Translation-invariant bilinear operators with positive kernels

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Abstract. We study L^r (or $L^{r,\infty}$) boundedness for bilinear translation-invariant operators with nonnegative kernels acting on functions on $\mathbb{R}^n$. We prove that if such operators are bounded on some products of Lebesgue spaces, then their kernels must necessarily be integrable functions on $\mathbb{R}^{2n}$, while via a counterexample we show that the converse statement is not valid. We provide certain necessary and some sufficient conditions on nonnegative kernels yielding boundedness for the corresponding operators on products of Lebesgue spaces. We also prove that, unlike the linear case where boundedness from L^1 to L^1 and from L^1 to $L^{1,\infty}$ are equivalent properties, boundedness from $L^1 \times L^1$ to $L^{1/2}$ and from $L^1 \times L^1$ to $L^{1/2,\infty}$ may not be equivalent properties for bilinear translation-invariant operators with nonnegative kernels.

Mathematics Subject Classification (2000). Primary 42A85; Secondary 47A07.

Keywords. Bilinear operators, convolution, positive kernels.

1. Introduction

For a nonnegative regular Borel measure μ on $\mathbb{R}^n \times \mathbb{R}^n$, we define the bilinear convolution operator:

$$T_\mu(f,g)(x) = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x-y)g(x-z)\, d\mu(y,z), \tag{1}$$

where $x \in \mathbb{R}^n$ and f, g are nonnegative functions on $\mathbb{R}^n$. If $d\mu(y,z) = K(y,z)\, dydz$, for some nonnegative function K, then we denote

$$T_K(f,g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-y)g(x-z)\, K(y,z)\, dydz,$$

The first author was supported by the NSF under grant DMS0400387.
The second author was partially supported by grants MTM2007-60500 and 2005SGR00556.

assuming no confusion occurs in the notation. We are interested in studying boundedness properties of these operators on different products of $L^p(\mathbb{R}^n)$ spaces and on more general rearrangement-invariant quasi-Banach function spaces.

We discuss necessary conditions for boundedness in terms of the range of the Lebesgue indices and of the kernels of such operators. A sufficient condition for boundedness is obtained in a particular case, see Theorem 3.2. Theorem 4.3 provides a characterization, in terms of the Lorentz space $L^{1/2n,1/2}(\mathbb{R}_+)$, of the boundedness of T_K from $L^1 \times L^1$ to $L^{1/2}$, if $K(y,z) = \varphi(|y|+|z|)$ and φ is decreasing.

The study of bilinear operators within the context of harmonic analysis was initiated by Coifman and Meyer [2, 3] in the late seventies but recent attention in the subject was rekindled by the breakthrough work of Lacey and Thiele [9, 10] on the bilinear Hilbert transform. The behavior of this operator is still not understood on spaces near $L^1 \times L^1$. Although the results obtained in this paper are not applicable to the bilinear Hilbert transform, they suggest that bilinear translation-invariant operators may exhibit behavior at the endpoint $L^1 \times L^1$ different from that of their linear counterparts on L^1 (see Theorems 3.4 and 4.1).

An interesting example of an operator of type (1) is given by the measure $\mu = \delta_0(y+z)\chi_{|y|\le 1}$ on $\mathbb{R}^n \times \mathbb{R}^n$, where δ_0 denotes the Dirac delta mass on the diagonal in $\mathbb{R}^n$. This operator (which appeared in the study of bilinear fractional integrals) can be written as

$$B(f,g)(x) = \int_{\mathbb{R}^n} f(x-y)g(x+y)\chi_{|y|\le 1}\,dy\,,$$

and maps $L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n)$ to $L^{1/2}(\mathbb{R}^n)$, as proved independently by Kenig and Stein [8] and Grafakos and Kalton [5]. The bilinear fractional integrals are also operators of the form (1) associated with the singular measures $\mu_\alpha = \delta_0(y+z)|y|^{-n+\alpha}$ on $\mathbb{R}^n \times \mathbb{R}^n$, where $0 < \alpha < n$, and they map $L^p \times L^q \to L^r$, when $1/p + 1/q = \alpha/n + 1/r$.

2. Necessary conditions

We begin by exhibiting a general restriction on a set of indices p, q, r for which an operator T_μ of the form (1) is bounded. The next result is analogous to Hörmander's [7] in the linear case; see also [6].

Proposition 2.1. *Let μ be a nonnegative regular Borel measure on $\mathbb{R}^n \times \mathbb{R}^n$. Suppose that the bilinear operator T_μ maps $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ for some $0 < p, q, r \le \infty$. Then one has $1/p + 1/q \ge 1/r$. In particular, if $T_\mu : L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, then $p \ge 1/2$.*

Proof. Fix $0 < p, q, r \le \infty$. By translating μ if necessary, we may assume that there exists a compact set $E \subset [1,M]^n \times [1,M]^n$ (for some $M > 1$), such that $0 < \mu(E) < \infty$.

Let $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$. Taking $f(x) = \prod_{j=1}^n |x_j|^{-\alpha}\chi_{(1,\infty)^n}(x)$, with $\alpha > 1/p$, and $g(x) = \prod_{j=1}^n |x_j|^{-\beta}\chi_{(1,\infty)^n}(x)$, with $\beta > 1/q$ we have, for $x_j > M+1$, $j = 1, \ldots, n$:

$$\begin{aligned} T_\mu(f,g)(x) &\geq \int_E f(x-y)g(x-z)\,d\mu(y,z) \\ &\geq \mu(E)\prod_{j=1}^n (x_j - 1)^{-(\alpha+\beta)}. \end{aligned}$$

Since $T_\mu(f,g) \in L^r(\mathbb{R}^n)$, this implies that $\alpha + \beta > 1/r$, for all $\alpha > 1/p$ and all $\beta > 1/q$; i.e., $1/p + 1/q \geq 1/r$. □

In the endpoint case $1/p + 1/q = 1/r$, we prove that the boundedness of the bilinear operator T_μ necessarily implies that the measure μ must be finite. In fact, this result is valid even under the weaker assumption that T_μ is of weak-type (p, q, r). We study this condition in detail in Section 3 where we give an example showing that, in general and contrary to what happens in the linear case, the finiteness of the measure (or the integrability of the kernel) is not a sufficient condition for the boundedness of the associated operator.

Proposition 2.2. *If μ is a nonnegative regular Borel measure and the operator $T_\mu : L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^{r,\infty}(\mathbb{R}^n)$ for some $0 < p, q \leq \infty$ satisfying $1/p + 1/q = 1/r$, then μ is a finite measure. In particular, if $K \geq 0$ and $T_K : L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^{r,\infty}(\mathbb{R}^n)$, for some $0 < p, q \leq \infty$ with $1/p + 1/q = 1/r$, then $K \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$.*

Proof. We consider first the case $0 < r < \infty$. Fix $R > 0$, such that $\mu(B_R \times B_R) > 0$, where B_R is the ball $B(0, R) \subset \mathbb{R}^n$. Then, for every $x \in B_R$ we have:

$$T_\mu(\chi_{B_{2R}}, \chi_{B_{2R}})(x) = \mu(B(x, 2R) \times B(x, 2R)) \geq \mu(B_R \times B_R) = \lambda > 0.$$

Therefore $B_R \subseteqq \big\{T_\mu(\chi_{B_{2R}}, \chi_{B_{2R}}) > \lambda/2\big\}$, and

$$\begin{aligned} |B_R| &\leq \big|\big\{T_\mu(\chi_{B_{2R}}, \chi_{B_{2R}}) > \lambda/2\big\}\big| \\ &\leq \frac{2^r C^r}{\lambda^r}|B_{2R}|^{r/p}|B_{2R}|^{r/q} \\ &= \frac{C^r 2^{r+n}}{\lambda^r}|B_R|. \end{aligned}$$

Hence, for every $R > 0$, we have that $\mu(B_R \times B_R) \leq 2^{1+n/r}C$, which proves the result when $r < \infty$ letting $R \to \infty$. When $r = \infty$ we have

$$\mu(B_R \times B_R) \leq \big\|T_\mu(\chi_{B_{2R}}, \chi_{B_{2R}})\big\|_{L^\infty} \leq C\big\|\chi_{B_{2R}}\big\|_{L^\infty}^2 = C,$$

and the conclusion follows letting $R \to \infty$ as well. □

3. Sufficient conditions

We now study certain sufficient conditions for boundedness of operators of the form (1). We start with a couple of observations:

If $K \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$, then $T_K : L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$, where $1 \le p, q \le \infty$ and $1/p + 1/q = 1/r \le 1$. In fact, this statement can be strengthened as follows:

Proposition 3.1. *If μ is a nonnegative regular Borel measure and $1/p + 1/q = 1/r \le 1$, then the following statements are equivalent:*

(a) $T_\mu : L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$.
(b) $T_\mu : L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^{r,\infty}(\mathbb{R}^n)$.
(c) μ is a finite measure.

Proof. Obviously (a) implies (b) while the fact that (b) implies (c) is proved in Proposition 2.2. Using Minkowski's integral inequality, we have:

$$\begin{aligned}
\|T_\mu(f,g)\|_r &\le \int_{\mathbb{R}^n \times \mathbb{R}^n} \|f(\cdot - y) g(\cdot - z)\|_r \, d\mu(y,z) \\
&\le \int_{\mathbb{R}^n \times \mathbb{R}^n} \|f(\cdot - y)\|_p \|g(\cdot - z)\|_q \, d\mu(y,z) \\
&= \mu(\mathbb{R}^n \times \mathbb{R}^n) \|f\|_p \|g\|_q .
\end{aligned}$$

□

It is interesting that this result is false, in general, when $0 < r < 1$. We show that there exists $K \ge 0$, $K \in L^1$ (in fact $K \in L^1 \cap L^\infty$) such that T_K does not map $L^1 \times L^1$ to $L^{1/2,\infty}$; see Theorem 3.4.

A second observation is that if a kernel K satisfies

$$|K(y,z)| \le K_1(y) K_2(z), \tag{2}$$

where $0 \le K_j \in L^1(\mathbb{R}^n)$, then $T_K : L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$, whenever $1 \le p, q \le \infty$ and $1/p + 1/q = 1/r$. In this case $r \ge 1/2$ and K lies in $L^1(\mathbb{R}^n \times \mathbb{R}^n)$, which is a necessary condition by Proposition 2.2.

We now provide a weaker sufficient condition than (2), that yields the boundedness of T_K in the nontrivial case $0 < r < 1$:

Theorem 3.2. *Suppose that $1/p + 1/q = 1/r \ge 1$ and φ is a nonnegative function on $\mathbb{R}_+ \times \mathbb{R}_+$, decreasing in each variable separately and obeying the estimate:*

$$\sum_{j_1 \in \mathbb{Z}} \sum_{j_2 \in \mathbb{Z}} (\varphi(2^{j_1}, 2^{j_2}) 2^{j_1 n} 2^{j_2 n})^r < \infty.$$

Let K be a function on $\mathbb{R}^n \times \mathbb{R}^n$ that satisfies

$$|K(y_1, y_2)| \le \varphi(|y_1|, |y_2|).$$

Then T_K maps $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$.

Proof. For each j_1, j_2 integers we set

$$K_{j_1,j_2}(y_1,y_2) = K(y_1,y_2)\chi_{I_{j_1}}(y_1)\chi_{I_{j_2}}(y_2),$$

where $I_{j_l} = \{2^{j_l} < |y_l| \leq 2^{j_l+1}\}$. Then we have

$$T(f_1,f_2)(x) \leq \sum_{j_1\in\mathbb{Z}}\sum_{j_2\in\mathbb{Z}} \varphi(2^{j_1},2^{j_2}) \prod_{l=1}^{2} \int_{I_{j_l}} |f_l(x-y_l)|\, dy_l.$$

We raise this expression to the power r and integrate over $\mathbb{R}^n$. As we can pass the power r inside the sum we obtain that

$$\begin{aligned}\int_{\mathbb{R}^n} |T(f_1,f_2)(x)|^r dx \leq{}& \sum_{j_1\in\mathbb{Z}}\sum_{j_2\in\mathbb{Z}} \varphi(2^{j_1},2^{j_2})^r \\ &\times \int_{\mathbb{R}^n} \left| \prod_{l=1}^{2} \int_{I_{j_l}} |f_l(x-y_l)|\, dy_l \right|^r dx\end{aligned}$$

and we apply Hölder's inequality to control the previous quantity by

$$\begin{aligned}& C \sum_{j_1\in\mathbb{Z}}\sum_{j_2\in\mathbb{Z}} \varphi(2^{j_1},2^{j_2})^r \left(\int_{\mathbb{R}^n} \left(\int_{I_{j_1}} |f_1(x-y_1)|\, dy_1 \right)^p dx \right)^{r/p} \\ &\qquad \times \left(\int_{\mathbb{R}^n} \left(\int_{I_{j_2}} |f_2(x-y_2)|\, dy_2 \right)^q dx \right)^{r/q} \\ &\leq C' \sum_{j_1\in\mathbb{Z}}\sum_{j_2\in\mathbb{Z}} \varphi(2^{j_1},2^{j_2})^r \left(\int_{\mathbb{R}^n} \int_{I_{j_1}} |f_1(x-y_1)|^p\, dy_1\, 2^{j_1 n(p-1)}\, dx \right)^{r/p} \\ &\qquad \times \left(\int_{\mathbb{R}^n} \int_{I_{j_2}} |f_2(x-y_2)|^q\, dy_2\, 2^{j_2 n(q-1)}\, dx \right)^{r/q} \\ &= C'' \sum_{j_1\in\mathbb{Z}}\sum_{j_2\in\mathbb{Z}} (\varphi(2^{j_1},2^{j_2}) 2^{j_1 n} 2^{j_2 n})^r (\|f_1\|_{L^p} \|f_2\|_{L^q})^r \\ &\leq C''' (\|f_1\|_{L^p} \|f_2\|_{L^q})^r.\end{aligned}$$

This proves the result. □

Remark 3.3. It is easy to see that the hypothesis on K can be equivalently written as

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|K(y_1,y_2)|^r}{(|y_1|^n |y_2|^n)^{1-r}}\, dy_1 dy_2 < \infty,$$

and in this case the monotonicity condition on φ is replaced by the condition: whenever $|y_1| \leq |y_1'|$ we have $|K(y_1,y_2)| \geq |K(y_1',y_2)|$ and whenever $|y_2| \leq |y_2'|$ we have $|K(y_1,y_2)| \geq |K(y_1,y_2')|$.

Under no extra conditions on K, and for the case $0 < r < 1$, no positive results can be obtained, as the following result indicates:

Theorem 3.4. *There exists a nonnegative function K on $\mathbb{R}^n \times \mathbb{R}^n$ such that, if X is an r.i. quasi Banach space, then $T_K : L^1 \times L^1 \to X$ if and only if L^∞ is a subspace of X.*

Proof. We work the details in the case $n = 1$, although the construction can be easily extended to $\mathbb{R}^n$ for $n \geq 2$. For $a < 0$ and $r > 0$ set

$$f_{a,r}(x) = \frac{1}{2r} \chi_{(a-r,a+r)}(x)\,.$$

Also let

$$\ell_a = \{(x-a, x) : x \in \mathbb{R}\}$$

be the line of slope 1 passing through the point $(0, a)$. Then for almost all $(x - a, x) \in \mathbb{R}^2$ we have

$$T_K(f_{a,r}, f_{0,r})(x) = \frac{1}{4r^2} \int_{(x-a-r,x-a+r)} \int_{(x-r,x+r)} K(y,z)\, dzdy$$

and from this we deduce that

$$T_K(f_{a,r}, f_{0,r})(x) \to K(x-a, x) \tag{3}$$

as $r \to 0$. In other words, (3) holds for almost every $a < 0$ and almost every point on the line ℓ_a with respect to one-dimensional Lebesgue measure.

For each $k \in \mathbb{N}$, we construct a sequence of disjoint rectangles R_k as in Figure 1 with base length equal to $1/k^3$, height equal to $2k$, and longest side parallel to the line ℓ_a. We arrange that all these rectangles touch each other and are contained in the right angle $-|x| \leq y \leq |x|$ on the (x, y) plane. We let $P(R_k)$ be the intersection of the smallest strip containing the longest side of R_k and the negative y-axis. Set

$$R = \bigcup_{k=1}^{\infty} R_k$$

and $K = \chi_R$. Then

$$\|K\|_1 = |R| = \sum_{k=1}^{\infty} 2/k^2 < \infty.$$

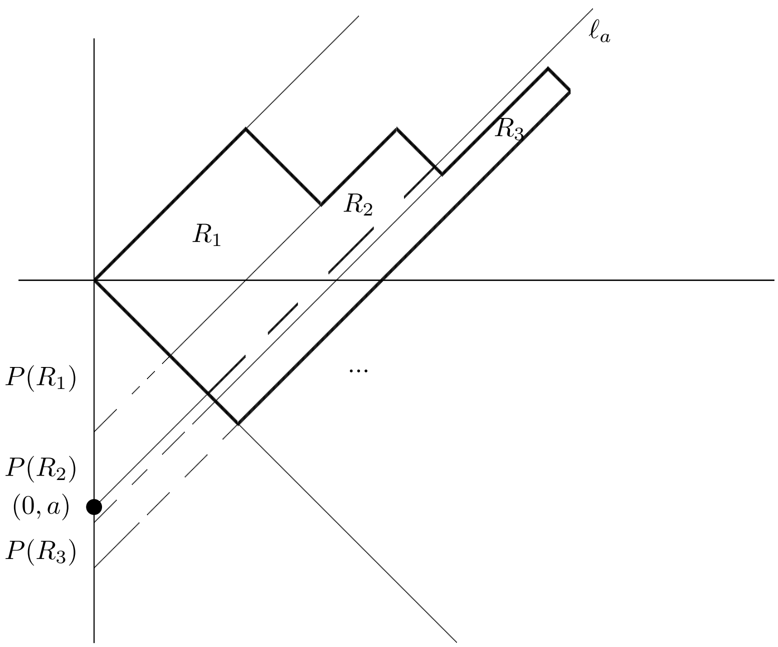

Figure 1.

Suppose that for this kernel K the following estimate holds:

$$\|T_K(f,g)\|_X \leq C\|f\|_1\|g\|_1$$

for all f,g nonnegative functions in $L^1(\mathbb{R}^n)$. Then for any $k \geq 1$, (3) holds for almost all $-\sqrt{2}\sum_{k=1}^{\infty} k^{-3} < a < 0$, with $(0,a)$ in $P(R_k)$ (in particular for one such a), and for almost all points $(x-a,x)$ in ℓ_a. Since

$$\chi_{(0,k)}(x) \leq \chi_{R_k}(x-a,x) \leq K(x-a,x),$$

for all real x, using Fatou's lemma and (3) we deduce that

$$\|\chi_{(0,k)}\|_X \leq \liminf_{r\to 0} \|T_K(f_{a,r},f_{0,r})\|_X \leq C\|f_{a,r}\|_1\|f_{0,r}\|_1 = C,$$

for every $k \in \mathbb{N}$. Thus, the fundamental function φ_X of X (see [1]) is bounded, which is equivalent to saying that L^∞ is a subspace of X.

Conversely, if L^∞ is a subspace of X and $K \in L^\infty$, then it is clear that $T_K : L^1 \times L^1 \to L^\infty$ and thus T_K maps $L^1 \times L^1$ to X. □

Remark 3.5. If $K \in L^\infty$, then we have just observed that, trivially, $T_K : L^1 \times L^1 \to L^\infty$. Therefore, if $K \in L^1 \cap L^\infty$, $1/p + 1/q \leq 1$ and $0 \leq \theta \leq 1$, using bilinear interpolation [1] for this estimate and Proposition 3.1, we obtain:

$$T_K : L^{p'/(p'-\theta)} \times L^{q'/(q'-\theta)} \to L^{pq/(\theta(p+q))}.$$

In particular, $T_K : L^p \times L^p \to L^{p/2}$ whenever for $2 \le p \le \infty$, and $T_K : L^p \times L^p \to L^{p'/2}$ whenever $1 \le p \le 2$.

Consequently, for $K \in L^1 \cap L^\infty$ such that $T_K : L^1 \times L^1 \to L^p$ for some $p \ge 1/2$ (cf. Proposition 2.1), the boundedness $T_K : L^1 \times L^1 \to L^q$ holds for every q in $[p, \infty]$. It is then an interesting question to determine the least possible value of p in the interval $[1/2, \infty]$, for which such an operator is bounded from $L^1 \times L^1$ to L^p. We have indicated that there are examples showing that we can have the best possible situation (boundedness on $L^{1/2}$ when K is a tensor product of two kernels in L^1) and also the worst case (only bounded on L^∞, as in Theorem 3.4). See also Proposition 4.2. Modifications of bilinear fractional integrals also provide examples in the intermediate cases.

4. Other examples and estimates

Well-known examples of bilinear singular integral operators, such as the bilinear Riesz transforms [6], indicate that boundedness from $L^1 \times L^1$ to $L^{1/2}$ may not hold although boundedness from $L^1 \times L^1$ to $L^{1/2,\infty}$ is valid. These operators have kernels that change sign but the next result shows that there exist positive measures that provide examples of kernels with the same property. This situation should be contrasted with its linear version that fails: if a convolution operator with a positive Borel measure on $\mathbb{R}^n$ maps $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, then the measure is finite and therefore the operator maps $L^1(\mathbb{R}^n)$ to itself!

Theorem 4.1. *There exists a nonnegative regular finite Borel measure μ on $\mathbb{R} \times \mathbb{R}$ with the property that T_μ maps $L^1 \times L^1$ to $L^{1/2,\infty}$ but does not map $L^1 \times L^1$ to $L^{1/2}$.*

Proof. We first observe that if we want $T_\mu : L^1 \times L^1 \to L^{1/2,\infty}$, then necessarily μ must be a finite measure (Proposition 2.2). We choose a positive sequence $\{\lambda_j\}_{j\in\mathbb{Z}} \in \ell^{1/2,\infty} \setminus \ell^{1/2}$ and define $\mu = \sum_j \lambda_j \delta_{a_j}$, where $a_j = (j, j)$ and δ_{a_j} is the Dirac mass at a_j. Clearly

$$\mu(\mathbb{R} \times \mathbb{R}) = \sum_j \lambda_j < \infty\,.$$

Then, $T_\mu(f, g)(x) = \sum_j \lambda_j f(x-j)g(x-j)$. Let also $D_\mu(h)(x) = \sum_j \lambda_j h(x-j)$. Using that $\{\lambda_j\}_{j\in\mathbb{Z}} \in \ell^{1/2,\infty}$ and [4, Lemma 3.5] we have that $D_\mu : L^{1/2} \to L^{1/2,\infty}$, and hence,

$$\|T_\mu(f, g)\|_{1/2,\infty} = \|D_\mu(fg)\|_{1/2,\infty} \le C\|fg\|_{1/2} \le C\|f\|_1\|g\|_1.$$

Now, since$\{\lambda_j\}_{j\in\mathbb{Z}} \notin \ell^{1/2}$, by [4, Theorem 3.1] (see also [11]), we have that D_μ is not of strong-type $L^{1/2}$ and, as before,

$$\|T_\mu\|_{L^1\times L^1\to L^{1/2}} \ge \sup_f \frac{\|T_\mu(f, f)\|_{1/2}}{\|f\|_1^2} = \sup_h \frac{\|D_\mu(h)\|_{1/2}}{\|h\|_{1/2}} = \infty.$$

□

We now consider some particular cases of kernels, defined in terms of a special function φ. The first example is $K(y,z)=\varphi(y+z)$, where $\varphi:\mathbb{R}^n\to\mathbb{R}_+$.

Proposition 4.2. *Let* $1\le\alpha\le\infty$ *and* $\varphi\in L^\alpha(\mathbb{R}^n)$ *be a positive function. Set* $K(y,z)=\varphi(y+z)$. *Then,*

$$T_K : L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)\to L^r(\mathbb{R}^n), \tag{4}$$

where

$$0\le\frac{1}{r}=\frac{1}{p}+\frac{1}{q}+\frac{1}{\alpha}-2\le 1.$$

Moreover, if $\varphi\in L^\alpha\cap L^\infty$, *then*

$$T_K : L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)\to L^r(\mathbb{R}^n), \tag{5}$$

for every $\alpha\le r\le\infty$ *and the result is false, in general, if* $r<\alpha$.

Proof. The main observation is that $T_K(f,g)(x)=(f*g*\varphi)(2x)$, and hence the result is a reformulation of Young's inequality:

$$\|T_K(f,g)\|_r\le\|f\|_p\|g*\varphi\|_\beta,$$

if $1\le p\le\beta'$ and $1/r=1/p+1/\beta-1$. Similarly,

$$\|g*\varphi\|_\beta\le\|g\|_q\|\varphi\|_\alpha,$$

if $1\le q\le\alpha'$ and $1/\beta=1/q+1/\alpha-1$, which proves (4).

If $p=q=1$ and $\varphi\in L^\infty$, then $T_K:L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)\to L^\alpha(\mathbb{R}^n)$ that, together with the estimate $T_K:L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)$, gives (5).

To finish, take $r<\alpha$, and define

$$\varphi(t)=t^{(-1-\varepsilon)/\alpha}\chi_{(1,\infty)}(t)\in L^\alpha\cap L^\infty,$$

where $0<\varepsilon<\alpha/r-1$. Set $f=g=\chi_{(0,1)}$. Then, if $x>3/2$:

$$\begin{aligned}
T_K(f,g)(x)&=\int_{x-1}^{x}\left(\int_{1-z}^{\infty}\chi_{(x-1,x)}(y)(z+y)^{(-1-\varepsilon)/\alpha}\,dy\right)dz\\
&\ge\int_{x-1}^{x}\left(\int_{1-z}^{\infty}\chi_{(x-1,x)}(y)\,dy\right)(z+x)^{(-1-\varepsilon)/\alpha}\,dz\\
&\ge\int_{x-1}^{x}(z+x)^{(-1-\varepsilon)/\alpha}\,dz\ge(2x)^{(-1-\varepsilon)/\alpha}.
\end{aligned}$$

Therefore $\|T_K(f,g)\|_r=\infty$. This proves the result if $n=1$. The n-dimensional case follows by adapting this idea. □

Another example of interest comes when the kernel is defined as $K(y,z)=\varphi(|y|+|z|)$, where $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is a decreasing function. We will study the behaviour of T_K at the endpoints $p=q=1$ and $r=1/2$, for which we give a complete characterization in terms of the Lorentz space $L^{1/2n,1/2}(\mathbb{R}_+)$:

Theorem 4.3. *Let* $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ *be a decreasing function and define* $K(y,z)=\varphi(|y|+|z|)$. *Then,* $T_K:L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)\to L^{1/2}(\mathbb{R}^n)$ *if and only if* $\varphi\in L^{1/2n,1/2}(\mathbb{R}_+)$.

Proof. Assume that $T_K : L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n) \to L^{1/2}(\mathbb{R}^n)$. Set

$$R_k = \{(y,z) \in \mathbb{R}^n \times \mathbb{R}^n : 2^k < |y| + |z| \le 2^{k+1}\},$$

so that $|R_k| \approx 2^{2kn}$. Fix $2^{j-1} < |x| \le 2^j$ and $\delta \le 2^l$, with $l \le j-2$. Consider also the functions $f(x) = g(x) = \chi_{\{|x|<\delta\}}(x)$. Then

$$\int_{\mathbb{R}^n} (T_K(f,g)(x))^{1/2}\, dx \le C\delta^n. \tag{6}$$

Observe that

$$\{y \in \mathbb{R}^n : |x-y| < \delta\} \times \{z \in \mathbb{R}^n : |x-z| < \delta\} \subset R_{j-1} \cup R_j \cup R_{j+1}, \tag{7}$$

since $|y| + |z| \le 2\delta + 2|x| \le 2^{j+2}$ and $|y| + |z| \ge 2|x| - 2\delta \ge 2^{j-1}$.

Discretizing the operator, and using (7), we obtain:

$$\begin{aligned} T_K(f,g)(x) &= \sum_{k\in\mathbb{Z}} \iint_{R_k} f(x-y)g(x-z)\varphi(|y|+|z|)\, dydz \\ &\ge \varphi(2^{j+2}) \sum_{k=j-1}^{j+1} \iint_{R_k} f(x-y)g(x-z)\, dydz \\ &= C_n \varphi(2^{j+2})\delta^{2n} \\ &\ge C_n \varphi(8|x|)\delta^{2n}. \end{aligned}$$

Thus, by (6) and the previous estimate:

$$C\delta^n \ge C \int_{\{|x|>2\delta\}} (T_K(f,g)(x))^{1/2}\, dx \ge C'\delta^n \int_{\{|x|>2\delta\}} \sqrt{\varphi(8|x|)}\, dx,$$

and hence,

$$\int_{16\delta}^{\infty} \sqrt{\varphi(t)}\, t^n \frac{dt}{t} \le C''.$$

Letting $\delta \to 0$ we finally obtain:

$$\|\varphi\|_{1/2n,1/2}^{1/2} = \int_0^\infty \sqrt{\varphi(t)}\, t^n \frac{dt}{t} < \infty.$$

Conversely, since $\varphi(|y|+|z|) \le \varphi(|y|)$ and $\varphi(|y|+|z|) \le \varphi(|z|)$, we have

$$\varphi(|y|+|z|) \le \sqrt{\varphi(|y|)}\sqrt{\varphi(|z|)}\,,$$

and therefore K is bounded from above by the tensor product of two functions in $L^1(\mathbb{R}^n)$, since

$$\int_{\mathbb{R}^n} \sqrt{\varphi(|y|)}\, dy = C\|\varphi\|_{1/2n,1/2}^{1/2} < \infty,$$

which implies the result (see (2)). $\square$

Remark 4.4. By Proposition 2.2 we know that the boundedness of T_K in the previous theorem would imply that $K \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$. This condition is, in fact, equivalent to $\varphi \in L^{1/2n,1}(\mathbb{R}_+)$:

$$\begin{aligned}
\|K\|_1 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi(|y|+|z|)\,dydz \\
&= C\int_0^\infty \left(\int_0^\infty \varphi(s+t)t^{n-1}\,dt\right)s^{n-1}\,ds \\
&= C\int_0^\infty \left(\int_s^\infty \varphi(u)(u-s)^{n-1}\,du\right)s^{n-1}\,ds \\
&= C\int_0^\infty \varphi(u)\left(\int_0^u (u-s)^{n-1}s^{n-1}\,ds\right)du \\
&\approx \int_0^\infty \varphi(u)u^{2n}\frac{du}{u} \\
&= \|\varphi\|_{1/2n,1}.
\end{aligned}$$

Since $L^{1/2n,1/2}(\mathbb{R}_+) \subsetneqq L^{1/2n,1}(\mathbb{R}_+)$, we observe that Theorem 4.3 gives a stronger condition.

We end by giving an analogous version of Proposition 3.1 in the case of linear convolution operators that, surprisingly enough, seems to be missing from the literature.

For $K \geq 0$, we define the averaging operator:

$$A(K)(x,r) = \frac{1}{|B(x,r)|}\int_{B(x,r)} K(y)\,dy.$$

We observe that $\|A(K)(x,\cdot)\|_{L^\infty_r} = M(K)(x)$, where M is the Hardy-Littlewood maximal function. We use the following notation for the mixed norm space $X[Y]$: $\|F\|_{X[Y]}$ denotes the quasinorm in X of the function $\|F(x,\cdot)\|_Y$. We consider first the case $p=1$:

Proposition 4.5. *Let $K \geq 0$, and*

$$T_K(f)(x) = \int_{\mathbb{R}^n} f(x-y)K(y)\,dy.$$

Then, the following statements are equivalent:

(a) $A(K) \in L^{1,\infty}_x[L^\infty_r]$.
(b) $A(K) \in L^\infty_r[L^{1,\infty}_x]$.
(c) $K \in L^1$.
(d) $T_K : L^1 \to L^{1,\infty}$.
(e) $T_K : L^1 \to L^1$.

Moreover, $\|A(K)\|_{L^\infty_r[L^{1,\infty}_x]} \approx \|A(K)\|_{L^{1,\infty}_x[L^\infty_r]} \approx \|K\|_1$.

Proof. It is well known that

$$\|A(K)\|_{L^\infty_r[L^{1,\infty}_x]} \leq \|A(K)\|_{L^{1,\infty}_x[L^\infty_r]} \approx \|K\|_1,$$

i.e., (a) $\Leftrightarrow$ (c) $\Rightarrow$ (b). Taking $r > 0$ such that $\int_{B(0,r/2)} K(y)\,dy > 0$ implies

$$r^n \leq C\left|\left\{T_K(\chi_{B(0,r)}) \geq \int_{B(0,r/2)} K\right\}\right| \leq \frac{C}{\int_{B(0,r/2)} K}\|A(K)(\cdot,r)\|_{1,\infty} r^n,$$

hence (b) $\Rightarrow$ (c). Clearly (c) $\Rightarrow$ (e) $\Rightarrow$ (d). Finally, if (d) holds, taking $f = \chi_{B(0,r)}$, we obtain that $T_K(f)(x) = Cr^n A(K)(x,r)$, thus

$$\|T_K(f)\|_{1,\infty} = Cr^n\|A(K)(\cdot,r)\|_{1,\infty} \leq Cr^n.$$

Therefore (d) $\Rightarrow$ (b). □

Remark 4.6. (i) It is easy to see that if $1 < p < \infty$, then we also have that:

$$K \in L^1 \Leftrightarrow T_K : L^p \to L^{p,\infty} \Leftrightarrow T_K : L^p \to L^p.$$

This should be compared to the bilinear case (cf. Theorem 4.1), where weak-type estimates do not imply, in general, the strong-type boundedness of the operator.

(ii) For an r.i. BFS X for which the maximal operator M maps X to itself (e.g., $X = L^p$, $1 < p \leq \infty$), the equivalences:

$$\|A(K)\|_{X_x[L^\infty_r]} \approx \|A(K)\|_{L^\infty_r[X_x]} \approx \|K\|_X,$$

are easy consequences of Fatou's Lemma.

Remark 4.7. The results of this article concerning positive bilinear operators easily adapt to the setting of m-linear positive convolution operators when $m \geq 3$. The precise formulation of these statements and their proofs are analogous to the case $m = 2$ and are omitted.

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