Carleson measures associated with families of multilinear operators: a Corrigendum

Loukas Grafakos
Department of Mathematics
University of Missouri
Columbia, MO 65211, USA
E-mail: grafakosl@missouri.edu

Lucas Oliveira
Department of Mathematics
Universidade Federal do Rio Grande do Sul
Porto Alegre, RS 91509-900, BRA
E-mail: lucas.oliveira@ufrgs.br

Abstract

We provide a modification for part of the proof of Theorem 1.2 given in [2], pages 85–89, under the multivariable $T(1)$ cancellation condition.

In this note we fix an erroneous derivation in [2]. We don't introduce any notation here but we adhere to the notation introduced in that article.

We reexamine the pointwise estimates for $L_{t,s_1,\dots,s_m}$, defined in equation (4.21) in [2] as the kernel of the m-linear operator $\Theta_t(Q_{s_1}f_1,\dots,Q_{s_m}f_m)$. We claimed in (4.25) in [2] that when $s_1,\dots,s_m \geq t$ we have

$$|\Theta_t(Q_{s_1}f_1,\dots,Q_{s_m}f_m)| \lesssim w(s_1,\dots,s_m,t)\prod_{i=1}^{m} M(f_i)\,, \tag{0.1}$$

where

$$w(s_1,\dots,s_m,t) = \prod_{i=1}^{m} \min\left(\frac{t}{s_i},\frac{s_i}{t}\right)^{\epsilon} \tag{0.2}$$

2010 *Mathematics Subject Classification*: Primary 42B99; Secondary 42B25.

for some $\epsilon > 0$. Although (0.1) holds for some function $w(s_1, \dots, s_m, t)$, it is not valid for the specific function in (0.2); in particular it is not the case that

$$\sup_t \int_0^\infty \cdots \int_0^\infty w(s_1, \dots, s_m, t) \frac{ds_1}{s_1} \cdots \frac{ds_m}{s_m} < \infty\,,$$

which is required to complete the method of the proof in [2].

In what follows, we fix this point providing an alternative argument, which resembles the approach in [3]. Basically, we need to prove the following inequality

$$|\Theta_t \Pi_{j,s}(f_1, \dots, f_m)(x)| \lesssim w(t,s) \prod_{i \neq j} M(f_i)(x) \sum_{k=1}^n M(Q_s^{2,k} f_j)(x).$$

We first state a proposition about the Calderón reproducing formulae for tensor products that will be useful in this revision.

Proposition 0.1. *Denote by*

$$(f_1 \otimes \cdots \otimes f_m)(x_1, \dots, x_m) = f_1(x_1) \dots f_m(x_m)$$

the tensor product of m functions and let $P_s f = \varphi_s \star f$ be a convolution operator with a nice function that satisfies $P_s^2 f \to f$ when $s \to 0$ and $P_s^2 f \to 0$ when $s \to \infty$ (convergence in L^p norm or in the sense of distributions). Then the following Calderón representation formulae hold (see [1] page 199 for the case $m = 1$) for Schwartz functions f_j:

$$\begin{aligned} f_1 \otimes \cdots \otimes f_m &= \lim_{\epsilon \to 0} P_\epsilon^2 f_1 \otimes \cdots \otimes P_\epsilon^2 f_m - P_{1/\epsilon}^2 f_1 \otimes \cdots \otimes P_{1/\epsilon}^2 f_m \\ &= \lim_{\epsilon \to 0} \int_\epsilon^{1/\epsilon} s \frac{d}{ds} \left(P_s^2 f_1 \otimes \cdots \otimes P_s^2 f_m \right) \frac{ds}{s} \\ &= \int_0^\infty s \frac{d}{ds} \left(P_s^2 f_1 \otimes \cdots \otimes P_s^2 f_m \right) \frac{ds}{s} \\ &= \sum_{j=1}^m \int_0^\infty \Pi_{j,s}(f_1, \dots, f_m) \frac{ds}{s}, \end{aligned}$$

where

$$\Pi_{j,s}(f_1, \dots, f_m) = P_s^2 f_1 \otimes \cdots \otimes \left(s \frac{d}{ds} P_s^2 f_j \right) \otimes \cdots \otimes P_s^2 f_m$$

and where $s\frac{d}{ds} P_s^2$ are operators of the type Q_s^2 introduced previously in [2] (page 75), that is, are squares of Littlewood-Paley projections.

The following formula, which can be found in [1], gives a explicit expression for the derivatives of squares of Littlewood-Paley projections (where now, we adopt the notation of the article [2], and write Q_s^2 instead of P_s^2 for the Littlewood-Paley projections):

$$s\frac{d}{ds}Q_s^2 = \sum_{k=1}^{n} Q_s^{k,1}Q_s^{k,2}.$$

In the preceding expression, $Q_s^{k,1}, Q_s^{k,2}$ are operators given by multiplication on the Fourier transform with bumps supported in balls and annuli, respectfully, of size comparable to s^{-1}.

We can use all this information together and obtain the following decomposition

$$\begin{aligned}\Theta_t(f_1,\dots,f_m) &= \widetilde{\Theta_t}(f_1\otimes\cdots\otimes f_m) = \widetilde{\Theta_t}\left(\int_0^\infty s\frac{d}{ds}\left(Q_s^2 f_1\otimes\cdots\otimes Q_s^2 f_m\right)\frac{ds}{s}\right)\\ &= \widetilde{\Theta_t}\left(\sum_{j=1}^m\int_0^\infty \Pi_{j,s}(f_1,\dots,f_m)\frac{ds}{s}\right)\\ &= \sum_{j=1}^m\int_0^\infty \Theta_t\left(\Pi_{j,s}(f_1,\dots,f_m)\right)\frac{ds}{s}.\end{aligned}$$

Applying duality

$$\|S(f_1,\dots,f_m)\|_p = \sup_{\|h\|_{p',2}\le 1}\left|\int_{\mathbb{R}^n}\int_0^\infty \Theta_t(f_1,\dots,f_m)(x)h(x,t)\,\frac{dt}{t}\,dx\right|. \tag{0.3}$$

Using the above expression we obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\int_0^\infty \Theta_t(f_1,\dots,f_m)(x)h(x,t)\,\frac{dt}{t}\,dx\right|\\ &= \left|\int_{\mathbb{R}^n}\int_0^\infty\left(\sum_{j=1}^m\int_0^\infty \Theta_t\Pi_{j,s}\left(f_1,\dots,f_m\right)(x)\frac{ds}{s}\right)h(x,t)\,\frac{dt}{t}\,dx\right|\\ &= \left|\sum_{j=1}^m\int_{\mathbb{R}^n}\int_0^\infty\int_0^\infty \Theta_t\Pi_{j,s}\left(f_1,\dots,f_m\right)(x)h(x,t)\frac{dt}{t}\frac{ds}{s}\,dx\right|\\ &\le \sum_{j=1}^m\left|\int_{\mathbb{R}^n}\int_0^\infty\int_0^\infty \Theta_t\Pi_{j,s}\left(f_1,\dots,f_m\right)(x)h(x,t)\frac{dt}{t}\frac{ds}{s}\,dx\right|\\ &\le \sum_{j=1}^m\left|\int_{\mathbb{R}^n}\left(\int_0^\infty\int_0^\infty \left|\Theta_t\Pi_{j,s}\left(f_1,\dots,f_m\right)(x)\right|^2 w(t,s)^{-1}\frac{ds}{s}\frac{dt}{t}\right)^{\frac{1}{2}}\right.\\ &\qquad\left.\times\left(\int_0^\infty\int_0^\infty |h(x,t)|^2 w(t,s)\frac{ds}{s}\frac{dt}{t}\right)^{\frac{1}{2}}dx\right|,\end{aligned} \tag{0.4}$$

where

$$w(t,s) = \min\left(\frac{t}{s}, \frac{s}{t}\right)^{\epsilon}$$

for some $\epsilon > 0$. An easy calculation allows us to deduce

$$\left(\int_{\mathbb{R}^n}\left(\int_0^\infty\int_0^\infty |h(x,t)|^2 w(t,s)\frac{ds}{s}\frac{dt}{t}\right)^{\frac{p'}{2}} dx\right)^{\frac{1}{p}'}$$

$$\lesssim \left(\int_{\mathbb{R}^n}\left(\int_0^\infty |h(x,t)|^2 \frac{dt}{t}\right)^{\frac{p'}{2}} dx\right)^{\frac{1}{p'}} = \|h\|_{p',2}\,.$$

Proceeding exactly as in the case of one-variable cancellation, we reduce the problem to showing that the following inequality is true

$$|\Theta_t \Pi_{j,s}(f_1,\dots,f_m)(x)| \lesssim w(t,s)\prod_{i\neq j} M(f_i)(x)\sum_{k=1}^{n} M(Q_s^{2,k} f_j)(x). \tag{0.5}$$

The validity of this inequality now follows using the same kind of idea applied to the one-variable case.

References

[1] L. Grafakos, *Modern Fourier Analysis*, Springer, Graduate Texts in Math., no 250, Springer, New York, 2008.

[2] L. Grafakos and L. Oliveira, *Carleson measures associated with families of multilinear operators*, Studia Math. 211 (2012), 71–94.

[3] J. Hart, *A New proof of the Bilinear $T(1)$ theorem*, Proc. Amer. Math. Soc. (2013), to appear.