# THE MULTILINEAR MARCINKIEWICZ INTERPOLATION THEOREM REVISITED: THE BEHAVIOR OF THE CONSTANT

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Abstract. We provide a self-contained proof of the multilinear extension of the Marcinkiewicz real method interpolation theorem with initial assumptions a set of restricted weak type estimates, considering possible degenerate situations that may arise. The advantage of this proof is that it yields a logarithmically convex bound for the norm of the operator on the intermediate spaces in terms of the initial restricted weak type bounds; it also provides an explicit estimate in terms of the exponents of the initial estimates: the constant blows up like a negative power of the distance from the intermediate point to the boundary of the convex hull of the initial points.

*In memory of Nigel Kalton*

## 1. Introduction

Multilinear interpolation is a powerful tool that yields intermediate estimates from a finite set of initial estimates for operators of several variables. In particular, the real multilinear interpolation method yields strong type bounds for multilinear (or multi-sublinear) operators as a consequence of initial weak type estimates. Versions of this theorem have been obtained in the literature by Strichartz [11], Sharpley [9], [10], Zafran [13], Christ [1], Janson [5], Grafakos and Kalton [3], and Grafakos and Tao [4]. In this article we give a version of Marcinkiewicz's real interpolation theorem for multilinear operators starting from a finite number of initial *restricted weak type estimates.* Our result is closest to the one in [3] but contains certain improvements. It yields a constant on the intermediate space that contains an optimal multiplicative factor in terms of the initial restricted weak type bounds and also describes an explicit behavior in terms of the location of the intermediate point inside the convex hull of the initial points. These elements were previously missing from the literature.

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Let $m \geq 1$ be an integer. For $1 \leq j \leq m$, let $(X_j, \mu_j)$ be measure spaces and let $(Y, \nu)$ be another measure space. All measures are assumed to be positive and $\sigma$-finite. For $0 < p \leq \infty$, we denote by $L^p(X_j, \mu_j)$ or simply by $L^p$ the Lebesgue space of all complex-valued functions whose $p$th power is integrable with respect to $\mu_j$ on the space $X_j$.

Let $\mathcal{S}(X_j)$ be the space of simple functions on $X_j$. Let $T$ be a map defined on $\mathcal{S}(X_1) \times \cdots \times \mathcal{S}(X_m)$ that takes values in the measure space $Y$. Then $T$ is called *multilinear* if for all $f_j$, $g_j$ in $\mathcal{S}(X_j)$ and all scalars $\lambda$ we have

$$T(f_1, \dots, \lambda f_j, \dots, f_m) = \lambda\, T(f_1, \dots, f_j, \dots, f_m)$$

and

$$T(\dots, f_j + g_j, \dots) = T(\dots, f_j, \dots) + T(\dots, g_j, \dots).$$

The operator $T$ is called *multi-quasilinear* if there is a constant $\mathbf{K} \geq 1$ such that for all $1 \leq j \leq m$, all $f_j$, $g_j$ in $\mathcal{S}(X_j)$, and all $\lambda \in \mathbf{C}$ we have

$$|T(f_1, \dots, \lambda f_j, \dots, f_m)| = |\lambda|\, |T(f_1, \dots, f_j, \dots, f_m)| \tag{1}$$

and also

$$|T(\dots, f_j + g_j, \dots)| \leq \mathbf{K}(|T(\dots, f_j, \dots)| + |T(\dots, g_j, \dots)|). \tag{2}$$

In the case where $\mathbf{K} = 1$, $T$ is called *multi-sublinear*.

Given a measure space $X$, we denote by $\Gamma(X)$ the space of all simple functions on $X$ that have the form $f = \sum_{i=n_1}^{n_2} 2^{-i} \chi_{E_i}$, where $E_i$ are subsets of $X$ of finite measure with $\mu(E_{n_1}) \neq 0$ and $\mu(E_{n_2}) \neq 0$, and $n_1$, $n_2$ are integers such that $n_1 < n_2$. We also denote by $\Gamma(X) - \Gamma(X)$ the set of functions of the form $f - g$, where $f, g \in \Gamma(X)$. This space is shown to be dense in the real Lorentz space $L^{p,s}(X, \mu)$ if $0 < p, s < \infty$, see [8]. Thus, the space $(\Gamma(X) - \Gamma(X)) + i(\Gamma(X) - \Gamma(X))$ of all functions of the form $f_1 + i f_2$, where $f_1, f_2 \in \Gamma(X) - \Gamma(X)$, is dense in the complex Lorentz space $L^{p,s}(X, \mu)$ with $0 < p, s < \infty$. Lorentz spaces in this paper will be complex-valued.

We introduce some notation. First, $1/q$ is defined to be zero when $q = \infty$. Let $m$ be a positive integer. For $1 \leq k \leq m+1$ and $1 \leq j \leq m$, we are given $p_{k,j}$ with $0 < p_{k,j} \leq \infty$ and $0 < q_k \leq \infty$. We define determinants $\gamma_j$ depending on these given numbers as follows:

$$\gamma_0 = \det \begin{pmatrix} 1/p_{1,1} & 1/p_{1,2} & \dots\dots & 1/p_{1,m} & 1 \\ 1/p_{2,1} & 1/p_{2,2} & \dots\dots & 1/p_{2,m} & 1 \\ \vdots & \vdots & \vdots\ \vdots & \vdots & \vdots \\ 1/p_{m,1} & 1/p_{m,2} & \dots\dots & 1/p_{m,m} & 1 \\ 1/p_{m+1,1} & 1/p_{m+1,2} & \dots\dots & 1/p_{m+1,m} & 1 \end{pmatrix}$$

and for each $j = 1, 2, \ldots, m$ we define

$$\gamma_j = \det \begin{pmatrix} 1/p_{1,1} & 1/p_{1,2} & \ldots & -1/q_1 & \ldots & 1/p_{1,m} & 1 \\ 1/p_{2,1} & 1/p_{2,2} & \ldots & -1/q_2 & \ldots & 1/p_{2,m} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1/p_{m,1} & 1/p_{m,2} & \ldots & -1/q_m & \ldots & 1/p_{m,m} & 1 \\ 1/p_{m+1,1} & 1/p_{m+1,2} & \ldots & -1/q_{m+1} & \ldots & 1/p_{m+1,m} & 1 \end{pmatrix}, \tag{3}$$

where the $j$th column of the determinant defining $\gamma_j$ is obtained by replacing the $j$th column of the determinant defining $\gamma_0$ by the vector $-(1/q_1, \ldots, 1/q_m, 1/q_{m+1})$.

We explain the geometric meaning of these determinants: for $k = 1, 2, \ldots, m+1$, let

$$\vec{P}_k := \Big(\frac{1}{p_{k,1}}, \frac{1}{p_{k,2}}, \ldots, \frac{1}{p_{k,m}}\Big)$$

be points in $\mathbf{R}^m$. Let $\mathbf{H}$ be the open convex hull of the points $\vec{P}_1, \ldots, \vec{P}_{m+1}$. Then $\mathbf{H}$ is an open subset of $\mathbf{R}^m$ whose $m$-dimensional volume is

$$\text{Volume}(\mathbf{H}) = m!\,|\gamma_0|\,.$$

Hence $\mathbf{H}$ is a nonempty set if and only if $\gamma_0 \neq 0$. Thus, the condition $\gamma_0 \neq 0$ is equivalent to the fact that the open convex hull of $\vec{P}_1, \ldots, \vec{P}_{m+1}$ is a nontrivial open simplex in $\mathbf{R}^m$. The boundary of $\mathbf{H}$ will be denoted by $\partial\mathbf{H}$.

Analogous geometric meaning is valid for the remaining $\gamma_j$'s. But it might be useful to think of each $\gamma_j$ as the $j$th dual of $\gamma_0$ in the following sense: suppose that for each $k = 1, 2, \ldots, m+1$, there is a correspondence of the form:

$$\Big(\frac{1}{p_{k,1}}, \frac{1}{p_{k,2}}, \ldots, \frac{1}{p_{k,m}}\Big) \longmapsto \frac{1}{q_k}\,.$$

Then the $j$th dual of this correspondence is

$$\Big(\frac{1}{p_{k,1}}, \ldots, \frac{1}{p_{k,j-1}}, 1 - \frac{1}{q_k}, \frac{1}{p_{k,j+1}}, \ldots, \frac{1}{p_{k,m}}\Big) \longmapsto 1 - \frac{1}{p_{k,j}}$$

for all $j = 1, 2, \ldots, m$. Then $\gamma_j$ plays the role of $\gamma_0$ for the $j$th dual of this correspondence of indices.

We now state the main result of this paper. It is a multilinear version of the Marcinkiewicz interpolation theorem with initial restricted weak-type conditions and multiplicative bounds for the intermediate spaces.

**Theorem 1.1.** *Let $m$ be a positive integer and let $T$ be a multi-quasilinear operator defined on $\mathcal{S}(X_1) \times \cdots \times \mathcal{S}(X_m)$ and taking values in the set of measurable functions of a space $(Y, \nu)$. For $1 \leq k \leq m+1$ and $1 \leq j \leq m$, we are given $p_{k,j}$ with $0 < p_{k,j} \leq \infty$, and $0 < q_k \leq \infty$. Suppose that the open convex hull of the points*

$$\vec{P}_k = \Big(\frac{1}{p_{k,1}}, \frac{1}{p_{k,2}}, \ldots, \frac{1}{p_{k,m}}\Big)$$

*is an open set in* $\mathbf{R}^m$*, in other words* $\gamma_0 \neq 0$*. Assume that* $T$ *satisfies*

$$\|T(\chi_{E_1}, \dots, \chi_{E_m})\|_{L^{q_k,\infty}} \le B_k \prod_{j=1}^{m} \mu_j(E_j)^{\frac{1}{p_{k,j}}}\,, \tag{4}$$

*for all* $1 \le k \le m+1$ *and for all subsets* $E_j$ *of* $X_j$ *with* $\mu_j(E_j) < \infty$*. Let*

$$\vec{P} = \Big(\frac{1}{p_1}, \dots, \frac{1}{p_m}\Big) = \sum_{k=1}^{m+1} \eta_k \vec{P}_k\,, \tag{5}$$

*for some* $\eta_k \in (0,1)$ *such that* $\sum_{k=1}^{m+1} \eta_k = 1$*, and define*

$$\frac{1}{q} = \sum_{k=1}^{m+1} \frac{\eta_k}{q_k}\,. \tag{6}$$

*For each* $j \in \{1, 2, \dots, m\}$ *let* $s_j$ *satisfy* $0 < s_j \le \infty$*, and let*

$$\frac{1}{s} = \sum_{\substack{1 \le j \le m \\ \gamma_j \neq 0}} \frac{1}{s_j}\,, \tag{7}$$

*with the understanding that if there is no* $j$ *with* $\gamma_j \neq 0$*, the sum in* (7) *is zero and thus* $s = \infty$*. Under these assumptions, there is a positive finite constant* $\mathbf{C}(m, \mathbf{K}, \delta, p_{k,i}, q_k, p_i, s_i)$ *such that*

$$\|T(f_1, \dots, f_m)\|_{L^{q,s}} \le \frac{\mathbf{C}(m, \mathbf{K}, \delta, p_{k,i}, q_k, p_i, s_i)}{\min(1, \mathrm{dist}\,(\vec{P}, \partial\mathbf{H}))^{\frac{m}{\delta}}} \Big(\prod_{k=1}^{m+1} B_k^{\eta_k}\Big)\Big(\prod_{j=1}^{m} \|f_j\|_{L^{p_j,s_j}}\Big) \tag{8}$$

*for all* $f_j \in \Gamma(X_j) - \Gamma(X_j) + i\big(\Gamma(X_j) - \Gamma(X_j)\big)$*. Here*

$$\begin{aligned}\mathbf{C}(m, \mathbf{K}, \delta, p_{k,i},\ & q_k, p_i, s_i) \\ &= \mathcal{C}_*(m, \mathbf{K}, \delta, p_{k,i}, q_k) \max(1, 2^{\frac{m(1-s)}{s}}) \prod_{\substack{1 \le j \le m \\ \gamma_j \neq 0}} \Big|\frac{\gamma_0}{\gamma_j}\Big|^{\frac{1}{s_j}} \prod_{\substack{1 \le j \le m \\ \gamma_j = 0}} \Big(\frac{s_j}{p_j}\Big)^{\frac{1}{s_j}}\,,\end{aligned}$$

*for some other constant* $\mathcal{C}_*(m, \mathbf{K}, \delta, p_{k,i}, q_k)$*, where*

$$0 < \delta < \min\Big(\frac{q_1}{2}, \frac{q_2}{2}, \dots, \frac{q_{m+1}}{2}, s_1, s_2, \dots, s_m, \frac{\ln 2}{\ln(2\mathbf{K})}\Big)\,. \tag{9}$$

The passage from $\Gamma(X_j) - \Gamma(X_j) + i\big(\Gamma(X_j) - \Gamma(X_j)\big)$ to the entire Lorentz space $L^{p_j,s_j}(X_j)$ is achieved by the following result:

**Proposition 1.1.** *Let* $T$ *be a multi-sublinear operator (i.e.,* $\mathbf{K} = 1$*) defined on* $\mathcal{S}(X_1) \times \cdots \times \mathcal{S}(X_m)$ *and taking values in the set of measurable functions of a space*

*$(Y,\nu)$. Let $0<q,s\le\infty$ and $0<p_j,t_j<\infty$ for all $1\le j\le m$. Suppose that the estimate holds:*

$$\|T(f_1,\dots,f_m)\|_{L^{q,s}}\le M\prod_{j=1}^m\|f_j\|_{L^{p_j,t_j}} \tag{10}$$

*for some fixed positive constant $M$ and all $f_j$ in $\Gamma(X_j)-\Gamma(X_j)+i\big(\Gamma(X_j)-\Gamma(X_j)\big)$. Then $T$ has a unique bounded extension from $L^{p_1,t_1}(X_1)\times\cdots\times L^{p_m,t_m}(X_m)$ to $L^{q,s}(Y,\nu)$ that satisfies (10) for all functions $f_j\in L^{p_j,t_j}(X_j)$.*

The proof of Proposition 1.1 uses the sublinearity of $T$ and the density of the space $\Gamma(X_j)-\Gamma(X_j)+i\big(\Gamma(X_j)-\Gamma(X_j)\big)$ in $L^{p_j,t_j}$ when $0<p_j,t_j<\infty$ and is given in Section 4. The following are consequences of Theorem 1.1 and of Proposition 1.1:

**Corollary 1.1.** *Suppose that in Theorem 1.1 we have all $\gamma_j\neq0$ and, instead of (7), the following holds:*

$$\frac1q\le\frac1{p_1}+\cdots+\frac1{p_m}\,. \tag{11}$$

*Then there is a positive constant $\mathcal{C}_{**}(m,\mathbf{K},p_{k,i},q_k)$ such that $T$ satisfies the strong bound*

$$\|T(f_1,\dots,f_m)\|_{L^q}\le\frac{\mathcal{C}_{**}(m,\mathbf{K},p_{k,i},q_k)}{\min(1,\operatorname{dist}(\vec P,\partial\mathbf{H}))^{\frac{m}{\delta_0}}}\Big(\prod_{k=1}^{m+1}B_k^{\eta_k}\Big)\Big(\prod_{j=1}^m\|f_j\|_{L^{p_j}}\Big) \tag{12}$$

*for all $f_j\in\Gamma(X_j)-\Gamma(X_j)+i\big(\Gamma(X_j)-\Gamma(X_j)\big)$, where*

$$0<\delta_0<\min\Big(\frac{q_1}{2},\frac{q_2}{2},\dots,\frac{q_{m+1}}{2},\frac{\ln2}{\ln(2\mathbf{K})},p_{1,1},\dots,p_{m+1,m}\Big)\,. \tag{13}$$

*Moreover, if $\mathbf{K}=1$, then $T$ has a unique bounded extension that satisfies (12) for all $f_j\in L^{p_j}(X_j)$.*

We prove this corollary. Using (5) we see that if $p_i=\infty$ for some $i$, then $\gamma_0=0$. Thus $p_j<\infty$ for all $j$ and in view of (11), we may take $s_j=p_j<\infty$ in (8) and define $s$ by $\frac1s=\frac1{p_1}+\cdots+\frac1{p_m}$. Since $q\ge s$ we have

$$\|T(f_1,\dots,f_m)\|_{L^q}\le\Big(\frac{s}{q}\Big)^{\frac1s-\frac1q}\|T(f_1,\dots,f_m)\|_{L^{q,s}}\le\|T(f_1,\dots,f_m)\|_{L^{q,s}}$$

and thus the required boundedness holds by Theorem 1.1.

As for the form of the constant in (12), using the observations that for $1\le j\le m$,

$$\frac1{p_j}\le\sum_{k=1}^{m+1}\frac1{p_{k,j}} \tag{14}$$

we can choose some $\delta_0>0$ satisfying (13) so that (9) holds. Also, observing that by (14), we have

$$\max(1,2^{\frac{m(1-s)}{s}})\le1+2^{\frac{m}{s}}\le1+2^{m\sum_{j=1}^m\sum_{k=1}^{m+1}\frac1{p_{k,j}}}$$

and

$$\prod_{\substack{1\le j\le m\\ \gamma_j\neq 0}}\left|\frac{\gamma_0}{\gamma_j}\right|^{\frac{1}{s_j}}\prod_{\substack{1\le j\le m\\ \gamma_j=0}}\left(\frac{s_j}{p_j}\right)^{\frac{1}{s_j}}\le\prod_{\substack{1\le j\le m\\ \gamma_j\neq 0}}\left|\frac{\gamma_0}{\gamma_j}\right|^{\sum_{k=1}^m\frac{1}{p_{k,j}}},$$

we conclude that the constant

$$\max(1,2^{\frac{m(1-s)}{s}})\prod_{\substack{1\le j\le m\\ \gamma_j\neq 0}}\left|\frac{\gamma_0}{\gamma_j}\right|^{\frac{1}{s_j}}\prod_{\substack{1\le j\le m\\ \gamma_j=0}}\left(\frac{s_j}{p_j}\right)^{\frac{1}{s_j}}$$

is bounded by another constant which depends only on $m$, $\mathbf{K}$, $p_{k,i}$, and $q_k$. (Recall $s_j=p_j$ here). In this way we derive a constant $\mathcal{C}_{**}(m,\mathbf{K},p_{k,i},q_k)$ in (12) independent of $\delta$ and of $p_i$.

The passage from $\Gamma(X_j)-\Gamma(X_j)+i\big(\Gamma(X_j)-\Gamma(X_j)\big)$ to $L^{p_j}(X_j)$ is obtained via Proposition 1.1 since $p_j<\infty$. A slightly more general version of this corollary (obtained in the same way) is the following:

**Corollary 1.2.** *Suppose that in Theorem 1.1, at least one $\gamma_j$ is nonzero, and instead of (7), we have*

$$\frac{1}{q}\le\sum_{\substack{1\le j\le m\\ \gamma_j\neq 0}}\frac{1}{p_j}.$$

*Then $T$ satisfies (12) for all $f_j$ in $\Gamma(X_j)-\Gamma(X_j)+i\big(\Gamma(X_j)-\Gamma(X_j)\big)$. Moreover, if $\mathbf{K}=1$, then $T$ has a unique extension that satisfies (12) for all $L^{p_j}(X_j)$.*

**Corollary 1.3.** *Suppose that $\gamma_j=0$ for all $j\in\{1,2,\dots,m\}$. Then we have $q_1=q_2=\cdots=q_{m+1}=q$. Moreover, there is a positive constant $\mathcal{C}_{***}(m,\mathbf{K},p_{k,i},q)$ such that $T$ satisfies*

$$\|T(f_1,\dots,f_m)\|_{L^{q,\infty}}\le\frac{\mathcal{C}_{***}(m,\mathbf{K},p_{k,i},q)}{\min(1,\operatorname{dist}(\vec{P},\partial\mathbf{H}))^{\frac{m}{\delta}}}\left(\prod_{k=1}^{m+1}B_k^{\eta_k}\right)\left(\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\infty}}\right),$$

*for all $f_j\in\Gamma(X_j)-\Gamma(X_j)+i\big(\Gamma(X_j)-\Gamma(X_j)\big)$, where $\delta$ satisfies*

$$0<\delta<\min\left(\frac{q}{2},\frac{\ln 2}{\ln(2\mathbf{K})}\right).$$

*Consequently, if $s_j<\infty$ for all $j\in\{1,2,\dots,m\}$ and $\mathbf{K}=1$, then $T$ has a unique bounded extension from $L^{p_1,s_1}(X_1)\times\cdots\times L^{p_m,s_m}(X_m)$ to $L^{q,\infty}(Y,\nu)$.*

Corollary 1.3 will be proved in Section 5. The assertion in last sentence is deduced from the trivial embedding $\|f_j\|_{L^{p_j,\infty}}\le(s_j/p_j)^{1/s_j}\|f_j\|_{L^{p_j,s_j}}$ (see [2, Proposition 1.4.10]) and from the fact that $\Gamma(X_j)-\Gamma(X_j)+i\big(\Gamma(X_j)-\Gamma(X_j)\big)$ is dense in $L^{p_j,s_j}(X_j)$. Note that the distinction between $s_j=\infty$ and $s_j<\infty$ is due to the fact that $\Gamma(X_j)-\Gamma(X_j)+i\big(\Gamma(X_j)-\Gamma(X_j)\big)$ may not be dense in $L^{p_j,\infty}(X_j)$.

## 2. Background and preliminary material

We first recall the definition of Lorentz spaces.

**Definition 2.1.** *The non-increasing rearrangement $f^*$ of a function $f$ on a measure space $(X,\mu)$ is given by*

$$f^*(t) = \inf\{s > 0,\ \mu(\{x \in X : |f(x)| > s\}) \leq t\}.$$

*Given $f$ a measurable function on a measure space and $0 < p, q \leq \infty$, define a quasi-norm*

$$\|f\|_{L^{p,q}(X,\mu)} = \begin{cases} \left(\displaystyle\int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } 0 < q < \infty, \\ \displaystyle\sup_{t>0} t^{\frac{1}{p}} f^*(t) & \text{if } q = \infty. \end{cases}$$

*The space $L^{p,q}(X,\mu)$ of all functions $f$ with $\|f\|_{L^{p,q}(X,\mu)} < \infty$ is called the Lorentz space with indices $p$ and $q$.*

The Lorentz space $L^{p,q}(X,\mu)$ is complete with respect to the quasi-norm previously defined and thus it is a quasi-Banach space.

We will make use of the following proposition due to Kalton (see page 56 in [2]), modified by Liang, Liu, and Yang [8].

**Proposition 2.1.** *Let $T$ be an operator defined on the set of simple functions of a measure space $(X,\mu)$ and taking values into the set of measurable functions of a measure space $(Y,\nu)$ that satisfies the conditions*

$$|T(f+g)| \leq \mathbf{K}\big(|T(f)| + |T(g)|\big)$$
$$|T(\lambda f)| = |\lambda|\,|T(f)|$$

*for some $\mathbf{K} \geq 1$ and for all simple functions $f$, $g$ on $X$ and all $\lambda \in \mathbf{C}$. Let $0 < p < \infty$ and $0 < q \leq \infty$. Suppose that for some constant $M > 0$ and for all measurable subsets $A$ of $X$ of finite measure we have*

$$\|T(\chi_A)\|_{L^{q,\infty}} \leq M\,\mu(A)^{\frac{1}{p}}.$$

*Fix $\delta_0 > 0$ such that $\delta_0 < q$ and $\delta_0 \leq \ln 2/\ln(2\mathbf{K})$. Then there exists a constant $C(p,q,\mathbf{K},\delta) < \infty$ such that for all $0 < \delta \leq \delta_0$ and all functions $f$ in $\Gamma(X) - \Gamma(X) + i\big(\Gamma(X) - \Gamma(X)\big)$, we have*

$$\|T(f)\|_{L^{q,\infty}} \leq C(p,q,\mathbf{K},\delta)\,M\,\|f\|_{L^{p,\delta}},$$

*where $C(p,q,\mathbf{K},\delta) = 100\,\mathbf{K}^3 4^{\frac{1}{p}+\frac{1}{q}} \big(\frac{q}{q-\delta}\big)^{\frac{2}{\delta}} (1-2^{-\delta})^{-\frac{1}{\delta}} (\ln 2)^{-\frac{1}{\delta}}$.*

An repeated application of this result yields its multilinear extension:

**Proposition 2.2.** *Let $T$ be an operator of $m$ variables defined on the set of simple functions of $(X_1,\mu_1) \times \cdots \times (X_m,\mu_m)$ and taking values into the set of measurable functions of a measure space $(Y,\nu)$ that satisfies (1) and (2) for some $\mathbf{K} \geq 1$. For*

*$j=1,\dots,m$, let $0<p_j<\infty$ and $0<q\le\infty$. Suppose that for some constant $M>0$ and for all measurable subsets $E_j$ of $X_j$ of finite measure we have*

$$\|T(\chi_{E_1},\dots,\chi_{E_m})\|_{L^{q,\infty}}\le M\prod_{j=1}^m \mu_j(E_j)^{\frac{1}{p_j}}.$$

*Fix $\delta_0>0$ such that $\delta_0<q$ and $\delta_0\le \ln 2/\ln(2\mathbf{K})$. Then there exists a constant $C_0(m,\mathbf{K},\delta,p_1,\dots,p_m,q)<\infty$ such that for all numbers $0<\delta\le\delta_0$ and all functions $f_j$ in $\Gamma(X_j)-\Gamma(X_j)+i\big(\Gamma(X_j)-\Gamma(X_j)\big)$ we have*

$$\|T(f_1,\dots,f_m)\|_{L^{q,\infty}}\le C_0(m,\mathbf{K},\delta,p_1,\dots,p_m,q)\,M\prod_{j=1}^m\|f_j\|_{L^{p_j,\delta}},$$

*where*

$$C_0(m,\mathbf{K},\delta,p_1,\dots,p_m,q)=\prod_{i=1}^m C(p_i,q,\mathbf{K},\delta),$$

*where $C(p_i,p,\mathbf{K},\delta)$ are the constants appearing in Proposition 2.1, i.e.,*

$$C_0(m,\mathbf{K},\delta,p_1,\dots,p_m,q)=\Big(100\,\mathbf{K}^3\big(\tfrac{q}{q-\delta}\big)^{\frac{2}{\delta}}(1-2^{-\delta})^{-\frac{1}{\delta}}(\ln 2)^{-\frac{1}{\delta}}\Big)^m 4^{\frac{1}{p_1}+\cdots+\frac{1}{p_m}+\frac{m}{q}}.$$

## 3. The proof of the main result

*Proof of Theorem 1.1.* If some $p_{j_0}=\infty$, then (5) implies that $p_{k,j_0}=\infty$ for all $k=1,2,\dots,m+1$, thus $\gamma_0=0$. Thus we have $0<p_j<\infty$ for all $j=1,2,\dots,m$.

Suppose that $0<\rho_k<1$ for all $1\le k\le m+1$, and $\sum_{k=1}^{m+1}\rho_k=1$. Let

$$\vec{R}=\Big(\frac{1}{r_1},\frac{1}{r_2}\,\dots,\frac{1}{r_m}\Big)=\sum_{k=1}^{m+1}\rho_k\vec{P}_k$$

be a point in $\mathbf{H}$ and define

$$\frac{1}{r}=\sum_{k=1}^{m+1}\frac{\rho_k}{q_k}.$$

It is a simple consequence of (4) that for all $E_j\subseteq X_j$, $1\le j\le m$ of finite measure we have

$$\prod_{k=1}^{m+1}\|T(\chi_{E_1},\dots,\chi_{E_m})\|_{L^{q_k,\infty}}^{\rho_k}\le\left(\prod_{k=1}^{m+1}B_k^{\rho_k}\right)\prod_{j=1}^m\mu_j(E_j)^{\frac{1}{r_j}}.$$

But for any measurable function $G$, by using $\sum_{k=1}^{m+1}\rho_k=1$ one has

$$\|G\|_{L^{r,\infty}}\le\prod_{k=1}^{m+1}\|G\|_{L^{q_k,\infty}}^{\rho_k},$$

and this implies that

$$\|T(\chi_{E_1},\dots,\chi_{E_m})\|_{L^{r,\infty}} \le \left(\prod_{k=1}^{m+1} B_k^{\rho_k}\right)\prod_{j=1}^{m}\mu_j(E_j)^{\frac{1}{r_j}}. \tag{15}$$

In other words, when restricted to characteristic functions, $T$ maps $L^{r_1,1}\times\cdots\times L^{r_m,1}$ to $L^{r,\infty}$ with the "correct" logarithmically convex bound in terms of the initial bounds $B_k$. It will be a considerable effort to extend this estimate to general functions keeping the multiplicative nature of the constant in (15).

In the sequel we will make use of the set

$$S_m = \big\{(\sigma_{\ell,1},\sigma_{\ell,2},\dots,\sigma_{\ell,m}):\ \ell=1,2,\dots,2^m\big\}$$

of all possible $m$-tuples of the form $(\pm1,\pm1,\dots,\pm1)$. Notice that elements of $S_m$ lie in different $2^m$-orthants of $\mathbf{R}^m$. Since all $p_j<\infty$ and $\vec{P}$ lies in the open convex hull $\mathbf{H}$, we choose $\varepsilon>0$ small enough such that $2\sqrt{m}\,\varepsilon$ is smaller than the distance from $\vec{P}$ to the boundary of the convex hull $\mathbf{H}$, i.e.,

$$\varepsilon < \min\left(1,\ \frac{\operatorname{dist}(\vec{P},\partial\mathbf{H})}{2\sqrt{m}}\right),$$

where $\partial\mathbf{H}$ is the set of all $(m-1)$-dimensional faces of $\mathbf{H}$. For all $\ell\in\{1,2,\dots,2^m\}$, we pick points

$$\vec{R}_\ell = \left(\frac{1}{r_{\ell,1}},\dots,\frac{1}{r_{\ell,m}}\right) = \sum_{k=1}^{m+1}\theta_{\ell,k}\vec{P}_k$$

such that $\sum_{k=1}^{m+1}\theta_{\ell,k}=1$ and, for all $j\in\{1,2,\dots,m\}$,

$$\frac{1}{r_{\ell,j}} - \frac{1}{p_j} = \varepsilon\sigma_{\ell,j}. \tag{16}$$

The choice of $\varepsilon$ implies that the cube of side length $2\varepsilon$ centered at $\vec{P}$ belongs to the open $\mathbf{H}$. Moreover, since $\mathbf{H}$ lies in the orthant $[0,\infty)^m$, it follows that for all $j\in\{1,2,\dots,m\}$

$$2\sqrt{m}\,\varepsilon < \operatorname{dist}(\vec{P},\partial\mathbf{H}) \le \frac{1}{p_j}. \tag{17}$$

From these and (16), we see that each $\vec{R}_\ell$ belongs to the open convex hull $\mathbf{H}$ and every $r_{\ell,j}$ is finite. Consequently, each $\theta_{\ell,k}\in(0,1)$.

Consider the system of equations

$$\begin{cases} \frac{1}{p_{1,1}}\theta_{\ell,1} + \frac{1}{p_{2,1}}\theta_{\ell,2} + \cdots + \frac{1}{p_{m+1,1}}\theta_{\ell,m+1} = \frac{1}{r_{\ell,1}} \\ \frac{1}{p_{1,2}}\theta_{\ell,1} + \frac{1}{p_{2,2}}\theta_{\ell,2} + \cdots + \frac{1}{p_{m+1,2}}\theta_{\ell,m+1} = \frac{1}{r_{\ell,2}} \\ \quad\vdots \qquad\qquad \vdots \qquad\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \frac{1}{p_{1,m}}\theta_{\ell,1} + \frac{1}{p_{2,m}}\theta_{\ell,2} + \cdots + \frac{1}{p_{m+1,m}}\theta_{\ell,m+1} = \frac{1}{r_{\ell,m}} \\ \qquad \theta_{\ell,1} + \qquad \theta_{\ell,2} + \cdots + \qquad \theta_{\ell,m+1} = 1\,, \end{cases}$$

which has a (unique) solution $(\theta_{\ell,1},\theta_{\ell,2},\dots,\theta_{\ell,m+1})$.

Denote by $A$ the matrix below

$$A=\begin{pmatrix} 1/p_{1,1} & 1/p_{2,1} & \dots & 1/p_{m+1,1}\\ 1/p_{1,2} & 1/p_{2,2} & \dots & 1/p_{m+1,2}\\ \vdots & \vdots & \vdots & \vdots\\ 1/p_{1,m} & 1/p_{2,m} & \dots & 1/p_{m+1,m}\\ 1 & 1 & \dots & 1 \end{pmatrix}.$$

For all $i,k\in\{1,2,\dots,m+1\}$, we denote by $D_{i,k}$ the determinant of the matrix obtained by deleting the $i$th row and $k$th column of the matrix $A$. Since $\gamma_0\neq 0$, it follows that not all these minor determinants are zero. Expanding the determinant (3) defining $\gamma_j$ along its $j$th column we obtain

$$\gamma_j=\sum_{k=1}^{m+1}(-1)^{j+k}\frac{1}{-q_k}D_{j,k}=-\sum_{k=1}^{m+1}(-1)^{j+k}\frac{1}{q_k}D_{j,k}\,. \tag{18}$$

For all $\ell=1,2,\dots,2^m$, in view of (5) and $\sum_{k=1}^{m+1}\eta_k=1$, we have that the $(m+1)$-tuple

$$(\theta_{\ell,1}-\eta_1,\theta_{\ell,2}-\eta_2,\dots,\theta_{\ell,m+1}-\eta_{m+1})$$

is a solution of the system

$$\begin{cases} \frac{1}{p_{1,1}}(\theta_{\ell,1}-\eta_1)+\dots+\frac{1}{p_{m+1,1}}(\theta_{\ell,m+1}-\eta_{m+1}) = \frac{1}{r_{\ell,1}}-\frac{1}{p_1}\\ \frac{1}{p_{1,2}}(\theta_{\ell,1}-\eta_1)+\dots+\frac{1}{p_{m+1,2}}(\theta_{\ell,m+1}-\eta_{m+1}) = \frac{1}{r_{\ell,2}}-\frac{1}{p_2}\\ \quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots\\ \frac{1}{p_{1,m}}(\theta_{\ell,1}-\eta_1)+\dots+\frac{1}{p_{m+1,m}}(\theta_{\ell,m+1}-\eta_{m+1}) = \frac{1}{r_{\ell,m}}-\frac{1}{p_m}\\ \qquad(\theta_{\ell,1}-\eta_1)+\dots+\qquad(\theta_{\ell,m+1}-\eta_{m+1}) = 0. \end{cases}$$

This unique solution can be expressed as the ratio

$$\theta_{\ell,k}-\eta_k=\frac{\det\begin{pmatrix} 1/p_{1,1} & 1/p_{2,1} & \dots & 1/r_{\ell,1}-1/p_1 & \dots & 1/p_{m+1,1}\\ 1/p_{1,2} & 1/p_{2,2} & \dots & 1/r_{\ell,2}-1/p_2 & \dots & 1/p_{m+1,2}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 1/p_{1,m} & 1/p_{2,m} & \dots & 1/r_{\ell,m}-1/p_m & \dots & 1/p_{m+1,m}\\ 1 & 1 & \dots & 0 & \dots & 1 \end{pmatrix}}{\det\begin{pmatrix} 1/p_{1,1} & 1/p_{2,1} & \dots & 1/p_{m+1,1}\\ 1/p_{1,2} & 1/p_{2,2} & \dots & 1/p_{m+1,2}\\ \vdots & \vdots & \vdots & \vdots\\ 1/p_{1,m} & 1/p_{2,m} & \dots & 1/p_{m+1,m}\\ 1 & 1 & \dots & 1 \end{pmatrix}},$$

where these determinants are different only in the $k$th column. Expanding the determinant in the numerator, we deduce that for all $k\in\{1,2,\dots,m+1\}$ and all

$\ell \in \{1, 2, \dots, 2^m\}$,

$$\theta_{\ell,k} - \eta_k = \sum_{j=1}^{m} \Big(\frac{1}{r_{\ell,j}} - \frac{1}{p_j}\Big)(-1)^{j+k}\frac{D_{j,k}}{\gamma_0}\,. \tag{19}$$

For any $\ell \in \{1, 2, \dots, 2^m\}$, we also define

$$\frac{1}{r_\ell} = \sum_{k=1}^{m+1} \frac{\theta_{\ell,k}}{q_k}\,. \tag{20}$$

Using these expressions and (6), we write

$$\begin{aligned}
\frac{1}{q} - \frac{1}{r_\ell} = \sum_{k=1}^{m+1} \frac{\eta_k - \theta_{\ell,k}}{q_k} &= -\sum_{k=1}^{m+1} \frac{1}{q_k} \sum_{j=1}^{m} \Big(\frac{1}{r_{\ell,j}} - \frac{1}{p_j}\Big)(-1)^{j+k}\frac{D_{j,k}}{\gamma_0} \\
&= -\sum_{j=1}^{m} \Big(\frac{1}{r_{\ell,j}} - \frac{1}{p_j}\Big) \sum_{k=1}^{m+1} \frac{1}{q_k}(-1)^{j+k}\frac{D_{j,k}}{\gamma_0} \\
&= \sum_{j=1}^{m} \Big(\frac{1}{r_{\ell,j}} - \frac{1}{p_j}\Big)\frac{\gamma_j}{\gamma_0}\,,
\end{aligned} \tag{21}$$

where the last identity follows from (18).

We introduce some more notation. For any $j \in \{1, 2, \dots, m\}$ and any $k$ in $\{1, 2, \dots, m+1\}$, set

$$t_{j,k} = (-1)^{j+k}\frac{D_{j,k}}{\gamma_0}$$

and then (19) can be written as

$$\eta_k = \theta_{\ell,k} - \sum_{j=1}^{m} \Big(\frac{1}{r_{\ell,j}} - \frac{1}{p_j}\Big) t_{j,k}\,. \tag{22}$$

Since the points $\vec{R}_\ell$ lie in the open convex hull $\mathbf{H}$, estimate (15) is valid for each $\vec{R}_\ell$ (with $\theta_{\ell,k}$ in the place of $\rho_k$). To simplify notation, set

$$\widetilde{B}_\ell = \prod_{k=1}^{m+1} B_k^{\theta_{\ell,k}}\,.$$

In view of (15) we have

$$\|T(\chi_{E_1}, \dots, \chi_{E_m})\|_{L^{r_\ell,\infty}} \le \widetilde{B}_\ell \prod_{j=1}^{m} \mu_j(E_j)^{\frac{1}{r_{\ell,j}}}$$

for all subsets $E_j$ of $X_j$ of finite measure. Let $\delta$ be a positive number satisfying (9). Observe that (9) and (20) imply

$$\delta < \min\Big(\frac{r_1}{2}, \frac{r_2}{2}, \dots, \frac{r_{2^m}}{2}, \frac{\ln 2}{\ln(2\mathbf{K})}\Big)\,. \tag{23}$$

Then, it follows from Proposition 2.2 that

$$\|T(f_1,\dots,f_m)\|_{L^{r_\ell,\infty}} \le C_0(m,\mathbf{K},\delta,r_{\ell,i},r_\ell)\,\widetilde{B}_\ell \prod_{j=1}^m \|f_j\|_{L^{r_{\ell,j},\delta}} \tag{24}$$

for all functions $f_j \in \Gamma(X_j)-\Gamma(X_j)+i\big(\Gamma(X_j)-\Gamma(X_j)\big)$, where

$$C_0(m,\mathbf{K},\delta,r_{\ell,i},r_\ell) = \Big(100\,\mathbf{K}^3\big(\tfrac{r_\ell}{r_\ell-\delta}\big)^{\frac{2}{\delta}}(1-2^{-\delta})^{-\frac{1}{\delta}}(\ln 2)^{-\frac{1}{\delta}}\Big)^m 4^{\frac{1}{r_{\ell,1}}+\cdots+\frac{1}{r_{\ell,m}}+\frac{m}{r_\ell}}.$$

Notice that (16) and (21) together with the fact $\varepsilon<1$ imply that

$$\begin{aligned} 4^{\frac{1}{r_{\ell,1}}+\cdots+\frac{1}{r_{\ell,m}}+\frac{m}{r_\ell}} &\le 4^{\frac{1}{p_1}+\cdots+\frac{1}{p_m}+\frac{m}{q}} 4^{|\frac{1}{r_{\ell,1}}-\frac{1}{p_1}|+\cdots+|\frac{1}{r_{\ell,m}}-\frac{1}{p_m}|+m|\frac{1}{r_\ell}-\frac{1}{q}|} \\ &\le 4^{\frac{1}{p_1}+\cdots+\frac{1}{p_m}+\frac{m}{q}} 4^{m+m\sum_{j=1}^m \frac{|\gamma_j|}{|\gamma_0|}} \\ &\le 4^{\sum_{j=1}^m\sum_{k=1}^{m+1}\frac{1}{p_{k,j}}+m\sum_{k=1}^{m+1}\frac{1}{q_k}} 4^{m+m\sum_{j=1}^m \frac{|\gamma_j|}{|\gamma_0|}}, \end{aligned}$$

where the last inequality is a consequence of the observation (14). Also, it follows from (23) that $\frac{r_\ell}{r_\ell-\delta}<2$ for all $1\le\ell\le 2^m$. Therefore, we can bound $C_0(m,\mathbf{K},\delta,r_{\ell,i},r_\ell)$ by

$$\Big(100\,\mathbf{K}^3 2^{\frac{2}{\delta}}(1-2^{-\delta})^{-\frac{1}{\delta}}(\ln 2)^{-\frac{1}{\delta}}\Big)^m 4^{\sum_{j=1}^m\sum_{k=1}^{m+1}\frac{1}{p_{k,j}}+m\sum_{k=1}^{m+1}\frac{1}{q_k}} 4^{m+m\sum_{j=1}^m \frac{|\gamma_j|}{|\gamma_0|}}. \tag{25}$$

for every $\ell$. We denote the constant in (25) by $C_0'(m,\mathbf{K},\delta,p_{k,i},q_k)$. From this and (24), we obtain that for all functions $f_j \in \Gamma(X_j)-\Gamma(X_j)+i\big(\Gamma(X_j)-\Gamma(X_j)\big)$,

$$\|T(f_1,\dots,f_m)\|_{L^{r_\ell,\infty}} \le C_0'(m,\mathbf{K},\delta,p_{k,i},q_k)\,\widetilde{B}_\ell \prod_{j=1}^m \|f_j\|_{L^{r_{\ell,j},\delta}}. \tag{26}$$

For all $j=1,2,\dots,m$, fix functions $f_j$ in $\Gamma(X_j)-\Gamma(X_j)+i\big(\Gamma(X_j)-\Gamma(X_j)\big)$ and for any $t>0$ write $f_j=f_{j,1,t}+f_{j,-1,t}$, by setting

$$f_{j,-1,t}=f_j\,\chi_{\{|f_j|>f_j^*(\lambda_j\,t^{-\frac{\gamma_j}{\gamma_0}})\}} \qquad\text{and}\qquad f_{j,1,t}=f_j\,\chi_{\{|f_j|\le f_j^*(\lambda_j\,t^{-\frac{\gamma_j}{\gamma_0}})\}} \tag{27}$$

for some $\lambda_j>0$ to be determined later.

Proposition 1.4.5 (6) in [2, p. 46] and Exercise 1.1.5 (c) in [2, p. 12] together with the multi-quasilinearity of the operator $T$ and of Lorentz norms imply

$$\begin{aligned} &\|T(f_1,\dots,f_m)\|_{L^{q,s}} \\ &= \|t^{\frac{1}{q}}T(f_1,\dots,f_m)^*(t)\|_{L^s(dt/t)} \\ &\le \mathbf{K}^m\Big\|t^{\frac{1}{q}}\Big(\sum_{i_1,\dots,i_m\in\{1,-1\}}|T(f_{1,i_1,t},\dots,f_{m,i_m,t})|\Big)^*(t)\Big\|_{L^s(dt/t)} \\ &\le \mathbf{K}^m\Big\|t^{\frac{1}{q}}\sum_{i_1,\dots,i_m\in\{1,-1\}}(|T(f_{1,i_1,t},\dots,f_{m,i_m,t})|)^*(t/2^m)\Big\|_{L^s(dt/t)} \\ &\le 2^{\frac{m}{q}}\max(1,2^{\frac{m(1-s)}{s}})\mathbf{K}^m\sum_{i_1,\dots,i_m\in\{1,-1\}}\|t^{\frac{1}{q}}(|T(f_{1,i_1,t},\dots,f_{m,i_m,t})|)^*(t)\|_{L^s(dt/t)} \end{aligned}$$

$$= 2^{\frac{m}{q}} \max(1, 2^{\frac{m(1-s)}{s}})\mathbf{K}^m \sum_{\ell=1}^{2^m} \|t^{\frac{1}{q}}(|T(f_{1,\sigma_{\ell,1},t}, \dots, f_{m,\sigma_{\ell,m},t})|)^*(t)\|_{L^s(dt/t)} ,$$

since each $m$-tuple $(i_1, \dots, i_m)$ with $i_j \in \{1, -1\}$ corresponds to a unique $\ell$ in $\{1, 2, \dots, 2^m\}$ such that $(i_1, \dots, i_m) = \sigma_\ell \in S_m$. It follows from (21) and (24) that for all $\ell \in \{1, 2, \dots, 2^m\}$ and $t > 0$,

$$\begin{aligned} & t^{\frac{1}{q}}(|T(f_{1,\sigma_{\ell,1},t}, \dots, f_{m,\sigma_{\ell,m},t})|)^*(t) \\ & \le t^{\frac{1}{q}-\frac{1}{r_\ell}} \sup_{s>0} s^{\frac{1}{r_\ell}}(|T(f_{1,\sigma_{\ell,1},t}, \dots, f_{m,\sigma_{\ell,m},t})|)^*(s) \\ & \le t^{\frac{1}{q}-\frac{1}{r_\ell}} \big\|T(f_{1,\sigma_{\ell,1},t}, \dots, f_{m,\sigma_{\ell,m},t})\big\|_{L^{r_\ell,\infty}} \\ & \le t^{\frac{1}{q}-\frac{1}{r_\ell}} C_0'(m, \mathbf{K}, \delta, p_{k,i}, q_k)\, \widetilde{B}_\ell \prod_{j=1}^m \|f_{j,\sigma_{\ell,j},t}\|_{L^{r_{\ell,j},\delta}} \\ & = C_0'(m, \mathbf{K}, \delta, p_{k,i}, q_k)\, \widetilde{B}_\ell \prod_{j=1}^m t^{\frac{\gamma_j}{\gamma_0}(\frac{1}{r_{\ell,j}}-\frac{1}{p_j})} \|f_{j,\sigma_{\ell,j},t}\|_{L^{r_{\ell,j},\delta}}. \end{aligned} \tag{28}$$

We now introduce sets

$$\Lambda = \{1 \le j \le m : \gamma_j \neq 0\}$$
$$\Lambda' = \{1 \le j \le m : \gamma_j = 0\}$$

and we rewrite (28) as

$$\begin{aligned} & t^{\frac{1}{q}}(|T(f_{1,\sigma_{\ell,1},t}, \dots, f_{m,\sigma_{\ell,m},t})|)^*(t) \\ & \le C_0'(m, \mathbf{K}, \delta, p_{k,i}, q_k)\widetilde{B}_\ell \Big(\prod_{j\in\Lambda} t^{\frac{\gamma_j}{\gamma_0}(\frac{1}{r_{\ell,j}}-\frac{1}{p_j})} \|f_{j,\sigma_{\ell,j},t}\|_{L^{r_{\ell,j},\delta}}\Big)\Big(\prod_{j\in\Lambda'} \|f_{j,\sigma_{\ell,j},1}\|_{L^{r_{\ell,j},\delta}}\Big) , \end{aligned} \tag{29}$$

where we made use of the observation that for $j \in \Lambda'$ we have $\gamma_j = 0$ and hence for all $t > 0$,

$$\|f_{j,\sigma_{\ell,j},t}\|_{L^{r_{\ell,j},\delta}} = \|f_{j,\sigma_{\ell,j},1}\|_{L^{r_{\ell,j},\delta}} .$$

To estimate the $L^s(dt/t)$ quasi-norm of (29), we need the following lemmas, whose proofs are presented in the next section.

**Lemma 3.1.** *For all $j \in \Lambda$ let $s_j$ satisfy $0 < s_j \le \infty$. Then for all $\ell$ in $\{1, 2, \dots, 2^m\}$, the following inequalities are valid: when $p_j > r_{\ell,j}$ we have*

$$\Big\| t^{\frac{\gamma_j}{\gamma_0}(\frac{1}{r_{\ell,j}}-\frac{1}{p_j})} \|f_{j,-1,t}\|_{L^{r_{\ell,j},\delta}} \Big\|_{L^{s_j}(\frac{dt}{t})} \le \frac{C_1(r_{\ell,j}, p_j, \delta)}{\big|\frac{\gamma_j}{\gamma_0}\big|^{\frac{1}{s_j}}} \lambda_j^{\frac{1}{r_{\ell,j}}-\frac{1}{p_j}} \|f_j\|_{L^{p_j,s_j}} \tag{30}$$

*and when $p_j < r_{\ell,j}$ we have*

$$\Big\| t^{\frac{\gamma_j}{\gamma_0}(\frac{1}{r_{\ell,j}}-\frac{1}{p_j})} \|f_{j,1,t}\|_{L^{r_{\ell,j},\delta}} \Big\|_{L^{s_j}(\frac{dt}{t})} \le \frac{C_1(r_{\ell,j}, p_j, \delta)}{\big|\frac{\gamma_j}{\gamma_0}\big|^{\frac{1}{s_j}}} \lambda_j^{\frac{1}{r_{\ell,j}}-\frac{1}{p_j}} \|f_j\|_{L^{p_j,s_j}} , \tag{31}$$

*where*

$$C_1(r_{\ell,j},p_j,\delta)=\left[\frac{\max(1,\ \frac{r_{\ell,j}}{p_j})}{\delta|\frac{1}{p_j}-\frac{1}{r_{\ell,j}}|}\right]^{\frac{1}{\delta}}=\left[\frac{\max(1,\ \frac{r_{\ell,j}}{p_j})}{\delta\varepsilon}\right]^{\frac{1}{\delta}}.$$

We note that each $C_1(r_{\ell,j},p_j,\delta)$ in Lemma 3.1 satisfies the following estimate:

$$C_1(r_{\ell,j},p_j,\delta)<\Big(\frac{2}{\delta\varepsilon}\Big)^{\frac{1}{\delta}}; \tag{32}$$

indeed, using (16) and the fact $\varepsilon p_j<\frac{1}{2\sqrt{m}}$ (see (17)) we have

$$\max\Big(1,\frac{r_{\ell,j}}{p_j}\Big)=\max\Big(1,\frac{1}{1+\varepsilon p_j\sigma_{\ell,j}}\Big)<\max\Big(1,\frac{1}{1-\frac{1}{2\sqrt{m}}}\Big)<2.$$

**Lemma 3.2.** *For all $j\in\Lambda'$ and all $\ell\in\{1,2,\dots,2^m\}$, when $p_j>r_{\ell,j}$ we have*

$$\|f_{j,-1,1}\|_{L^{r_{\ell,j},\delta}}\le C_1(r_{\ell,j},p_j,\delta)\,\lambda_j^{\frac{1}{r_{\ell,j}}-\frac{1}{p_j}}\,\|f_j\|_{L^{p_j,\infty}} \tag{33}$$

*and when $p_j<r_{\ell,j}$ we have*

$$\|f_{j,1,1}\|_{L^{r_{\ell,j},\delta}}\le C_1(r_{\ell,j},p_j,\delta)\,\lambda_j^{\frac{1}{r_{\ell,j}}-\frac{1}{p_j}}\|f_j\|_{L^{p_j,\infty}}, \tag{34}$$

*where $C_1(r_{\ell,j},p_j,\delta)$ is as in Lemma 3.1.*

Then, we take the $L^s(dt/t)$ quasi-norm of (29), by virtue of Hölder's inequality with exponents $\frac{1}{s}=\sum_{j\in\Lambda}\frac{1}{s_j}$, and use Lemma 3.1 when $j\in\Lambda$ or Lemma 3.2 when $j\in\Lambda'$. Summing over $\ell$ and invoking (32), we obtain that for all functions $f_j$ in $\Gamma(X_j)-\Gamma(X_j)+i\big(\Gamma(X_j)-\Gamma(X_j)\big)$ the expression $\|T(f_1,\dots,f_m)\|_{L^{q,s}}$ is bounded by

$$2^{\frac{m}{q}}\max(1,2^{\frac{m(1-s)}{s}})\mathbf{K}^m\sum_{\ell=1}^{2^m}C_0'(m,\mathbf{K},\delta,p_{k,i},q_k)\widetilde{B}_\ell$$
$$\left\{\Big(\prod_{j\in\Lambda}\Big(\frac{2}{\delta\varepsilon}\Big)^{\frac{1}{\delta}}\Big|\frac{\gamma_0}{\gamma_j}\Big|^{\frac{1}{s_j}}\lambda_j^{\frac{1}{r_{\ell,j}}-\frac{1}{p_j}}\|f_j\|_{L^{p_j,s_j}}\Big)\Big(\prod_{j\in\Lambda'}\Big(\frac{2}{\delta\varepsilon}\Big)^{\frac{1}{\delta}}\lambda_j^{\frac{1}{r_{\ell,j}}-\frac{1}{p_j}}\|f_j\|_{L^{p_j,\infty}}\Big)\right\}.$$

To obtain (8), for each $j\in\{1,2,\dots,m\}$ we choose

$$\lambda_j=\Big(B_1^{t_{j,1}}B_2^{t_{j,2}}\cdots B_{m+1}^{t_{j,m+1}}\Big)^{-1}.$$

Then, for each $1\le k\le m+1$, the dependence of the preceding expression on the $B_k$'s is

$$\prod_{k=1}^{m+1}B_k^{\theta_{\ell,k}-\sum_{j\in\Lambda}\big(\frac{1}{r_{\ell,j}}-\frac{1}{p_j}\big)t_{j,k}-\sum_{j\in\Lambda'}\big(\frac{1}{r_{\ell,j}}-\frac{1}{p_j}\big)t_{j,k}}=\prod_{k=1}^{m+1}B_k^{\eta_k},$$

in view of (22).

From this, we conclude that for all $f_j \in \Gamma(X_j) - \Gamma(X_j) + i\big(\Gamma(X_j) - \Gamma(X_j)\big)$, the expression $\|T(f_1,\dots,f_m)\|_{L^{q,s}}$ is at most

$$\mathcal{C}'_*(m,\mathbf{K},\delta,p_{k,i},q_k,s_j,s)\varepsilon^{-m/\delta}\Big(\prod_{k=1}^{m+1} B_k^{\eta_k}\Big)\Big(\prod_{j\in\Lambda}\|f_j\|_{L^{p_j,s_j}}\Big)\Big(\prod_{j\in\Lambda'}\|f_j\|_{L^{p_j,\infty}}\Big),$$

where $\mathcal{C}'_*(m,p_{k,j},q_k,\eta_k,s_j,s,\mathbf{K})$ is equal to

$$2^{\frac{m}{q}}\max(1,2^{\frac{m(1-s)}{s}})\mathbf{K}^m\,2^m C'_0(m,\mathbf{K},\delta,p_{k,i},q_k)\Big(\frac{2}{\delta}\Big)^{\frac{m}{\delta}}\prod_{j\in\Lambda}\Big|\frac{\gamma_0}{\gamma_j}\Big|^{\frac{1}{s_j}}.$$

If $j\in\Lambda'$ then it is a simple fact (see [2, Proposition 1.4.10]) that for any $s_j\in(0,\infty]$ we have

$$\|f_j\|_{L^{p_j,\infty}} \le \Big(\frac{s_j}{p_j}\Big)^{\frac{1}{s_j}}\|f_j\|_{L^{p_j,s_j}}.$$

Thus for all functions $f_j \in \Gamma(X_j) - \Gamma(X_j) + i\big(\Gamma(X_j) - \Gamma(X_j)\big)$ we conclude

$$\|T(f_1,\dots,f_m)\|_{L^{q,s}} \le \mathcal{C}''_*(m,\mathbf{K},\delta,p_{k,i},q_k,s_i,s)\varepsilon^{-m/\delta}\Big(\prod_{k=1}^{m+1} B_k^{\eta_k}\Big)\prod_{j=1}^{m}\|f_j\|_{L^{p_j,s_j}}, \tag{35}$$

where $\mathcal{C}''_*(m,\mathbf{K},\delta,p_{k,i},q_k,s_i,s)$ is equal to

$$2^{\frac{m}{q}}\max(1,2^{\frac{m(1-s)}{s}})\mathbf{K}^m\,2^m C'_0(m,\mathbf{K},\delta,p_{k,i},q_k)\Big(\frac{2}{\delta}\Big)^{\frac{m}{\delta}}\prod_{j\in\Lambda}\Big|\frac{\gamma_0}{\gamma_j}\Big|^{\frac{1}{s_j}}\prod_{j\in\Lambda'}\Big(\frac{s_j}{p_j}\Big)^{\frac{1}{s_j}}.$$

Since (35) is valid for all $\varepsilon < \min(1,\frac{\operatorname{dist}(\vec{P},\partial\mathbf{H})}{2\sqrt{m}})$, letting $\varepsilon\to\min(1,\frac{\operatorname{dist}(\vec{P},\partial\mathbf{H})}{2\sqrt{m}})$, and noticing that $\frac{1}{q}\le\sum_{k=1}^{m+1}\frac{1}{q_k}$, we then obtain estimate (8) for all functions $f_j$ in $\Gamma(X_j) - \Gamma(X_j) + i\big(\Gamma(X_j) - \Gamma(X_j)\big)$, $1\le j\le m$, where

$$\mathcal{C}_*(m,\mathbf{K},\delta,p_{k,i},q_k) = 2^{m\sum_{k=1}^{m+1}\frac{1}{q_k}}\mathbf{K}^m\,2^m C'_0(m,\mathbf{K},\delta,p_{k,i},q_k)\Big(\frac{2}{\delta}\Big)^{\frac{m}{\delta}}\big(2\sqrt{m}\big)^{m/\delta}.$$

This concludes the proof of Theorem 1.1. □

## 4. The proof of Proposition 1.1

We need to show that (10) is valid for general functions in $L^{p_1,t_1}\times\cdots\times L^{p_m,t_m}$. For any $j=1,2,\dots,m$ and $f_j\in L^{p_j,t_j}$, since $\Gamma(X_j) - \Gamma(X_j) + i\big(\Gamma(X_j) - \Gamma(X_j)\big)$ is dense in $L^{p_j,t_j}$ when $0<t_j<\infty$, there exists a sequence $\{f_j^{(n)}\}_{n=1}^\infty$ contained in $\Gamma(X_j) - \Gamma(X_j) + i\big(\Gamma(X_j) - \Gamma(X_j)\big)$ such that

$$\lim_{n\to\infty}\|f_j^{(n)} - f_j\|_{L^{p_j,t_j}} = 0$$

and

$$\|f_j^{(n)}\|_{L^{p_j,t_j}} \le 2\|f_j\|_{L^{p_j,t_j}}$$

for all $n \geq 1$. For all positive integers $n$ and $i$, we use the multi-sublinearity to deduce that

$$\begin{aligned}
&|T(f_1^{(n)}, \dots, f_m^{(n)}) - T(f_1^{(i)}, \dots, f_m^{(i)})| \\
&\quad \leq \sum_{j=1}^{m} |T(f_1^{(i)}, \dots, f_{j-1}^{(i)}, f_j^{(n)}, \dots, f_m^{(n)}) - T(f_1^{(i)}, \dots, f_j^{(i)}, f_{j+1}^{(n)} \dots, f_m^{(n)})| \\
&\quad \leq \sum_{j=1}^{m} |T(f_1^{(i)}, \dots, f_{j-1}^{(i)}, f_j^{(n)} - f_j^{(i)}, f_{j+1}^{(n)}, \dots, f_m^{(n)})|\,,
\end{aligned}$$

where the $j$th entry is $f_j^{(n)} - f_j^{(i)}$. This implies that

$$\begin{aligned}
&\|T(f_1^{(n)}, \dots, f_m^{(n)}) - T(f_1^{(i)}, \dots, f_m^{(i)})\|_{L^{q,s}} \\
&\quad \leq \max(1, 2^{\frac{m(1-s)}{s}}) \sum_{j=1}^{m} \|T(f_1^{(i)}, \dots, f_{j-1}^{(i)}, f_j^{(n)} - f_j^{(i)}, f_{j+1}^{(n)}, \dots, f_m^{(n)})\|_{L^{q,s}} \\
&\quad \leq \max(1, 2^{\frac{m(1-s)}{s}}) M \sum_{j=1}^{m} \|f_j^{(n)} - f_j^{(i)}\|_{L^{q_j,t_j}} \prod_{\substack{1 \leq k \leq m \\ k \neq j}} 2\|f_k\|_{L^{q_k,t_k}}\,,
\end{aligned}$$

which tends to 0 as $n, i \to \infty$. Thus, $\{T(f_1^{(n)}, \dots, f_m^{(n)})\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^{q,s}$ and it converges to some element in $L^{q,s}$, so it makes sense to define

$$\widetilde{T}(f_1, \dots, f_m) = \lim_{n \to \infty} T(f_1^{(n)}, \dots, f_m^{(n)}) \quad \text{in } L^{q,s}\,.$$

Similar arguments show that if, for $j = 1, 2, \dots, m$, $\{g_j^{(n)}\}_{n=1}^{\infty}$ is another sequence contained in $\Gamma(X_j) - \Gamma(X_j) + i\big(\Gamma(X_j) - \Gamma(X_j)\big)$ that converges to $f_j$ in $L^{q_j,t_j}$, then

$$\widetilde{T}(f_1, \dots, f_m) = \lim_{n \to \infty} T(g_1^{(n)}, \dots, g_m^{(n)}) \quad \text{in } L^{q,s}\,.$$

Therefore, $\widetilde{T}$ is a well-defined multi-sublinear operator. Consequently, for all functions $(f_1, \dots, f_m) \in L^{p_1,t_1} \times \cdots \times L^{p_m,t_m}$, we have

$$\begin{aligned}
\|\widetilde{T}(f_1, \dots, f_m)\|_{L^{q,s}} &\leq \lim_{n \to \infty} \|T(f_1^{(n)}, \dots, f_m^{(n)})\|_{L^{q,s}} \\
&\leq M \lim_{n \to \infty} \prod_{j=1}^{m} \|f_j^{(n)}\|_{L^{p_j,t_j}} \\
&= M \prod_{j=1}^{m} \|f_j\|_{L^{p_j,t_j}}\,.
\end{aligned}$$

This concludes the proof of Proposition 1.1. □

## 5. Proof of Corollary 1.3

We first show that if $\gamma_j = 0$ for all $j$, then $q_1 = \cdots = q_{m+1}$. We define vectors

$$\vec{1} = (1, 1, \ldots, 1), \qquad \vec{Q} = (1/q_1, \ldots, 1/q_{m+1}),$$

and for each $j \in \{1, 2, \ldots, m\}$, we also define

$$\vec{A}_j = (1/p_{1,j}, 1/p_{2,j}, \ldots, 1/p_{m+1,j}).$$

Then $(\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_m, \vec{1})$ is linearly independent since $\gamma_0 \neq 0$. If all $\gamma_j = 0$, this means that for each $j \in \{1, 2, \ldots, m\}$,

$$(\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_{j-1}, \vec{Q}, \vec{A}_{j+1}, \ldots, \vec{A}_m, \vec{1})$$

is linearly dependent. Therefore, for any $j \in \{1, 2, \ldots, m\}$, we can write

$$\vec{Q} = \sum_{1 \le i \le m,\, i \neq j} a_i^{(j)} \vec{A}_i + c^{(j)} \vec{1},$$

where $a_i^{(j)}$ and $c^{(j)}$ are constants. Equivalently,

$$\begin{cases} \vec{Q} &= \quad 0 \quad + a_2^{(1)}\vec{A}_2 + a_3^{(1)}\vec{A}_3 + \cdots + a_{m-1}^{(1)}\vec{A}_{m-1} + a_m^{(1)}\vec{A}_m + c^{(1)}\vec{1} \\ \vec{Q} &= a_1^{(2)}\vec{A}_1 + \quad 0 \quad + a_3^{(2)}\vec{A}_3 + \cdots + a_{m-1}^{(2)}\vec{A}_{m-1} + a_m^{(2)}\vec{A}_m + c^{(2)}\vec{1} \\ \vdots & \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \quad \vdots \\ \vec{Q} &= a_1^{(m)}\vec{A}_1 + a_2^{(m)}\vec{A}_2 + a_3^{(m)}\vec{A}_3 + \cdots + a_{m-1}^{(m)}\vec{A}_{m-1} + \quad 0 + c^{(m)}\vec{1}. \end{cases}$$

Consider $j = 1$ and $j = 2$. Then

$$\vec{0} = \vec{Q} - \vec{Q} = -a_1^{(2)}\vec{A}_1 + a_2^{(1)}\vec{A}_2 + \sum_{i=3}^{m} (a_i^{(1)} - a_i^{(2)})\vec{A}_i + (c^{(1)} - c^{(2)})\vec{1},$$

which combined with the fact that $(\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_m, \vec{1})$ is linearly independent implies that

$$a_2^{(1)} = 0.$$

Likewise, by considering $j = 1$ and $j = 3$, we obtain

$$\vec{0} = \vec{Q} - \vec{Q} = -a_1^{(3)}\vec{A}_1 + a_3^{(1)}\vec{A}_3 + \sum_{1 \le i \le m,\, i \neq 1, i \neq 3} (a_i^{(1)} - a_i^{(3)})\vec{A}_i + (c^{(1)} - c^{(3)})\vec{1},$$

and consequently

$$a_3^{(1)} = 0.$$

Repeating the above process implies that

$$a_4^{(1)} = \cdots = a_m^{(1)} = 0.$$

Therefore, $\vec{Q}$ is a constant multiple of the vector $\vec{1}$, that is, $q_1 = \cdots = q_{m+1}$. Then $q$ is equal to these numbers as well.

The remaining assertions in the corollary are already proved in Section 3 and Section 4. □

## 6. Proofs of Lemmas 3.1 and 3.2

For each $j \in \{1, 2, \dots, m\}$ and $f_j \in \Gamma(X_j) - \Gamma(X_j) + i\big(\Gamma(X_j) - \Gamma(X_j)\big)$, with $f_{j,-1,t}$ and $f_{j,1,t}$ defined as in (27), it is easy to show that the following inequalities are valid:

$$f^*_{j,-1,t}(v) \le \begin{cases} f^*_j(v) & \text{if } 0 < v < \lambda_j\, t^{-\frac{\gamma_j}{\gamma_0}}, \\ 0 & \text{if } v \ge \lambda_j\, t^{-\frac{\gamma_j}{\gamma_0}}, \end{cases} \tag{36}$$

and

$$f^*_{j,1,t}(v) \le \begin{cases} f^*_j(\lambda_j\, t^{-\frac{\gamma_j}{\gamma_0}}) & \text{if } 0 < v < \lambda_j\, t^{-\frac{\gamma_j}{\gamma_0}}, \\ f^*_j(v) & \text{if } v \ge \lambda_j\, t^{-\frac{\gamma_j}{\gamma_0}}\,. \end{cases} \tag{37}$$

First we prove Lemma 3.1.

*Proof of Lemma 3.1.* We first prove (30). In view of (36) we have

$$\begin{aligned} &\left\| t^{\frac{\gamma_j}{\gamma_0}(\frac{1}{r_{\ell,j}}-\frac{1}{p_j})} \|f_{j,-1,t}\|_{L^{r_{\ell,j},\delta}} \right\|_{L^{s_j}(\frac{dt}{t})} \\ &= \left[ \int_0^\infty t^{s_j\frac{\gamma_j}{\gamma_0}(\frac{1}{r_{\ell,j}}-\frac{1}{p_j})} \left\{ \int_0^{\lambda_j t^{-\frac{\gamma_j}{\gamma_0}}} (f^*_{j,-1,t}(v))^\delta v^{\frac{\delta}{r_{\ell,j}}} \,\frac{dv}{v} \right\}^{\frac{s_j}{\delta}} \frac{dt}{t} \right]^{\frac{1}{s_j}} . \end{aligned}$$

Change variables $u = \lambda_j\, t^{-\frac{\gamma_j}{\gamma_0}}$ and use (36) to estimate the preceding expression by

$$\left|\frac{\gamma_0}{\gamma_j}\right|^{\frac{1}{s_j}} \lambda_j^{\frac{1}{r_{\ell,j}}-\frac{1}{p_j}} \left[ \left\{ \int_0^\infty u^{-s_j(\frac{1}{r_{\ell,j}}-\frac{1}{p_j})} \left( \int_0^u (f^*_j(v))^\delta v^{\frac{\delta}{r_{\ell,j}}} \frac{dv}{v} \right)^{\frac{s_j}{\delta}} \frac{du}{u} \right\}^{\frac{\delta}{s_j}} \right]^{\frac{1}{\delta}} . \tag{38}$$

We now use the following inequality of Hardy (valid for $0 < \beta < \infty$, $1 \le p < \infty$)

$$\left( \int_0^\infty \left( \int_0^x |f(t)|\, dt \right)^p x^{-\beta} \,\frac{dx}{x} \right)^{\frac{1}{p}} \le \frac{p}{\beta} \left( \int_0^\infty |f(t)|^p \, t^{p-\beta} \,\frac{dt}{t} \right)^{\frac{1}{p}}$$

with $\beta = s_j(\frac{1}{r_{\ell,j}} - \frac{1}{p_j}) > 0$ and $p = \frac{s_j}{\delta}$ since $p_j > r_{\ell,j}$ and $\delta \le s_j$. We obtain that (38) is at most

$$\begin{aligned} &\left( \frac{1}{\delta|\frac{1}{r_{\ell,j}} - \frac{1}{p_j}|} \right)^{\frac{1}{\delta}} \left|\frac{\gamma_0}{\gamma_j}\right|^{\frac{1}{s_j}} \lambda_j^{\frac{1}{r_{\ell,j}}-\frac{1}{p_j}} \left( \int_0^\infty ((f^*_j(v))^\delta v^{\frac{\delta}{r_{\ell,j}}} v^{-1})^{\frac{s_j}{\delta}} v^{\frac{s_j}{\delta} - s_j(\frac{1}{r_{\ell,j}}-\frac{1}{p_j})} \,\frac{dv}{v} \right)^{\frac{1}{s_j}} \\ &= \left( \frac{1}{\delta|\frac{1}{r_{\ell,j}} - \frac{1}{p_j}|} \right)^{\frac{1}{\delta}} \left|\frac{\gamma_0}{\gamma_j}\right|^{\frac{1}{s_j}} \lambda_j^{\frac{1}{r_{\ell,j}}-\frac{1}{p_j}} \|f_j\|_{L^{p_j,s_j}}\,. \end{aligned}$$

We now prove (31). We begin with

$$\Big\| t^{\frac{\gamma_j}{\gamma_0}(\frac{1}{r_{\ell,j}}-\frac{1}{p_j})} \|f_{j,1,t}\|_{L^{r_{\ell,j},\delta}} \Big\|_{L^{s_j}(\frac{dt}{t})} = \Bigg[ \int_0^\infty t^{s_j \frac{\gamma_j}{\gamma_0}(\frac{1}{r_{\ell,j}}-\frac{1}{p_j})} \Bigg( \int_0^{\lambda_j t^{-\frac{\gamma_j}{\gamma_0}}} (f_{j,1,t}^*(v))^\delta v^{\frac{\delta}{r_{\ell,j}}} \frac{dv}{v} + \int_{\lambda_j t^{-\frac{\gamma_j}{\gamma_0}}}^\infty (f_{j,1,t}^*(v))^\delta v^{\frac{\delta}{r_{\ell,j}}} \frac{dv}{v} \Bigg)^{\frac{s_j}{\delta}} \frac{dt}{t} \Bigg]^{\frac{1}{s_j}} .$$

In both integrals we first use (37) and then perform a change of variables $u = \lambda_j t^{-\frac{\gamma_j}{\gamma_0}}$ to estimate the preceding expression by

$$\Big|\frac{\gamma_0}{\gamma_j}\Big|^{\frac{1}{s_j}} \lambda_j^{\frac{1}{r_{\ell,j}}-\frac{1}{p_j}} \times \Bigg[ \int_0^\infty u^{-s_j(\frac{1}{r_{\ell,j}}-\frac{1}{p_j})} \Big\{ (f_j^*(u))^\delta \int_0^u v^{\frac{\delta}{r_{\ell,j}}} \frac{dv}{v} + \int_u^\infty (f_j^*(v))^\delta v^{\frac{\delta}{r_{\ell,j}}} \frac{dv}{v} \Big\}^{\frac{s_j}{\delta}} \frac{du}{u} \Bigg]^{\frac{\delta}{s_j}\frac{1}{\delta}} ,$$

which by Minkowski's inequality is at most

$$\Big|\frac{\gamma_0}{\gamma_j}\Big|^{\frac{1}{s_j}} \lambda_j^{\frac{1}{r_{\ell,j}}-\frac{1}{p_j}} \Bigg[ \Bigg\{ \int_0^\infty u^{-s_j(\frac{1}{r_{\ell,j}}-\frac{1}{p_j})} \Big( (f_j^*(u))^\delta \int_0^u v^{\frac{\delta}{r_{\ell,j}}} \frac{dv}{v} \Big)^{\frac{s_j}{\delta}} \frac{du}{u} \Bigg\}^{\frac{\delta}{s_j}} + \Bigg\{ \int_0^\infty u^{-s_j(\frac{1}{r_{\ell,j}}-\frac{1}{p_j})} \Big( \int_u^\infty (f_j^*(v))^\delta v^{\frac{\delta}{r_{\ell,j}}} \frac{dv}{v} \Big)^{\frac{s_j}{\delta}} \frac{du}{u} \Bigg\}^{\frac{\delta}{s_j}} \Bigg]^{\frac{1}{\delta}} . \tag{39}$$

The first term of the sum is easily evaluated while for the second of the sum we use the folllowing inequality of Hardy (valid for $0 < \beta < \infty$, $1 \le p < \infty$)

$$\Big( \int_0^\infty \Big( \int_x^\infty |f(t)|\, dt \Big)^p x^\beta \frac{dx}{x} \Big)^{\frac{1}{p}} \le \frac{p}{\beta} \Big( \int_0^\infty |f(t)|^p\, t^{p+\beta} \frac{dt}{t} \Big)^{\frac{1}{p}} ,$$

with $\beta = -(\frac{1}{r_{\ell,j}} - \frac{1}{p_j})s_j > 0$ and $p = \frac{s_j}{\delta}$ since $p_j < r_{\ell,j}$ and $\delta \le s_j$. Then (39) can be estimated by

$$\Big|\frac{\gamma_0}{\gamma_j}\Big|^{\frac{1}{s_j}} \lambda_j^{\frac{1}{r_{\ell,j}}-\frac{1}{p_j}} \Bigg[ \frac{1}{\frac{\delta}{r_{\ell,j}}} \Big\{ \int_0^\infty u^{\frac{s_j}{p_j}} (f_j^*(u))^{s_j} \frac{du}{u} \Big\}^{\frac{\delta}{s_j}} + \frac{1}{\delta(\frac{1}{p_j} - \frac{1}{r_{\ell,j}})} \Bigg\{ \int_0^\infty \Big( (f_j^*(v))^\delta v^{\frac{\delta}{r_{\ell,j}}} v^{-1} \Big)^{\frac{s_j}{\delta}} v^{\frac{s_j}{\delta}+(\frac{1}{p_j}-\frac{1}{r_{\ell,j}})s_j} \frac{dv}{v} \Bigg\}^{\frac{\delta}{s_j}} \Bigg]^{\frac{1}{\delta}}$$

$$= \left|\frac{\gamma_0}{\gamma_j}\right|^{\frac{1}{s_j}} \lambda_j^{\frac{1}{r_{\ell,j}}-\frac{1}{p_j}} \left[\frac{1}{\frac{\delta}{r_{\ell,j}}}\|f\|_{L^{p_j,s_j}}^{\delta} + \frac{1}{\delta(\frac{1}{p_j}-\frac{1}{r_{\ell,j}})}\|f\|_{L^{p_j,s_j}}^{\delta}\right]^{\frac{1}{\delta}}$$

$$= \left|\frac{\gamma_0}{\gamma_j}\right|^{\frac{1}{s_j}} \lambda_j^{\frac{1}{r_{\ell,j}}-\frac{1}{p_j}} \left[\frac{\frac{r_{\ell,j}}{p_j}}{\delta|\frac{1}{p_j}-\frac{1}{r_{\ell,j}}|}\right]^{\frac{1}{\delta}} \|f\|_{L^{p_j,s_j}} ,$$

which proves (31).

We now consider the case $s_j = \infty$. If $p_j > r_{\ell,j}$, then we change variables $u = \lambda_j t^{-\frac{\gamma_j}{\gamma_0}}$ and use (36) to obtain that for all $t > 0$,

$$\begin{aligned} t^{\frac{\gamma_j}{\gamma_0}(\frac{1}{r_{\ell,j}}-\frac{1}{p_j})}\|f_{j,-1,t}\|_{L^{r_{\ell,j},\delta}} &\le \lambda_j^{\frac{1}{r_{\ell,j}}-\frac{1}{p_j}} u^{-(\frac{1}{r_{\ell,j}}-\frac{1}{p_j})} \left(\int_0^u (f_j^*(v))^{\delta} v^{\frac{\delta}{r_{\ell,j}}}\,\frac{dv}{v}\right)^{\frac{1}{\delta}} \\ &\le \lambda_j^{\frac{1}{r_{\ell,j}}-\frac{1}{p_j}} u^{-(\frac{1}{r_{\ell,j}}-\frac{1}{p_j})} \left(\int_0^u v^{\frac{\delta}{r_{\ell,j}}-\frac{\delta}{p_j}}\frac{dv}{v}\right)^{\frac{1}{\delta}} \|f_j\|_{L^{p_j,\infty}} \\ &= \left(\frac{1}{\delta|\frac{1}{r_{\ell,j}}-\frac{1}{p_j}|}\right)^{\frac{1}{\delta}} \lambda_j^{\frac{1}{r_{\ell,j}}-\frac{1}{p_j}} \|f_j\|_{L^{p_j,\infty}} , \end{aligned}$$

which implies (33).

If $p_j < r_{\ell,j}$, again by the same change of variables $u = \lambda_j t^{-\frac{\gamma_j}{\gamma_0}}$ and via (37) we obtain for all $t > 0$,

$$\begin{aligned} & t^{\frac{\gamma_j}{\gamma_0}(\frac{1}{r_{\ell,j}}-\frac{1}{p_j})}\|f_{j,1,t}\|_{L^{r_{\ell,j},\delta}} \\ &\le \lambda_j^{\frac{1}{r_{\ell,j}}-\frac{1}{p_j}} u^{-(\frac{1}{r_{\ell,j}}-\frac{1}{p_j})} \left(\int_0^u (f_j^*(u))^{\delta} v^{\frac{\delta}{r_{\ell,j}}}\,\frac{dv}{v} + \int_u^\infty (f_j^*(v))^{\delta} v^{\frac{\delta}{r_{\ell,j}}}\,\frac{dv}{v}\right)^{\frac{1}{\delta}} \\ &\le \lambda_j^{\frac{1}{r_{\ell,j}}-\frac{1}{p_j}} u^{-(\frac{1}{r_{\ell,j}}-\frac{1}{p_j})} \left(\int_0^u u^{-\frac{\delta}{p_j}} v^{\frac{\delta}{r_{\ell,j}}}\,\frac{dv}{v} + \int_u^\infty v^{\frac{\delta}{r_{\ell,j}}-\frac{\delta}{p_j}}\,\frac{dv}{v}\right)^{\frac{1}{\delta}} \|f_j\|_{L^{p_j,\infty}} \\ &\le \lambda_j^{\frac{1}{r_{\ell,j}}-\frac{1}{p_j}} \left(\frac{1}{\frac{\delta}{r_{\ell,j}}} + \frac{1}{\delta(\frac{1}{p_j}-\frac{1}{r_{\ell,j}})}\right)^{\frac{1}{\delta}} \|f_j\|_{L^{p_j,\infty}} \\ &= \lambda_j^{\frac{1}{r_{\ell,j}}-\frac{1}{p_j}} \left(\frac{\frac{r_{\ell,j}}{p_j}}{\delta|\frac{1}{p_j}-\frac{1}{r_{\ell,j}}|}\right)^{\frac{1}{\delta}} \|f_j\|_{L^{p_j,\infty}} . \end{aligned}$$

This concludes the proof of Lemma 3.1. □

*Proof of Lemma 3.2.* When $j \in \Lambda'$ we have $\gamma_j = 0$ and

$$f_{j,-1,1} = f_j\,\chi_{\{|f_j|>f_j^*(\lambda_j)\}}, \qquad f_{j,1,1} = f_j\,\chi_{\{|f_j|\le f_j^*(\lambda_j)\}}$$

and

$$f^*_{j,-1,1}(v) \le \begin{cases} f^*_j(v) & \text{if } 0 < v < \lambda_j, \\ 0 & \text{if } v \ge \lambda_j, \end{cases} \tag{40}$$

and

$$f^*_{j,1,1}(v) \le \begin{cases} f^*_j(\lambda_j) & \text{if } 0 < v < \lambda_j, \\ f^*_j(v) & \text{if } v \ge \lambda_j. \end{cases} \tag{41}$$

If $p_j > r_{\ell,j}$, then by (40) we obtain

$$\begin{aligned} \|f_{j,-1,1}\|_{L^{r_{\ell,j},\delta}} &\le \left[\int_0^{\lambda_j} v^{\frac{\delta}{r_{\ell,j}}} f^*_j(v)^\delta \, \frac{dv}{v}\right]^{1/\delta} \\ &\le \left[\int_0^{\lambda_j} v^{\frac{\delta}{r_{\ell,j}} - \frac{\delta}{p_j}} \, \frac{dv}{v}\right]^{1/\delta} \|f_j\|_{L^{p_j,\infty}} = \frac{\lambda_j^{\frac{1}{r_{\ell,j}} - \frac{1}{p_j}}}{|\frac{\delta}{r_{\ell,j}} - \frac{\delta}{p_j}|^{1/\delta}} \|f_j\|_{L^{p_j,\infty}}, \end{aligned}$$

which proves (33). Now we suppose $p_j < r_{\ell,j}$ and show (34). To this end, applying (41) yields that

$$\begin{aligned} \|f_{j,1,1}\|_{L^{r_{\ell,j},\delta}} &\le \left[\int_0^{\lambda_j} v^{\frac{\delta}{r_{\ell,j}}} \lambda_j^{-\frac{\delta}{p_j}} \lambda_j^{\frac{\delta}{p_j}} f^*_j(\lambda_j)^\delta \, \frac{dv}{v} + \int_{\lambda_j}^\infty v^{\frac{\delta}{r_{\ell,j}} - \frac{\delta}{p_j}} v^{\frac{\delta}{p_j}} f^*_j(v)^\delta \, \frac{dv}{v}\right]^{1/\delta} \\ &\le \left[\frac{\lambda_j^{\frac{\delta}{r_{\ell,j}} - \frac{\delta}{p_j}}}{\frac{\delta}{r_{\ell,j}}} + \frac{\lambda_j^{\frac{\delta}{r_{\ell,j}} - \frac{\delta}{p_j}}}{\frac{\delta}{p_j} - \frac{\delta}{r_{\ell,j}}}\right]^{1/\delta} \|f_j\|_{L^{p_j,\infty}} \\ &= \left[\frac{\frac{r_{\ell,j}}{p_j}}{\delta|\frac{1}{p_j} - \frac{1}{r_{\ell,j}}|}\right]^{\frac{1}{\delta}} \lambda_j^{\frac{1}{r_{\ell,j}} - \frac{1}{p_j}} \|f_j\|_{L^{p_j,\infty}}, \end{aligned}$$

and hence (34) holds. This concludes the proof of Lemma 3.2. □

## 7. Remarks and applications

Previous proofs of Theorem 1.1 yielded a constant in (8) that was additive in the $B_k$'s, i.e., it had the form

$$\sum_{k=1}^{m+1} c_k B_k$$

for some $c_k > 0$. Obviously, a constant of the form

$$B_1^{\eta_1} B_2^{\eta_2} \dots B_{m+1}^{\eta_{m+1}}$$

is advantageous since it becomes small when only one $B_{k_0}$ is small and the other remain bounded.

Moreover, previous proofs of Theorem 1.1 did not yield a constant in (8) that was explicit in terms of the initial points $\vec{P}_k$. Our proof shows the explicit behavior

$$\mathrm{dist}(\vec{P}, \partial\mathbf{H})^{-m/\delta}$$

as $\vec{P}$ tends to the boundary of $\mathbf{H}$. This behavior was used in the study of the Calderón problem by Thiele [12]. We describe the details of the argument. Using the notation in [12] we consider the operator

$$B_\alpha(f_1,f_2)(x) = \mathrm{p.v.}\int_{\mathbf{R}} f_1(x-\alpha t)f_2(x+(1-\alpha)t)\,\frac{dt}{t}$$

where $f_1$, $f_2$ are Schwartz functions on the line, and $x, \alpha$ are real numbers.

Let us assume that $|\alpha| \le 1/2$. The proof in [6] and [7] shows that for all $\lambda > 0$,

$$\lambda^{\frac{1}{r}}|\{x\in\mathbf{R}:\ |B_\alpha(f_1,f_2)(x)|>\lambda\}| \le C\,|\alpha|^{-M}\|f_1\|_{L^{r_1}}\|f_2\|_{L^{r_2}} \tag{42}$$

when $(\frac{1}{r_1},\frac{1}{r_2})$ is a point with $\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}<\frac{3}{2}$, $1<r_1,r_2<\infty$, for some constant $M$ possibly depending on $r_1, r_2$. In particular, there is a constant $M>0$ such that (42) holds when $(\frac{1}{r_1},\frac{1}{r_2})$ is one of $(\frac{1}{2},\frac{3}{4})$, $(\frac{3}{4},\frac{1}{2})$, $(\frac{1}{4},\frac{1}{4})$. Theorem 1.1 in [12] claims that

$$\lambda|\{x\in\mathbf{R}:\ |B_\alpha(f_1,f_2)(x)|>\lambda\}| \le C\,\|f_1\|_{L^2}\|f_2\|_{L^2} \tag{43}$$

for some constant $C$ independent of $|\alpha|\le 1/2$. Interpolation between (42) and (43) yields the three bounds

$$\lambda^{\frac{1}{r_i}}|\{x\in\mathbf{R}:\ |B_\alpha(f_1,f_2)(x)|>\lambda\}| \le C'\,|\alpha|^{-M\varepsilon}\|f_1\|_{L^{r_{i,1}}}\|f_2\|_{L^{r_{i,2}}}\,,$$

where $i=1,2,3$ and

$$\Big(\frac{1}{r_{1,1}},\frac{1}{r_{1,2}},\frac{1}{r_1}\Big) = (1-\varepsilon)\Big(\frac{1}{2},\frac{1}{2},\frac{1}{1}\Big)+\varepsilon\Big(\frac{1}{2},\frac{3}{4},\frac{5}{4}\Big)$$
$$\Big(\frac{1}{r_{2,1}},\frac{1}{r_{2,2}},\frac{1}{r_2}\Big) = (1-\varepsilon)\Big(\frac{1}{2},\frac{1}{2},\frac{1}{1}\Big)+\varepsilon\Big(\frac{3}{4},\frac{1}{2},\frac{5}{4}\Big)$$
$$\Big(\frac{1}{r_{3,1}},\frac{1}{r_{3,2}},\frac{1}{r_3}\Big) = (1-\varepsilon)\Big(\frac{1}{2},\frac{1}{2},\frac{1}{1}\Big)+\varepsilon\Big(\frac{1}{4},\frac{1}{4},\frac{1}{2}\Big).$$

Choosing

$$\varepsilon = \frac{\log\big(10+\log\frac{1}{|\alpha|}\big)}{M\log\frac{1}{|\alpha|}}\,,$$

we obtain that

$$\lambda^{\frac{1}{r_i}}|\{x\in\mathbf{R}:\ |B_\alpha(f_1,f_2)(x)|>\lambda\}| \le C''\,\Big(10+\log\frac{1}{|\alpha|}\Big)\,\|f_1\|_{L^{r_{i,1}}}\|f_2\|_{L^{r_{i,2}}}\,.$$

Clearly the point $(\frac{1}{2},\frac{1}{2})$ lies in the interior of the convex hull of the points $(\frac{1}{r_{i,1}},\frac{1}{r_{i,2}})$, $i=1,2,3$. Corollary 1.1 yields the strong bound

$$\|B_\alpha(f_1,f_2)\|_{L^1} \le C'''\,\mathbf{d}^{-\frac{2}{\delta_0}}\Big(10+\log\frac{1}{|\alpha|}\Big)\,\|f_1\|_{L^2}\|f_2\|_{L^2}\,,$$

where $0 < \delta_0 < 2/5$ and $\mathbf{d}$ denotes the distance from the point $(\frac{1}{2}, \frac{1}{2})$ to the boundary of the convex hull of the points $(\frac{1}{2}, \frac{3}{4})$, $(\frac{3}{4}, \frac{1}{2})$, $(\frac{1}{4}, \frac{1}{4})$. But $\mathbf{d}$ is proportional to $\varepsilon$ and thus one obtains the estimate

$$\|B_\alpha(f_1, f_2)\|_{L^1} \le C''' \left( \frac{M \log \frac{1}{|\alpha|}}{\log\left(10 + \log \frac{1}{|\alpha|}\right)} \right)^{\frac{2}{\delta_0}} \left(10 + \log \frac{1}{|\alpha|}\right) \|f_1\|_{L^2} \|f_2\|_{L^2} .$$

This estimate is integrable on $[0, 1/2]$ and a symmetric estimate shows that the constant is also integrable on $[1/2, 1]$. These estimates allow one to obtain the $L^2$ boundedness of the first commutator of Calderón by expressing it as an average of the operators $B_\alpha$ over the interval $[0, 1]$; see [12] for the remaining details.

## References


[1] M. Christ, *On the restriction of the Fourier transform to curves: endpoint results and the degenerate case*, Trans. Amer. Math. Soc. **287** (1985), 223–238.

[2] L. Grafakos, Classical Fourier analysis, Second edition, Graduate Texts in Math., no 249, Springer, New York, 2008.

[3] L. Grafakos and N. Kalton, *Some remarks on multilinear maps and interpolation*, Math. Ann. **319** (2001), 151–180.

[4] L. Grafakos and T. Tao, *Multilinear interpolation between adjoint operators*, J. Func. Anal. **199** (2003), 379–385.

[5] S. Janson, *On interpolation of multilinear operators*, Function Spaces and Applications (Lund, 1986), pp. 290–302, Lect. Notes in Math. 1302, Springer, Berlin–New York, 1988.

[6] M. T. Lacey and C. M. Thiele, *$L^p$ bounds for the bilinear Hilbert transform, $p > 2$*, Ann. of Math. **146** (1997), 693–724.

[7] M. Lacey and C. Thiele, *On Calderón's conjecture*, Ann. of Math. **149** (1999), 475–496.

[8] Y. Liang, L. Liu, D. Yang, *An off-diagonal Marcinkiewicz interpolation theorem on Lorentz spaces*, Acta Math. Sinica, **27** (2011), 1477–1488.

[9] R. Sharpley, *Interpolation of $n$ pairs and counterexamples employing indices*, J. Approx. Theory **13** (1975), 117–127.

[10] R. Sharpley, *Multilinear weak type interpolation of $mn$-tuples with applications*, Studia Math. **60** (1977), 179–194.

[11] R. Strichartz, *A multilinear version of the Marcinkiewicz interpolation theorem*, Proc. Amer. Math. Soc. **21** (1969), 441–444.

[12] C. Thiele, *A uniform estimate*, Ann. of Math. **157** (2002), 1–45.

[13] M. Zafran, *A multilinear interpolation theorem*, Studia Math. **62** (1978), 107–124.

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