Interpolation for analytic families of multilinear operators on metric measure spaces

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Abstract

Let (X_j, d_j, μ_j), $j = 0, 1, \dots, m$ be metric measure spaces. Given $0 < p^\kappa \le \infty$ for $\kappa = 1, \dots, m$ and an analytic family of multilinear operators

$$T_z : L^{p^1}(X_1) \times \cdots \times L^{p^m}(X_m) \to L^1_{\mathrm{loc}}(X_0),$$

for z in the complex unit strip, we prove a theorem in the spirit of Stein's complex interpolation for analytic families. Analyticity and our admissibility condition are defined in the weak (integral) sense and relax the pointwise definitions given in [9]. Continuous functions with compact support are natural dense subspaces of Lebesgue spaces over metric measure spaces and we assume the operators T_z are initially defined on them. Our main lemma concerns the approximation of continuous functions with compact support by similar functions that depend analytically in an auxiliary parameter z. An application of the main theorem concerning bilinear estimates for Schrödinger operators on L^p is included.

Keywords: multilinear operators, analytic families of operators, interpolation, bilinear estimates for Schrödinger operators.

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1 Introduction

Interpolation between function spaces plays a fundamental role in many areas of analysis such as harmonic, complex, and functional analysis, as well as in PDE. The most common interpolation theorems are the ones of Riesz-Thorin, Marcinkiewicz, and Stein. Unlike the first two results which concern a single linear operator, Stein's interpolation theorem for analytic families of linear operators is formulated for families which vary analytically in an auxiliary parameter. In this way it covers and supersedes the case of a single operator, it is more flexible, and finds a variety of applications.

In recent years, there is an increasing interest for multilinear analysis. In this setting, it is of interest to have interpolation theorems analogous to those for linear operators. The primary purpose of this article is to prove a version of Stein's interpolation theorem for multilinear operators. Our interpolation result is given in the context of Lebesgue spaces over metric measure spaces (X_j, d_j, μ_j), $j = 0, 1, \dots, m$ in which balls have finite measure. Such spaces have nice subspace of dense functions such as the spaces of continuous functions with compact support $\mathcal{C}_c(X_j)$. We consider a family of multilinear operators T_z for z in the unit strip $\mathbf{S} = \{z \in \mathbb{C} : \ 0 \le \mathrm{Re}\ z \le 1\}$. This family is analytic in the following sense: For every f^j in $\mathcal{C}_c(X_j)$, $j = 1, \dots, m$ and w bounded function with compact support on X_0 the mapping

$$z \mapsto \int_{X_0} T_z(f^1, \dots, f^m)\, w\, d\mu_0 \tag{1.1}$$

is analytic in $\mathbf{S}$ and continuous on its closure. The operators T_z are taking values in the space of locally integrable functions on X_0 and satisfy the admissibility condition: there exists a constant γ with $0 \le \gamma < \pi$ and an $s \in [1, \infty]$ such that for any f^j in $\mathcal{C}_c(X_j)$ and every compact subset K of X_0 there is a constant $C(f^1, \dots, f^m, K)$ such that

$$\log \left[\int_K |T_z(f^1, \dots, f^m)|^s d\mu_0 \right]^{1/s} \le C(f^1, \dots, f^m, K)\, e^{\gamma |\mathrm{Im}\, z|}, \quad z \in \overline{\mathbf{S}}. \tag{1.2}$$

The initial estimates are of the form

$$\left\| T_{j+iy}(f^1, \dots, f^m) \right\|_{L^{q_j}(X_0)} \le \ B_j M_j(y) \prod_{\kappa=1}^{m} \|f^\kappa\|_{L^{p_j^\kappa}(X_j)}, \quad j \in \{0, 1\}, \ \ y \in \mathbb{R},$$

where $B_j > 0$ and $0 < p_j^\kappa \le \infty$. Then we prove that for $\theta \in (0, 1)$ and

$$\frac{1}{p^\kappa} = \frac{1-\theta}{p_0^\kappa} + \frac{\theta}{p_1^\kappa} \qquad \text{and} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

the multilinear operator

$$T_\theta : L^{p^1}(X_1) \times \cdots \times L^{p^m}(X_m) \to L^q(X_0)$$

is bounded with an appropriate estimate on its norm. We refer to Theorem 3.2 for the full statement. The reader easily recognizes the resemblance to the Stein's complex interpolation in the linear context.

In the multilinear setting, this type of results already appeared in the work of Grafakos and Mastyło [9]. The theorem in [9] was proved in the more general setting of quasi-Banach spaces. However the analyticity and admissibility required there were in the pointwise sense. The admissibility used there is that for every $(f^1, \dots, f^m) \in L^{p^1}(X_1) \times \cdots \times L^{p^m}(X_m)$ and a.e. $y \in X_0$, the mapping

$$z \mapsto T_z(\varphi_1, \dots, \varphi_m)(y)$$

is of admissible growth. Unlike the integral condition (1.2), the pointwise admissibility is not easy to check when the operators are not explicit. The extension to more general rearrangement invariant spaces over X_j is not important in our applications and is not pursued here.

Section 4 is devoted to some bilinear estimates for Schrödinger operators. We consider $L = -\mathrm{div}(A\nabla) + V$, where $A = (a_{kl})_{1\le k,l\le n}$ is a symmetric matrix with real-valued and bounded measurable entries and V is a nonnegative locally integrable potential on $\mathbb{R}^n$. We prove that for every $p \in (1,\infty)$ and $\alpha, \beta \in [0,\infty)$, there exists a constant $C(\alpha,\beta,\gamma,p)$, independent of the dimension n, such that

$$\int_0^\infty \int_{\mathbb{R}^n} |\mathbf{\Gamma} L^\alpha e^{-tL} f(x) \cdot \mathbf{\Gamma} L^\beta e^{-tL} g(x)|\, dx\, t^{\alpha+\beta} dt \le C(\alpha,\beta,\gamma,p)\|f\|_{L^p}\|g\|_{L^{p'}}.$$

Here $\mathbf{\Gamma}$ is either ∇ or multiplication by $\sqrt{V}$. The result for $\alpha = \beta = 0$ is due to Dragicevic and Volberg [8]. Our proof relies heavily on their result as well as the interpolation theorem applied to an appropriate analytic family of bilinear operators.

Finally, we provide the reader with some useful results on log-subharmonic functions in the Section 5 (Appendix).

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2 Some preliminary facts

Throughout this section, (X, d, μ) will be a metric space equipped with a metric d and with a positive measure μ on a σ-algebra $\mathcal{A}$ of subsets of X. Let $x \in X$ and $r > 0$. A ball $B(x,r)$ is the set of points $B(x,r) = \{y \in X : \ d(x,y) < r\}$. We assume the following mild assumptions on (X, d, μ):

(i) $\mu(B(x,r)) < \infty$ for any $x \in X$ and $r > 0$,

(ii) μ is a regular measure with respect to the topology of X, i.e., for any $A \in \mathcal{A}$ with $\mu(A) < \infty$ one has

$$\begin{aligned} \mu(A) &= \sup\{\mu(K) : \ K \text{ is compact subset of } A\} \\ \mu(A) &= \inf\{\mu(U) : \ U \text{ is open subset of } X \text{ and contains } A\}. \end{aligned}$$

Simple functions on X have the form: $\sum_{j=1}^N \lambda_j \chi_{A_j}$, where λ_j are complex numbers and $A_j \in \mathcal{A}$ are pairwise disjoint and satisfy $\mu(A_j) < \infty$. Simple functions are dense in $L^p(X)$ for any $0 < p < \infty$ (as (X,μ) is σ-finite). Moreover, as the sets A_j can be approximated from below by compact sets, simple functions with A_j being compact sets are dense in $L^p(X)$ for $0 < p < \infty$. We denote by $L^1_{\mathrm{loc}}(X)$ the space of all measurable functions on X that are integrable over any compact subset of X. We also denote by $\mathcal{C}_c(X)$ the space of all continuous functions with compact support in X. The subsequent lemma guarantees the abundance of such functions.

Lemma 2.1. *Given K compact and U open subsets of X such that $K \subset U$, there exists $h \in \mathcal{C}_c(X)$ such that*

$$\chi_K \le h \le \chi_U.$$

Proof. The sets K and $X \setminus U$ are closed and disjoint, so by Urysohn's lemma (which is applicable on metric spaces) there is a continuous function $h : X \to [0,1]$ that is equal to 1 on K and 0 on $X \setminus U$. This function h satisfies the claim. □

The following result, inspired by [1], allows us to approximate $\mathcal{C}_c(X)$ functions by functions that are analytic in a new auxiliary variable z.

Lemma 2.2. *Let $0 < p_0 \le p_1 \le \infty$ satisfy $p_0 < \infty$, and define p via $1/p = (1-\theta)/p_0 + \theta/p_1$, where $0 < \theta < 1$. Given $f \in \mathcal{C}_c(X)$ and $\varepsilon > 0$, there exist N_ε and $h_j^\varepsilon \in \mathcal{C}_c(X)$ supported in pairwise disjoint open sets U_j^ε, $j = 1, \dots, N_\varepsilon$, and there exist nonzero complex constants c_j^ε such that the functions*

$$f_z^\varepsilon = \sum_{j=1}^{N_\varepsilon} |c_j^\varepsilon|^{\frac{p}{p_0}(1-z)+\frac{p}{p_1}z}\, h_j^\varepsilon \tag{2.1}$$

satisfy

$$\big\|f_\theta^\varepsilon - f\big\|_{L^{p_0}} \le \varepsilon, \qquad \begin{cases} \big\|f_\theta^\varepsilon - f\big\|_{L^{p_1}} \le \varepsilon & \text{if } p_1 < \infty \\ \\ \big\|f_\theta^\varepsilon\big\|_{L^\infty} \le \big\|f\big\|_{L^\infty} + \varepsilon & \text{if } p_1 = \infty, \end{cases} \tag{2.2}$$

and

$$\big\|f_{it}^\varepsilon\big\|_{L^{p_0}}^{p_0} \le \big\|f\big\|_{L^p}^p + \varepsilon', \quad \big\|f_{1+it}^\varepsilon\big\|_{L^{p_1}} \le \left(\big\|f\big\|_{L^p}^p + \varepsilon'\right)^{\frac{1}{p_1}}, \tag{2.3}$$

where ε' depends on $\varepsilon, p, \|f\|_{L^p}$ and tends to zero as $\varepsilon \to 0$.

Proof. Given $f \in \mathcal{C}_c(X)$ and $\varepsilon > 0$, let $E = \operatorname{supp} f$. Let $E' = \bigcup_{x \in E} B(x,1)$ and notice that in view of the compactness of E, the set E' has finite measure. By the uniform continuity of f there is a δ in $(0,1)$ such that

$$x, y \in X, \quad d(x,y) < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2^{\max(1,\frac{1}{p_0})}}\left(\frac{1}{1+\mu(E')}\right)^{\frac{1}{p_0}}.$$

Then we cover the support of f by finitely many balls $B_1, \dots, B_{N'_\varepsilon}$ of radius $\delta/2$. We find pairwise disjoint measurable subsets A_j of B_j that satisfy $B_1 \cup \dots \cup B_{N'_\varepsilon} = A_1 \cup \dots \cup A_{N_\varepsilon}$; notice that this union contains E and is contained in E'. Suppose that N_ε of the A_j are nonempty, without loss of generality assume these are the first N_ε; this way we have $A_j \neq \emptyset$ for all $j \le N_\varepsilon \le N'_\varepsilon$. We now let $c_j^\varepsilon = f(x_j)$, where x_j is any fixed point in A_j. As a consequence of these choices one has

$$\big\|f - \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \chi_{A_j}\big\|_{L^\infty} \le \frac{\varepsilon}{2^{\max(1,\frac{1}{p_0})}}\left(\frac{1}{1+\mu(E')}\right)^{\frac{1}{p_0}}.$$

It follows from this that if $p_1 = \infty$ then

$$\sup_{1 \le j \le N_\varepsilon} |c_j^\varepsilon| = \big\|\sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \chi_{A_j}\big\|_{L^\infty} \le \big\|f\big\|_{L^\infty} + \varepsilon, \tag{2.4}$$

while if $p_1 < \infty$ then

$$\|f - \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \chi_{A_j}\|_{L^{p_\kappa}}^{\min(1,p_\kappa)} \le \Big[\frac{\varepsilon}{2^{\max(1,\frac{1}{p_0})}}\Big(\frac{1}{1+\mu(E')}\Big)^{\frac{1}{p_0}} \mu\Big(\bigcup_{j=1}^{N_\varepsilon} A_j\Big)^{\frac{1}{p_\kappa}}\Big]^{\min(1,p_\kappa)} \le \frac{\varepsilon^{\min(1,p_\kappa)}}{2},$$

where $p_\kappa \in \{p_0, p_1, p\}$.

By the regularity of μ we pick compact sets K_j contained in A_j such that $\mu(A_j \backslash K_j) < \frac{\eta}{2}$, for some $\eta > 0$ chosen to satisfy

$$\max_{\kappa\in\{0,1\}} \Big(\sum_{j=1}^{N_\varepsilon} |2c_j^\varepsilon|^{\min(1,p_\kappa)}\Big)\eta^{\min(1,\frac{1}{p_\kappa})} < \frac{\varepsilon^{\min(1,p_\kappa)}}{2}.$$

Then the compact sets K_j are pairwise disjoint, so

$$\min_{j\neq k} \big(\mathrm{dist}(K_j, K_k)\big) = \rho > 0.$$

Now let

$$U_j' = \bigcup_{x\in K_j} B(x, \frac{\rho}{3})$$

and choose U_j'' open such that $A_j \subset U_j''$ and $\mu(U_j'' \setminus A_j) < \frac{\eta}{2}$ by the regularity of μ. Then define

$$U_j^\varepsilon = U_j' \cap U_j'', \qquad\qquad j = 1, 2, \ldots, N_\varepsilon.$$

The sets U_j^ε are open and pairwise disjoint. Also each U_j^ε contains the compact set K_j. By Lemma 2.1 we pick $g_j^\varepsilon \in \mathcal{C}_c(X)$ with values in $[0,1]$ satisfying $\chi_{K_j} \le g_j^\varepsilon \le \chi_{U_j^\varepsilon}$. Then if $p_1 < \infty$ by the subadditivity of $\|\cdot\|_p^{\min(1,p)}$ we write

$$\begin{aligned}
\Big\|f - \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon g_j^\varepsilon\Big\|_{L^{p_\kappa}}^{\min(1,p_\kappa)} &\le \Big\|f - \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \chi_{A_j}\Big\|_{L^{p_\kappa}}^{\min(1,p_\kappa)} + \Big\|\sum_{j=1}^{N_\varepsilon} c_j^\varepsilon(\chi_{A_j} - g_j^\varepsilon)\Big\|_{L^{p_\kappa}}^{\min(1,p_\kappa)} \\
&\le \frac{\varepsilon^{\min(1,p_\kappa)}}{2} + \sum_{j=1}^{N_\varepsilon} |2c_j^\varepsilon|^{\min(1,p_\kappa)}\eta^{\min(1,\frac{1}{p_\kappa})} \\
&\le \varepsilon^{\min(1,p_\kappa)},
\end{aligned}$$

as the $\chi_{A_j} - g_j^\varepsilon$ is bounded by 2 and supported in $U_j^\varepsilon \setminus K_j$ which has measure at most η. This proves (2.2) when $p_1 < \infty$. Note that the same argument shows that

$$\Big\|f - \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \chi_{U_j^\varepsilon}\Big\|_{L^{p_\kappa}} \le \varepsilon, \qquad \kappa \in \{0,1\}. \tag{2.5}$$

We set $h_j^\varepsilon = e^{i\phi_j^\varepsilon} g_j^\varepsilon$, where ϕ_j^ε is the argument of the complex number c_j^ε. Then h_j^ε is that function claimed in (2.1). Observe that

$$f_\theta^\varepsilon = \sum_{j=1}^{N_\varepsilon} |c_j^\varepsilon| h_j^\varepsilon = \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon g_j^\varepsilon$$

satisfies (2.2) when $p_1 < \infty$; in the case $p_1 = \infty$ we have

$$|f_\theta^\varepsilon| \le \sum_{j=1}^{N_\varepsilon} |c_j^\varepsilon| \chi_{U_j^\varepsilon} \le \sup_j |c_j^\varepsilon| \le \|f\|_{L^\infty} + \varepsilon$$

by (2.4). Thus (2.2) holds when $p_1 = \infty$. We now write

$$\|f_{it}^\varepsilon\|_{L^{p_0}}^{p_0} \le \sum_{j=1}^{N_\varepsilon} |c_j^\varepsilon|^p \mu(U_j^\varepsilon) = \Big\| \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \chi_{U_j^\varepsilon} \Big\|_{L^p}^p \le \Big(\varepsilon^{\min(1,p)} + \|f\|_{L^p}^{\min(1,p)}\Big)^{\frac{p}{\min(1,p)}},$$

having used (2.5).

We set $\varepsilon' = \varepsilon^p$ if $p \le 1$ and $\varepsilon' = (\varepsilon + \|f\|_{L^p})^p - \|f\|_{L^p}^p$ when $1 < p < \infty$. Then $\varepsilon' \to 0$ as $\varepsilon \to 0$ and this proves (2.3) for p_0 and analogously for p_1 when $p_1 < \infty$; now if $p_1 = \infty$ then $\|f_{1+it}^\varepsilon\|_{L^\infty} \le 1$ and the right hand side of the second inequality in (2.3) is equal to 1, so the inequality is still valid. □

Throughout this paper $\mathbf{S}$ will denote the open unit strip $\mathbf{S} = \{z \in \mathbb{C} : \ 0 < \operatorname{Re} z < 1\}$ and $\overline{\mathbf{S}}$ its closure, i.e., the closed unit strip. As the boundary of $\mathbf{S}$ has two disjoint pieces and integration over each piece will be written separately, we introduce the "half" Poisson kernel Ω on $\overline{\mathbf{S}} \setminus \{1\}$ via:

$$\Omega(x, y) = \frac{1}{2} \frac{\sin(\pi x)}{\cosh(\pi y) + \cos(\pi x)} \tag{2.6}$$

where $0 \le x \le 1$ and $-\infty < y < \infty$ but $(x, y) \ne (1, 0)$. This function is nonnegative and satisfies

$$\int_{-\infty}^{+\infty} \Omega(x, t)\, dt = x \qquad \text{for all } 0 \le x < 1. \tag{2.7}$$

The next result due to Hirschman [11, Lemma 1] is fundamental in complex interpolation.

Proposition 2.3. *Let F be a continuous function on the closed unit strip $\overline{\mathbf{S}}$ such that* $\log |F|$ *is subharmonic in* $\mathbf{S}$ *that satisfies*

$$\sup_{0 \le x \le 1} \log |F(x + iy)| \le C\, e^{a\,|y|}, \qquad -\infty < y < \infty, \tag{2.8}$$

for some fixed $C, a > 0$ with $a < \pi$. If N_0, N_1 are continuous functions on the line that satisfy $N_0(y) \ge \log |F(iy)|$ and $N_1(y) \ge \log |F(1 + iy)|$ for all $y \in (-\infty, \infty)$, then for any $\theta \in (0, 1)$ we have

$$\log |F(\theta)| \le \int_{-\infty}^{+\infty} \Omega(1 - \theta, t) N_0(t)\, dt + \int_{-\infty}^{+\infty} \Omega(\theta, t) N_1(t)\, dt. \tag{2.9}$$

3 Interpolation for analytic families of multilinear operators

Throughout this section (X_j, d_j, μ_j), $0 \le j \le m$, are metric measure spaces that satisfy assumptions (i) and (ii).

Definition 3.1. *Suppose that for every $z \in \overline{\mathbf{S}}$ there is an associated m-linear operator T_z defined on $\mathcal{C}_c(X_1) \times \cdots \times \mathcal{C}_c(X_m)$ and taking values in $L^1_{\rm loc}(X_0)$. We call $\{T_z\}_z$ an* analytic family *if for all $(\varphi_1, \dots, \varphi_m)$ in $\mathcal{C}_c(X_1) \times \cdots \times C_c(X_m)$ and w bounded function with compact support on X_0 the mapping*

$$z \mapsto \int_{X_0} T_z(\varphi_1, \dots, \varphi_m)\, w \, d\mu_0 \tag{3.1}$$

is analytic in the open strip $\mathbf{S}$ and continuous on its closure. The analytic family $\{T_z\}_z$ is called of admissible growth *if there is a constant γ with $0 \le \gamma < \pi$ and an s satisfying $1 \le s \le \infty$, such that for any $(\varphi_1, \dots, \varphi_m)$ in $\mathcal{C}_c(X_1) \times \cdots \times C_c(X_m)$ and every compact subset K of X_0 there is a constant $C(\varphi_1, \dots, \varphi_m, K)$ such that*

$$\log \left[\int_K |T_z(\varphi_1, \dots, \varphi_m)|^s d\mu_0 \right]^{1/s} \le C(\varphi_1, \dots, \varphi_m, K)\, e^{\gamma |Im\, z|}, \quad \text{for all } z \in \overline{\mathbf{S}}. \tag{3.2}$$

Now we state the main result on interpolation of analytic multilinear operators.

Theorem 3.2. *For $z \in \overline{\mathbf{S}}$, let T_z be an m-linear operator on $\mathcal{C}_c(X_1) \times \cdots \times \mathcal{C}_c(X_m)$ with values in $L^1_{\rm loc}(X_0)$ that form an analytic family of admissible growth. For $\kappa \in \{1, \dots, m\}$ let $0 < p_0^\kappa, p_1^\kappa \le \infty$, $0 < q_0, q_1 \le \infty$, fix $0 < \theta < 1$, and define p^κ, q by the equations*

$$\frac{1}{p^\kappa} = \frac{1-\theta}{p_0^\kappa} + \frac{\theta}{p_1^\kappa} \qquad \textit{and} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}\,. \tag{3.3}$$

Suppose that for all $(f^1, \dots, f^m) \in \mathcal{C}_c(X_1) \times \cdots \times \mathcal{C}_c(X_m)$ we have

$$\big\| T_{iy}(f^1, \dots, f^m) \big\|_{L^{q_0}(X_0)} \le B_0 M_0(y) \prod_{\kappa=1}^m \| f^\kappa \|_{L^{p_0^\kappa}(X_\kappa)}\,, \tag{3.4}$$

$$\big\| T_{1+iy}(f^1, \dots, f^m) \big\|_{L^{q_1}(X_0)} \le B_1 M_1(y) \prod_{\kappa=1}^m \| f^\kappa \|_{L^{p_1^\kappa}(X_\kappa)}\,, \tag{3.5}$$

where M_0 and M_1 are nonnegative continuous functions on the real line that satisfy

$$M_0(y) \le e^{c\, e^{\tau |y|}}, \qquad M_1(y) \le e^{c\, e^{\tau |y|}} \tag{3.6}$$

for some $c, \tau \ge 0$ with $\tau < \pi$, and $B_0, B_1 > 0$. Then for all f^j in $\mathcal{C}_c(X_j)$, $1 \le j \le m$, we have

$$\big\| T_\theta(f^1, \dots, f^m) \big\|_{L^q(X_0)} \le B_0^{1-\theta} B_1^\theta M(\theta) \prod_{\kappa=1}^m \big\| f^\kappa \big\|_{L^{p^\kappa}(X_\kappa)}, \tag{3.7}$$

where

$$M(\theta) = \exp \left\{ \int_{-\infty}^{\infty} \Big[\Omega(1-\theta, y) \log M_0(y) + \Omega(\theta, y) \log M_1(y) \Big]\, dy \right\}.$$

Proof. **Case I**: $\min(q_0, q_1) > 1$.

This assumption forces $q_0', q_1' < \infty$ and so $q' < \infty$ as well. Given T_z as in the statement of the theorem, for $f^j \in \mathcal{C}_c(X_j)$, $1 \le j \le m$, and $g \in \mathcal{C}_c(X_0)$ one may be tempted to consider the family of operators

$$H(z) = \int_{X_0} T_z(f^1, \dots, f^m)\, g\, d\mu_0$$

which is analytic in $\mathbf{S}$, continuous and bounded in $\overline{\mathbf{S}}$ and satisfies the hypotheses of Proposition 2.3 with bounds

$$|H(iy)| \le B_0\, M_0(y) \prod_{\kappa=1}^m \|f^\kappa\|_{L^{p_0^\kappa}} \|g\|_{L^{q_0'}}, \quad |H(1+iy)| \le B_1\, M_1(y) \prod_{\kappa=1}^m \|f^\kappa\|_{L^{p_1^\kappa}} \|g\|_{L^{q_1'}}$$

for all real y. Applying the result of Proposition 2.3 and identity (2.7) (with $x = 1-\theta$ and $x = \theta$) yields for all $f^1, \dots, f^m \in \mathcal{C}_c(X)$ and $g \in \mathcal{C}_c(X_0)$

$$\left| \int_{X_0} T_\theta(f^1, \dots, f^m)\, g\, d\mu_0 \right| \le M(\theta) \big(B_0 \prod_{\kappa=1}^m \|f^\kappa\|_{L^{p_0}} \|g\|_{L^{q_0'}} \big)^{1-\theta} \big(B_1 \prod_{\kappa=1}^m \|f^\kappa\|_{L^{p_1}} \|g\|_{L^{q_1'}} \big)^{\theta}. \quad (3.8)$$

Unfortunately this bound does not provide the claimed assertion; it supplies, however, a useful continuity estimate for the operator T_θ.

To improve (3.8), let us first consider the situation where $\min(p_0^\kappa, p_1^\kappa) < \infty$ for some $\kappa \in \{1, \dots, m\}$, which forces $p^\kappa < \infty$ for the same κ. Fix $f^j \in \mathcal{C}_c(X_j)$, $g \in \mathcal{C}_c(X_0)$ and $\varepsilon > 0$. By Lemma 2.2 we can find $f_z^{1,\varepsilon}, \dots, f_z^{m,\varepsilon}$ and g_z^ε such that

$$f_z^{\kappa,\varepsilon} = \sum_{j_\kappa=1}^{N_\varepsilon^\kappa} |c_{j_\kappa}^{\kappa,\varepsilon}|^{\frac{p^\kappa}{p_0^\kappa}(1-z) + \frac{p^\kappa}{p_1^\kappa} z} u_{j_\kappa}^{\kappa,\varepsilon}, 1 \le \kappa \le m, \quad g_z^\varepsilon = \sum_{k=1}^{M_\varepsilon} |d_k^\varepsilon|^{\frac{q'}{q_0'}(1-z) + \frac{q'}{q_1'} z} v_k^\varepsilon,$$

where $(u_{j_1}^{1,\varepsilon}, \dots, u_{j_m}^{m,\varepsilon})$ lies in $\mathcal{C}_c(X_1) \times \cdots \times \mathcal{C}_c(X_m)$, v_k^ε in $\mathcal{C}_c(X_0)$, and

$$\|f_\theta^{\kappa,\varepsilon} - f^\kappa\|_{L^{p_0}} < \varepsilon,\ \|g_\theta^\varepsilon - g\|_{L^{q_0'}} < \varepsilon,\ \|f_\theta^{\kappa,\varepsilon} - f^\kappa\|_{L^{p_1}} < \varepsilon,\ \|g_\theta^\varepsilon - g\|_{L^{q_1'}} < \varepsilon \quad (3.9)$$

$$\|f_{it}^{\kappa,\varepsilon}\|_{L^{p_0}} \le \big(\|f^\kappa\|_{L^p} + \varepsilon' \big)^{\frac{p}{p_0}}, \quad \|g_{it}^\varepsilon\|_{L^{q_0'}} \le \big(\|g\|_{L^{q'}} + \varepsilon' \big)^{\frac{q'}{q_0'}}, \quad (3.10)$$

$$\|f_{1+it}^{\kappa,\varepsilon}\|_{L^{p_1}} \le \big(\|f^\kappa\|_{L^p} + \varepsilon' \big)^{\frac{p}{p_1}}, \quad \|g_{1+it}^\varepsilon\|_{L^{q_1'}} \le \big(\|g\|_{q'} + \varepsilon' \big)^{\frac{q'}{q_1'}}, \quad (3.11)$$

where $\|f_\theta^{\kappa,\varepsilon} - f^\kappa\|_{L^{p_1^\kappa}} < \varepsilon$ in (3.9) is replaced by $\|f_\theta^{\kappa,\varepsilon}\|_{L^\infty} \le \|f^\kappa\|_{L^\infty} + \varepsilon$, if $p_1^\kappa = \infty$ and analogously if $p_0^\kappa = \infty$.

Now consider the function defined on the closure of the unit strip

$$F(z) = \int_{X_0} T_z(f_z^{1,\varepsilon}, \dots, f_z^{m,\varepsilon})\, g_z^\varepsilon\, d\mu_0$$

$$= \sum_{\substack{1 \le j_1 \le N_\varepsilon^1 \\ \vdots \\ 1 \le j_m \le N_\varepsilon^m}} \sum_{k=1}^{M_\varepsilon} \left\{ \left[\prod_{\kappa=1}^m |c_{j_\kappa}^{\kappa,\varepsilon}|^{\frac{p^\kappa}{p_0^\kappa}(1-z) + \frac{p^\kappa}{p_1^\kappa} z} |d_k^\varepsilon|^{\frac{q'}{q_0'}(1-z) + \frac{q'}{q_1'} z} \right] \int_{X_0} T_z(u_{j_1}^{1,\varepsilon}, \dots, u_{j_m}^{m,\varepsilon}) v_k^\varepsilon\, d\mu_0 \right\}.$$

Applying Hölder's inequality with exponents s and s' to $\int_{X_0} T_z(u_{j_1}^{1,\varepsilon},\dots,u_{j_m}^{m,\varepsilon})v_k^\varepsilon\,d\mu_0$ and using condition (3.2) we obtain for any z in $\overline{\mathbf{S}}$

$$|F(z)| \le \left[\prod_{\kappa=1}^m \sum_{j_\kappa=1}^{N_\varepsilon^\kappa} |c_{j_\kappa}^{\kappa,\varepsilon}|^{\frac{p^\kappa}{p_0^\kappa}+\frac{p^\kappa}{p_1^\kappa}} \sum_{k=1}^{M_\varepsilon} |d_k^\varepsilon|^{\frac{q'}{q_0'}+\frac{q'}{q_1'}} \|v_k^\varepsilon\|_{L^{s'}}\right] e^{[\max_{j_1,\dots,j_m,k} C(u_{j_1}^{1,\varepsilon},\dots,u_{j_m}^{m,\varepsilon},\mathrm{supp}\ v_k^\varepsilon)]e^{\gamma|\mathrm{Im}\, z|}}$$

$$\le e^{C' e^{\gamma|\mathrm{Im}\, z|}},$$

where C' equals $\max_{j_1,\dots,j_m,k} C(u_{j_1}^{1,\varepsilon},\dots,u_{j_m}^{m,\varepsilon},\mathrm{supp}\ v_k^\varepsilon)$ plus the logarithm of the double sum in the square brackets. Thus F satisfies the hypothesis of Proposition 2.3, as $\gamma<\pi$.

Hölder's inequality, hypothesis (3.4) and (3.10) give for y real

$$|F(iy)| \le B_0 M_0(y) \prod_{\kappa=1}^m \|f_{iy}^{\kappa,\varepsilon}\|_{L^{p_0^\kappa}} \|g_{iy}^\varepsilon\|_{L^{q_0'}} \le B_0 M_0(y) \prod_{\kappa=1}^m \big(\|f^\kappa\|_{L^p}+\varepsilon'\big)^{\frac{p^\kappa}{p_0^\kappa}} \big(\|g\|_{L^{q'}}+\varepsilon'\big)^{\frac{q'}{q_0'}}.$$

Likewise, Hölder's inequality, the hypothesis (3.5) and (3.11) imply for y real

$$\begin{aligned} |F(1+iy)| &\le B_1 M_1(y) \prod_{\kappa=1}^m \|f_{1+iy}^{\kappa,\varepsilon}\|_{L^{p_1^\kappa}} \|g_{1+iy}^\varepsilon\|_{L^{q_1'}} \\ &\le B_1 M_1(y) \prod_{\kappa=1}^m \big(\|f^\kappa\|_{L^{p^\kappa}}+\varepsilon'\big)^{\frac{p^\kappa}{p_1^\kappa}} \big(\|g\|_{L^{q'}}+\varepsilon'\big)^{\frac{q'}{q_1'}}. \end{aligned}$$

As $\log|F|$ is subharmonic in $\mathbf{S}$, applying Proposition 2.3 we obtain

$$\log|F(\theta)| \le \int_{-\infty}^{+\infty} \Omega(1-\theta,t)\log[M_0(t)\,Q_0]\,dt + \int_{-\infty}^{+\infty} \Omega(\theta,t)\log[M_1(t)\,Q_1]\,dt,$$

where Ω is the Poisson kernel on the strip [defined in (2.6)] and

$$Q_0 = B_0 \prod_{\kappa=1}^m \big(\|f^\kappa\|_{L^p}+\varepsilon'\big)^{\frac{p^\kappa}{p_0^\kappa}} \big(\|g\|_{L^{q'}}+\varepsilon'\big)^{\frac{q'}{q_0'}}, \quad Q_1 = B_1 \prod_{\kappa=1}^m \big(\|f^\kappa\|_{L^p}+\varepsilon'\big)^{\frac{p^\kappa}{p_1^\kappa}} \big(\|g\|_{L^{q'}}+\varepsilon'\big)^{\frac{q'}{q_1'}}.$$

Using identity (2.7) (with $x=1-\theta$ and $x=\theta$) and the fact that

$$Q_0^{1-\theta}Q_1^\theta = B_0^{1-\theta}B_1^\theta \prod_{\kappa=1}^m \big(\|f^\kappa\|_{L^p}+\varepsilon'\big)\big(\|g\|_{L^{q'}}+\varepsilon'\big)$$

we obtain (with $M(\theta)$ as in the statement of the theorem) that

$$\left|\int_{X_0} T_\theta(f_\theta^{1,\varepsilon},\dots,f_\theta^{m,\varepsilon})g_\theta^\varepsilon\,d\mu_0\right| = |F(\theta)| \le M(\theta)B_0^{1-\theta}B_1^\theta \prod_{\kappa=1}^m \big(\|f^\kappa\|_{L^p}+\varepsilon'\big)\big(\|g\|_{L^{q'}}+\varepsilon'\big). \tag{3.12}$$

An application of the triangle inequality gives

$$\begin{aligned} &\left|\int_{X_0} T_\theta(f^1,\dots,f^m)\,g\,d\mu_0 - \int_{X_0} T_\theta(f_\theta^{1,\varepsilon},\dots,f_\theta^{m,\varepsilon})\,g_\theta^\varepsilon\,d\mu_0\right| \qquad (3.13) \\ &\le \sum_{\kappa=1}^m \left|\int_{X_0} T_\theta(f_\theta^1,\dots,f_\theta^{\kappa-1},f^\kappa-f_\theta^\kappa,f^{\kappa+1},\dots,f^m)\,g\,d\mu_0\right| \\ &\quad + \left|\int_{X_0} T_\theta(f_\theta^{1,\varepsilon},\dots,f_\theta^{m,\varepsilon})\,(g-g_\theta^\varepsilon)\,d\mu_0\right|. \end{aligned}$$

We now apply (3.8) in each of the terms on the right side of the inequality and we use (3.9). We deduce that (3.13) tends to zero as $\varepsilon \to 0$. We conclude

$$\left| \int_{X_0} T_\theta(f^1, \dots, f^m)\, g\, d\mu_0 \right| \le M(\theta) B_0^{1-\theta} B_1^\theta \prod_{\kappa=1}^m \|f^\kappa\|_{L^p} \|g\|_{L^{q'}}. \tag{3.14}$$

Finally we obtain (3.7) by taking the supremum in (3.14) over all g in $\mathcal{C}_c(X_0)$ with $L^{q'}$ norm equal to 1.

Suppose now that $p_0^\kappa = p_1^\kappa = \infty$ for some κ. This forces $p^\kappa = \infty$ for these κ. Without loss of generality assume that $p_0^\kappa = p_1^\kappa = \infty$ for all $\kappa \le \lambda$ and $\min(p_0^\kappa, p_1^\kappa) < \infty$ for all $\kappa \in \{\lambda+1, \dots, m\}$. We repeat the preceding argument working with the analytic function

$$F(z) = \int_{X_0} T_z(f^1, \dots, f^\lambda, f_z^{\lambda+1,\varepsilon}, \dots, f_z^{m,\varepsilon})\, g_z^\varepsilon\, d\mu_0$$

on $\mathbf{S}$ which is multilinear of a lower degree and satisfies the initial estimates

$$\big|F(iy)\big| \le B_0 \Big(\prod_{\kappa=1}^\lambda \big\|f^\kappa\big\|_{L^\infty} \Big) M_0(y) \Big(\prod_{\kappa=\lambda+1}^m \big\|f^\kappa\big\|_{L^{p_0^\kappa}} \Big) \big\|g\big\|_{L^{q'}}.$$

and

$$\big|F(1+iy)\big| \le B_1 \Big(\prod_{\kappa=1}^\lambda \big\|f^\kappa\big\|_{L^\infty} \Big) M_1(y) \Big(\prod_{\kappa=\lambda+1}^m \big\|f^\kappa\big\|_{L^{p_1^\kappa}} \Big) \big\|g\big\|_{L^{q'}}.$$

The argument in the previous case using Proposition 2.3 yields

$$\left| \int_{X_0} T_\theta(f^1, \dots, f^m)\, g\, d\mu_0 \right| \le B_0^{1-\theta} B_1^\theta \Big(\prod_{\kappa=1}^\lambda \big\|f^\kappa\big\|_{L^\infty} \Big) M(\theta) \Big(\prod_{\kappa=\lambda+1}^m \big\|f^\kappa\big\|_{L^{p^\kappa}} \Big) \big\|g\big\|_{L^{q'}}.$$

Finally we take the supremum of the integrals over all g in $\mathcal{C}_c(X)$ with $L^{q'}$ norm equal to 1, to deduce (3.7).

Case II: $\min(q_0, q_1) \le 1$.

Assume first that $\min(p_0^\kappa, p_1^\kappa) < \infty$ for all κ. Choose $r > 1$ such that $r \min(q_0, q_1) > q$. Let us fix a nonnegative step function g with $\|g\|_{L^{r'}(X_0)} = 1$. Assume that $g = \sum_{k=1}^K a_k \chi_{E_k}$, where $a_k > 0$ and E_k are pairwise disjoint measurable compact subsets of X_0 (hence of finite measure). It suffices to work with such dense subsets of $L^{r'}(X_0)$ in view of the assumption that X_0 is a σ-finite metric space. For $z \in \mathbb{C}$ set

$$g^z = \sum_{k=1}^K a_k^{R(z)} \chi_{E_k},$$

where we set

$$R(z) = r' \Big[1 - \frac{q}{rq_0}(1-z) - \frac{q}{rq_1} z \Big].$$

Notice that $R(\theta)=1$. We fix $f^\kappa \in \mathcal{C}_c(X)$ and $\varepsilon>0$. Let $f_z^{\kappa,\varepsilon}$ be as in Case I obtained by Lemma 2.2. Define the function

$$G(z)=\int_{X_0}\left|T_z(f_z^{1,\varepsilon},\dots,f_z^{m,\varepsilon})\right|^{\frac{q}{r}}|g^z|\;d\mu_0=\sum_{k=1}^{K}\int_{E_k}\left|F_k(z,x)\right|^{\frac{q}{r}}\;d\mu_0(x). \tag{3.15}$$

where

$$F_k(x,z)=a_k^{\frac{r}{q}R(z)}\sum_{\substack{1\le j_1\le N_\varepsilon^1\\ \vdots\\ 1\le j_m\le N_\varepsilon^m}}\left[\prod_{\kappa=1}^{m}\left(|c_{j_\kappa}^{\kappa,\varepsilon}|^{\frac{p^\kappa}{p_0^\kappa}(1-z)+\frac{p^\kappa}{p_1^\kappa}z}\right)T_z(u_{j_1}^{1,\varepsilon},\dots,u_{j_m}^{m,\varepsilon})(x)\right].$$

If we knew that each term of the sum on the right in (3.15) is log-subharmonic, it would follow from Lemma 5.1 that so is G. To achieve this we use Lemma 5.3, which requires knowing that for each k, the mapping $z\mapsto F_k(\cdot,z)$ is analytic from $\mathbf{S}$ to $L^1(E_k)$. To show this, in view of Theorem 5.2, it suffices to show that for any bounded function w supported in E_k the function $z\mapsto\int_{E_k}F_k(z,x)w(x)\,d\mu_0(x)$ is analytic in $\mathbf{S}$ and continuous on its closure; but this condition is guaranteed by the definition of analytic families.

We plan to apply Proposition 2.3 to G and we verify its hypotheses. Using Hölder's inequality with indices $\frac{rq_0}{q}$ and $\left(\frac{rq_0}{q}\right)'$, (3.4), and the fact $\|g\|_{L^{r'}}=1$ we obtain

$$\begin{aligned}G(it)&\le\left\{\int_{X_0}\left|T_{it}(f_{it}^{1,\varepsilon},\dots,f_{it}^{m,\varepsilon})\right|^{q_0}d\mu_0\right\}^{\frac{q}{rq_0}}\left\|g^{it}\right\|_{L^{(\frac{rq_0}{q})'}}\\&\le\left[B_0M_0(t)\prod_{\kappa=1}^{m}\left(\left\|f^\kappa\right\|_{L^{p^\kappa}}^{p^\kappa}+\varepsilon'\right)^{\frac{1}{p^\kappa}}\right]^{\frac{q}{r}}.\end{aligned}$$

Similarly, we obtain the estimate

$$G(1+it)\le\left[B_1M_1(t)\prod_{\kappa=1}^{m}\left(\left\|f^\kappa\right\|_{L^{p^\kappa}}^{p^\kappa}+\varepsilon'\right)^{\frac{1}{p^\kappa}}\right]^{\frac{q}{r}}.$$

Finally we verify condition (2.8) for G. Let E be a compact set that contains all E_k. We apply Hölder's inequality with indices $\frac{rs}{q}$ and $\left(\frac{rs}{q}\right)'$ to obtain for $z\in\overline{\mathbf{S}}$

$$\begin{aligned}&G(z)\\&\le\left\|T_z(f_z^\varepsilon)\chi_E\right\|_{L^s}^{\frac{q}{r}}\left\|g^z\right\|_{L^{(\frac{rs}{q})'}}\\&\le\left[\sum_{\substack{1\le j_1\le N_\varepsilon^1\\ \vdots\\ 1\le j_m\le N_\varepsilon^m}}\prod_{\kappa=1}^{m}|c_{j_\kappa}^{\kappa,\varepsilon}|^{\frac{p^\kappa}{p_0^\kappa}+\frac{p^\kappa}{p_1^\kappa}}\left\|T_z(u_{j_1}^{1,\varepsilon},\dots,u_{j_m}^{m,\varepsilon})\right\|_{L^s(E)}\right]^{\frac{q}{r}}\left[\sum_{k=1}^{K}|d_k|^{r'[1+\frac{q}{r}(\frac{1}{q_0}+\frac{1}{q_1})]}\left\|\chi_{E_k}\right\|_{L^{(\frac{rs}{q})'}}\right]\\&\le e^{\frac{q}{r}\sup_{j_1,\dots,j_m}C(u_{j_1}^{1,\varepsilon},\dots,u_{j_m}^{m,\varepsilon},E)e^{\gamma|\mathrm{Im}\,z|}}\left[\sum_{\substack{1\le j_1\le N_\varepsilon^1\\ \vdots\\ 1\le j_m\le N_\varepsilon^m}}\prod_{\kappa=1}^{m}|c_{j_\kappa}^{\kappa,\varepsilon}|^{\frac{p^\kappa}{p_0^\kappa}+\frac{p^\kappa}{p_1^\kappa}}\right]^{\frac{q}{r}}\left[\sum_{k=1}^{K}|a_k|^{r'[1+\frac{q}{r}(\frac{1}{q_0}+\frac{1}{q_1})]}\left\|\chi_{E_k}\right\|_{L^{(\frac{rs}{q})'}}\right]\end{aligned}$$

having used (3.2). Taking the logarithm we deduce condition (2.8) for G.

As $g^\theta = g$, by Proposition 2.3 we conclude

$$\int_{X_0} \left|T_\theta(f^\varepsilon_{1,\theta},\dots,f^\varepsilon_{m,\theta})\right|^{\frac{q}{r}} g\, d\mu_0 = G(\theta) \le \left(B_0^{1-\theta}B_1^\theta M(\theta)\prod_{\kappa=1}^m \left(\left\|f^\kappa\right\|_{L^{p^\kappa}}^{p^\kappa}+\varepsilon'\right)^{\frac{1}{p^\kappa}}\right)^{\frac{q}{r}}. \quad (3.16)$$

Inequality (3.16) implies that

$$\begin{aligned} &\left\|T_\theta(f_\theta^{1,\varepsilon},\dots,f_\theta^{m,\varepsilon})\right\|_{L^q} \\ &= \left\| \left|T_\theta(f_\theta^{1,\varepsilon},\dots,f_\theta^{m,\varepsilon})\right|^{\frac{q}{r}} \right\|_{L^r}^{\frac{r}{q}} \\ &= \sup\left\{\int_{X_0} \left|T_\theta(f_\theta^{1,\varepsilon},\dots,f_\theta^{m,\varepsilon})\right|^{\frac{q}{r}} g d\mu_0 \colon\, g=\textstyle\sum_{k=1}^K a_k\chi_{E_k},\ a_k>0,\ E_k \text{ compact},\ \|g\|_{L^{r'}}=1\right\}^{\frac{r}{q}} \\ &\le B_0^{1-\theta}B_1^\theta M(\theta)\prod_{\kappa=1}^m\left(\left\|f^\kappa\right\|_{L^{p^\kappa}}^{p^\kappa}+\varepsilon'\right)^{\frac{1}{p^\kappa}}. \end{aligned} \quad (3.17)$$

We also note that a similar argument applied to the log-subharmonic function

$$H(z) = \int_{X_0} |T_z(f_1,\dots,f_m)|^{\frac{q}{r}} |g^z|\, d\mu_0$$

yields the estimate

$$|H(\theta)| = \left|\int_{X_0} |T_\theta(f_1,\dots,f_m)|^{\frac{q}{r}} g\, d\mu_0\right| \le \left(B_0^{1-\theta}B_1^\theta M(\theta)\prod_{\kappa=1}^m \|f^\kappa\|_{L^{p_0^\kappa}}^{1-\theta}\|f^\kappa\|_{L^{p_1^\kappa}}^{\theta}\right)^{\frac{q}{r}},$$

from which it follows that

$$\left\|T_\theta(f^1,\dots,f^m)\right\|_{L^q} \le B_0^{1-\theta}B_1^\theta M(\theta)\prod_{\kappa=1}^m \|f^\kappa\|_{L^{p_0^\kappa}}^{1-\theta}\|f^\kappa\|_{L^{p_1^\kappa}}^{\theta}, \quad (3.18)$$

via a duality argument similar to that leading to (3.17).

We now make use of the triangle inequality

$$\|T_\theta(f^1,\dots,f^m)\|_{L^q}^{\min(1,q)} \le \sum_{\kappa=1}^m \|T_\theta(\dots,f^\kappa - f_\theta^{\kappa,\varepsilon},\dots)\|_{L^q}^{\min(1,q)} + \|T_\theta(f_\theta^{1,\varepsilon},\dots,f_\theta^{m,\varepsilon})\|_{L^q}^{\min(1,q)}.$$

For the second term on the right above we use (3.17), while the first term is bounded by a constant multiple of $(\varepsilon^{1-\theta})^{\min(1,q)}$ in view of (3.18), and hence it tends to zero as $\varepsilon \to 0$. We deduce (3.7) by letting $\varepsilon \to 0$.

Finally, if $p_0^\kappa = p_1^\kappa = \infty$ for certain κ we factor these κ's and we consider another multilinear operator of lower degree. For instance if $p_0^\kappa = p_1^\kappa = \infty$ exactly when $\kappa \le \lambda$, we consider the operator

$$(f^{\lambda+1},\dots,f^m) \mapsto T_z(f^1,\dots,f^\lambda,f^{\lambda+1},\dots,f^m)$$

which satisfies the initial assumptions with constants B_0 and B_1 replaced by the original ones multiplied by $\prod_{\kappa=1}^\lambda \|f^\kappa\|_{L^\infty}$. □

As we already mentioned in the introduction, an interpolation theorem for analytic families of multilinear operators was proved in [9]. The main difference between these results is that in [9] the concepts of analyticity and admissibility condition are in the pointwise while ours are in the integral sense, as mandated by applications (see next section). Unlike (3.2) this pointwise admissibility condition is not easy to check in general, especially when the operators involved do not have explicit formulae.

4 A bilinear estimate for Schrödinger operators

We consider the self-adjoint operator

$$L = -\operatorname{div}(A\nabla) + V$$

on $L^2(\mathbb{R}^n)$ where $A = (a_{kl})$ is a symmetric matrix with real-valued and bounded measurable entries. It is assumed to be elliptic with ellipticity constant $\gamma > 0$, that is

$$\sum_{k,l} a_{kl}(x)\xi_k\xi_l \geq \gamma|\xi|^2, \quad a.e.\, x \in \mathbb{R}^n,\ \forall \xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n.$$

The potential V is assumed to be nonnegative and locally integrable.
By the standard sesquilinear form technique, one constructs a self-adjoint realization of L. The following theorem was proved by Dragicevic and Volberg [8].

Theorem 4.1. *Let $\boldsymbol{\Gamma}$ be either ∇ or multiplication by $\sqrt{V}$. Let $p \in (1, \infty)$ and p' its conjugate number. Then there exists a constant C_γ, independent of the dimension n, such that*

$$\int_0^\infty \int_{\mathbb{R}^n} |\boldsymbol{\Gamma} e^{-tL} f(x)||\boldsymbol{\Gamma} e^{-tL} g(x)|\, dx\, dt \leq C_\gamma \max(p, p')\|f\|_{L^p}\|g\|_{L^{p'}}. \tag{4.1}$$

The constant C_γ can be taken to be $C\max(1, \frac{1}{\gamma})$ with C an absolute constant.

The aim of this section is to prove, under the same assumptions as before, the following result.

Proposition 4.2. *Let $\alpha, \beta \in [0, \infty)$ and $1 < p < \infty$. Then there exists a constant $C(\alpha, \beta, \gamma, p)$, independent of n, such that*

$$\int_0^\infty \int_{\mathbb{R}^n} |\boldsymbol{\Gamma} L^\alpha e^{-tL} f(x) \cdot \boldsymbol{\Gamma} L^\beta e^{-tL} g(x)|\, dx\, t^{\alpha+\beta} dt \leq C(\alpha, \beta, \gamma, p)\|f\|_{L^p}\|g\|_{L^{p'}}. \tag{4.2}$$

This proposition can be viewed as a weighted version of the bilinear estimate stated in the previous theorem. More precisely, let $\omega : (0, \infty) \to (0, \infty)$ such that $\omega(t) \sim t^\eta$ for some $\eta > 0$. Then (4.2) can be rewritten as

$$\int_0^\infty \int_{\mathbb{R}^n} |\boldsymbol{\Gamma} e^{-tL} f(x) \cdot \boldsymbol{\Gamma} e^{-tL} g(x)|\, dx\, \omega(t)\, dt \leq C(\alpha, \eta, \gamma, p)\|L^{-\alpha} f\|_{L^p}\|L^{-(\eta-\alpha)} g\|_{L^{p'}} \tag{4.3}$$

for $\alpha \in [0, \eta]$.

Proof of Proposition 4.2. Define

$$T_{\alpha,\beta}(f,g)(x,t) = \mathbf{\Gamma}(tL)^{\alpha} e^{-tL} f(x) \cdot \mathbf{\Gamma}(tL)^{\beta} e^{-tL} g(x).$$

The above proposition can be rephrased as

$$T_{\alpha,\beta} : L^p(\mathbb{R}^n) \times L^{p'}(\mathbb{R}^n) \to L^1(\mathbb{R}^n \times (0,\infty), dxdt)$$

is a bounded bilinear operator with norm estimated by $C(\alpha, \beta, \gamma, p)$.

For complex z, we define the bilinear operator

$$T_z(f,g)(x,t) = \mathbf{\Gamma}(tL)^{\alpha' z} e^{-tL} f(x) \cdot \mathbf{\Gamma}(tL)^{\beta' z} e^{-tL} g(x),$$

where $\alpha', \beta' \in [0,\infty)$ will be specified later.

We show that the family (T_z) is analytic in the sense of Definition 3.1. Let $f, g \in \mathcal{C}_c(\mathbb{R}^n)$ and $w \in L^\infty(\mathbb{R}^n \times (0,\infty))$ a bounded function with compact support K. We prove that

$$z \mapsto \int_0^\infty \int_{\mathbb{R}^n} T_z(f,g)(x,t) w(x,t)\, dx\, dt \tag{4.4}$$

is analytic on $\mathbf{S}$ and continuous on $\overline{\mathbf{S}}$.

Note that there exist a compact set K_0 of $\mathbb{R}^n$ and $0 < a < b < \infty$ such that $K \subset K_0 \times [a,b]$. This can be seen by taking $K_0 = p_1(K)$ and $[a,b] = p_2(K)$ where $p_1 : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}^n$ and $p_2 : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ are the first and second projections. These functions are continuous and hence $p_1(K)$ and $p_2(K)$ are compact sets of $\mathbb{R}^n$ and $(0,\infty)$, respectively. By arguing by contradiction, it is easy to see that $a > 0$. In particular, the function in (4.4) coincides with

$$z \mapsto \int_a^b \left\langle \mathbf{\Gamma}(tL)^{\alpha' z} e^{-tL} f, w(\cdot,t) \mathbf{\Gamma}(tL)^{\beta' z} e^{-tL} g \right\rangle_{L^2} dt.$$

Note that by ellipticity and the fact that V is nonnegative,

$$\|\nabla u\|_{L^2}^2 \le \frac{1}{\gamma} \int_{\mathbb{R}^n} Lu\, \overline{u}\, dx = \frac{1}{\gamma} \big\| L^{1/2} u \big\|_{L^2}^2 \quad \text{and} \quad \big\| \sqrt{V} u \big\|_{L^2}^2 \le \big\| L^{1/2} u \big\|_{L^2}^2.$$

Hence

$$\|\mathbf{\Gamma} u\|_{L^2}^2 \le \max\left(1, \frac{1}{\gamma}\right) \big\| L^{1/2} u \big\|_{L^2}^2. \tag{4.5}$$

Recall that for every $h \in D(L)$, the function $z \mapsto L^z h$ is analytic on $\mathbf{S}$ and continuous on $\overline{\mathbf{S}}$ (see e.g. [10], Proposition 3.1.1, b)). Since the operator Γe^{-tL} is bounded on $L^2(\mathbb{R}^n)$ for every $t > 0$ (see (4.5)), it follows that the function

$$z \mapsto \left\langle \mathbf{\Gamma}(tL)^{\alpha' z} e^{-tL} f, w(\cdot,t) \mathbf{\Gamma}(tL)^{\beta' z} e^{-tL} g \right\rangle_{L^2}$$

is analytic on $\mathbf{S}$ and continuous on $\overline{\mathbf{S}}$. It remains to bound in a neighborhood of each $z_0 \in \overline{\mathbf{S}}$ this function by some function $\psi(t)$ which is integrable on $[a,b]$ and then obtain the desired conclusion for the function in (4.4).

By the Cauchy-Schwarz inequality and (4.5) we write

$$\begin{aligned} &\left|\left\langle \mathbf{\Gamma}(tL)^{\alpha' z}e^{-tL}f, w(\cdot,t)\mathbf{\Gamma}(tL)^{\beta' z}e^{-tL}g\right\rangle_{L^2}\right| \qquad (4.6)\\ \le\ & \|w\|_{L^\infty}\left\|\mathbf{\Gamma}(tL)^{\alpha'\operatorname{Re} z}e^{-tL}(tL)^{i\alpha'\operatorname{Im} z}f\right\|_{L^2}\left\|\mathbf{\Gamma}(tL)^{\beta'\operatorname{Re} z}e^{-tL}(tL)^{i\beta'\operatorname{Im} z}g\right\|_{L^2}\\ \le\ & \|w\|_{L^\infty}\max\left(1,\frac{1}{\gamma}\right)\left\|L^{1/2}(tL)^{\alpha'\operatorname{Re} z}e^{-tL}(tL)^{i\alpha'\operatorname{Im} z}f\right\|_{L^2}\left\|L^{1/2}(tL)^{\beta'\operatorname{Re} z}e^{-tL}(tL)^{i\beta'\operatorname{Im} z}g\right\|_{L^2}. \end{aligned}$$

The standard functional calculus for self-adjoint operators, i.e.,

$$\|\phi(L)h\|_{L^2} \le \sup_{\lambda>0}|\phi(\lambda)|\|h\|_{L^2},$$

gives

$$\|(tL)^{\alpha'\operatorname{Re} z}e^{-tL}h\|_{L^2} \le e^{\alpha'\operatorname{Re} z(\log(\alpha'\operatorname{Re} z)-1)}\|h\|_{L^2}. \qquad (4.7)$$

Clearly the term on the right hand side of (4.7) is uniformly bounded in z in a bounded neighborhood W_0 of a fixed $z_0 \in \overline{\mathbf{S}}$. It follows from this and the estimates in (4.6) that

$$\left|\left\langle \mathbf{\Gamma}(tL)^{\alpha' z}e^{-tL}f, w(\cdot,t)\mathbf{\Gamma}(tL)^{\beta' z}e^{-tL}g\right\rangle_{L^2}\right| \le \frac{C\|w\|_{L^\infty}}{t}\|f\|_{L^2}\|g\|_{L^2}, \quad z \in W_0.$$

This function is integrable on $[a,b]$ and hence by the dominated convergence theorem we obtain that the function in (4.4) is analytic at $z_0 \in \mathbf{S}$ and continuous at $z_0 \in \overline{\mathbf{S}}$.

Next, we prove the admissibility condition. For $f, g \in L^2(\mathbb{R}^n)$ and $z = r + is \in \overline{\mathcal{S}}$,

$$\begin{aligned} &\|T_z(f,g)\|_{L^1(\mathbb{R}^n\times(0,\infty))}\\ =\ & \int_0^\infty\int_{\mathbb{R}^n}|\mathbf{\Gamma}(tL)^{\alpha' z}e^{-tL}f(x)\cdot\mathbf{\Gamma}(tL)^{\beta' z}e^{-tL}g(x)|\,dx\,dt\\ =\ & \int_0^\infty\int_{\mathbb{R}^n}|\mathbf{\Gamma}(tL)^{\alpha' r}e^{-tL}L^{is\alpha'}f\cdot\mathbf{\Gamma}(tL)^{r\beta'}e^{-tL}L^{is\beta'}g|\,dx\,dt\\ \le\ & \left\|\left(\int_0^\infty|\mathbf{\Gamma}(tL)^{\alpha' r}e^{-tL}L^{is\alpha'}f|^2\,dt\right)^{1/2}\right\|_{L^2}\left\|\left(\int_0^\infty|\mathbf{\Gamma}(tL)^{\beta' r}e^{-tL}L^{is\beta'}f|^2\,dt\right)^{1/2}\right\|_{L^2}. \end{aligned}$$

We estimate the latest terms using the standard functional calculus for the self-adjoint operator L on $L^2(\mathbb{R}^n)$. Using (4.5) we have

$$\begin{aligned} &\left\|\left(\int_0^\infty|\mathbf{\Gamma}(tL)^{\alpha' r}e^{-tL}L^{is\alpha'}f|^2\,dt\right)^{1/2}\right\|_{L^2}^2\\ =\ & \int_0^\infty\left\|\mathbf{\Gamma}(tL)^{\alpha' r}e^{-tL}L^{is\alpha'}f\right\|_{L^2}^2\,dt\\ \le\ & \max\left(1,\frac{1}{\gamma}\right)\int_0^\infty\left\|L^{1/2}(tL)^{\alpha' r}e^{-tL}L^{is\alpha'}f\right\|_{L^2}^2\,dt\\ =\ & \max\left(1,\frac{1}{\gamma}\right)\int_0^\infty\left\langle L^{1/2}(tL)^{\alpha' r}e^{-tL}L^{is\alpha'}f, L^{1/2}(tL)^{\alpha' r}e^{-tL}L^{is\alpha'}f\right\rangle_{L^2}\,dt\\ =\ & \max\left(1,\frac{1}{\gamma}\right)\int_0^\infty\left\langle (tL)^{2\alpha' r+1}e^{-2tL}L^{is\alpha'}f, L^{is\alpha'}f\right\rangle_{L^2}\frac{dt}{t} \end{aligned}$$

$$
\begin{aligned}
&= \max\left(1,\frac{1}{\gamma}\right)\left\langle \int_0^\infty (tL)^{2\alpha' r+1} e^{-2tL} L^{is\alpha'} f \,\frac{dt}{t}, L^{is\alpha'} f\right\rangle_{L^2} \\
&\le \max\left(1,\frac{1}{\gamma}\right)\left\| \int_0^\infty (tL)^{2\alpha' r+1} e^{-2tL} L^{is\alpha'} f \,\frac{dt}{t}\right\|_{L^2} \left\| L^{is\alpha'} f\right\|_{L^2}.
\end{aligned}
$$

Using again the functional calculus, we have $\|L^{is\alpha'}f\|_2 = \|f\|_2$ and

$$
\begin{aligned}
\left\| \int_0^\infty (tL)^{2\alpha' r+1} e^{-2tL} L^{is\alpha'} f \,\frac{dt}{t}\right\|_{L^2} &\le \sup_{\lambda>0}\left| \int_0^\infty (t\lambda)^{2\alpha' r+1} e^{-2t\lambda} \frac{dt}{t}\right| \|L^{is\alpha'} f\|_{L^2} \\
&= \int_0^\infty t^{2\alpha' r} e^{-2t}\, dt \, \|f\|_{L^2} \\
&= 2^{-2\alpha' r-1}\Gamma(2\alpha' r+1)\, \|f\|_{L^2}.
\end{aligned}
$$

Thus we obtain for all $z = r + is \in \overline{\mathcal{S}}$

$$
\|T_z(f,g)\|_{L^1(\mathbb{R}^n\times(0,\infty))} \le \frac{\max(1,\frac{1}{\gamma})}{2^{(\alpha'+\beta')r+1}}\sqrt{\Gamma(2\alpha' r+1)\Gamma(2\beta' r+1)}\,\|f\|_{L^2}\|g\|_{L^2}. \tag{4.8}
$$

In particular,

$$
\|T_z(f,g)\|_{L^1(\mathbb{R}^n\times(0,\infty))} \le \max\left(1,\frac{1}{\gamma}\right)\sqrt{\Gamma(2\alpha'+1)\Gamma(2\beta'+1)}\,\|f\|_{L^2}\|g\|_{L^2} \tag{4.9}
$$

for all $f, g \in L^2(\mathbb{R}^n)$ and all $z \in \overline{\mathcal{S}}$. This proves that the analytic family of bilinear operators T_z is of admissible growth in the sense of Definition 3.1.

The particular case of (4.8) for $z = 1 + is$ yields

$$
\|T_{1+is}(f,g)\|_{L^1(\mathbb{R}^n\times(0,\infty))} \le \frac{\max(1,\frac{1}{\gamma})}{2^{(\alpha'+\beta')+1}}\sqrt{\Gamma(2\alpha'+1)\Gamma(2\beta'+1)}\,\|f\|_{L^2}\|g\|_{L^2}. \tag{4.10}
$$

Next, we estimate the L^1-norm of $T_{is}(f,g)$. Let $p_1 \in (1,2)$ be a fixed number and let $f \in L^{p_1}(\mathbb{R}^n)$ and $g \in L^{p_1'}(\mathbb{R}^n)$. By Theorem 4.1 we obtain

$$
\begin{aligned}
\|T_{is}(f,g)\|_{L^1(\mathbb{R}^n\times(0,\infty))} &= \int_0^\infty \int_{\mathbb{R}^n} |\mathbf{\Gamma} e^{-tL} L^{is\alpha'} f(x) \cdot \mathbf{\Gamma} e^{-tL} L^{is\beta'} g(x)|\, dx\, dt \\
&\le C_\gamma p_1' \|L^{is\alpha'} f\|_{L^{p_1}} \|L^{is\beta'} g\|_{L^{p_1'}}.
\end{aligned}
$$

Since the semigroup (e^{-tL}) is sub-Markovian and symmetric, L has a holomorphic functional calculus on $L^q(\mathbb{R}^n)$ for all $q \in (1,\infty)$ (cf. [6], or [2]). For imaginary powers, it follows from these last two references that there exists a constant $C(p_1)$, independent of n, such that for all $s \in \mathbb{R}$,

$$
\|L^{is\alpha'} f\|_{L^{p_1}} \le C(p_1) e^{\frac{\pi}{2}|s|\alpha'} \|f\|_{L^{p_1}}. \tag{4.11}
$$

Therefore,

$$
\|T_{is}(f,g)\|_{L^1(\mathbb{R}^n\times(0,\infty))} \le C(\gamma,p_1) e^{\frac{\pi}{2}(\alpha'+\beta')|s|} \|f\|_{L^{p_1}} \|g\|_{L^{p_1'}}. \tag{4.12}
$$

We are now in the position to apply Theorem 3.2. It follows from (4.10) and (4.12) that for $\theta \in (0,1)$ and $\frac{1}{p_\theta} = \frac{\theta}{2} + \frac{1-\theta}{p_1}$ we have

$$
\|T_\theta(f,g)\|_{L^1(\mathbb{R}^n\times(0,\infty))} \le c_\theta M(\theta) \|f\|_{L^{p_\theta}} \|g\|_{L^{p_\theta'}} \tag{4.13}
$$

with

$$c_\theta = \left(\max\left(1, \frac{1}{\gamma}\right) 2^{-\alpha'-\beta'-1}\sqrt{\Gamma(2\alpha'+1)\Gamma(2\beta'+1)}\right)^\theta C(\gamma, p_1)^{1-\theta}$$

and

$$M(\theta) = \exp\left\{\frac{\sin(\pi\theta)}{2}\frac{\pi}{2}(\alpha'+\beta')\int_{-\infty}^{+\infty}\frac{|s|}{\cosh(\pi s)+\cos(\pi\theta)}\,ds\right\}.$$

Finally, for any $p \in (1,2)$ we choose $p_1 < p$, $\alpha' = \frac{\alpha}{\theta}$, $\beta' = \frac{\beta}{\theta}$ and set $\theta = \frac{p-p_1}{2-p_1}\frac{2}{p}$ so that $p_\theta = p$ and $T_\theta = T_{\alpha,\beta}$. The proposition follows from (4.13). □

Remark 4.3. *It is an interesting question to understand for which functions F and G one has*

$$\int_0^\infty \int_{\mathbb{R}^n} |\mathbf{\Gamma} F(tL)f(x) \cdot \mathbf{\Gamma} G(tL)g(x)|\,dx\,dt \le C(F,G,p)\|f\|_{L^p}\|g\|_{L^{p'}}.$$

Note that the term on the left hand side is bounded by the product

$$\left\|\left(\int_0^\infty |\mathbf{\Gamma} F(tL)f|^2\,dt\right)^{1/2}\right\|_{L^p}\left\|\left(\int_0^\infty |\mathbf{\Gamma} G(tL)g|^2\,dt\right)^{1/2}\right\|_{L^{p'}}.$$

The Littlewood-Paley-Stein functional $\left(\int_0^\infty |\mathbf{\Gamma} F(tL)f|^2\,dt\right)^{1/2}$ is bounded on $L^p(\mathbb{R}^n)$ for $p \in (1,2]$ as soon as F is holomorphic in a certain sector (with angle depending on p) and decays faster that $\frac{1}{\sqrt{|z|}}$ at ∞ (see [4]). Thus the first term in the above product is fine for $p \in (1,2]$. However the second term could be unbounded on $L^{p'}(\mathbb{R}^n)$ even if $L = \Delta + V$ ($V \neq 0$) and $G(z) = e^{-z}$. See again [4].

5 Appendix: Log-subharmonic functions on the plane

A locally integrable function f on an open subset O of the complex plane with values in $[-\infty, \infty)$ is called subharmonic if it is upper semicontinuous, i.e., $\limsup_{w\to z} f(w) \le f(z)$ for every $z \in O$ and satisfies

$$f(z) \le \frac{1}{|B(z,r)|}\int_{B(z,r)} f(w)\,dw \tag{5.1}$$

for any $z \in O$ and every $r > 0$ such that $B(z,r) \subset O$. If $f \in \mathcal{C}^2$, then the above condition is equivalent to $\Delta f \ge 0$. A function is called log-subharmonic if it is nonnegative and its logarithm is subharmonic.

Lemma 5.1. *The sum of two log-subharmonic functions is log-subharmonic.*

Proof. Let $\varphi(x,y) = \log(e^x + e^y)$ defined on $\mathbb{R}^2$. Then φ is obviously increasing in each variable and is a convex function of both variables.

Suppose that F, G are subharmonic functions on an open subset of the complex plane. Then the fact that φ is increasing in each variable and Jensen's inequality (which can be used since φ is convex) gives

$$\begin{aligned}\varphi(F(z),G(z)) &\leq \varphi\left(\frac{1}{|B(z,r)|}\int_{B(z,r)} F(w)\,dw \ , \ \frac{1}{|B(z,r)|}\int_{B(z,r)} G(w)\,dw\right)\\ &\leq \frac{1}{|B(z,r)|}\int_{B(z,r)} \varphi(F(w),G(w))\,dw\,,\end{aligned}$$

which implies that $\varphi(F(z),G(z))$ is subharmonic. Now let f,g be log-subharmonic functions. Writing $f=e^F$ and $g=e^G$, then $\log(f+g)=\varphi(F,G)$. But $\varphi(F,G)$ was shown to be subharmonic, thus $\log(f+g)$ is also subharmonic. □

We review a couple of facts from the theory of analytic functions with values in Banach spaces. Let $\mathcal{B}$ be a Banach space and let $\mathbf{f}$ be a mapping from an open subset U of $\mathbb{C}$ to $\mathcal{B}$. We say that $\mathbf{f}$ is analytic if

$$\mathbf{f}'(z_0)=\lim_{z\to z_0}\frac{\mathbf{f}(z)-\mathbf{f}(z_0)}{z-z_0}$$

exists in the norm of $\mathcal{B}$.

Theorem 5.2. *Let* $\mathbf{f}$ *be a mapping from an open subset* U *of* $\mathbb{C}$ *to a Banach space* $\mathcal{B}$*. Then* $\mathbf{f}$ *is analytic if and only if for every bounded linear functional* Λ *in* $\mathcal{B}$ *we have*

$$\lim_{z\to z_0}\Lambda\left(\frac{\mathbf{f}(z)-\mathbf{f}(z_0)}{z-z_0}\right)$$

exists in $\mathbb{C}$.

Log-subharmonic can be generated from $L^1(X)$-valued analytic functions in terms of the subsequent lemma.

Lemma 5.3. *[15, Lemma 2] Let* (X,μ) *be a measure space with* $\mu(X)<\infty$ *and let* V *be a complex-valued function defined on* $X\times S$ *such that the mapping* $z\mapsto V(\cdot,z)$ *from* S *to* $L^1(X)$ *is a Banach-valued analytic function. Then the function*

$$z\mapsto F(z)=\int_X |V(x,z)|^q\,d\mu(x)$$

is log-subharmonic for any $0<q\leq 1$.

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