# A LIMITED-RANGE CALDERÓN-ZYGMUND THEOREM

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Abstract. For a limited range of indices $p$, we obtain $L^p(\mathbb{R}^n)$ boundedness for singular integral operators whose kernels satisfy a condition weaker than the typical Hörmander smoothness estimate. These operators are assumed to be bounded (or weakly bounded) on $L^s(\mathbb{R}^n)$ for some index $s$. Our estimates are obtained via interpolation from the appropriate weak-type estimates. We provide two proofs of this result. One proof is based on the Calderón-Zygmund decomposition, while the other uses ideas of Nazarov, Treil, and Volberg.

## 1. Introduction

The classical theory of singular integral operators was introduced by Calderón and Zygmund in [3] and says that for certain kernels defined on $\mathbb{R}^n \setminus \{0\}$, the weak-type $(1,1)$ bound holds for the associated singular integral operator, assuming that an $L^s(\mathbb{R}^n)$ bound is known for some $1 < s \le \infty$. Hörmander extended this theory in [9] to more general kernels $K$ satisfying the smoothness condition

$$[K]_H := \sup_{y\in\mathbb{R}^n} \int_{|x|\ge 2|y|} |K(x-y) - K(x)|\, dx < \infty.$$

The Hörmander condition is an $L^1(\mathbb{R}^n)$-type smoothness condition and has some variants. For example, Watson introduced the following $L^r(\mathbb{R}^n)$ versions in [18]: for $1 \le r \le \infty$, we say a kernel $K$ is in the class $H^r$ if

$$[K]_{H^r} := \sup_{R>0} \sup_{\substack{y\in\mathbb{R}^n \\ |y|\le R}} \sum_{m=1}^{\infty} (2^m R)^{\frac{n}{r'}} \left[ \int\limits_{\substack{|x|\ge 2^m R \\ |x|<2^{m+1}R}} |K(x-y) - K(x)|^r dx \right]^{\frac{1}{r}} < \infty,$$

where $r'$ is the Hölder conjugate of $r$. Observe that Watson's condition coincides with Hörmander's condition when $r = 1$, and for $r_1, r_2 \in [1,\infty]$ with $r_1 \le r_2$,

$$H^{r_2} \subseteq H^{r_1} \subseteq H^1 = H.$$

In this paper, we focus on a different set of $L^r(\mathbb{R}^n)$-adapted conditions defined as follows.

**Definition 1.** Let $1 \le r \le \infty$. A kernel $K$ defined on $\mathbb{R}^n \setminus \{0\}$ is in the class $H_r$ if

$$[K]_{H_r} := \sup_{R>0} \left[ \frac{1}{v_n R^n} \int_{|y|\le R} \left( \int_{|x|\ge 2R} |K(x-y) - K(x)|\, dx \right)^r dy \right]^{\frac{1}{r}} < \infty,$$

where $v_n$ is the volume of the unit ball $B(0,1)$ in $\mathbb{R}^n$.

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Notice that this condition coincides with the Hörmander condition when $r = \infty$. Moreover, for $r_1, r_2 \in [1, \infty]$ with $r_1 \leq r_2$,

$$H = H_\infty \subseteq H_{r_2} \subseteq H_{r_1},$$

meaning the $H_r$ conditions are weaker than Hörmander's smoothness condition.

We prove boundedness results for the associated singular integral operators.

**Definition 2.** Let $K \in H_r$ for some $1 \leq r \leq \infty$ and suppose $K$ satisfies the size estimate $|K(x)| \leq \frac{A}{|x|^n}$ for all $x \neq 0$. We associate $K$ with a linear operator $T$ given by

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)\,dy$$

for smooth functions $f$ and $x \notin \operatorname{supp} f$.

Notice that this definition also makes sense if $f$ is an integrable, compactly supported function and $x \notin \operatorname{supp} f$. Moreover, there is no unique way to define $Tf$ in terms of $K$ for general functions $f$ (see the relevant discussions in [5, 6, 14]).

If $K \in H = H_\infty$, Hörmander proved that given $1 < s \leq \infty$, $L^s(\mathbb{R}^n)$ bounds for $T$ imply the weak-type $(1,1)$ bound, and hence $L^p(\mathbb{R}^n)$ bounds for all $1 < p < \infty$. In this note, we prove the following variant of this result, where weak-type $(1,1)$ is replaced by weak-type $(q,q)$.

**Theorem 1.** Let $1 \leq q < \infty$, $K \in H_{q'}$, and $|K(x)| \leq \frac{A}{|x|^n}$ for all $x \neq 0$. If the associated singular integral operator $T$ is bounded on $L^s(\mathbb{R}^n)$ for some $s \in (q, \infty]$ with bound $B$, then $T$ maps $L^q(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$ with bound at most a constant multiple of $B + [K]_{H_{q'}}$. That is,

$$\|Tf\|_{L^{q,\infty}(\mathbb{R}^n)} := \sup_{\alpha > 0} \alpha |\{|Tf| > \alpha\}|^{\frac{1}{q}} \leq C_{n,s,q}(B + [K]_{H_{q'}})\|f\|_{L^q(\mathbb{R}^n)}$$

for all $f \in L^q(\mathbb{R}^n)$.

We give two proofs of Theorem 1. The first proof uses the $L^q(\mathbb{R}^n)$ version of the Calderón-Zygmund decomposition and is an adaptation of the classical proof given in [3]. The second proof is motivated by Nazarov, Treil, and Volberg's proof for the weak-type $(1,1)$ inequality in the nonhomogeneous setting, given in [11]. Adaptations of the proof in the nonhomogeneous setting are needed in our setting; some modifications include ideas that can be found in [14]. See [15–17] for other applications of the Nazarov, Treil, and Volberg technique to multilinear and weighted settings. Refer to [8, 10, 12] for related results regarding multilinear and weighted Calderón-Zygmund theory.

By interpolation we obtain the following corollary.

**Corollary 1.** Under the hypotheses of Theorem 1, the operator $T$ is bounded on $L^p(\mathbb{R}^n)$ for $p$ in the interval $\big(\min(s', q), \max(q', s)\big)$.

**Remark 1.** The constant $A$ does not appear in the conclusion of Theorem 1. The estimate $|K(x)| \leq \frac{A}{|x|^n}$ is only needed to ensure that the operator $T$ is well-defined for a dense class of functions.

If $q > 1$ and $s < \infty$, then the interval $\big(\min(s', q), \max(q', s)\big)$ is properly contained in $(1, \infty)$. Hence in this case, we obtain $L^p(\mathbb{R}^n)$ estimates for a limited-range of values of $p$. Prior to this work, other "limited-range" versions of the Calderón-Zygmund theorem appeared in Baernstein and Sawyer [1], Carbery [4], Seeger [13], and Grafakos, Honzík, Ryabogin [7].

## 2. Calderón-Zygmund Decomposition Method

The first proof of Theorem 1 relies on the $L^q(\mathbb{R}^n)$ version of the Calderón-Zygmund decomposition. See [5, 6, 14] for details on the decomposition.

*Proof.* Fix $f \in L^q(\mathbb{R}^n)$ and $\alpha > 0$. We will show that

$$|\{|Tf| > \alpha\}| \le C_{n,s,q}(B + [K]_{H^{q'}})^q \alpha^{-q} \|f\|^q_{L^q(\mathbb{R}^n)}.$$

Apply the $L^q(\mathbb{R}^n)$-form of the Calderón-Zygmund decomposition to $f$ at height $\gamma\alpha$ (the constant $\gamma > 0$ will be chosen later), to write $f = g + b = g + \sum_{j=1}^\infty b_j$, where

(1) $\|g\|_{L^\infty(\mathbb{R}^n)} \le 2^{\frac{n}{q}}\gamma\alpha$ and $\|g\|_{L^q(\mathbb{R}^n)} \le \|f\|_{L^q(\mathbb{R}^n)}$,

(2) the $b_j$ are supported on pairwise disjoint cubes $Q_j$ satisfying $\sum_{j=1}^\infty |Q_j| \le (\gamma\alpha)^{-q}\|f\|^q_{L^q(\mathbb{R}^n)}$,

(3) $\|b_j\|^q_{L^q(\mathbb{R}^n)} \le 2^{n+q}(\gamma\alpha)^q|Q_j|$,

(4) $\int_{Q_j} b_j(x)\,dx = 0$, and

(5) $\|b\|_{L^q(\mathbb{R}^n)} \le 2^{\frac{n+q}{q}}\|f\|_{L^q(\mathbb{R}^n)}$ and $\|b\|_{L^1(\mathbb{R}^n)} \le 2(\gamma\alpha)^{1-q}\|f\|^q_{L^q(\mathbb{R}^n)}$.

Now,

$$|\{|Tf| > \alpha\}| \le \left|\left\{|Tg| > \frac{\alpha}{2}\right\}\right| + \left|\left\{|Tb| > \frac{\alpha}{2}\right\}\right|.$$

Assume first that $s < \infty$. Choose $\gamma = (B + [K]_{H_{q'}})^{-1}$. Using Chebyshev's inequality, the bound of $T$ on $L^s(\mathbb{R}^n)$, property (1), and trivial estimates, we have that

$$\begin{aligned}\left|\left\{|Tg| > \frac{\alpha}{2}\right\}\right| &\le 2^s\alpha^{-s}\|Tg\|^s_{L^s(\mathbb{R}^n)}\\ &\le (2B)^s\alpha^{-s}\|g\|^s_{L^s(\mathbb{R}^n)}\\ &\le 2^{s-n+\frac{ns}{q}}B^s\alpha^{-s}(\gamma\alpha)^{s-q}\|g\|^q_{L^q(\mathbb{R}^n)}\\ &\le 2^{s-n+\frac{ns}{q}}(B + [K]_{H_{r'}})^q\alpha^{-q}\|f\|^q_{L^q(\mathbb{R}^n)}.\end{aligned}$$

We next control the second term. Let $c_j$ denote the center of $Q_j$, let $Q_j^* := Q(c_j, 2\sqrt{n}l(Q_j))$ be the cube centered at $c_j$ and having side length $2\sqrt{n}$ times the side length of $Q_j$, and set $\Omega^* := \bigcup_{j=1}^\infty Q_j^*$. Then

$$\left|\left\{|Tb| > \frac{\alpha}{2}\right\}\right| \le |\Omega^*| + \left|\left\{x \in \mathbb{R}^n \setminus \Omega^* : |Tb(x)| > \frac{\alpha}{2}\right\}\right|.$$

Notice that since $|Q_j^*| = (2\sqrt{n})^n|Q_j|$ and by property (2), we have

$$|\Omega^*| \le \sum_{j=1}^\infty |Q_j^*| = (2\sqrt{n})^n\sum_{j=1}^\infty |Q_j| \le (2\sqrt{n})^n(B + [K]_{H_q'})^q\alpha^{-q}\|f\|^q_{L^q(\mathbb{R}^n)}.$$

It remains to control the last term. Use Chebyshev's inequality, property (4), Fubini's theorem, Hölder's inequality, property (3), and property (2) to estimate

$$\begin{aligned}
\left|\left\{\mathbb{R}^n \setminus \Omega^* : |Tb| > \frac{\alpha}{2}\right\}\right| &\le 2\alpha^{-1} \int_{\mathbb{R}^n\setminus\Omega^*} |Tb(x)|\,dx \\
&\le 2\alpha^{-1} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n\setminus\Omega^*} |Tb_j(x)|dx \\
&\le 2\alpha^{-1} \sum_{j=1}^{\infty} \int_{Q_j} \left[\int_{\mathbb{R}^n\setminus\Omega^*} |K(x-y)-K(x-c_j)|\,dx\right] |b_j(y)|\,dy \\
&\le 2\alpha^{-1} \sum_{j=1}^{\infty} \left\| \int_{\mathbb{R}^n\setminus\Omega^*} |K(x-\cdot)-K(x-c_j)|\,dx \right\|_{L^{q'}(Q_j)} \|b_j\|_{L^q} \\
&\le 2\alpha^{-1} \sup_{j\in\mathbb{N}} \left\| \int_{\mathbb{R}^n\setminus\Omega^*} |K(x-\cdot)-K(x-c_j)|\,dx \right\|_{L^{q'}\left(Q_j, \frac{dy}{|Q_j|}\right)} \sum_{j=1}^{\infty} |Q_j|^{\frac{1}{q'}} \|b_j\|_{L^q} \\
&\le 2^{\frac{n}{q}+2}\gamma \sup_{j\in\mathbb{N}} \left\| \int_{\mathbb{R}^n\setminus\Omega^*} |K(x-\cdot)-K(x-c_j)|\,dx \right\|_{L^{q'}\left(Q_j, \frac{dy}{|Q_j|}\right)} \sum_{j=1}^{\infty} |Q_j| \\
&\le 2^{\frac{n}{q}+2}\gamma^{1-q}\alpha^{-q}\|f\|_{L^q(\mathbb{R}^n)}^q \sup_{j\in\mathbb{N}} \left\| \int_{\mathbb{R}^n\setminus\Omega^*} |K(x-\cdot)-K(x-c_j)|\,dx \right\|_{L^{q'}\left(Q_j, \frac{dy}{|Q_j|}\right)}.
\end{aligned}$$

For each $j$, setting $R_j = \frac{\sqrt{n}}{2} l(Q_j)$, we have

$$Q_j \subseteq B(c_j, R_j) \subseteq B(c_j, 2R_j) \subseteq Q_j^*,$$

where $B(x,r)$ denotes the ball centered at $x$ and with radius $r$. Then the factor involving the supremum is less than or equal to

$$\sup_{j\in\mathbb{N}} \left[ \int_{B(c_j,R_j)} \left( \int_{\mathbb{R}^n\setminus B(c_j,2R_j)} |K(x-y)-K(x-c_j)|dx \right)^{q'} \frac{dy}{|Q_j|} \right]^{\frac{1}{q'}},$$

which is bounded by $\left(\frac{\sqrt{n}}{2}\right)^n v_n [K]_{H_{q'}}$ by changing variables $x' = x - c_j$, $y' = y - c_j$ and by replacing the supremum over $R_j$ by the supremum over all $R > 0$.

Putting all of the estimates together, we get

$$|\{|Tf| > \alpha\}| \le \left(2^{s-n+\frac{ns}{q}} + (2\sqrt{n})^n + 2^{\frac{n}{q}+2-n} n^{\frac{n}{2}}\right) (B + [K]_{H_{q'}})^q \alpha^{-q} \|f\|_{L^q(\mathbb{R}^n)}^q.$$

When $s = \infty$, set $\gamma = 2^{-\frac{n}{q}}(4([K]_{H_{q'}} + B))^{-1}$. Then

$$\|Tg\|_{L^\infty(\mathbb{R}^n)} \le B\|g\|_{L^\infty(\mathbb{R}^n)} \le 2^{\frac{n}{q}} B\gamma\alpha \le \frac{\alpha}{4},$$

so

$$\left|\left\{|Tg| > \frac{\alpha}{2}\right\}\right| = 0.$$

The part of the argument involving $\left\{|Tb| > \frac{\alpha}{2}\right\}$ is the same as in the case $s < \infty$. $\square$

## 3. Method of Nazarov, Treil, and Volberg

We provide a second proof of Theorem 1. This proof is motivated by the argument given by Nazarov, Treil, and Volberg in [11]. See also [15–17] for other applications of this technique.

*Proof.* Fix $f \in L^q(\mathbb{R}^n)$ and $\alpha > 0$. We will show that

$$|\{|Tf| > \alpha\}| \le C_{n,s,q}(B + [K]_{H_{q'}})^q \alpha^{-q} \|f\|^q_{L^q(\mathbb{R}^n)}.$$

By density, we may assume $f$ is a nonnegative continuous function with compact support. Set

$$\Omega := \{M(f^q) > (\gamma\alpha)^q\}$$

where $\gamma > 0$ is to be chosen later and where $M$ denotes the Hardy-Littlewood maximal operator. Apply a Whitney decomposition to write

$$\Omega = \bigcup_{j=1}^{\infty} Q_j,$$

a disjoint union of dyadic cubes where

$$2\mathrm{diam}(Q_j) \le d(Q_j, \mathbb{R}^n \setminus \Omega) \le 8\mathrm{diam}(Q_j).$$

Put

$$g := f\mathbb{1}_{\mathbb{R}^n\setminus\Omega}, \qquad b := f\mathbb{1}_\Omega, \qquad \text{and} \qquad b_j := f\mathbb{1}_{Q_j}.$$

Then

$$f = g + b = g + \sum_{j=1}^{\infty} b_j,$$

where we claim that

(1) $\|g\|_{L^\infty(\mathbb{R}^n)} \le \gamma\alpha$ and $\|g\|_{L^q(\mathbb{R}^n)} \le \|f\|_{L^q(\mathbb{R}^n)}$,

(2) the $b_j$ are supported on pairwise disjoint cubes $Q_j$ satisfying

$$\sum_{j=1}^{\infty} |Q_j| \le 3^n(\gamma\alpha)^{-q}\|f\|^q_{L^q(\mathbb{R}^n)},$$

(3) $\|b_j\|^q_{L^q(\mathbb{R}^n)} \le (17\sqrt{n})^n(\gamma\alpha)^q|Q_j|$, and

(4) $\|b\|_{L^q(\mathbb{R}^n)} \le \|f\|_{L^q(\mathbb{R}^n)}$ and $\|b\|_{L^1(\mathbb{R}^n)} \le (17\sqrt{n})^{\frac{n}{q}} 3^n(\gamma\alpha)^{1-q}\|f\|^q_{L^q(\mathbb{R}^n)}$.

Indeed, since for any $x \notin \Omega$, we have

$$|g(x)|^q = |f(x)|^q \le M(f^q)(x) \le (\gamma\alpha)^q,$$

it follows that $\|g\|_{L^\infty(\mathbb{R}^n)} \le \gamma\alpha$. Since $g$ is a restriction of $f$, we have $\|g\|_{L^q(\mathbb{R}^n)} \le \|f\|_{L^q(\mathbb{R}^n)}$, and so (1) holds. Using the weak-type $(1,1)$ bound for $M$ with $\|M\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)} \le 3^n$, we obtain property (2) as follows

$$\sum_{j=1}^{\infty} |Q_j| = |\Omega| \le 3^n(\gamma\alpha)^{-q}\|f\|^q_{L^q(\mathbb{R}^n)}.$$

Addressing (3) and (4), let $Q_j^* := Q(c_j, 17\sqrt{n}l(Q_j))$ be the cube with the same center as $Q_j$ but side length $17\sqrt{n}$ times as large. Then $Q_j^* \cap (\mathbb{R}^n \setminus \Omega) \ne \emptyset$, so there is a point $x \in Q_j^*$ such that $M(f^q)(x) \leq (\gamma\alpha)^q$. In particular, $\int_{Q_j^*} |f(y)|^q dy \leq (\gamma\alpha)^q |Q_j^*|$. Since $|Q_j^*| = (17\sqrt{n})^n |Q_j|$, we have

$$\|b_j\|_{L^q(\mathbb{R}^n)}^q = \int_{Q_j} |f(y)|^q dy \leq \int_{Q_j^*} |f(y)|^q dy \leq (\gamma\alpha)^q |Q_j^*| = (17\sqrt{n})^n (\gamma\alpha)^q |Q_j|.$$

This proves (3). We use Hölder's inequality, property (3), and property and (2) to justify property (4)

$$\begin{aligned}\|b\|_{L^1(\mathbb{R}^n)} = \sum_{j=1}^{\infty} \|b_j\|_{L^1(\mathbb{R}^n)} &\leq \sum_{j=1}^{\infty} \|b_j\|_{L^q(\mathbb{R}^n)} |Q_j|^{\frac{1}{q'}} \leq (17\sqrt{n})^{\frac{n}{q}} (\gamma\alpha) \sum_{j=1}^{\infty} |Q_j| \\ &\leq (17\sqrt{n})^{\frac{n}{q}} 3^n (\gamma\alpha)^{1-q} \|f\|_{L^q(\mathbb{R}^n)}^q.\end{aligned}$$

Now,

$$|\{|Tf| > \alpha\}| \leq \left|\left\{|Tg| > \frac{\alpha}{2}\right\}\right| + \left|\left\{|Tb| > \frac{\alpha}{2}\right\}\right|.$$

Assume first that $s < \infty$. Choose $\gamma = (B + [K]_{H_{q'}})^{-1}$. Use Chebyshev's inequality, the bound of $T$ on $L^s(\mathbb{R}^n)$, and property (1) to see

$$\begin{aligned}\left|\left\{|Tg| > \frac{\alpha}{2}\right\}\right| &\leq 2^s \alpha^{-s} \|Tg\|_{L^s(\mathbb{R}^n)}^s \\ &\leq (2B)^s \alpha^{-s} \|g\|_{L^s(\mathbb{R}^n)}^s \\ &\leq (2B)^s (\gamma\alpha)^{s-q} \alpha^{-s} \|g\|_{L^q(\mathbb{R}^n)}^q \\ &\leq 2^s (B + [K]_{H_{q'}})^q \alpha^{-q} \|f\|_{L^q(\mathbb{R}^n)}^q.\end{aligned}$$

We will now control the second term. Let $E_j$ be a concentric dilate of $Q_j$; precisely,

$$E_j := Q(c_j, r_j),$$

where $c_j$ is the center of $Q_j$ and $r_j > 0$ is chosen so that $|E_j| = \frac{1}{(17\sqrt{n})^{\frac{n}{q}}\gamma\alpha} \int_{Q_j} b_j(x)\,dx$. Note that such $E_j$ exist since the function $r \mapsto |Q(x,r)|$ is continuous for each $x \in \mathbb{R}^n$. Applying Hölder's inequality and property (3), we have

$$|E_j| = \frac{1}{(17\sqrt{n})^{\frac{n}{q}}\gamma\alpha} \int_{Q_j} b_j(x)\,dx \leq \frac{1}{(17\sqrt{n})^{\frac{n}{q}}\gamma\alpha} |Q_j|^{\frac{1}{q'}} \|b_j\|_{L^q(\mathbb{R}^n)} \leq |Q_j|.$$

Since $E_j$ is a cube with the same center as $Q_j$ and since $|E_j| \leq |Q_j|$, the containment $E_j \subseteq Q_j$ holds. In particular, the $E_j$ are pairwise disjoint. Set

$$E := \bigcup_{j=1}^{\infty} E_j.$$

Then

$$\left|\left\{|Tb| > \frac{\alpha}{2}\right\}\right| \leq \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where

$$\begin{aligned}\mathrm{I} &= |\Omega|, \\ \mathrm{II} &= \left|\left\{x \in \mathbb{R}^n \setminus \Omega : \left|T\left(b - (17\sqrt{n})^{\frac{n}{q}}\gamma\alpha \mathbb{1}_E\right)(x)\right| > \frac{\alpha}{4}\right\}\right|, \text{ and} \\ \mathrm{III} &= \left|\left\{(17\sqrt{n})^{\frac{n}{q}}\gamma\alpha |T(\mathbb{1}_E)| > \frac{\alpha}{4}\right\}\right|.\end{aligned}$$

The control of I follows from property (2),

$$|\Omega| = \sum_{j=1}^{\infty} \leq 3^n (B + [K]_{H_{q'}}) \|f\|_{L^q(\mathbb{R}^n)}^q.$$

For II, use Chebyshev's inequality, the fact that $\int_{Q_j} b_j(y) - (17\sqrt{n})^{\frac{n}{q}} \gamma\alpha \mathbb{1}_{E_j}(y)\, dy = 0$, Fubini's theorem, and Hölder's inequality to estimate

$$\begin{aligned}
\mathrm{II} &\leq 4\alpha^{-1} \int_{\mathbb{R}^n \setminus \Omega} \left| T\left( b - (17\sqrt{n})^{\frac{n}{q}} \gamma\alpha \mathbb{1}_E \right)(x) \right| dx \\
&\leq 4\alpha^{-1} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n \setminus \Omega} \left| T\left( b_j - (17\sqrt{n})^{\frac{n}{q}} \gamma\alpha \mathbb{1}_{E_j} \right)(x) \right| dx \\
&\leq 4\alpha^{-1} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n \setminus \Omega} \int_{Q_j} |K(x-y) - K(x-c_j)| \left| b_j(y) - (17\sqrt{n})^{\frac{n}{q}} \gamma\alpha \mathbb{1}_{E_j}(y) \right| dy dx \\
&= 4\alpha^{-1} \sum_{j=1}^{\infty} \int_{Q_j} \left( \int_{\mathbb{R}^n \setminus \Omega} |K(x-y) - K(x-c_j)| dx \right) \left| b_j(y) - (17\sqrt{n})^{\frac{n}{q}} \gamma\alpha \mathbb{1}_{E_j}(y) \right| dy \\
&\leq 4\alpha^{-1} \sum_{j=1}^{\infty} \left\| \int_{\mathbb{R}^n \setminus \Omega} |K(x-y) - K(x-c_j)| dx \right\|_{L^{q'}(Q_j)} \left\| b_j - (17\sqrt{n})^{\frac{n}{q}} \gamma\alpha \mathbb{1}_{E_j} \right\|_{L^q(\mathbb{R}^n)} \\
&\leq 4\alpha^{-1} \sup_{j\in\mathbb{N}} \left\| \int_{\mathbb{R}^n \setminus \Omega} |K(x-y) - K(x-c_j)| dx \right\|_{L^{q'}\left(Q_j, \frac{dy}{|Q_j|}\right)} \\
&\quad \times \sum_{j=1}^{\infty} |Q_j|^{\frac{1}{q'}} \left\| b_j - (17\sqrt{n})^{\frac{n}{q}} \gamma\alpha \mathbb{1}_{E_j} \right\|_{L^q(\mathbb{R}^n)}.
\end{aligned}$$

Using the triangle inequality, property (3), and the fact that $|E_j| \leq |Q_j|$, we have

$$\left\| b_j - (17\sqrt{n})^{\frac{n}{q}} \gamma\alpha \mathbb{1}_{E_j} \right\|_{L^q(\mathbb{R}^n)} \leq \|b_j\|_{L^q(\mathbb{R}^n)} + (17\sqrt{n})^{\frac{n}{q}} \gamma\alpha |E_j|^{\frac{1}{q}} \leq 2(17\sqrt{n})^{\frac{n}{q}} \gamma\alpha |Q_j|^{\frac{1}{q}}.$$

Using the above estimate and property (2), we control

$$\begin{aligned}
\mathrm{II} &\leq 8(17\sqrt{n})^{\frac{n}{q}} \gamma \sup_{j\in\mathbb{N}} \left\| \int_{\mathbb{R}^n \setminus \Omega} |K(x-y) - K(x-c_j)| dx \right\|_{L^{q'}\left(Q_j, \frac{dy}{|Q_j|}\right)} \sum_{j=1}^{\infty} |Q_j| \\
&\leq 8(17\sqrt{n})^{\frac{n}{q}} 3^n \gamma^{1-q} \alpha^{-q} \|f\|_{L^q(\mathbb{R}^n)}^q \sup_{j\in\mathbb{N}} \left\| \int_{\mathbb{R}^n \setminus \Omega} |K(x-y) - K(x-c_j)| dx \right\|_{L^{q'}\left(Q_j, \frac{dy}{|Q_j|}\right)}.
\end{aligned}$$

For each $j$, setting $R_j = \frac{\sqrt{n}}{2} l(Q_j)$, we have

$$Q_j \subseteq B(c_j, R_j) \subseteq B(c_j, 2R_j) \subseteq \Omega.$$

Then the supremum is bounded by

$$\sup_{j\in\mathbb{N}} \left[ \int_{B(c_j,R_j)} \left( \int_{\mathbb{R}^n \setminus B(c_j,2R_j)} |K(x-y) - K(x-c_j)| dx \right)^{q'} \frac{dy}{|Q_j|} \right]^{\frac{1}{q'}},$$

which is bounded by $\left(\frac{\sqrt{n}}{2}\right)^n v_n[K]_{H_{q'}}$ by changing variables $x' = x - c_j$, $y' = y - c_j$ and by replacing the supremum over $R_j$ by the supremum over all $R > 0$. Therefore

$$\mathrm{II} \leq 8(17\sqrt{n})^{\frac{n}{q}} \left(\frac{3\sqrt{n}}{2}\right)^n v_n(B + [K]_{H_{q'}})^q \alpha^{-q} \|f\|^q_{L^q(\mathbb{R}^n)}.$$

To control III, use Chebyshev's inequality, the bound of $T$ on $L^s(\mathbb{R}^n)$, the fact that $|E| \leq |\Omega|$, and property (2) to estimate

$$\begin{aligned}
\mathrm{III} &\leq 4^s(17\sqrt{n})^{\frac{ns}{q}} \gamma^s \int_{\mathbb{R}^n} |T(\mathbb{1}_E)(x)|^s \, dx \\
&\leq 4^s(17\sqrt{n})^{\frac{ns}{q}} \gamma^s B^s |E| \\
&\leq 4^s(17\sqrt{n})^{\frac{ns}{q}} |\Omega| \\
&\leq 4^s(17\sqrt{n})^{\frac{ns}{q}} 3^n (B + [K]_{H_{q'}})^q \alpha^{-q} \|f\|^q_{L^q(\mathbb{R}^n)}.
\end{aligned}$$

Putting the estimates together, we get

$$|\{|Tf| > \alpha\}| \leq \left(2^s + 3^n + 8(17\sqrt{n})^{\frac{n}{q}} \left(\frac{3\sqrt{n}}{2}\right)^n v_n + 4^s(17\sqrt{n})^{\frac{ns}{q}} 3^n\right) \frac{(B + [K]_{H_{q'}})^q}{\alpha^q} \|f\|^q_{L^q(\mathbb{R}^n)}.$$

Since we assumed that $f$ was nonnegative, we must double the constant above to prove the statement for general $f \in L^q(\mathbb{R}^n)$.

When $s = \infty$, set $\gamma = (4(B + [K]_{H_{q'}}))^{-1}$. Then

$$\|Tg\|_{L^\infty(\mathbb{R}^n)} \leq B\|g\|_{L^\infty(\mathbb{R}^n)} \leq B\gamma\alpha \leq \frac{\alpha}{4},$$

so $\left|\left\{|Tg| > \frac{\alpha}{2}\right\}\right| = 0$. The part of the argument involving the set $\left\{|Tb| > \frac{\alpha}{2}\right\}$ is the same as in the case $s < \infty$. □

## 4. Conclusion

We end with some remarks and an open question.

**Remark 3.** The conclusions of Theorem 1 and Corollary 1 also follow under the weaker hypothesis that $T$ is bounded from $L^{s,1}(\mathbb{R}^n)$ to $L^{s,\infty}(\mathbb{R}^n)$. Here $L^{s,r}(\mathbb{R}^n)$ is the usual Lorentz space.

**Remark 4.** As in the case $q = 1$, there are natural vector-valued extensions of Theorem 1 and Corollary 1, in the spirit of [2].

**Remark 5.** Theorem 1 and Corollary 1 are also valid if the original kernel is not of convolution type. In this setting, we say a kernel $K$ defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, y) : x = y\}$ is in $H_r$ if

$$\sup_{R>0} \left[\frac{1}{v_n R^n} \int_{|y-y'|\leq R} \left(\int_{|x-y|\geq 2R} |K(x, y) - K(x, y')| \, dx\right)^r dy\right]^{\frac{1}{r}} < \infty,$$

and

$$\sup_{R>0} \left[\frac{1}{v_n R^n} \int_{|x-x'|\leq R} \left(\int_{|x-y|\geq 2R} |K(x, y) - K(x', y)| \, dx\right)^r dy\right]^{\frac{1}{r}} < \infty,$$

where $v_n$ is the volume of the unit ball $B(0, 1)$ in $\mathbb{R}^n$.

As stated in Remark 2 in the introduction, if $q > 1$ and $s < \infty$, then $T$ satisfies strong $L^p(\mathbb{R}^n)$ estimates for $p \in \big(\min(s', q), \max(q', s)\big)$, and in this case, the interval $\big(\min(s', q), \max(q', s)\big)$ is properly contained in $(1, \infty)$.

Let $q > 1$ and $s < \infty$. As of this writing, we are unable to establish whether the interval $\big(\min(s', q), \max(q', s)\big)$ is the largest interval $(a, b)$ for which an operator $T$ with kernel in $H_{q'}$ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (a, b)$. This certainly relates to the existence of examples of kernels in $H_{q_1}$ but not in $H_{q_2}$ for $q_1 < q_2$.

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