu\in\mathbb{Z}$ we define

$$K_\mu^\gamma(\vec{\boldsymbol{y}}):=\Phi_{\mu+\gamma}^{(m)}*K^\gamma(\vec{\boldsymbol{y}})=2^{\gamma mn}[\Phi_\mu^{(m)}*K^0](2^\gamma\vec{\boldsymbol{y}})=2^{\gamma mn}K_\mu^0(2^\gamma\vec{\boldsymbol{y}}) \tag{3.1}$$

and

$$K_\mu(\vec{\boldsymbol{y}}):=\sum_{\gamma\in\mathbb{Z}}K_\mu^\gamma(\vec{\boldsymbol{y}}).$$

This decomposition yields

$$K = \sum_{\mu\in\mathbb{Z}} K_\mu, \qquad \mathcal{L}_\Omega = \sum_{\mu\in\mathbb{Z}} L_\mu,$$

where L_μ is a multilinear operator associated with the kernel K_μ defined by

$$\mathcal{L}_\mu\big(f_1,\dots,f_m\big)(x) := \int_{(\mathbb{R}^n)^m} K_\mu(\vec{\boldsymbol{y}}) \prod_{j=1}^m f_j(x-y_j)\, d\vec{\boldsymbol{y}}, \quad x\in\mathbb{R}^n.$$

Then we have

$$\begin{aligned}\big\|\mathcal{L}_\Omega(f_1,\dots,f_m)\big\|_{L^p(\mathbb{R}^n)} \lesssim \Big\|\sum_{\mu<0}\mathcal{L}_\mu(f_1,\dots,f_m)\Big\|_{L^p(\mathbb{R}^n)} \\ + \Big\|\sum_{\mu\ge 0}\mathcal{L}_\mu(f_1,\dots,f_m)\Big\|_{L^p(\mathbb{R}^n)}.\end{aligned} \tag{3.2}$$

First, we recall the following estimate mentioned in [18, (30)]: for $f_j\in\mathcal{S}(\mathbb{R}^n)$

$$\big\|\mathcal{L}_\mu(f_1,\dots,f_m)\big\|_{L^p(\mathbb{R}^n)} \lesssim \|\Omega\|_{L^q(\mathbb{S}^{mn-1})}\Big(\prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)}\Big)\begin{cases}2^{(mn-\delta)\mu}, & \mu\ge 0\\ 2^{(1-\delta)\mu}, & \mu<0\end{cases} \tag{3.3}$$

for all $1<q<\infty$ and $0<\delta<1/q'$. For the sake of completeness, we sketch the proof of (3.3): We first see that Duoandikoetxea and Rubio de Francia [14] proved that if $1<q<\infty$ and $0<\delta<1/q'$, then

$$\begin{aligned}\big|\widehat{K^0}(\vec{\boldsymbol{\xi}}\,)\big| &\lesssim \|\Omega\|_{L^q(\mathbb{S}^{mn-1})}\min\big\{|\vec{\boldsymbol{\xi}}\,|, |\vec{\boldsymbol{\xi}}\,|^{-\delta}\big\}\\ \big|\partial^\alpha\widehat{K^0}(\vec{\boldsymbol{\xi}}\,)\big| &\lesssim \|\Omega\|_{L^q(\mathbb{S}^{mn-1})}\min\big\{1, |\vec{\boldsymbol{\xi}}\,|^{-\delta}\big\}, \qquad \alpha\ne\vec{0}\end{aligned}$$

and this proves that

$$\begin{aligned}\big|\widehat{K_\mu}(\vec{\boldsymbol{\xi}}\,)\big| &\lesssim \|\Omega\|_{L^q(\mathbb{S}^{mn-1})}\min\big\{2^\mu, 2^{-\delta\mu}\big\}\\ \big|\partial^\alpha\widehat{K_\mu}(\vec{\boldsymbol{\xi}}\,)\big| &\lesssim \|\Omega\|_{L^q(\mathbb{S}^{mn-1})}\min\big\{2^{\mu|\alpha|}, 2^{\mu(mn-\delta)}\big\}, \qquad 1\le|\alpha|\le mn.\end{aligned}$$

Finally, we have

$$\big|\partial^\alpha\widehat{K_\mu}(\vec{\boldsymbol{\xi}}\,)\big| \lesssim \|\Omega\|_{L^q(\mathbb{S}^{mn-1})}\begin{cases}2^{(mn-\delta)\mu}, & \mu\ge 0\\ 2^{(1-\delta)\mu}, & \mu<0\end{cases}$$

for all multi-indices α with $|\alpha|\le mn$. Since $\mathcal{L}_\mu$ is m-linear multiplier operator associated with $\widehat{K_\mu}$, a multi-linear version of Mihlin's multiplier theory (see [8, 23]) yields (3.3). We may also refer to [16, Theorems 7.5.3 and 7.5.5] for the multilinear multiplier theories. Then (3.3) implies that

$$\Big\|\sum_{\mu<0}\mathcal{L}_\mu(f_1,\dots,f_m)\Big\|_{L^p(\mathbb{R}^n)} \lesssim \|\Omega\|_{L^2(\mathbb{S}^{mn-1})}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\mathbb{R}^n)}.$$

It remains to estimate the term (3.2). This can be reduced to proving that for $\mu\ge 0$, there exists $\delta_0>0$, possibly depending on $p_1,\dots,p_m$, such that

$$\big\|\mathcal{L}_\mu(f_1,\dots,f_m)\big\|_{L^p(\mathbb{R}^n)} \lesssim 2^{-\delta_0\mu}\|\Omega\|_{L^{\frac{1}{1-s}}(\mathbb{S}^{mn-1})}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\mathbb{R}^n)}, \tag{3.4}$$

which improves (3.3) for $\mu \geq 0$. It is already proved in [18, (31)] that

$$\big\|\mathcal{L}_\mu(f_1,\dots,f_m)\big\|_{L^{2/m}(\mathbb{R}^n)} \lesssim_{\widetilde{\delta}} 2^{-\widetilde{\delta}\mu}\|\Omega\|_{L^2(\mathbb{S}^{mn-1})}\prod_{j=1}^m \|f_j\|_{L^2(\mathbb{R}^n)} \tag{3.5}$$

for some $\widetilde{\delta} > 0$. In order to achieve the estimate (3.4), we shall use interpolation methods between (3.5) and the estimates in the following proposition.

Proposition 3.1. *Let $1/2 \leq s < 1$, $1/m < p < \infty$, and $1 < p_j < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$ and $(1/p_1,\dots,1/p_m) \in \mathcal{H}^m(s)$. Suppose that $\mu \geq 0$, $\Omega \in L^{\frac{1}{1-s}}(\mathbb{S}^{mn-1})$, and $\int_{\mathbb{S}^{mn-1}} \Omega\, d\sigma = 0$. Then for any $0 < \epsilon < 1$, there exists a constant $C_\epsilon > 0$ such that*

$$\big\|\mathcal{L}_\mu(f_1,\dots,f_m)\big\|_{L^p(\mathbb{R}^n)} \leq C_\epsilon 2^{\epsilon\mu}\|\Omega\|_{L^{\frac{1}{1-s}}(\mathbb{S}^{mn-1})}\prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)} \tag{3.6}$$

for Schwartz functions $f_1,\dots,f_m$ on $\mathbb{R}^n$.

The proof of the proposition will be provided in the last section.

We present a multilinear version of the Marcinkiewicz interpolation theorem, which is a straightforward corollary of [21, Theorem 1.1] or [28, Theorem 3].

Lemma 3.2. [21, 28] *Let $0 < p_j^{\mathrm{i}} \leq \infty$ for each $j \in J_m$ and $\mathrm{i} = 0,1,\dots,m$, and $0 < p^{\mathrm{i}} \leq \infty$ satisfy $1/p^{\mathrm{i}} = 1/p_1^{\mathrm{i}} + \cdots + 1/p_m^{\mathrm{i}}$ for $\mathrm{i} = 0,1,\dots,m$. Suppose that T is an m-linear operator having the mapping properties*

$$\big\|T(f_1,\dots,f_m)\big\|_{L^{p^{\mathrm{i}},\infty}(\mathbb{R}^n)} \leq M_{\mathrm{i}}\prod_{j=1}^m \|f_j\|_{L^{p_j^{\mathrm{i}}}(\mathbb{R}^n)}, \quad \mathrm{i} = 0,1,\dots,m$$

for Schwartz functions $f_1,\dots,f_m$ on $\mathbb{R}^n$. Given $0 < \theta_{\mathrm{i}} < 1$ with $\sum_{\mathrm{i}=0}^m \theta_{\mathrm{i}} = 1$, set

$$\frac{1}{p_j} = \sum_{\mathrm{i}=0}^m \frac{\theta_{\mathrm{i}}}{p_j^{\mathrm{i}}}, \quad j \in J_m, \qquad \frac{1}{p} = \sum_{\mathrm{i}=0}^m \frac{\theta_{\mathrm{i}}}{p^{\mathrm{i}}}.$$

Then for Schwartz functions $f_1,\dots,f_m$ on $\mathbb{R}^n$ we have

$$\big\|T(f_1,\dots,f_m)\big\|_{L^{p,\infty}(\mathbb{R}^n)} \lesssim M_0^{\theta_0}\cdots M_m^{\theta_m}\prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)}.$$

Also, if the points $(\frac{1}{p_1^{\mathrm{i}}},\dots,\frac{1}{p_m^{\mathrm{i}}})$, $0 \leq \mathrm{i} \leq m$, form a non trivial open simplex in $\mathbb{R}^m$, then

$$\big\|T(f_1,\dots,f_m)\big\|_{L^p(\mathbb{R}^n)} \lesssim M_0^{\theta_0}\cdots M_m^{\theta_m}\prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)}.$$

Now taking Proposition 3.1 temporarily for granted, let us complete the proof of (3.4). We first fix $p_1,\dots,p_m$ such that $P := (1/p_1,\dots,1/p_m) \in \mathcal{H}^m(s)$ is not equal to $T := (1/2,\dots,1/2)$. Then there exists the unique point $Q := (1/q_1,\dots,1/q_m)$ on the boundary of $\mathcal{H}^m(s)$ such that

$$(1-\theta)T + \theta Q = P$$

for some $0 < \theta < 1$. Now let $R := (1/r_1,\dots,1/r_m)$ be the middle point of P and Q. We note that R is inside $\mathcal{H}^m(s)$ because $\mathcal{H}^m(s)$ is convex. Since $R = \frac{1}{2}P + \frac{1}{2}Q$, we have

$$(1-\widetilde{\theta})T + \widetilde{\theta}R = P \tag{3.7}$$

where

$$0 < \widetilde{\theta} := \frac{2}{1/\theta + 1} < 1.$$

Here, $\widetilde{\theta}$ definitely depends on the point P as θ does. Moreover, since R is contained in the open set $\mathcal{H}^m(s)$, we may choose m distinct points $R^1, \dots, R^m \in \mathcal{H}^m(s)$ such that $R \neq R^{\mathrm{i}}$ for all $\mathrm{i} \in J_m$ and

$$R = \theta_1 R^1 + \cdots + \theta_m R^m$$

for some $0 < \theta_1, \dots, \theta_m < 1$ with $\theta_1 + \cdots + \theta_m = 1$. This, together with (3.7), clearly yields that

$$P = (1-\widetilde{\theta})T + \widetilde{\theta}\theta_1 R^1 + \cdots + \widetilde{\theta}\theta_m R^m \tag{3.8}$$

where $0 < 1-\widetilde{\theta} < 1$, $0 < \widetilde{\theta}\theta_{\mathrm{i}} < 1$, and $(1-\widetilde{\theta}) + \widetilde{\theta}\theta_1 + \cdots + \widetilde{\theta}\theta_m = 1$. From the estimate (3.5), it follows that

$$\|\mathcal{L}_\mu\|_{L^2 \times \cdots \times L^2 \to L^{2/m}} \lesssim 2^{-\widetilde{\delta}\mu} \|\Omega\|_{L^{\frac{1}{1-s}}(\mathbb{S}^{mn-1})} \quad \text{at } \ T = (1/2, \dots, 1/2) \tag{3.9}$$

where the embedding $L^{\frac{1}{1-s}}(\mathbb{S}^{mn-1}) \hookrightarrow L^2(\mathbb{S}^{mn-1})$ is applied. On the other hand, letting $\epsilon_P := \frac{(1-\widetilde{\theta})\widetilde{\delta}}{2\widetilde{\theta}} > 0$ (which can be made less than 1), Proposition 3.1 implies that for each $\mathrm{i} \in J_m$ one has

$$\|\mathcal{L}_\mu\|_{L^{r_1^{\mathrm{i}}} \times \cdots \times L^{r_m^{\mathrm{i}}} \to L^{r^{\mathrm{i}}}} \lesssim 2^{\epsilon_P \mu} \|\Omega\|_{L^{\frac{1}{1-s}}(\mathbb{S}^{mn-1})} \quad \text{at } \ R^{\mathrm{i}} = (1/r_1^{\mathrm{i}}, \dots, 1/r_m^{\mathrm{i}}) \in \mathcal{H}(s) \tag{3.10}$$

where $1/r^{\mathrm{i}} = 1/r_1^{\mathrm{i}} + \cdots + 1/r_m^{\mathrm{i}}$. Now interpolating, via Lemma 3.2, between (3.9) and m points R^{i} satisfying (3.10) yields

$$\|\mathcal{L}_\mu\|_{L^{p_1} \times \cdots \times L^{p_m} \to L^p} \lesssim 2^{-\mu[(1-\widetilde{\theta})\widetilde{\delta} - \widetilde{\theta}\epsilon_P]} \|\Omega\|_{L^{\frac{1}{1-s}}(\mathbb{S}^{mn-1})},$$

in view of (3.8). Here, a straightforward computation shows that

$$(1-\widetilde{\theta})\widetilde{\delta} - \widetilde{\theta}\epsilon_P = \frac{(1-\widetilde{\theta})\widetilde{\delta}}{2}.$$

See Figure 2 for the interpolation.

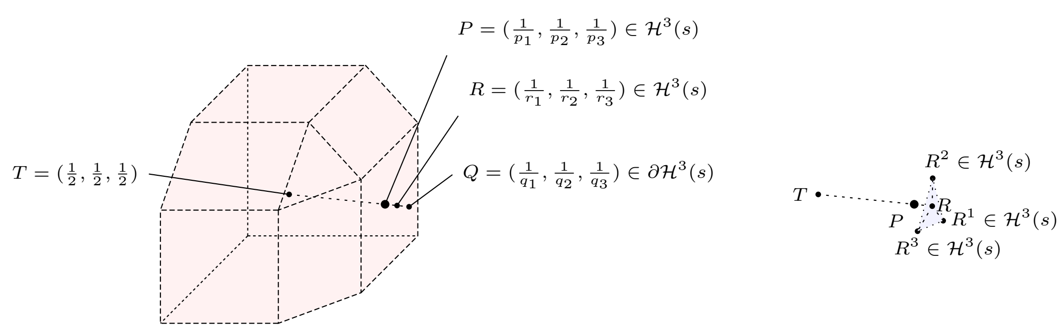

FIGURE 2. $(1-\widetilde{\theta})T + \widetilde{\theta}R = P$ and $\theta_1 R^1 + \theta_2 R^2 + \theta_3 R^3 = R$ for $m = 3$

Finally, (3.4) follows from choosing $\delta_0 = \frac{(1-\widetilde{\theta})\widetilde{\delta}}{2}$ and this completes the proof of Theorem 1.1.

4. Preliminaries for Proposition 3.1

4.1. **Maximal inequalities.** Let $\mathcal{M}$ be the Hardy-Littlewood maximal operator, defined by

$$\mathcal{M}f(x) := \sup_{Q:x\in Q} \frac{1}{|Q|}\int_Q |f(y)|dy$$

where the supremum ranges over all cubes containing x, and for $0<t<\infty$ let $\mathcal{M}_t f := \big(\mathcal{M}(|f|^t)\big)^{1/t}$. Then the maximal operator $\mathcal{M}_t$ is bounded on $L^p(\mathbb{R}^n)$ for $t<p\le\infty$ and more generally, for $t<p,q<\infty$, we have

$$\Big\| \Big(\sum_{\gamma\in\mathbb{Z}} (\mathcal{M}_t f_\gamma)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \lesssim \Big\| \Big(\sum_{\gamma\in\mathbb{Z}} |f_\gamma|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \tag{4.1}$$

for any sequence of Schwartz functions $\{f_\gamma\}_{\gamma\in\mathbb{Z}}$. See [15, Theorem 5.6.6]. The inequality (4.1) also holds for $0<p\le\infty$ and $q=\infty$.

4.2. **Compactly supported wavelets.** For any fixed $L\in\mathbb{N}$ one can construct real-valued compactly supported functions ψ_F, ψ_M in $\mathcal{C}^L(\mathbb{R})$ satisfying the following properties: $\|\psi_F\|_{L^2(\mathbb{R})} = \|\psi_M\|_{L^2(\mathbb{R})} = 1$, $\int_{\mathbb{R}} x^\alpha \psi_M(x)dx = 0$ for all $0\le\alpha\le L$, and moreover, if $\Psi_{\vec{\boldsymbol{G}}}$ is a function on $\mathbb{R}^{mn}$, defined by

$$\Psi_{\vec{\boldsymbol{G}}}(\vec{\boldsymbol{x}}) := \psi_{g_1}(x_1)\cdots\psi_{g_{mn}}(x_{mn})$$

for $\vec{\boldsymbol{x}} := (x_1,\dots,x_{mn})\in\mathbb{R}^{mn}$ and $\vec{\boldsymbol{G}} := (g_1,\dots,g_{mn})$ in the set

$$\mathcal{I} := \big\{ \vec{\boldsymbol{G}} := (g_1,\dots,g_{mn}) : g_{\mathrm{i}} \in \{F,M\} \big\},$$

then the family of functions

$$\bigcup_{\lambda\in\mathbb{N}_0} \bigcup_{\vec{\boldsymbol{k}}\in\mathbb{Z}^{mn}} \big\{ 2^{\lambda mn/2}\Psi_{\vec{\boldsymbol{G}}}(2^\lambda \vec{\boldsymbol{x}} - \vec{\boldsymbol{k}}) : \vec{\boldsymbol{G}} \in \mathcal{I}^\lambda \big\}$$

forms an orthonormal basis of $L^2(\mathbb{R}^{mn})$, where $\mathcal{I}^0 := \mathcal{I}$ and for $\lambda\ge 1$, we set $\mathcal{I}^\lambda := \mathcal{I}\setminus\{(F,\dots,F)\}$.

It is known in [30, Theorem 1.64] that if L is sufficiently large, then every $H\in L^q(\mathbb{R}^{mn})$ with $1<q<\infty$ can be represented as

$$H(\vec{\boldsymbol{x}}) = \sum_{\lambda\in\mathbb{N}_0} \sum_{\vec{\boldsymbol{G}}\in\mathcal{I}^\lambda} \sum_{\vec{\boldsymbol{k}}\in\mathbb{Z}^{mn}} b^\lambda_{\vec{\boldsymbol{G}},\vec{\boldsymbol{k}}} 2^{\lambda mn/2} \Psi_{\vec{\boldsymbol{G}}}(2^\lambda\vec{\boldsymbol{x}} - \vec{\boldsymbol{k}}) \tag{4.2}$$

where the right hand side converges in $\mathcal{S}'(\mathbb{R}^{mn})$, and

$$\Big\| \Big(\sum_{\lambda\in\mathbb{N}_0} \sum_{\vec{G}\in\mathcal{I}^\lambda} \sum_{\vec{k}\in\mathbb{Z}^{mn}} \big| b^\lambda_{\vec{\boldsymbol{G}},\vec{\boldsymbol{k}}} \Psi^\lambda_{\vec{\boldsymbol{G}},\vec{\boldsymbol{k}}} \big|^2 \Big)^{1/2} \Big\|_{L^q(\mathbb{R}^{mn})} \lesssim \|H\|_{L^q(\mathbb{R}^{mn})} \tag{4.3}$$

where $\Psi^\lambda_{\vec{\boldsymbol{G}},\vec{\boldsymbol{k}}}(\vec{x}) = 2^{\lambda mn/2}\Psi_{\vec{\boldsymbol{G}}}(2^\lambda\vec{\boldsymbol{x}} - \vec{\boldsymbol{k}})$,

$$b^\lambda_{\vec{\boldsymbol{G}},\vec{\boldsymbol{k}}} := \int_{\mathbb{R}^{mn}} H(\vec{\boldsymbol{x}}) \Psi^\lambda_{\vec{\boldsymbol{G}},\vec{\boldsymbol{k}}}(\vec{x}) d\vec{\boldsymbol{x}}.$$

Moreover, it follows from (4.3) and the disjoint support property of the $\Psi^\lambda_{\vec{\boldsymbol{G}},\vec{\boldsymbol{k}}}$'s that

$$\big\| \{ b^\lambda_{\vec{\boldsymbol{G}},\vec{\boldsymbol{k}}} \}_{\vec{\boldsymbol{k}}\in\mathbb{Z}^{mn}} \big\|_{\ell^q} \approx \Big(2^{\lambda mn(1-q/2)} \int_{\mathbb{R}^{mn}} \Big(\sum_{\vec{k}} \big| b^\lambda_{\vec{\boldsymbol{G}},\vec{\boldsymbol{k}}} \Psi^\lambda_{\vec{\boldsymbol{G}},\vec{\boldsymbol{k}}}(\vec{x}) \big|^2 \Big)^{q/2} d\vec{x} \Big)^{1/q}$$

$$\lesssim 2^{-\lambda mn(1/2-1/q)}\|H\|_{L^q(\mathbb{R}^{mn})}. \tag{4.4}$$

Throughout, we will consistently use the notation $G_j := (g_{(j-1)n+1}, \dots, g_{jn})$ for an element of $\{F, M\}^n$ and $\Psi_{G_j}(\xi_j) := \psi_{g_{(j-1)n+1}}(\xi_j^1) \cdots \psi_{g_{jn}}(\xi_j^n)$ for $\xi_j := (\xi_j^1, \dots, \xi_j^n) \in \mathbb{R}^n$ so that $\vec{\boldsymbol{G}} = (G_1, \dots, G_m) \in (\{F, M\}^n)^m$ and $\Psi_{\vec{\boldsymbol{G}}}(\vec{\boldsymbol{\xi}}\,) = \Psi_{G_1}(\xi_1) \cdots \Psi_{G_m}(\xi_m)$. For each $\vec{\boldsymbol{k}} := (k_1, \dots, k_m) \in (\mathbb{Z}^n)^m$ and $\lambda \in \mathbb{N}_0$, let

$$\Psi^\lambda_{G_j,k_j}(\xi_j) := 2^{\lambda n/2}\Psi_{G_j}(2^\lambda \xi_j - k_j), \qquad 1 \le j \le m$$

and

$$\Psi^\lambda_{\vec{\boldsymbol{G}},\vec{\boldsymbol{k}}}(\vec{\boldsymbol{\xi}}\,) := \Psi^\lambda_{G_1,k_1}(\xi_1) \cdots \Psi^\lambda_{G_m,k_m}(\xi_m).$$

We also assume that the support of ψ_{g_j} is contained in $\{\xi \in \mathbb{R} : |\xi| \le C_0\}$ for some $C_0 > 1$, which implies that

$$\operatorname{supp}(\Psi^\lambda_{G_j,k_j}) \subset \big\{\xi_j \in \mathbb{R}^n : |2^\lambda \xi_j - k_j| \le C_0\sqrt{n}\big\}. \tag{4.5}$$

In other words, the support of $\Psi^\lambda_{G_j,k_j}$ is contained in the ball centered at $2^{-\lambda}k_j$ and radius $C_0\sqrt{n}2^{-\lambda}$.

4.3. **Columns and Projections.** We now introduce a few notions and related combinatorial properties. For a fixed $\vec{\boldsymbol{k}} \in (\mathbb{Z}^n)^m$, $l \in J_m = \{1, 2, \dots, m\}$, and $1 \le j_1 < \cdots < j_l \le m$ let

$$\vec{\boldsymbol{k}}^{j_1,\dots,j_l} := (k_{j_1}, \dots, k_{j_l})$$

denote the vector in $(\mathbb{Z}^n)^l$ consisting of the $j_1, \dots, j_l$ components of $\vec{\boldsymbol{k}}$ and $\vec{\boldsymbol{k}}^{*j_1,j_2,\dots,j_l}$ stand for the vector in $(\mathbb{Z}^n)^{m-l}$, consisting of $\vec{\boldsymbol{k}}$ except for the $j_1, \dots, j_l$ components (e.g. $\vec{\boldsymbol{k}}^{*1,\dots,j} = \vec{\boldsymbol{k}}^{j+1,\dots,m} = (k_{j+1}, \dots, k_m) \in (\mathbb{Z}^n)^{m-j}$). For any sets $\mathcal{U}$ in $(\mathbb{Z}^n)^m$, $j \in J_m$, and $1 \le j_1 < \cdots < j_l \le m$ let

$$\mathcal{P}_j\mathcal{U} := \big\{k_j \in \mathbb{Z}^n : \vec{\boldsymbol{k}} \in \mathcal{U} \ \text{ for some } \vec{\boldsymbol{k}}^{*j} \in (\mathbb{Z}^n)^{m-1}\big\}$$

$$\mathcal{P}_{*j_1,\dots,j_l}\mathcal{U} := \big\{\vec{\boldsymbol{k}}^{*j_1,\dots,j_l} \in (\mathbb{Z}^n)^{m-l} : \vec{\boldsymbol{k}} \in \mathcal{U} \ \text{ for some } k_{j_1}, \dots, k_{j_l} \in \mathbb{Z}^n\big\}$$

be the projections of $\mathcal{U}$ onto the k_j-column and $\vec{\boldsymbol{k}}^{*j_1,\dots,j_l}$-plane, respectively. For a fixed $\vec{\boldsymbol{k}}^{*j_1,\dots,j_l} \in \mathcal{P}_{*j_1,\dots,j_l}\mathcal{U}$, we define

$$Col^{\mathcal{U}}_{\vec{\boldsymbol{k}}^{*j_1,\dots,j_l}} := \{\vec{\boldsymbol{k}}^{j_1,\dots,j_l} \in (\mathbb{Z}^n)^l : \vec{\boldsymbol{k}} = (k_1, \dots, k_m) \in \mathcal{U}\}.$$

Then we observe that

$$\sum_{\vec{\boldsymbol{k}} \in \mathcal{U}} \cdots = \sum_{\vec{\boldsymbol{k}}^{*j_1,\dots,j_l} \in \mathcal{P}_{*j_1,\dots,j_l}\mathcal{U}} \Big(\sum_{\vec{\boldsymbol{k}}^{j_1,\dots,j_l} \in Col^{\mathcal{U}}_{\vec{\boldsymbol{k}}^{*j_1,\dots,j_l}}} \cdots \Big). \tag{4.6}$$

For more details of these notations and their applications, we refer to [18], while similar ideas go back to [17].

5. Key Lemmas for the proof of Proposition 3.1

Let ϕ be a Schwartz function on $\mathbb{R}^n$ whose Fourier transform is supported in the annulus $\{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\}$ and satisfies $\sum_{\gamma\in\mathbb{Z}} \widehat{\phi_\gamma}(\xi) = 1$ for $\xi \neq 0$, where $\phi_\gamma := 2^{\gamma n}\phi(2^\gamma \cdot)$. For each $\gamma \in \mathbb{Z}$, we define the convolution operator Λ_γ by $\Lambda_\gamma f := \phi_\gamma * f$.

Let C_0 be the constant that appeared in (4.5). For $\lambda \in \mathbb{N}_0$ satisfying $C_0\sqrt{n} \le 2^{\lambda+1}$, let

$$\mathcal{W}^\lambda := \big\{k \in \mathbb{Z}^n : 2C_0\sqrt{n} \le |k| \le 2^{\lambda+2}\big\}.$$

For $\lambda \in \mathbb{N}_0$, $G \in \{F, M\}^n$, $k \in \mathbb{Z}^n$, and $\gamma \in \mathbb{Z}$, we define the operator $L_{G,k}^{\lambda,\gamma}$ via the Fourier transform by

$$\big(L_{G,k}^{\lambda,\gamma} f\big)^\wedge(\xi) := \Psi_{G,k}^\lambda(\xi/2^\gamma)\widehat{f}(\xi), \qquad \gamma \in \mathbb{Z}. \tag{5.1}$$

Then we observe that

$$\big|L_{G,k}^{\lambda,\gamma} f(x)\big| \lesssim 2^{\lambda n/2}\mathcal{M}f(x) \quad \text{uniformly in the parameters } \lambda, G, k, \gamma \tag{5.2}$$

and for $k \in \mathcal{W}^{\lambda+\mu}$ with $C_0\sqrt{n} \le 2^{\lambda+\mu+1}$,

$$L_{G,k}^{\lambda,\gamma} f = L_{G,k}^{\lambda,\gamma} f^{\lambda,\gamma,\mu} \tag{5.3}$$

due to the support of Ψ_G, where

$$f^{\lambda,\gamma,\mu} := \sum_{j=-\lambda+c_0}^{\mu+3} \Lambda_{\gamma+j} f$$

for some $c_0 \in \mathbb{N}$, depending on C_0 and n. It is easy to check that for $1 < p < \infty$

$$\Big\|\Big(\sum_{\gamma\in\mathbb{Z}} \big|f^{\lambda,\gamma,\mu}\big|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \le \sum_{j=-\lambda+c_0}^{\mu+3} \Big\|\Big(\sum_{\gamma\in\mathbb{Z}} \big|\Lambda_{\gamma+j} f\big|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \lesssim (\mu+\lambda+4)\|f\|_{L^p(\mathbb{R}^n)},$$

where the triangle inequality and the Littlewood-Paley theory are applied in the inequalities.

Lemma 5.1. *Let $2 \le p < \infty$, $1 < t < 2$, $u \in \mathbb{Z}^n$, and $s \ge 0$. Then we have*

$$\Big\|\Big(\sum_{\gamma\in\mathbb{Z}} \big\|\Lambda_\gamma f(x - 2^{s-\gamma}\cdot)\ \Psi_G^\vee\big\|_{L^t(u+[0,1)^n)}^2\Big)^{1/2}\Big\|_{L^p(x)} \lesssim_M \frac{1}{(1+|u|)^M}\|f\|_{L^p(\mathbb{R}^n)} \tag{5.4}$$

uniformly in $s \ge 0$.

Proof. Using integration by parts we can show that

$$\sup_{y\in u+[0,1)^n} |\Psi_G^\vee(y)| \lesssim_{M,t} \frac{1}{(1+|u|)^{M+n/t}} \qquad \text{for any } \ 0 < M < L - \tfrac{n}{t},$$

where L is the order of derivatives of compactly supported wavelets. It follows from this observation that

$$\begin{aligned}\big\|\Lambda_\gamma f(x - 2^{s-\gamma}\cdot)\ \Psi_G^\vee\big\|_{L^t(u+[0,1)^n)} &= \Big(\int_{u+[0,1)^n} \big|\Lambda_\gamma f(x - 2^{s-\gamma}y)\big|^t |\Psi_G^\vee(y)|^t dy\Big)^{1/t} \\ &\lesssim \frac{1}{(1+|u|)^{M+n/t}}\Big(\int_{u+[0,1)^n} \big|\Lambda_\gamma f(x - 2^{s-\gamma}y)\big|^t dy\Big)^{1/t}.\end{aligned}$$

Moreover, using a change of variables, we have

$$\Big(\int_{u+[0,1)^n} \big|\Lambda_\gamma f(x - 2^{s-\gamma}y)\big|^t dy\Big)^{1/t} \lesssim \Big(\frac{1}{2^{(s-\gamma)n}} \int_{|y|\le\sqrt{n}(1+|u|)2^{s-\gamma}} \big|\Lambda_\gamma f(x-y)\big|^t dy\Big)^{1/t}$$

$$\lesssim (1+|u|)^{n/t}\mathcal{M}_t\Lambda_\gamma f(x),$$

which proves

$$\big\|\Lambda_\gamma f(x-2^{s-\gamma}\cdot)\;\Psi_G^\vee\big\|_{L^t(u+[0,1)^n)} \lesssim_{M,t} \frac{1}{(1+|u|)^M}\mathcal{M}_t\Lambda_\gamma f(x).$$

Now the left-hand side of (5.4) is less than a constant times

$$\frac{1}{(1+|u|)^M}\big\|\{\mathcal{M}_t\Lambda_\gamma f\}_{\gamma\in\mathbb{Z}}\big\|_{L^p(\ell^2)} \lesssim \frac{1}{(1+|u|)^M}\big\|\{\Lambda_\gamma f\}_{\gamma\in\mathbb{Z}}\big\|_{L^p(\ell^2)} \sim \frac{1}{(1+|u|)^M}\|f\|_{L^p(\mathbb{R}^n)}$$

by using the maximal inequality (4.1) and the Littlewood-Paley theory. □

Lemma 5.2. *Let $2\le p<\infty$, $0<\epsilon<1$ and $\lambda,\mu\in\mathbb{Z}$ with $2^{\lambda+\mu+1}\ge C_0\sqrt{n}$. Suppose that $E^{\lambda+\mu}$ is a subset of $\mathcal{W}^{\lambda+\mu}$. Let $\{b_k^\gamma\}_{k\in\mathbb{Z}^n}$ be a sequence of complex numbers and*

$$\mathcal{B}_2 := \sup_{\gamma\in\mathbb{Z}}\big\|\{b_k^\gamma\}_{k\in\mathbb{Z}^n}\big\|_{\ell^2} \quad \text{and} \quad \mathcal{B}_\infty := \sup_{\gamma\in\mathbb{Z}}\big\|\{b_k^\gamma\}_{k\in\mathbb{Z}^n}\big\|_{\ell^\infty}.$$

Then there exists $C_\epsilon>0$ such that

$$\Big\|\Big(\sum_{\gamma\in\mathbb{Z}}\Big|\sum_{k\in E^{\lambda+\mu}} b_k^\gamma L_{G,k}^{\lambda,\gamma}f\Big|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \le C_\epsilon 2^{\lambda n/2}(\lambda+\mu+4)\mathcal{B}_2^{1-\epsilon}\mathcal{B}_\infty^\epsilon|E^{\lambda+\mu}|^\epsilon\|f\|_{L^p(\mathbb{R}^n)} \tag{5.5}$$

for $f\in\mathcal{S}(\mathbb{R}^n)$.

Proof. Using (5.3), the left-hand side of (5.5) is less than

$$\sum_{j=-\lambda+c_0}^{\mu+3}\Big\|\Big(\sum_{\gamma\in\mathbb{Z}}\Big|\sum_{k\in E^{\lambda+\mu}} b_k^\gamma L_{G,k}^{\lambda,\gamma}\Lambda_{\gamma+j}f\Big|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}.$$

Let $t:=\frac{2}{1+\epsilon}$ so that $1<t<2<t'=\frac{2}{1-\epsilon}$. Then we apply Hölder's inequality to obtain

$$\begin{aligned}
&\Big|\sum_{k\in E^{\lambda+\mu}} b_k^\gamma L_{G,k}^{\lambda,\gamma}\Lambda_{\gamma+j}f(x)\Big|\\
&\le 2^{\lambda n/2}\int_{\mathbb{R}^n}\big|B_{E^{\lambda+\mu}}^\gamma(y)\big|\big|\Lambda_{\gamma+j}f(x-2^{\lambda-\gamma}y)\big|\big|\Psi_G^\vee(y)\big|dy\\
&= 2^{\lambda n/2}\sum_{u\in\mathbb{Z}^n}\int_{u+[0,1)^n}\big|B_{E^{\lambda+\mu}}^\gamma(y)\big|\big|\Lambda_{\gamma+j}f(x-2^{\lambda-\gamma}y)\big|\big|\Psi_G^\vee(y)\big|dy\\
&\le 2^{\lambda n/2}\sum_{u\in\mathbb{Z}^n}\big\|B_{E^{\lambda+\mu}}^\gamma\big\|_{L^{t'}(u+[0,1)^n)}\big\|\Lambda_{\gamma+j}f(x-2^{\lambda-\gamma}\cdot)\Psi_G^\vee\big\|_{L^t(u+[0,1)^n)}
\end{aligned}$$

where

$$B_{E^{\lambda+\mu}}^\gamma(x) := \sum_{k\in E^{\lambda+\mu}} b_k^\gamma e^{2\pi i\langle x,k\rangle}.$$

We first observe that

$$\begin{aligned}
\big\|B_{E^{\lambda+\mu}}^\gamma\big\|_{L^{t'}(u+[0,1)^n)} &= \big\|B_{E^{\lambda+\mu}}^\gamma\big\|_{L^{t'}([0,1)^n)}\\
&\le \big\|B_{E^{\lambda+\mu}}^\gamma\big\|_{L^2([0,1)^n)}^{2/t'}\big\|B_{E^{\lambda+\mu}}^\gamma\big\|_{L^\infty([0,1]^n)}^{1-2/t'} \le \mathcal{B}_2^{1-\epsilon}\mathcal{B}_\infty^\epsilon|E^{\lambda+\mu}|^\epsilon.
\end{aligned} \tag{5.6}$$

Therefore, the left-hand side of (5.5) is dominated by a constant times

$$2^{\lambda n/2}\mathcal{B}_2^{1-\epsilon}\mathcal{B}_\infty^\epsilon|E^{\lambda+\mu}|^\epsilon\sum_{j=-\lambda+c_0}^{\mu+3}\Big\|\Big(\sum_{\gamma\in\mathbb{Z}}\Big(\sum_{u\in\mathbb{Z}^n}\big\|\Lambda_{\gamma+j}f(x-2^{\lambda-\gamma}\cdot)\Psi_G^\vee\big\|_{L^t(u+[0,1)^n)}\Big)^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}$$

$$\leq 2^{\lambda n/2}\mathcal{B}_2^{1-\epsilon}\mathcal{B}_\infty^{\epsilon}|E^{\lambda+\mu}|^{\epsilon}\sum_{u\in\mathbb{Z}^n}\sum_{j=-\lambda+c_0}^{\mu+3}\Big\|\Big(\sum_{\gamma\in\mathbb{Z}}\big\|\Lambda_\gamma f(x-2^{\lambda+j-\gamma}\cdot)\Psi_G^{\vee}\big\|_{L^t(u+[0,1)^n)}^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}.$$

Now it follows from Lemma 5.1 that the preceding expression is controlled by a constant multiple of

$$2^{\lambda n/2}\mathcal{B}_2^{1-\epsilon}\mathcal{B}_\infty^{\epsilon}|E^{\lambda+\mu}|^{\epsilon}(\lambda+\mu+4)\|f\|_{L^p(\mathbb{R}^n)}\sum_{u\in\mathbb{Z}^n}\frac{1}{(1+|u|)^M}$$

for $M>n$. The sum over $u\in\mathbb{Z}^n$ is obviously finite and this completes the proof of Lemma 5.2. □

Lemma 5.3. *Let $2\leq l\leq m$, $2\leq p_1,\dots,p_l<\infty$, and $0<p<\infty$ with $1/p_1+\cdots+1/p_l=1/p$. Let $0<\epsilon<1$ and $\lambda,\mu\in\mathbb{Z}$ with $2^{\lambda+\mu+1}\geq C_0\sqrt{n}$. Suppose that $E_l^{\lambda+\mu}$ is a subset of $(\mathcal{W}^{\lambda+\mu})^l$. Let $\{b_{\vec{\boldsymbol{k}}}^{\gamma}\}_{\vec{\boldsymbol{k}}\in(\mathbb{Z}^n)^l}$ be a sequence of complex numbers and*

$$\mathcal{D}_2:=\sup_{\gamma\in\mathbb{Z}}\big\|\{b_{\vec{\boldsymbol{k}}}^{\gamma}\}_{\vec{\boldsymbol{k}}\in(\mathbb{Z}^n)^l}\big\|_{\ell^2}\quad\textit{and}\quad\mathcal{D}_\infty:=\sup_{\gamma\in\mathbb{Z}}\big\|\{b_{\vec{\boldsymbol{k}}}^{\gamma}\}_{\vec{\boldsymbol{k}}\in(\mathbb{Z}^n)^l}\big\|_{\ell^\infty}.$$

Then there exists $C_\epsilon>0$ such that

$$\begin{aligned}&\Big\|\sum_{\gamma\in\mathbb{Z}}\Big|\sum_{\vec{\boldsymbol{k}}\in E_l^{\lambda+\mu}}b_{\vec{\boldsymbol{k}}}^{\gamma}\prod_{j=1}^{l}L_{G_j,k_j}^{\lambda,\gamma}f_j\Big|\Big\|_{L^p(\mathbb{R}^n)}\\&\quad\leq C_\epsilon 2^{\lambda ln/2}(\lambda+\mu+4)^{l/\min\{1,p\}}\mathcal{D}_2^{1-\epsilon}\mathcal{D}_\infty^{\epsilon}|E^{\lambda+\mu}|^{\epsilon}\prod_{j=1}^{l}\|f_j\|_{L^{p_j}(\mathbb{R}^n)}\end{aligned}\tag{5.7}$$

for $f_1,\dots,f_l\in\mathcal{S}(\mathbb{R}^n)$.

Proof. Using (5.3), the left-hand side of (5.7) is less than

$$\bigg(\sum_{\mathrm{i}_1=-\lambda+c_0}^{\mu+3}\cdots\sum_{\mathrm{i}_l=-\lambda+c_0}^{\mu+3}\Big\|\sum_{\gamma\in\mathbb{Z}}\Big|\sum_{\vec{\boldsymbol{k}}\in E_l^{\lambda+\mu}}b_{\vec{\boldsymbol{k}}}^{\gamma}\,L_{G_1,k_1}^{\lambda,\gamma}\Lambda_{\gamma+\mathrm{i}_1}f_1\cdots L_{G_l,k_l}^{\lambda,\gamma}\Lambda_{\gamma+\mathrm{i}_l}f_l\Big|\Big\|_{L^p(\mathbb{R}^n)}^{\min\{1,p\}}\bigg)^{1/\min\{1,p\}}.$$

Here one uses the convexity of $\|\cdot\|_{L^p}$ when $p\geq1$ and of $\|\cdot\|_{L^p}^p$ when $p<1$. Choose $t:=\frac{2}{1+\epsilon}$ so that $1<t<2<t'=\frac{2}{1-\epsilon}$ as in the proof of Lemma 5.2. Then it follows from Hölder's inequality that

$$\begin{aligned}&\Big|\sum_{\vec{\boldsymbol{k}}\in E_l^{\lambda+\mu}}b_{\vec{\boldsymbol{k}}}^{\gamma}\,L_{G_1,k_1}^{\lambda,\gamma}\Lambda_{\gamma+\mathrm{i}_1}f_1(x)\cdots L_{G_l,k_l}^{\lambda,\gamma}\Lambda_{\gamma+\mathrm{i}_l}f_l(x)\Big|\\&\leq 2^{\lambda ln/2}\int_{(\mathbb{R}^n)^l}\big|B_{E_l^{\lambda+\mu}}^{\gamma}(\vec{\boldsymbol{y}})\big|\prod_{j=1}^{l}\big|\Lambda_{\gamma+\mathrm{i}_j}f(x-2^{\lambda-\gamma}y_j)\Psi_{G_j}^{\vee}(y_j)\big|d\vec{\boldsymbol{y}}\\&=2^{\lambda ln/2}\sum_{\vec{\boldsymbol{u}}\in(\mathbb{Z}^n)^l}\int_{\vec{\boldsymbol{u}}+[0,1)^{nl}}\big|B_{E_l^{\lambda+\mu}}^{\gamma}(\vec{\boldsymbol{y}})\big|\prod_{j=1}^{l}\big|\Lambda_{\gamma+\mathrm{i}_j}f(x-2^{\lambda-\gamma}y_j)\Psi_{G_j}^{\vee}(y_j)\big|d\vec{\boldsymbol{y}}\\&\leq 2^{\lambda ln/2}\sum_{\vec{\boldsymbol{u}}\in(\mathbb{Z}^n)^l}\big\|B_{E_l^{\lambda+\mu}}^{\gamma}\big\|_{L^{t'}(\vec{\boldsymbol{u}}+[0,1)^{nl})}\prod_{j=1}^{l}\big\|\Lambda_{\gamma+\mathrm{i}_j}f(x-2^{\lambda-\gamma}\cdot)\Psi_{G_j}^{\vee}\big\|_{L^t(u_j+[0,1)^n)}\end{aligned}$$

where $\vec{\boldsymbol{y}} := (y_1, \dots, y_l) \in (\mathbb{R}^n)^l$, $\vec{\boldsymbol{u}} := (u_1, \dots, u_l) \in (\mathbb{Z}^n)^l$, and

$$B^{\gamma}_{E_l^{\lambda+\mu}}(\vec{\boldsymbol{y}}) := \sum_{\vec{\boldsymbol{k}} \in E_l^{\lambda+\mu}} b^{\gamma}_{\vec{\boldsymbol{k}}} e^{2\pi i \langle \vec{\boldsymbol{y}}, \vec{\boldsymbol{k}} \rangle}.$$

Similar to (5.6), we have

$$\big\| B^{\gamma}_{E_l^{\lambda+\mu}} \big\|_{L^{t'}(\vec{\boldsymbol{u}}+[0,1)^{nl})} \lesssim \mathcal{D}_2^{1-\epsilon} \mathcal{D}_\infty^{\epsilon} |E_l^{\lambda+\mu}|^{\epsilon}.$$

Thus, the left-hand side of (5.7) is controlled by a constant times

$$2^{\lambda l n/2} \mathcal{D}_2^{1-\epsilon} \mathcal{D}_\infty^{\epsilon} \big|E_l^{\lambda+\mu}\big| \Big(\sum_{\vec{\boldsymbol{u}} \in (\mathbb{Z}^n)^l} \sum_{\mathrm{i}_1 = -\lambda + c_0}^{\mu+3} \cdots \sum_{\mathrm{i}_l = -\lambda + c_0}^{\mu+3}$$
$$\Big\| \sum_{\gamma \in \mathbb{Z}} \prod_{j=1}^{l} \big\| \Lambda_{\gamma + \mathrm{i}_j} f(x - 2^{\lambda - \gamma} \cdot) \Psi^{\vee}_{G_j} \big\|_{L^t(u_j + [0,1)^n)} \Big\|_{L^p(dx)}^{\min\{1,p\}} \Big)^{1/\min\{1,p\}}$$

and, using the inequality $\sum_\gamma |a^1_\gamma \cdots a^l_\gamma| \le (\sum_\gamma |a^1_\gamma \cdots a^l_\gamma|^{2/l})^{l/2} \le \prod_{j=1}^l (\sum_\gamma |a^j_\gamma|^2)^{1/2}$ for $l \ge 2$, the $L^p(dx)$ norm is less than

$$\Big\| \prod_{j=1}^{l} \Big(\sum_{\gamma \in \mathbb{Z}} \big\| \Lambda_\gamma f(x - 2^{\lambda + \mathrm{i}_j - \gamma} \cdot) \Psi^{\vee}_{G_j} \big\|^2_{L^t(u_j + [0,1)^n)} \Big)^{1/2} \Big\|_{L^p(dx)}$$
$$\le \prod_{j=1}^{l} \Big\| \Big(\sum_{\gamma \in \mathbb{Z}} \big\| \Lambda_\gamma f(x - 2^{\lambda + \mathrm{i}_j - \gamma} \cdot) \Psi^{\vee}_{G_j} \big\|^2_{L^t(u_j + [0,1)^n)} \Big)^{1/2} \Big\|_{L^{p_j}(dx)}$$
$$\lesssim \prod_{j=1}^{l} \frac{1}{(1 + |u_j|)^M} \|f_j\|_{L^{p_j}(\mathbb{R}^n)}$$

for $M > n$, where the Cauchy-Schwarz inequality, Hölder's inequality, and Lemma 5.1 are applied in the inequalities. This concludes that the left-hand side of (5.7) is dominated by

$$2^{\lambda l n/2} (\lambda + \mu + 4)^{l/\min\{1,p\}} \mathcal{D}_2^{1-\epsilon} \mathcal{D}_\infty^{\epsilon} \big|E_l^{\lambda+\mu}\big| \Big(\prod_{j=1}^{l} \|f_j\|_{L^{p_j}(\mathbb{R}^n)} \Big) \sum_{\vec{\boldsymbol{u}} \in (\mathbb{Z}^n)^l} \prod_{j=1}^{l} \frac{1}{(1 + |u_j|)^M}$$
$$\lesssim 2^{\lambda l n/2} (\lambda + \mu + 4)^{l/\min\{1,p\}} \mathcal{D}_2^{1-\epsilon} \mathcal{D}_\infty^{\epsilon} \big|E_l^{\lambda+\mu}\big| \prod_{j=1}^{l} \|f_j\|_{L^{p_j}(\mathbb{R}^n)},$$

which completes the proof. □

We recall that, for $l \in J_m$,

(5.8) $$\mathcal{V}_l^m(s) = \{(t_1, \dots, t_m) : 0 < t_l < 1 \quad \text{and} \quad 0 < t_j < s \quad \text{for} \quad j \neq l\}.$$

We prove a useful geometric lemma concerning the convex hull of $\mathcal{V}_l^m(s)$ below.

Lemma 5.4. *Let $0 < s < 1$. Let $\mathbb{H}^m(s)$ be the convex hull of $\mathcal{V}_1^m(s), \dots, \mathcal{V}_m^m(s)$. Then we have*

(5.9) $$\mathcal{H}^m(s) = \mathbb{H}^m(s).$$

Proof. Clearly, $\mathcal{H}^m(s)$ is open and convex as each $\mathcal{H}_J(s)$, $J \subseteq J_m$, is an open convex set. It is easy to see that $\mathcal{V}_l^m(s) \subset \mathcal{H}^m(s)$ for all $l \in J_m$ and thus we have

$$\mathcal{H}^m(s) \supseteq \mathbb{H}^m(s) \quad \text{for all} \;\; m \geq 2.$$

Now let's us prove the opposite direction by using induction on the degree m of multilinearity. We first note that

$$\mathcal{H}^m(s) = \Big\{ (t_1, \dots, t_m) \in (0,1)^m : \sum_{j=1}^m \min\{s - t_j, 0\} > s - 1 \Big\}. \tag{5.10}$$

To verify (5.10), we denote by $H^m(s)$ the right-hand side of (5.10). Suppose that $t = (t_1, \dots, t_m) \in \mathcal{H}^m(s)$ and let $J^{(t)} := \{ j \in J_m : \; t_j > s\}$. Then by the definition of $\mathcal{H}^m(s)$ in (1.3), we have

$$s - 1 < \sum_{j \in J^{(t)}} (s - t_j) = \sum_{j=1}^m \min\{s - t_j, 0\},$$

which implies $\mathcal{H}^m(s) \subset H^m(s)$. Moreover, if $t = (t_1, \dots, t_m) \in H^m(s)$, then

$$s - 1 < \sum_{j \in J^{(t)}} (s - t_j) \leq \sum_{j \in J \cap J^{(t)}} (s - t_j) \leq \sum_{j \in J \cap J^{(t)}} (s - t_j) + \sum_{j \in J \setminus J^{(t)}} (s - t_j) = \sum_{j \in J} (s - t_j)$$

for any $J \subset J_m$. This gives that $H^m(s) \subset \mathcal{H}^m(s) \big(= \cap_J \mathcal{H}_J^m(s) \big)$.

We now return to the proof of $\mathcal{H}^m(s) \subseteq \mathbb{H}^m(s)$ for $m \geq 2$.The case $m = 2$ is obvious from a simple geometric observation, but we provide an explicit approach. If $(t_1, t_2) \in \mathcal{H}^2(s)$, then we have $0 < t_1, t_2 < 1$ and $0 < t_1 + t_2 < 1 + s$. When either t_1 or t_2 is less than s, then (t_1, t_2) belongs to one of $\mathcal{V}_1^m(s)$ or $\mathcal{V}_2^m(s)$ by definition. When $s \leq t_1, t_2 < 1$, we choose $0 < \epsilon < 1$ such that

$$0 < \epsilon < 1 + s - (t_1 + t_2).$$

Then the point (t_1, t_2) lies on the segment joining $(t_1 + t_2 - s + \epsilon, s - \epsilon) \in \mathcal{V}_1^2(s)$ and $(s - \epsilon, t_1 + t_2 - s + \epsilon) \in \mathcal{V}_2^2(s)$ as $0 < s - \epsilon < s$ and $0 < t_1 + t_2 - s + \epsilon < 1$. This shows

$$\mathcal{H}^2(s) \subseteq \mathbb{H}^2(s).$$

Now suppose that $\mathcal{H}^m(s) \subseteq \mathbb{H}^m(s)$ is true for some $m \geq 2$, and let $(t_1, \dots, t_{m+1}) \in \mathcal{H}^{m+1}(s)$. If $0 < t_{m+1} < s$, then

$$\sum_{j=1}^m \min\{s - t_j, 0\} = \sum_{j=1}^{m+1} \min\{s - t_j, 0\} > s - 1$$

so that $(t_1, \dots, t_m) \in \mathcal{H}^m(s) \subseteq \mathbb{H}^m(s)$ by applying the induction hypothesis. Therefore, the point $(t_1, \dots, t_m, t_{m+1})$ belongs to the convex hull of the following m sets:

$$\mathcal{V}_l^m(s) \times (0, s) = \mathcal{V}_l^{m+1}(s), \qquad l \in J_m.$$

This implies $(t_1, \dots, t_{m+1}) \in \mathbb{H}^{m+1}(s)$. Similarly, the same conclusion also holds if $0 < t_j < s$ for some $j \in J_m$. For the remaining cases, we assume that $s \leq t_1, \dots, t_{m+1} < 1$. Since $(t_1, \dots, t_{m+1}) \in \mathcal{H}^{m+1}(s)$, we see that $t_1 + \cdots + t_{m+1} < 1 + ms$, and thus there exists $0 < \epsilon < 1$ so that

$$0 < m\epsilon < 1 + ms - (t_1 + \cdots + t_{m+1}). \tag{5.11}$$

Then the point $(t_1, \dots, t_{m+1})$ is clearly located on the convex hull of the points

$$(t_1 + \cdots + t_{m+1} - ms + m\epsilon, s - \epsilon, \dots, s - \epsilon) \in \mathcal{V}_1^{m+1}(s)$$

$$\vdots$$

$$(s-\epsilon,\dots,s-\epsilon,t_1+\cdots+t_{m+1}-ms+m\epsilon)\in\mathcal{V}^{m+1}_{m+1}(s)$$

as $0<t_1+\cdots+t_{m+1}-ms+m\epsilon<1$, because of (5.11). This proves that $(t_1,\dots,t_{m+1})\in\mathbb{H}^{m+1}(s)$ and completes that proof of $\mathcal{H}^{m+1}(s)\subseteq\mathbb{H}^{m+1}(s)$.

By induction, we finally have

$$\mathcal{H}^m(s)\subseteq\mathbb{H}^m(s)\quad\text{ for general }\; m\geq 2.$$

$\square$

6. Proof of Proposition 3.1

It suffices to prove (3.6) for μ such that $2^{\mu-10}>C_0\sqrt{mn}$ in view of (3.3), which takes care of $\mu<0$ and of finitely many $\mu\geq 0$. The proof will be based on mathematical induction starting with the estimate in the following proposition.

Proposition 6.1. *Let $2\leq p_1,\dots,p_m<\infty$ and $2/m\leq p<\infty$ with $1/p_1+\cdots+1/p_m=1/p$. Suppose that $0<\epsilon<1$ and $2^{\mu-10}>C_0\sqrt{mn}$. Then there exists $C_\epsilon>0$ such that*

$$\big\|\mathcal{L}_\mu(f_1,\dots,f_m)\big\|_{L^p(\mathbb{R}^n)}\lesssim C_\epsilon 2^{\epsilon\mu}\|\Omega\|_{L^2(\mathbb{S}^{mn-1})}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\mathbb{R}^n)}. \tag{6.1}$$

The proof of the above proposition will be presented below.

In order to describe the induction argument, for $0<s<1$ and $l\in J_m$, we define

$$\mathcal{R}^m_l(s):=\{(t_1,\dots,t_m):t_l=1\quad\text{and}\quad 0\leq t_j<s\;\;\text{for}\;\; j\neq l\}.$$

and let

$$\mathcal{C}^m(s):=\{(t_1,\dots,t_m):0<t_j<s,\quad j\in J_m\}$$

be the open cube of side length s with the lower left corner $(0,\dots,0)$.

Claim $X(s)$. *Let $1/m<p<\infty$ and $(1/p_1,\dots,1/p_m)\in\mathcal{C}^m(s)$ with $1/p_1+\cdots+1/p_m=1/p$. Suppose that $0<\epsilon<1$ and $2^{\mu-10}>C_0\sqrt{mn}$. Then there exists $C_\epsilon>0$ such that*

$$\big\|\mathcal{L}_\mu(f_1,\dots,f_m)\big\|_{L^p(\mathbb{R}^n)}\leq C_\epsilon\|\Omega\|_{L^{\frac{1}{1-s}}(\mathbb{S}^{mn-1})}2^{\epsilon\mu}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\mathbb{R}^n)}.$$

Claim $Y(s)$. *Let $1/m<p<1$ and $(1/p_1,\dots,1/p_m)\in\bigcup_{l=1}^m\mathcal{R}^m_l(s)$ with $1/p_1+\cdots+1/p_m=1/p$. Suppose that $0<\epsilon<1$ and $2^{\mu-10}>C_0\sqrt{mn}$. Then there exists $C_\epsilon>0$ such that*

$$\big\|\mathcal{L}_\mu(f_1,\dots,f_m)\big\|_{L^{p,\infty}(\mathbb{R}^n)}\leq C_\epsilon\|\Omega\|_{L^{\frac{1}{1-s}}(\mathbb{S}^{mn-1})}2^{\epsilon\mu}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\mathbb{R}^n)}$$

Claim $Z(s)$. *Let $1/m<p<\infty$ and $(1/p_1,\dots,1/p_m)\in\bigcup_{l=1}^m\mathcal{V}^m_l(s)$, where $\mathcal{V}^m_l(s)$ is defined in (1.4), with $1/p_1+\cdots+1/p_m=1/p$. Suppose that $0<\epsilon<1$ and $2^{\mu-10}>C_0\sqrt{mn}$. Then there exists $C_\epsilon>0$ such that*

$$\big\|\mathcal{L}_\mu(f_1,\dots,f_m)\big\|_{L^p(\mathbb{R}^n)}\leq C_\epsilon\|\Omega\|_{L^{\frac{1}{1-s}}(\mathbb{S}^{mn-1})}2^{\epsilon\mu}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\mathbb{R}^n)}$$

Claim $\Sigma(s)$. *Let $1/m < p < \infty$ and $(1/p_1, \dots, 1/p_m) \in \mathcal{H}^m(s)$ with $1/p_1 + \cdots + 1/p_m = 1/p$. Suppose that $0 < \epsilon < 1$ and $2^{\mu-10} > C_0\sqrt{mn}$. Then there exists $C_\epsilon > 0$ such that*

$$\big\| \mathcal{L}_\mu(f_1, \dots, f_m) \big\|_{L^p(\mathbb{R}^n)} \le C_\epsilon \|\Omega\|_{L^{\frac{1}{1-s}}(\mathbb{S}^{mn-1})} 2^{\epsilon\mu} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)}$$

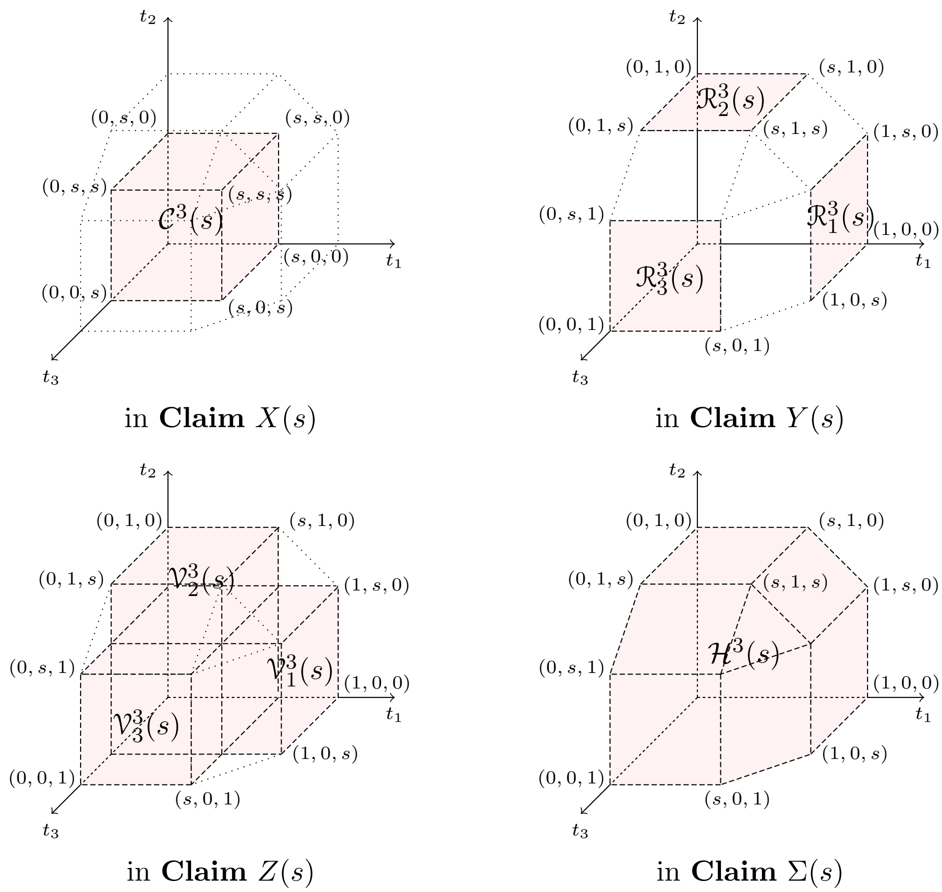

FIGURE 3. The trilinear case $m = 3$: the range of $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3})$

Then the following proposition will play an essential role in the induction steps.

Proposition 6.2. *Let $0 < s < 1$. Then*

$$\textbf{Claim } X(s) \Rightarrow \textbf{Claims } X(s) \text{ and } Y(s) \Rightarrow \textbf{Claim } Z(s) \Rightarrow \textbf{Claim } \Sigma(s).$$

The proof of Proposition 6.2 will be given below.

We now complete the proof of Proposition 3.1, using Propositions 6.1 and 6.2.

Proof of Proposition 3.1. For $\nu \in \mathbb{N}_0$, let

$$a_\nu := 1 - \frac{1}{2}\Big(1 - \frac{1}{m}\Big)^\nu,$$

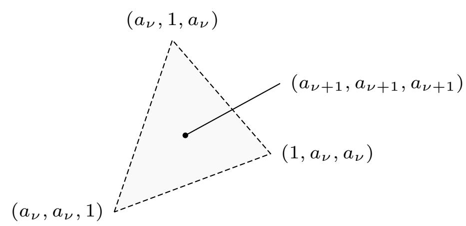

FIGURE 4. $(a_{\nu+1}, a_{\nu+1}, a_{\nu+1})$ when $m = 3$

for which $(a_{\nu+1}, \dots, a_{\nu+1}) \in \mathbb{R}^m$ is the center of the $(m-1)$ simplex with m vertices $(1, a_\nu, a_\nu, \dots, a_\nu)$, $(a_\nu, 1, a_\nu, \dots, a_\nu)$, $\dots$, $(a_\nu, \dots, a_\nu, 1, a_\nu)$, and $(a_\nu, \dots, a_\nu, a_\nu, 1)$. We notice that $a_0 = 1/2$, $a_{\nu+1} = \frac{a_\nu(m-1)+1}{m}$, and $a_\nu \nearrow 1$ as $\nu \to \infty$. Moreover, by the definition of $\mathcal{H}^m(a_\nu)$ we have

$$\mathcal{C}^m(a_{\nu+1}) \subset \mathcal{H}^m(a_\nu) \quad \text{for all} \;\; \nu \in \mathbb{N}_0,$$

which implies

$$\textbf{Claim } \Sigma(a_\nu) \Rightarrow \textbf{Claim } X(a_{\nu+1}) \quad \text{for all} \;\; \nu \in \mathbb{N}_0 \tag{6.2}$$

as $L^{\frac{1}{1-a_{\nu+1}}}(\mathbb{S}^{mn-1}) \hookrightarrow L^{\frac{1}{1-a_\nu}}(\mathbb{S}^{mn-1})$. Now Proposition 6.1 implies that **Claim** $X(a_0)$

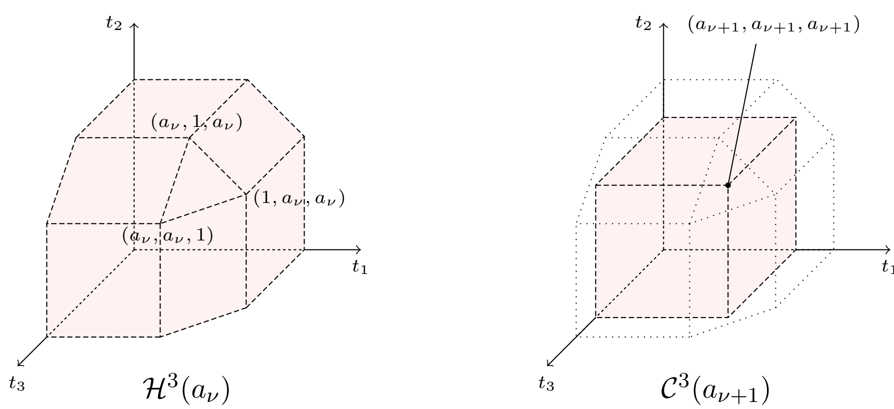

FIGURE 5. The trilinear case $m = 3$: $\mathcal{H}^3(a_\nu)$ and $\mathcal{C}^3(a_{\nu+1})$

holds, and consequently, **Claim** $\Sigma(a_\nu)$ should be also true for all $\nu \in \mathbb{N}_0$ with the aid of Proposition 6.2 and (6.2).

When $s = 1/2$ $(= a_0)$, the asserted estimate (3.6) is exactly **Claim** $\Sigma(a_0)$. If $a_\nu < s \le a_{\nu+1}$ for some $\nu \in \mathbb{N}_0$, then $\mathcal{C}^m(s) \subset \mathcal{H}^m(a_\nu)$. This yields that **Claim** $X(s)$ holds since $L^{\frac{1}{1-s}}(\mathbb{S}^{mn-1}) \hookrightarrow L^{\frac{1}{1-a_\nu}}(\mathbb{S}^{mn-1})$, and accordingly, Proposition 6.2 shows that **Claim** $\Sigma(s)$ works. This finishes the proof of Proposition 3.1. $\square$

Observing that $\widehat{K_\mu^0} \in L^2((\mathbb{R}^n)^m)$, we apply the wavelet decomposition (4.2) to write

$$\widehat{K_\mu^0}(\vec{\boldsymbol{\xi}}) = \sum_{\lambda \in \mathbb{N}_0} \sum_{\vec{\boldsymbol{G}} \in \mathcal{I}^\lambda} \sum_{\vec{\boldsymbol{k}} \in (\mathbb{Z}^n)^m} b^{\lambda,\mu}_{\vec{\boldsymbol{G}},\vec{\boldsymbol{k}}} \Psi^\lambda_{G_1,k_1}(\xi_1) \cdots \Psi^\lambda_{G_m,k_m}(\xi_m) \tag{6.3}$$

where

$$b_{\vec{G},\vec{k}}^{\lambda,\mu} := \int_{(\mathbb{R}^n)^m} \widehat{K_\mu^0}(\vec{\boldsymbol{\xi}}\,)\Psi_{\vec{G},\vec{k}}^{\lambda}(\vec{\boldsymbol{\xi}}\,)d\vec{\boldsymbol{\xi}}\,. \tag{6.4}$$

Concerning the ℓ^p norms of $\{b_{\vec{G},\vec{k}}^{\lambda,\mu}\}_{\vec{k}\in(\mathbb{Z}^n)^m}$, we have the following results playing essential roles in the sequel.

Lemma 6.3. *Let $\Omega \in L^2(\mathbb{S}^{mn-1})$, and $\{b_{\vec{G},\vec{k}}^{\lambda,\mu}\}_{\vec{k}\in(\mathbb{Z}^n)^m}$ be defined in (6.4). Then*

$$\big\|\{b_{\vec{G},\vec{k}}^{\lambda,\mu}\}_{\vec{k}\in(\mathbb{Z}^n)^m}\big\|_{\ell^2} \lesssim \big\|\widehat{K_\mu^0}\big\|_{L^2((\mathbb{R}^n)^m)} \lesssim \|\Omega\|_{L^2(\mathbb{S}^{mn-1})}, \tag{6.5}$$

and, for any $0<\delta<1/2$,

$$\big\|\{b_{\vec{G},\vec{k}}^{\lambda,\mu}\}_{\vec{k}\in(\mathbb{Z}^n)^m}\big\|_{\ell^\infty} \lesssim 2^{-\delta\mu}2^{-\lambda(L+1+mn)}\|\Omega\|_{L^2(\mathbb{S}^{mn-1})}, \tag{6.6}$$

where L is the number of vanishing moments of $\Psi_{\vec{G}}$; this number L can be chosen sufficiently large.

Inequality (6.5) follows from (4.4) and Plancherel's identity. Moreover, (6.6) was proved in [17, Lemma 7].

Proof of Proposition 6.1. Using (3.1) and (6.3), we can write

$$\begin{aligned}\mathcal{L}_\mu\big(f_1,\dots,f_m\big)(x) &= \sum_{\gamma\in\mathbb{Z}}\int_{(\mathbb{R}^n)^m} 2^{\gamma mn}K_\mu^0(2^\gamma\vec{\boldsymbol{y}})\prod_{j=1}^m f_j(x-y_j)\,d\vec{\boldsymbol{y}}\\ &= \sum_{\gamma\in\mathbb{Z}}\int_{(\mathbb{R}^n)^m} \widehat{K_\mu^0}(\vec{\boldsymbol{\xi}}\,/2^\gamma)e^{2\pi i\langle x,\xi_1+\cdots+\xi_m\rangle}\prod_{j=1}^m \widehat{f_j}(\xi_j)\,d\vec{\boldsymbol{\xi}}\\ &= \sum_{\lambda\in\mathbb{N}_0}\sum_{\vec{G}\in\mathcal{I}^\lambda}\sum_{\gamma\in\mathbb{Z}}\sum_{\vec{k}\in(\mathbb{Z}^n)^m} b_{\vec{G},\vec{k}}^{\lambda,\mu}\prod_{j=1}^m L_{G_j,k_j}^{\lambda,\gamma}f_j(x)\end{aligned} \tag{6.7}$$

where $L_{G,k}^{\lambda,\gamma}$ is defined in (5.1).

When $2^{\mu-10} > C_0\sqrt{mn}$, we may replace $\sum_{\vec{k}\in(\mathbb{Z}^n)^m}$ in (6.7) by $\sum_{2^{\lambda+\mu-2}\le|\vec{k}|\le 2^{\lambda+\mu+2}}$, due to the compact supports of $\widehat{K_\mu^0}$ and $\Psi_{\vec{G},\vec{k}}^{\lambda}$. In addition, by symmetry, it suffices to focus only on the case $|k_1|\ge\cdots\ge|k_m|$. Therefore the estimate (6.1) can be reduced to the inequality

$$\Big\|\sum_{\lambda\in\mathbb{N}_0}\sum_{\vec{G}\in\mathcal{I}^\lambda}\sum_{\gamma\in\mathbb{Z}}\sum_{\vec{k}\in\mathcal{U}^{\lambda+\mu}} b_{\vec{G},\vec{k}}^{\lambda,\mu}\prod_{j=1}^m L_{G_j,k_j}^{\lambda,\gamma}f_j\Big\|_{L^p(\mathbb{R}^n)} \lesssim_\epsilon 2^{\epsilon\mu}\|\Omega\|_{L^2(\mathbb{S}^{mn-1})}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\mathbb{R}^n)} \tag{6.8}$$

where

$$\mathcal{U}^{\lambda+\mu} := \big\{\vec{k}\in(\mathbb{Z}^n)^m : 2^{\lambda+\mu-2}\le|\vec{k}|\le 2^{\lambda+\mu+2},\ |k_1|\ge\cdots\ge|k_m|\big\}.$$

We split $\mathcal{U}^{\lambda+\mu}$ into the following m disjoint subsets:

$$\begin{aligned}\mathcal{U}_1^{\lambda+\mu} &:= \{\vec{k}\in\mathcal{U}^{\lambda+\mu} : |k_1|\ge 2C_0\sqrt{n} > |k_2|\ge\cdots\ge|k_m|\}\\ \mathcal{U}_2^{\lambda+\mu} &:= \{\vec{k}\in\mathcal{U}^{\lambda+\mu} : |k_1|\ge|k_2|\ge 2C_0\sqrt{n} > |k_3|\ge\cdots\ge|k_m|\}\\ &\ \ \vdots\\ \mathcal{U}_m^{\lambda+\mu} &:= \{\vec{k}\in\mathcal{U}^{\lambda+\mu} : |k_1|\ge\cdots\ge|k_m|\ge 2C_0\sqrt{n}\}.\end{aligned}$$

Then the left-hand side of (6.8) is estimated by

$$\Big(\sum_{l=1}^{m}\sum_{\lambda\in\mathbb{N}_0}\sum_{\vec{\boldsymbol{G}}\in\mathcal{I}^\lambda}\Big\|\sum_{\gamma\in\mathbb{Z}}\mathcal{T}_{\vec{\boldsymbol{G}},l}^{\lambda,\gamma,\mu}(f_1,\dots,f_m)\Big\|_{L^p(\mathbb{R}^n)}^{\min\{1,p\}}\Big)^{1/\min\{1,p\}}$$

where the operator $\mathcal{T}_{\vec{\boldsymbol{G}},l}^{\lambda,\gamma,\mu}$ is defined by

$$\mathcal{T}_{\vec{\boldsymbol{G}},l}^{\lambda,\gamma,\mu}(f_1,\dots,f_m):=\sum_{\vec{\boldsymbol{k}}\in\mathcal{U}_l^{\lambda+\mu}}b_{\vec{\boldsymbol{G}},\vec{\boldsymbol{k}}}^{\lambda,\mu}\prod_{j=1}^{m}L_{G_j,k_j}^{\lambda,\gamma}f_j.$$

We claim that for each $l\in J_m$, there exists $M_0>0$, depending on $p_1,\dots,p_m$, such that

$$\Big\|\sum_{\gamma\in\mathbb{Z}}\mathcal{T}_{\vec{\boldsymbol{G}},l}^{\lambda,\gamma,\mu}(f_1,\dots,f_m)\Big\|_{L^p(\mathbb{R}^n)}\lesssim_\epsilon 2^{\epsilon\mu}2^{-\lambda M_0}\|\Omega\|_{L^2(\mathbb{S}^{mn-1})}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(\mathbb{R}^n)},\tag{6.9}$$

which clearly concludes (6.8). Therefore it remains to prove (6.9).

The proof of (6.9) for the case $l=1$ relies on the fact that if $\widehat{g_\gamma}$ is supported in the set $\{\xi\in\mathbb{R}^n: C^{-1}2^{\gamma+\mu}\le|\xi|\le C2^{\gamma+\mu}\}$ for some $C>1$ and $\mu\in\mathbb{Z}$, then

$$\Big\|\Big\{\Lambda_j\Big(\sum_{\gamma\in\mathbb{Z}}g_\gamma\Big)\Big\}_{j\in\mathbb{Z}}\Big\|_{L^p(\ell^q)}\lesssim_C\big\|\{g_j\}_{j\in\mathbb{Z}}\big\|_{L^p(\ell^q)}\quad\text{uniformly in }\ \mu\tag{6.10}$$

for $0<p<\infty$ and $0<q\le\infty$. The proof of (6.10) is elementary and standard, so it is omitted here. See [22, (3.9)] and [31, Theorem 3.6] for a related argument. Note that if $\vec{\boldsymbol{k}}\in\mathcal{U}_1^{\lambda+\mu}$ and $2^{\mu-10}\ge C_0\sqrt{mn}$, then

$$2^{\lambda+\mu-3}\le2^{\lambda+\mu-2}-2C_0\sqrt{mn}\le|\vec{\boldsymbol{k}}|-(|k_2|^2+\cdots+|k_m|^2)^{1/2}\le|k_1|\le2^{\lambda+\mu+2},$$

and this implies that

$$\operatorname{supp}\big(\Psi_{G_1,k_1}^{\lambda}(\cdot/2^\gamma)\big)\subset\{\xi\in\mathbb{R}^n:2^{\gamma+\mu-4}\le|\xi|\le2^{\gamma+\mu+3}\}.$$

Moreover, since $|k_j|\le2C_0\sqrt{n}$ for $2\le j\le m$ and $2^{\mu-10}>C_0\sqrt{mn}$,

$$\operatorname{supp}\big(\Psi_{G_j,k_j}^{\lambda}(\cdot/2^\gamma)\big)\subset\{\xi\in\mathbb{R}^n:|\xi|\le m^{-1/2}2^{\gamma+\mu-8}\}.$$

Therefore, the Fourier transform of $\mathcal{T}_{\vec{\boldsymbol{G}},1}^{\lambda,\gamma,\mu}(f_1,\dots,f_m)$ for $2^{\mu-10}\ge C_0\sqrt{mn}$ is supported in the set $\{\xi\in\mathbb{R}^n:2^{\gamma+\mu-5}\le|\xi|\le2^{\gamma+\mu+4}\}$. Now, using the Littlewood-Paley theory for Hardy spaces, we have

$$\Big\|\sum_{\gamma\in\mathbb{Z}}\mathcal{T}_{\vec{\boldsymbol{G}},1}^{\lambda,\gamma,\mu}(f_1,\dots,f_m)\Big\|_{L^p(\mathbb{R}^n)}\sim\Big\|\Big\{\Lambda_j\Big(\sum_{\gamma\in\mathbb{Z}}\mathcal{T}_{\vec{\boldsymbol{G}},1}^{\lambda,\gamma,\mu}(f_1,\dots,f_m)\Big)\Big\}_{j\in\mathbb{Z}}\Big\|_{L^p(\ell^2)}$$

and then (6.10) yields that the above $L^p(\ell^2)$-norm is dominated by a constant multiple of

$$\Big\|\Big(\sum_{\gamma\in\mathbb{Z}}\big|\mathcal{T}_{\vec{\boldsymbol{G}},1}^{\lambda,\gamma,\mu}(f_1,\dots,f_m)\big|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}.\tag{6.11}$$

Using (4.6) and (5.2), we see that

$$\big|\mathcal{T}_{\vec{\boldsymbol{G}},1}^{\lambda,\gamma,\mu}(f_1,\dots,f_m)(x)\big|$$

$$\le \sum_{\vec{\boldsymbol{k}}^{*1}\in\mathcal{P}_{*1}\mathcal{U}_1^{\lambda+\mu}} \prod_{j=2}^{m} \big|L_{G_j,k_j}^{\lambda,\gamma} f_j(x)\big| \Big| \sum_{k_1\in Col_{\vec{\boldsymbol{k}}^{*1}}^{\mathcal{U}_1^{\lambda+\mu}}} b_{\vec{\boldsymbol{G}},\vec{\boldsymbol{k}}}^{\lambda,\mu} L_{G_1,k_1}^{\lambda,\gamma} f_1(x)\Big|$$

$$\lesssim 2^{\lambda n(m-1)/2} \prod_{j=2}^{m} \mathcal{M} f_j(x) \sum_{\vec{\boldsymbol{k}}^{*1}\in\mathcal{P}_{*1}\mathcal{U}_1^{\lambda+\mu}} \Big| \sum_{k_1\in Col_{\vec{\boldsymbol{k}}^{*1}}^{\mathcal{U}_1^{\lambda+\mu}}} b_{\vec{\boldsymbol{G}},\vec{\boldsymbol{k}}}^{\lambda,\mu} L_{G_1,k_1}^{\lambda,\gamma} f_1(x)\Big|.$$

Then it follows from Hölder's inequality and the maximal inequality for $\mathcal{M}$ that (6.11) is bounded by

$$2^{\lambda n(m-1)/2}\Big(\prod_{j=2}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^n)}\Big) \sum_{\vec{\boldsymbol{k}}^{*1}\in\mathcal{P}_{*1}\mathcal{U}_1^{\lambda+\mu}} \Big\| \Big(\sum_{\gamma\in\mathbb{Z}} \Big| \sum_{k_1\in Col_{\vec{\boldsymbol{k}}^{*1}}^{\mathcal{U}_1^{\lambda+\mu}}} b_{\vec{\boldsymbol{G}},\vec{\boldsymbol{k}}}^{\lambda,\mu} L_{G_1,k_1}^{\lambda,\gamma} f_1\Big|^2\Big)^{1/2}\Big\|_{L^{p_1}(\mathbb{R}^n)}.$$

Now let $0<\epsilon_0<1$ be a sufficiently small number to be chosen later. Then, as $p_1\in[2,\infty)$, Lemma 5.2, together with (6.6) and (6.5), yields that

$$\begin{aligned}
&\Big\| \Big(\sum_{\gamma\in\mathbb{Z}} \Big| \sum_{k_1\in Col_{\vec{\boldsymbol{k}}^{*1}}^{\mathcal{U}_1^{\lambda+\mu}}} b_{\vec{\boldsymbol{G}},\vec{\boldsymbol{k}}}^{\lambda,\mu} L_{G_1,k_1}^{\lambda,\gamma} f_1\Big|^2\Big)^{1/2}\Big\|_{L^{p_1}(\mathbb{R}^n)} \\
&\lesssim 2^{\lambda n/2}(\lambda+\mu+4)\|\Omega\|_{L^2(\mathbb{S}^{mn-1})} 2^{-\delta\mu\epsilon_0} 2^{-\lambda(L+1+mn)\epsilon_0} 2^{(\lambda+\mu)n\epsilon_0}\|f_1\|_{L^{p_1}(\mathbb{R}^n)} \\
&\lesssim 2^{\lambda n/2} 2^{\mu\epsilon_0(n-\delta)} 2^{-\lambda\epsilon_0(L+1+mn-n)}(\lambda+\mu+4)\|\Omega\|_{L^2(\mathbb{S}^{mn-1})}\|f_1\|_{L^{p_1}(\mathbb{R}^n)}
\end{aligned}$$

as the cardinality of $Col_{\vec{\boldsymbol{k}}^{*1}}^{\mathcal{U}_1^{\lambda+\mu}}$ is less than $2^{(\lambda+\mu)n}$. Finally, we have

$$\Big\| \sum_{\gamma\in\mathbb{Z}} \mathcal{T}_{\vec{\boldsymbol{G}},1}^{\lambda,\gamma,\mu}(f_1,\dots,f_m)\Big\|_{L^p(\mathbb{R}^n)} \lesssim_{M_0} 2^{\epsilon\mu} 2^{-\lambda M_0}\|\Omega\|_{L^2(\mathbb{S}^{mn-1})} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^n)},$$

by choosing ϵ_0 and L such that

$$\epsilon=\epsilon_0 n, \quad M_0<\epsilon_0(L+1+mn-n)-mn/2.$$

This shows (6.9) for the case $l=1$.

Now we suppose that $2\le l\le m$. Using (4.6) and (5.2), we write

$$\begin{aligned}
&\big|\mathcal{T}_{\vec{\boldsymbol{G}},l}^{\lambda,\gamma,\mu}(f_1,\dots,f_m)(x)\big| \\
&\lesssim 2^{\lambda n(m-l)/2} \sum_{\vec{\boldsymbol{k}}^{*1,\dots,l}\in\mathcal{P}_{*1,\dots,l}\mathcal{U}_l^{\lambda+\mu}} \Big(\prod_{j=l+1}^{m} \big|\mathcal{M} f_j(x)\big|\Big) \Big| \sum_{\vec{\boldsymbol{k}}^{1,\dots,l}\in Col_{\vec{\boldsymbol{k}}^{*1,\dots,l}}^{\mathcal{U}_l^{\lambda+\mu}}} b_{\vec{\boldsymbol{G}},\vec{\boldsymbol{k}}}^{\lambda,\mu} \prod_{j=1}^{l} L_{G_j,k_j}^{\lambda,\gamma} f_j(x)\Big|
\end{aligned}$$

and thus it follows from Hölder's inequality and the maximal inequality for $\mathcal{M}$ that the left-hand side of (6.9) is less than

$$2^{\lambda n(m-l)/2}\Big(\prod_{j=l+1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^n)}\Big)$$

$$\times \Big(\sum_{\vec{\boldsymbol{k}}^{*1,\dots,l} \in \mathcal{P}_{*1,\dots,l}\mathcal{U}_l^{\lambda+\mu}} \Big\| \sum_{\gamma\in\mathbb{Z}} \Big| \sum_{\vec{\boldsymbol{k}}^{1,\dots,l} \in Col^{\mathcal{U}_l^{\lambda+\mu}}_{\vec{\boldsymbol{k}}^{*1,\dots,l}}} b^{\lambda,\mu}_{\vec{\boldsymbol{G}},\vec{\boldsymbol{k}}} \prod_{j=1}^{l} L^{\lambda,\gamma}_{G_j,k_j} f_j \Big| \Big\|_{L^{q_l}(\mathbb{R}^n)}^{\min\{1,p\}} \Big)^{1/\min\{1,p\}}$$

where $1/q_l := 1/p_1 + \cdots + 1/p_l$. Note that $Col^{\mathcal{U}_l^{\lambda+\mu}}_{\vec{\boldsymbol{k}}^{*1,\dots,l}}$ is a subset of $(\mathcal{W}^{\lambda+\mu})^l$ and thus

$$\big| Col^{\mathcal{U}_l^{\lambda+\mu}}_{\vec{\boldsymbol{k}}^{*1,\dots,l}} \big| \lesssim 2^{(\lambda+\mu)nl}.$$

Accordingly, Lemma 5.3, (6.6), and (6.5) yields that

$$\begin{aligned} &\Big\| \sum_{\gamma\in\mathbb{Z}} \Big| \sum_{\vec{\boldsymbol{k}}^{1,\dots,l} \in Col^{\mathcal{U}_l^{\lambda+\mu}}_{\vec{\boldsymbol{k}}^{*1,\dots,l}}} b^{\lambda,\mu}_{\vec{\boldsymbol{G}},\vec{\boldsymbol{k}}} \prod_{j=1}^{l} L^{\lambda,\gamma}_{G_j,k_j} f_j \Big| \Big\|_{L^{q_l}(\mathbb{R}^n)} \\ &\lesssim \|\Omega\|_{L^2(\mathbb{S}^{mn-1})} 2^{\mu\epsilon_0(n-\delta)} 2^{\lambda nl/2} (\lambda+\mu+4)^{l/\min\{1,q_l\}} 2^{-\lambda\epsilon_0(L+1+mn-nl)} \prod_{j=1}^{l} \|f_j\|_{L^{p_j}(\mathbb{R}^n)} \\ &\lesssim \|\Omega\|_{L^2(\mathbb{S}^{mn-1})} 2^{\epsilon\mu} 2^{-\lambda(M_0+n(m-l)/2)} \prod_{j=1}^{l} \|f_j\|_{L^{p_j}(\mathbb{R}^n)} \end{aligned}$$

choosing $0<\epsilon_0<1$ and $L>0$ so that

$$\epsilon = \epsilon_0 n \quad \text{and} \quad M_0 + mn/2 < \epsilon_0(L+1).$$

This conclude that (6.9) holds for $2 \le l \le m$. □

Proof of Proposition 6.2. Let $0<s<1$. We first note that the direction

$$\textbf{Claims } X(s) \text{ and } Y(s) \Rightarrow \textbf{Claim } Z(s)$$

follows from the (linear) Marcinkiewicz interpolation method. Here, we apply the interpolation separately m times and in each interpolation, $m-1$ parameters among $p_1,\dots,p_m$ are fixed. Moreover, the direction

$$\textbf{Claim } Z(s) \Rightarrow \textbf{Claim } \Sigma(s)$$

also holds due to Lemmas 3.2 and 5.4.

Therefore we need to prove the remaining direction **Claim** $X(s) \Rightarrow$ **Claim** $Y(s)$. The proof is based on the idea in [25]. We are only concerned with the case $(1/p_1,\dots,1/p_m) \in \mathfrak{R}_1^m(s)$ as a symmetric argument is applicable to the other cases. Assume that $p_1 = 1$, $1/s < p_2,\dots,p_m < \infty$, and

$$1 + 1/p_2 + \cdots + 1/p_m = 1/p. \tag{6.12}$$

Without loss of generality, we may also assume $\|f_1\|_{L^1(\mathbb{R}^n)} = \|f_2\|_{L^{p_2}(\mathbb{R}^n)} = \cdots = \|f_m\|_{L^{p_m}(\mathbb{R}^n)} = \|\Omega\|_{L^{\frac{1}{1-s}}(\mathbb{S}^{mn-1})} = 1$ and then it is enough to prove

$$\Big| \Big\{ x\in\mathbb{R}^n : \big| \mathcal{L}_\mu(f_1,\dots,f_m)(x) \big| > t \Big\} \Big| \lesssim_\epsilon 2^{\epsilon\mu p} t^{-p}. \tag{6.13}$$

We shall use the Calderón-Zygmund decomposition of f_1 at height t^p. Then f_1 can be expressed as

$$f_1 = g_1 + \sum_{Q\in\mathcal{A}} b_{1,Q}$$

where $\mathcal{A}$ is a subset of disjoint dyadic cubes, $\big|\bigcup_{Q\in\mathcal{A}} Q\big| \lesssim t^{-p}$, $\mathrm{supp}(b_{1,Q}) \subset Q$, $\int b_{1,Q}(y)dy = 0$, $\|b_{1,Q}\|_{L^1(\mathbb{R}^n)} \lesssim t^p|Q|$, and $\|g_1\|_{L^r(\mathbb{R}^n)} \lesssim t^{(1-1/r)p}$ for all $1 \le r \le \infty$. Then the left-hand side of (6.13) is less than

$$\Big|\Big\{x \in \mathbb{R}^n : \big|\mathcal{L}_\mu(g_1, f_2, \dots, f_m)(x)\big| > t/2\Big\}\Big| \\ + \Big|\Big\{x \in \mathbb{R}^n : \Big|\mathcal{L}_\mu\Big(\sum_{Q\in\mathcal{A}} b_{1,Q}, f_2, \dots, f_m\Big)(x)\Big| > t/2\Big\}\Big| =: \Xi_1^\mu + \Xi_2^\mu$$

For the estimation of the first term, we choose $1/s < p_0 < \infty$ and $\widetilde{p} > p$ with

$$1/p_0 + 1/p_2 + \cdots + 1/p_m = 1/\widetilde{p} \tag{6.14}$$

and set $\epsilon_0 := \epsilon p/\widetilde{p}$ so that $0 < \epsilon_0 < 1$. Then the assumption **Claim** $X(s)$ yields that

$$\big\|\mathcal{L}_\mu(g_1, f_2, \dots, f_m)\big\|_{L^{\widetilde{p}}(\mathbb{R}^n)} \lesssim_{\epsilon_0} 2^{\epsilon_0\mu}\|g_1\|_{L^{p_0}(\mathbb{R}^n)} \lesssim 2^{\epsilon_0\mu} t^{(1-1/p_0)p}. \tag{6.15}$$

Now, using Chebyshev's inequality and the estimate (6.15), the first term Ξ_1^μ is clearly dominated by

$$t^{-\widetilde{p}}\big\|\mathcal{L}_\mu(g_1, f_2, \dots, f_m)\big\|_{L^{\widetilde{p}}(\mathbb{R}^n)}^{\widetilde{p}} \lesssim 2^{\epsilon_0\mu\widetilde{p}} t^{-\widetilde{p}(1-p(1-1/p_0))} = 2^{\epsilon\mu p} t^{-p}$$

since $\widetilde{p}(1 - p(1 - 1/p_0)) = p$ by (6.12) and (6.14).

Moreover, the remaining term Ξ_2^μ is estimated by the sum of $\big|\bigcup_{Q\in\mathcal{A}} Q^*\big|$ and

$$\Gamma_\mu := \Big|\Big\{x \in \Big(\bigcup_{Q\in\mathcal{A}} Q^*\Big)^c : \Big|\mathcal{L}_\mu\Big(\sum_{Q\in\mathcal{A}} b_{1,Q}, f_2, \dots, f_m\Big)(x)\Big| > t/2\Big\}\Big|$$

where Q^* is the concentric dilate of Q with $\ell(Q^*) = 10^2\sqrt{n}\ell(Q)$. Since $\big|\bigcup_{Q\in\mathcal{A}} Q^*\big| \lesssim t^{-p}$, the proof of (6.13) can be reduced to the inequality

$$\Gamma_\mu \lesssim_\epsilon 2^{\epsilon\mu p} t^{-p}. \tag{6.16}$$

We apply Chebyshev's ineqaulity to deduce

$$\begin{aligned} \Gamma_\mu &\lesssim t^{-p}\int_{(\bigcup_{Q\in\mathcal{A}} Q^*)^c} \Big(\sum_{Q\in\mathcal{A}}\sum_{\gamma\in\mathbb{Z}} \big|T_{K_\mu^\gamma}\big(b_{1,Q}, f_2, \dots, f_m\big)(x)\big|\Big)^p dx \\ &\le t^{-p}\int_{(\bigcup_{Q\in\mathcal{A}} Q^*)^c} \Big(\sum_{Q\in\mathcal{A}}\sum_{\gamma:2^\gamma\ell(Q)\ge 1} \big|T_{K_\mu^\gamma}\big(b_{1,Q}, f_2, \dots, f_m\big)(x)\big|\Big)^p dx \\ &\qquad + t^{-p}\int_{\mathbb{R}^n} \Big(\sum_{Q\in\mathcal{A}}\sum_{\gamma:2^\gamma\ell(Q)<1} \big|T_{K_\mu^\gamma}\big(b_{1,Q}, f_2, \dots, f_m\big)(x)\big|\Big)^p dx \\ &=: \Gamma_\mu^1 + \Gamma_\mu^2 \end{aligned}$$

where $T_{K_\mu^\gamma}$ is the multilinear operator associated with the kernel K_μ^γ so that

$$T_{K_\mu^\gamma}\big(b_{1,Q}, f_2, \dots, f_m\big)(x) = \int_{(\mathbb{R}^n)^m} K_\mu^\gamma(x - y_1, \dots, x - y_m) b_{1,Q}(y_1)\prod_{j=2}^m f_j(y_j)\, d\vec{\boldsymbol{y}}.$$

To estimate Γ_μ^1, we see that

$$\begin{aligned}
&\big|T_{K_\mu^\gamma}\big(b_{1,Q},f_2,\dots,f_m\big)(x)\big|\\
&\lesssim \int_{(\mathbb{R}^n)^m}\int_{|\vec{\boldsymbol{z}}|\sim 2^{-\gamma}} 2^{\gamma mn}\big|\Omega(\vec{\boldsymbol{z}}\,')\big|\big|\Phi_{\mu+\gamma}(x-y_1-z_1,\dots,x-y_m-z_m)\big|\\
&\qquad\qquad\qquad\qquad \times |b_{1,Q}(y_1)|\Big(\prod_{j=2}^m |f_j(y_j)|\Big)\, d\vec{\boldsymbol{z}}\, d\vec{\boldsymbol{y}}\\
&\lesssim_L \int_{|\vec{\boldsymbol{z}}|\sim 2^{-\gamma}} 2^{\gamma mn}\big|\Omega(\vec{\boldsymbol{z}}\,')\big|\Big(\int_{y_1\in Q}\frac{2^{(\mu+\gamma)n}}{(1+2^{\mu+\gamma}|x-y_1-z_1|)^L}|b_{1,Q}(y_1)|dy_1\Big)\\
&\qquad\qquad \times \prod_{j=2}^m\Big(\int_{\mathbb{R}^n}\frac{2^{(\mu+\gamma)n}}{(1+2^{\mu+\gamma}|x-y_j-z_j|)^L}|f_j(y_j)|dy_j\Big)\, d\vec{\boldsymbol{z}}
\end{aligned}$$

for all $L>n$. Clearly, we have

$$\int_{\mathbb{R}^n}\frac{2^{(\mu+\gamma)n}}{(1+2^{\mu+\gamma}|x-y_j-z_j|)^L}|f_j(y_j)|dy_j \lesssim \mathcal{M}f_j(x-z_j), \qquad j=2,\dots,m \tag{6.17}$$

and for $2^\gamma\ell(Q)\ge 1$ and $|z_1|\le 2^{-\gamma+1}$,

$$\int_{y_1\in Q}\frac{2^{(\mu+\gamma)n}}{(1+2^{\mu+\gamma}|x-y_1-z_1|)^L}|b_{1,Q}(y_1)|dy_1 \lesssim \frac{2^{(\mu+\gamma)n}}{(1+2^{\mu+\gamma}|x-c_Q|)^L}\|b_{1,Q}\|_{L^1(\mathbb{R}^n)}$$

because $|x-y_1-z_1|\gtrsim |x-c_Q|$. Therefore, we have

$$\begin{aligned}
&\big|T_{K_\mu^\gamma}\big(b_{1,Q},f_2,\dots,f_m\big)(x)\big|\\
&\lesssim \frac{2^{(\mu+\gamma)n}}{(1+2^{\mu+\gamma}|x-c_Q|)^L}\|b_{1,Q}\|_{L^1(\mathbb{R}^n)}\int_{|\vec{\boldsymbol{z}}|\sim 2^{-\gamma}} 2^{\gamma mn}\big|\Omega(\vec{\boldsymbol{z}}\,')\big|\Big(\prod_{j=2}^m \mathcal{M}f_j(x-z_j)\Big)d\vec{\boldsymbol{z}}.
\end{aligned}$$

Now Hölder's inequality yields

$$\begin{aligned}
&\int_{|\vec{\boldsymbol{z}}|\sim 2^{-\gamma}} 2^{\gamma mn}\big|\Omega(\vec{\boldsymbol{z}}\,')\big|\Big(\prod_{j=2}^m \mathcal{M}f_j(x-z_j)\Big)\, d\vec{\boldsymbol{z}}\\
&\le \Big(\int_{|\vec{\boldsymbol{z}}|\sim 2^{-\gamma}} 2^{\gamma mn}\big|\Omega(\vec{\boldsymbol{z}}\,')\big|^{\frac{1}{1-s}}d\vec{\boldsymbol{z}}\Big)^{1-s}\Big(\int_{|\vec{\boldsymbol{z}}|\sim 2^{-\gamma}} 2^{\gamma mn}\Big(\prod_{j=2}^m \mathcal{M}f_j(x-z_j)\Big)^{\frac{1}{s}}d\vec{\boldsymbol{z}}\,'\Big)^s\\
&\le \|\Omega\|_{L^{\frac{1}{1-s}}(\mathbb{S}^{mn-1})}\prod_{j=2}^m\Big(2^{\gamma n}\int_{|z_j|\lesssim 2^{-\gamma}}\big|\mathcal{M}f_j(x-z_j)\big|^{\frac{1}{s}}dz_j\Big)^s \lesssim \prod_{j=2}^m \mathcal{M}_{\frac{1}{s}}\mathcal{M}f_j(x)
\end{aligned}$$

and thus

$$\big|T_{K_\mu^\gamma}\big(b_{1,Q},f_2,\dots,f_m\big)(x)\big| \lesssim \frac{2^{(\mu+\gamma)n}}{(1+2^{\mu+\gamma}|x-c_Q|)^L}\|b_{1,Q}\|_{L^1(\mathbb{R}^n)}\prod_{j=2}^m \mathcal{M}_{\frac{1}{s}}\mathcal{M}f_j(x).$$

This, together with Hölder's inequality, deduces that Γ_μ^1 is dominated by a constant times

$$t^{-p}\int_{(\bigcup_{Q\in\mathcal{A}}Q^*)^c}\Big(\prod_{j=2}^m \mathcal{M}_{\frac{1}{s}}\mathcal{M}f_j(x)\Big)^p\Big(\sum_{Q\in\mathcal{A}}\sum_{\gamma:2^\gamma\ell(Q)\ge 1}\frac{2^{(\mu+\gamma)n}}{(1+2^{\mu+\gamma}|x-c_Q|)^L}\|b_{1,Q}\|_{L^1(\mathbb{R}^n)}\Big)^p dx$$

$$\le t^{-p}\Big(\prod_{j=2}^{m}\|\mathcal{M}_{\frac{1}{s}}\mathcal{M}f_j\|_{L^{p_j}(\mathbb{R}^n)}\sum_{Q\in\mathcal{A}}\sum_{\gamma:2^\gamma\ell(Q)\ge1}\Big\|\frac{2^{(\mu+\gamma)n}}{(1+2^{\mu+\gamma}|\cdot-c_Q|)^L}\Big\|_{L^1((Q^*)^c)}\|b_{1,Q}\|_{L^1(\mathbb{R}^n)}\Big)^p.$$

Since $1/s<p_2,\dots,p_m<\infty$, each L^{p_j} norm is controlled by $\|f_j\|_{L^{p_j}(\mathbb{R}^n)}=1$, using the L^{p_j} boundedness of both $\mathcal{M}_{\frac{1}{s}}$ and $\mathcal{M}$. Moreover, using the fact that for $2^{\mu-10}\ge C_0\sqrt{mn}$,

$$\Big\|\frac{2^{(\mu+\gamma)n}}{(1+2^{\mu+\gamma}|\cdot-c_Q|)^L}\Big\|_{L^1((Q^*)^c)}\lesssim 2^{-\mu(L-n)}\big(2^\gamma\ell(Q)\big)^{-(L-n)}\le\big(2^\gamma\ell(Q)\big)^{-(L-n)},$$

we have

$$\sum_{Q\in\mathcal{A}}\sum_{\gamma:2^\gamma\ell(Q)\ge1}\Big\|\frac{2^{(\mu+\gamma)n}}{(1+2^{\mu+\gamma}|\cdot-c_Q|)^L}\Big\|_{L^1((Q^*)^c)}\|b_{1,Q}\|_{L^1(\mathbb{R}^n)}\lesssim\sum_{Q\in\mathcal{A}}\|b_{1,Q}\|_{L^1(\mathbb{R}^n)}\lesssim 1.$$

This concludes

$$\Gamma_\mu^1\lesssim t^{-p}.$$

Next, let us deal with the other term Γ_μ^2. By using the vanishing moment condition of $b_{1,Q}$, we have

$$\begin{aligned}&\big|T_{K_\mu^\gamma}\big(b_{1,Q},f_2,\dots,f_m\big)(x)\big|\\ &\lesssim\int_{|\vec{\boldsymbol{z}}|\sim2^{-\gamma}}2^{\gamma mn}\big|\Omega(\vec{\boldsymbol{z}}')\big|\bigg(\int_{(\mathbb{R}^n)^m}\big|\Phi_{\mu+\gamma}(x-y_1-z_1,\dots,x-y_m-z_m)\\ &\qquad-\Phi_{\mu+\gamma}(x-c_Q-z_1,x-y_2-z_2,\dots,x-y_m-z_m)\big|\big|b_{1,Q}(y_1)\big|\Big(\prod_{j=2}^m|f_j(y_j)|\Big)d\vec{\boldsymbol{y}}\bigg)\,d\vec{\boldsymbol{z}}.\end{aligned}\tag{6.18}$$

We observe that

$$\begin{aligned}&\big|\Phi_{\mu+\gamma}(x-y_1-z_1,\dots,x-y_m-z_m)-\Phi_{\mu+\gamma}(x-c_Q-z_1,x-y_2-z_2,\dots,x-y_m-z_m)\big|\\ &\lesssim 2^{(\mu+\gamma)}\ell(Q)\;V_{\mu+\gamma}^L(x-z_1,y_1,c_Q)\;\Big(\prod_{j=2}^m\frac{2^{(\mu+\gamma)n}}{(1+2^{\mu+\gamma}|x-y_j-z_j|)^L}\Big)\end{aligned}$$

where

$$V_{\mu+\gamma}^L(x,y_1,c_Q):=\int_0^1\frac{2^{(\mu+\gamma)n}}{(1+2^{\mu+\gamma}|x-ty_1-(1-t)c_Q|)^L}dt.$$

Furthermore,

$$\begin{aligned}&\big|\Phi_{\mu+\gamma}(x-y_1-z_1,\dots,x-y_m-z_m)-\Phi_{\mu+\gamma}(x-c_Q-z_1,x-y_2-z_2,\dots,x-y_m-z_m)\big|\\ &\lesssim_L W_{\mu+\gamma}^L(x-z_1,y_1,c_Q)\;\Big(\prod_{j=2}^m\frac{2^{(\mu+\gamma)n}}{(1+2^{\mu+\gamma}|x-y_j-z_j|)^L}\Big)\end{aligned}$$

where

$$W_{\mu+\gamma}^L(x,y_1,c_Q):=\frac{2^{(\mu+\gamma)n}}{(1+2^{\mu+\gamma}|x-y_1|)^L}+\frac{2^{(\mu+\gamma)n}}{(1+2^{\mu+\gamma}|x-c_Q|)^L}.$$

By averaging these two estimates and letting

$$U_{\mu+\gamma}^{L,\epsilon}(x,y_1,c_Q):=\big(V_{\mu+\gamma}^L(x,y_1,c_Q)\big)^\epsilon\big(W_{\mu+\gamma}^L(x,y_1,c_Q)\big)^{1-\epsilon},$$

we obtain

$$\big|\Phi_{\mu+\gamma}(x-y_1-z_1,\dots,x-y_m-z_m)-\Phi_{\mu+\gamma}(x-c_Q-z_1,x-y_2-z_2,\dots,x-y_m-z_m)\big|$$

(6.19)

$$\lesssim_{L,\epsilon} 2^{\epsilon\mu}\big(2^{\gamma}\ell(Q)\big)^{\epsilon} U^{L,\epsilon}_{\mu+\gamma}(x-z_1,y_1,c_Q)\,\Big(\prod_{j=2}^{m}\frac{2^{(\mu+\gamma)n}}{(1+2^{\mu+\gamma}|x-y_j-z_j|)^L}\Big).$$

Here, we note that

$$\big\|U^{L,\epsilon}_{\mu+\gamma}(\cdot,y_1,c_Q)\big\|_{L^1(\mathbb{R}^n)} \le \big\|V^{L}_{\mu+\gamma}(\cdot,y_1,c_Q)\big\|^{\epsilon}_{L^1(\mathbb{R}^n)}\big\|W^{L}_{\mu+\gamma}(\cdot,y_1,c_Q)\big\|^{1-\epsilon}_{L^1(\mathbb{R}^n)} \lesssim 1.$$

By plugging (6.19) into (6.18), we obtain

$$\begin{aligned}
&\big|T_{K_\mu^\gamma}\big(b_{1,Q},f_2,\dots,f_m\big)(x)\big|\\
&\lesssim 2^{\epsilon\mu}\big(2^{\gamma}\ell(Q)\big)^{\epsilon}\int_{|\vec{\boldsymbol{z}}|\sim 2^{-\gamma}} 2^{\gamma mn}\big|\Omega(\vec{\boldsymbol{z}}\,')\big|\Big(\int_{\mathbb{R}^n} U^{L,\epsilon}_{\mu+\gamma}(x-z_1,y_1,c_Q)|b_{1,Q}(y_1)|dy_1\Big)\\
&\qquad\qquad\times\Big(\prod_{j=2}^{m}\int_{\mathbb{R}^n}\frac{2^{(\mu+\gamma)n}}{(1+2^{\mu+\gamma}|x-y_j-z_j|)^L}|f_j(y_j)|dy_j\Big)\,d\vec{\boldsymbol{z}}\\
&\lesssim 2^{\epsilon\mu}\big(2^{\gamma}\ell(Q)\big)^{\epsilon}\int_{|z_1|\lesssim 2^{-\gamma}}\int_{\mathbb{R}^n} U^{L,\epsilon}_{\mu+\gamma}(x-z_1,y_1,c_Q)|b_{1,Q}(y_1)|dy_1\\
&\qquad\qquad\times\Big(\int_{|(z_2,\dots,z_m)|\lesssim 2^{-\gamma}} 2^{\gamma mn}\big|\Omega(\vec{\boldsymbol{z}}\,')\big|\prod_{j=2}^{m}\mathcal{M}f_j(x-z_j)\,dz_2\cdots dz_m\Big)\,dz_1
\end{aligned}$$

where (6.17) is applied. The innermost integral is, via Hölder's inequality, bounded by

$$2^{\gamma n}\Big(\int_{|(z_2,\dots,z_m)|\lesssim 2^{-\gamma}} 2^{\gamma(m-1)n}\big|\Omega(\vec{\boldsymbol{z}}\,')\big|^{\frac{1}{1-s}}dz_2\cdots dz_m\Big)^{1-s}\prod_{j=2}^{m}\mathcal{M}_{\frac{1}{s}}\mathcal{M}f_j(x)$$

and thus we have

$$\begin{aligned}
\big|T_{K_\mu^\gamma}\big(b_{1,Q},f_2,\dots,f_m\big)(x)\big| &\lesssim 2^{\epsilon\mu}\big(2^{\gamma}\ell(Q)\big)^{\epsilon}\ \prod_{j=2}^{m}\mathcal{M}_{\frac{1}{s}}\mathcal{M}f_j(x)\int_{\mathbb{R}^n}|b_{1,Q}(y_1)|\\
\times\int_{|z_1|\lesssim 2^{-\gamma}} 2^{\gamma n}U^{L,\epsilon}_{\mu+\gamma}(x-z_1,y_1,c_Q)&\Big(\int_{|(z_2,\dots,z_m)|\lesssim 2^{-\gamma}} 2^{\gamma(m-1)n}\big|\Omega(\vec{\boldsymbol{z}}\,')\big|^{\frac{1}{1-s}}\ dz_2\cdots dz_m\Big)^{1-s}dz_1dy_1
\end{aligned}$$

Now, by using Hölder's inequality and the maximal inequality for $\mathcal{M}_{\frac{1}{s}}$ and $\mathcal{M}$, we have

$$\begin{aligned}
\Gamma_\mu^2 \lesssim t^{-p}2^{\epsilon\mu p}\Big\|\sum_{Q\in\mathcal{A}}\sum_{\gamma:2^{\gamma}\ell(Q)<1}\big(2^{\gamma}\ell(Q)\big)^{\epsilon}\int_{\mathbb{R}^n}|b_{1,Q}(y_1)|\int_{|z_1|\lesssim 2^{-\gamma}} 2^{\gamma n}U^{L,\epsilon}_{\mu+\gamma}(\cdot-z_1,y_1,c_Q)\\
\times\Big(\int_{|(z_2,\dots,z_m)|\lesssim 2^{-\gamma}} 2^{\gamma(m-1)n}\big|\Omega(\vec{\boldsymbol{z}}\,')\big|^{\frac{1}{1-s}}\ dz_2\cdots dz_m\Big)^{1-s}dz_1dy_1\Big\|^{p}_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

Moreover, the L^1 norm in the last displayed expression is bounded by

$$\begin{aligned}
&\sum_{Q\in\mathcal{A}}\sum_{\gamma:2^{\gamma}\ell(Q)<1}\big(2^{\gamma}\ell(Q)\big)^{\epsilon}\int_{\mathbb{R}^n}|b_{1,Q}(y_1)|\big\|U^{L,\epsilon}_{\mu+\gamma}(\cdot,y_1,c_Q)\big\|_{L^1(\mathbb{R}^n)}\\
&\qquad\times\int_{|z_1|\lesssim 2^{-\gamma}} 2^{\gamma n}\Big(\int_{|(z_2,\dots,z_m)|\lesssim 2^{-\gamma}} 2^{\gamma(m-1)n}\big|\Omega(\vec{\boldsymbol{z}}\,')\big|^{\frac{1}{1-s}}\ dz_2\cdots dz_m\Big)^{1-s}dz_1\ dy_1\\
&\lesssim\sum_{Q\in\mathcal{A}}\sum_{\gamma:2^{\gamma}\ell(Q)<1}\big(2^{\gamma}\ell(Q)\big)^{\epsilon}\int_{\mathbb{R}^n}|b_{1,Q}(y_1)|\Big(\int_{|\vec{\boldsymbol{z}}|\lesssim 2^{-\gamma}} 2^{\gamma mn}\big|\Omega(\vec{\boldsymbol{z}}\,')\big|^{\frac{1}{1-s}}d\vec{\boldsymbol{z}}\Big)^{1-s}dy_1
\end{aligned}$$

$$\lesssim \sum_{Q\in\mathcal{A}} \|b_{1,Q}\|_{L^1(\mathbb{R}^n)} \sum_{\gamma:2^\gamma\ell(Q)<1} \big(2^\gamma\ell(Q)\big)^\epsilon \lesssim 1$$

where the first inequality follows from Hölder's inequality. This proves

$$\Gamma_\mu^2 \lesssim t^{-p}2^{\epsilon\mu p},$$

which finally completes the proof of (6.16). □

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Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
Email address: grafakosl@umsystem.edu

School of Mathematical Sciences, Fudan University, People's Republic of China
Email address: hedanqing@fudan.edu.cn

Department of Mathematics, Charles University, 116 36 Praha 1, Czech Republic
Email address: honzik@gmail.com

B. Park, Department of Mathematics, Sungkyunkwan University, Suwon 16419, Republic of Korea
Email address: bpark43@skku.edu