

A SHARP VERSION OF THE HÖRMANDER MULTIPLIER THEOREM

LOUKAS GRAFAKOS AND LENKA SLAVÍKOVÁ

ABSTRACT. We provide an improvement of the Hörmander multiplier theorem in which the Sobolev space $L_s^r(\mathbb{R}^n)$ with integrability index r and smoothness index $s > n/r$ is replaced by the Sobolev space with smoothness s built upon the Lorentz space $L^{n/s,1}(\mathbb{R}^n)$.

1. INTRODUCTION

Given a bounded function σ on \mathbb{R}^n , we define a linear operator

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)\sigma(\xi)e^{2\pi i x \cdot \xi} d\xi$$

acting on Schwartz functions f on \mathbb{R}^n ; here $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x \cdot \xi} dx$ is the Fourier transform of f . An old problem in harmonic analysis is to find optimal sufficient conditions on σ to be an L^p Fourier multiplier, i.e., for the operator T_σ to admit a bounded extension from $L^p(\mathbb{R}^n)$ to itself for a given $p \in (1, \infty)$.

Mikhlin's [13] classical multiplier theorem states that if the condition

$$(1.1) \quad |\partial^\alpha \sigma(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}, \quad \xi \neq 0,$$

holds for all multi-indices α with size $|\alpha| \leq [n/2]+1$, then T_σ admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself for all $1 < p < \infty$. This theorem is well suited for dealing with multipliers whose derivatives have a singularity at one point, such as functions which are homogeneous of degree zero and indefinitely differentiable on the unit sphere.

An extension of the Mikhlin theorem was obtained by Hörmander [12]. It asserts the following: for $s > 0$ let $(I - \Delta)^{s/2}$ denote the operator given on the Fourier transform by multiplication by $(1 + 4\pi^2|\xi|^2)^{s/2}$ and let Ψ be a Schwartz function whose

2010 *Mathematics Subject Classification.* Primary 42B15. Secondary 42B25.

Key words and phrases. Multiplier theorems, Sobolev spaces, Lorentz spaces.

The first author acknowledges the support of the Simons Foundation and of the University of Missouri Research Board.

Fourier transform is supported in the annulus $\{\xi : 1/2 < |\xi| < 2\}$ and which satisfies $\sum_{j \in \mathbb{Z}} \widehat{\Psi}(2^{-j}\xi) = 1$ for all $\xi \neq 0$. If for some $1 \leq r \leq 2$ and $s > n/r$, σ satisfies

$$(1.2) \quad \sup_{k \in \mathbb{Z}} \|(I - \Delta)^{s/2} [\widehat{\Psi}\sigma(2^k \cdot)]\|_{L^r(\mathbb{R}^n)} < \infty,$$

then T_σ admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself for all $1 < p < \infty$.

It is natural to ask whether condition (1.2) can still guarantee that σ is an L^p Fourier multiplier for some $p \in (1, \infty)$ if $s \leq \frac{n}{2}$. Via an interpolation argument, Calderón and Torchinsky [2, Theorem 4.6] showed that T_σ is bounded from $L^p(\mathbb{R}^n)$ to itself whenever condition (1.2) holds with p satisfying $|\frac{1}{p} - \frac{1}{2}| < \frac{s}{n}$ and $|\frac{1}{p} - \frac{1}{2}| = \frac{1}{r}$. It was observed in [9] that the assumption $|\frac{1}{p} - \frac{1}{2}| = \frac{1}{r}$ can be replaced by a weaker one, namely, by $\frac{1}{r} < \frac{s}{n}$. Moreover, it is known that if T_σ is bounded from $L^p(\mathbb{R}^n)$ to itself for every σ satisfying (1.2), then $|\frac{1}{p} - \frac{1}{2}| \leq \frac{s}{n}$, see Hirschman [11], Wainger [21], Miyachi [14], Miyachi and Tomita [15], Grafakos, He, Honzík, and Nguyen [9]. In other words, when $rs > n$, then the condition $|\frac{1}{p} - \frac{1}{2}| < \frac{s}{n}$ is essentially optimal for assumption (1.2). Observe also that the condition $rs > n$ is dictated by the embedding of $L_s^r(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$. It is still unknown to us if L^p boundedness holds on the line $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$. Positive endpoint results on L^p and on H^1 involving Besov spaces can be found in Seeger [16], [17], [18].

Unlike the Mihlin multiplier theorem, the Hörmander and Calderón-Torchinsky theorems can treat multipliers whose derivatives have infinitely many singularities, such as the multiplier

$$(1.3) \quad \sigma(x) = \sum_{k \in \mathbb{Z}} \phi(2^{-k}x) |2^{-k}x - a_k|^\beta,$$

where $\beta > 0$, ϕ is a smooth function supported in the set $\{x \in \mathbb{R}^n : \frac{1}{2} < |x| < 2\}$ and, for every $k \in \mathbb{N}$, $a_k \in \mathbb{R}^n$ belongs to the same set.

In this paper, we improve the result of [2, Theorem 4.6] by replacing the Lebesgue space $L^r(\mathbb{R}^n)$, $r > \frac{n}{s}$, in condition (1.2) by the locally larger Lorentz space $L_{s,1}^{\frac{n}{s}}(\mathbb{R}^n)$, defined in terms of the norm

$$\|f\|_{L_{s,1}^{\frac{n}{s}}(\mathbb{R}^n)} = \int_0^\infty f^*(r) r^{\frac{s}{n}-1} dr.$$

Here, f^* stands for the nonincreasing rearrangement of the function f , namely, for the unique nonincreasing left-continuous function on $(0, \infty)$ equimeasurable with f , given by the explicit expression

$$f^*(t) = \inf \{r \geq 0 : |\{y \in \mathbb{R}^n : |f(y)| > r\}| < t\}.$$

We point out that the Lorentz space $L^{\frac{n}{s},1}(\mathbb{R}^n)$ appears naturally in this context, since it is known to be, at least for integer values of s , locally the largest rearrangement-invariant function space such that membership of $(I - \Delta)^{\frac{s}{2}} f$ to this space forces f to be bounded, see [20, 3].

Theorem 1.1. *Let Ψ be a Schwartz function on \mathbb{R}^n whose Fourier transform is supported in the annulus $1/2 < |\xi| < 2$ and satisfies $\sum_{j \in \mathbb{Z}} \widehat{\Psi}(2^{-j}\xi) = 1$, $\xi \neq 0$. Let $p \in (1, \infty)$, $n \in \mathbb{N}$, and let $s \in (0, n)$ satisfy*

$$\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{s}{n}.$$

Then for all functions f in the Schwartz class of \mathbb{R}^n we have the a priori estimate

$$(1.4) \quad \|T_\sigma f\|_{L^p(\mathbb{R}^n)} \leq C \sup_{j \in \mathbb{Z}} \left\| (I - \Delta)^{\frac{s}{2}} [\widehat{\Psi}\sigma(2^j \cdot)] \right\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.$$

As an application of Theorem 1.1 we show that the function σ from (1.3) continues to be an L^p Fourier multiplier for any $p \in (1, \infty)$ if $|2^{-k}x - a_k|$ is replaced by $\log \frac{e4^n}{|2^{-k}x - a_k|^n}$. In fact, we can even allow an arbitrary iteration of logarithms in this example.

Example 1.2. *Assume that $n \in \mathbb{N}$, $n \geq 2$, and $\beta < 0$. Let ϕ be a smooth function supported in the set $A = \{x \in \mathbb{R}^n : 1/2 < |x| < 2\}$ and let $a_k \in A$, $k \in \mathbb{Z}$. Then the function*

$$(1.5) \quad \sigma(x) = \sum_{k \in \mathbb{Z}} \phi(2^{-k}x) \left(\log \frac{e4^n}{|2^{-k}x - a_k|^n} \right)^\beta$$

is an L^p Fourier multiplier for any $p \in (1, \infty)$.

To verify the statement of Example 1.2, we fix a positive integer s and observe that for any $j \in \mathbb{Z}$,

$$\begin{aligned} \|(I - \Delta)^{\frac{s}{2}} [\widehat{\Psi}\sigma(2^j \cdot)]\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)} &\leq \left\| (I - \Delta)^{\frac{s}{2}} \left[\widehat{\Psi}(x)\phi(x) \left(\log \frac{e4^n}{|x - a_j|^n} \right)^\beta \right] \right\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)} \\ &+ \left\| (I - \Delta)^{\frac{s}{2}} \left[\widehat{\Psi}(x)\phi(2x) \left(\log \frac{e4^n}{|2x - a_{j-1}|^n} \right)^\beta \right] \right\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)} \\ &+ \left\| (I - \Delta)^{\frac{s}{2}} \left[\widehat{\Psi}(x)\phi\left(\frac{x}{2}\right) \left(\log \frac{e4^n}{|\frac{x}{2} - a_{j+1}|^n} \right)^\beta \right] \right\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)}. \end{aligned}$$

In what follows, let us deal with the first term only, since the last two terms can be estimated in a similar way.

Fix $j \in \mathbb{Z}$ and denote

$$f_j(x) = \widehat{\Psi}(x)\phi(x) \left(\log \frac{e4^n}{|x - a_j|^n} \right)^\beta.$$

Also, for any multiindex α satisfying $|\alpha| \geq 1$, let $\frac{\partial^\alpha}{\partial x^\alpha} f_j$ stand for the weak derivative of f_j with respect to α . We have

$$\left| \frac{\partial^\alpha}{\partial x^\alpha} f_j(x) \right| \leq C\chi_A(x) \left(\log \frac{e4^n}{|x - a_j|^n} \right)^{\beta-1} |x - a_j|^{-|\alpha|}.$$

Since $|A| \leq 2^n \omega_n$, where ω_n stands for the volume of the unit ball in \mathbb{R}^n , the previous estimate implies

$$\left(\frac{\partial^\alpha}{\partial x^\alpha} f_j \right)^* (t) \leq C\chi_{(0, 2^n \omega_n)}(t) \left(\log \frac{e4^n \omega_n}{t} \right)^{\beta-1} t^{-\frac{|\alpha|}{n}},$$

where the constant C is independent of j . Therefore, if s is a positive integer and α is a multiindex with $1 \leq |\alpha| \leq s$, then

$$\left(\frac{\partial^\alpha}{\partial x^\alpha} f_j \right)^* (t) \leq C\chi_{(0, 2^n \omega_n)}(t) \left(\log \frac{e4^n \omega_n}{t} \right)^{\beta-1} t^{-\frac{s}{n}}.$$

Consequently,

$$(1.6) \quad \sup_{1 \leq |\alpha| \leq s} \left\| \frac{\partial^\alpha}{\partial x^\alpha} f_j \right\|_{L^{\frac{n}{s}, 1}(\mathbb{R}^n)} \leq C \int_0^{2^n \omega_n} \left(\log \frac{e4^n \omega_n}{t} \right)^{\beta-1} t^{-1} dt < \infty.$$

Since each $|f_j|$ is bounded by a constant independent of j and compactly supported in the set A , we also have

$$\|f_j\|_{L^{\frac{n}{s}}(\mathbb{R}^n)} \leq C < \infty.$$

It remains to observe that the quantity $\|(I - \Delta)^{\frac{s}{2}} f_j\|_{L^{\frac{n}{s}, 1}(\mathbb{R}^n)}$ is equivalent to

$$\sum_{|\alpha| \leq s} \left\| \frac{\partial^\alpha}{\partial x^\alpha} f_j \right\|_{L^{\frac{n}{s}, 1}(\mathbb{R}^n)}.$$

This can be proved in exactly the same way as the corresponding result for the Lebesgue spaces, see, e.g., [19, Theorem 3, Chapter 5]. Therefore, we deduce that

$$\sup_{j \in \mathbb{Z}} \|(I - \Delta)^{\frac{s}{2}} [\widehat{\Psi}\sigma(2^j \cdot)]\|_{L^{\frac{n}{s}, 1}(\mathbb{R}^n)} < \infty$$

for any positive integer s . Theorem 1.1 now yields that σ is an L^p Fourier multiplier for any $p \in (1, \infty)$.

Finally, notice that we can in fact replace the logarithm in (1.5) by any iteration of logarithms, namely, we can consider the more general symbol

$$\sigma(x) = \sum_{k \in \mathbb{Z}} \phi(2^{-k}x) \left(\underbrace{\log \cdots \log}_{\ell\text{-times}} \frac{4^n \underbrace{e^{\cdots e}}_{\ell\text{-times}}}{|2^{-k}x - a_k|^n} \right)^\beta,$$

where ℓ is any positive integer. A computation similar to the one we performed above shows that σ is an L^p Fourier multiplier for any $p \in (1, \infty)$ as well.

2. THE MAIN ESTIMATE

In this section we show that inequality (1.4) holds for any $p \in (1, \infty)$ provided that $s \in (n/2, n)$, see Theorem 2.2 below. This estimate will serve as one endpoint in the interpolation argument leading to the proof of Theorem 1.1. The interpolation is the content of the next section.

Let us start by recalling the definitions of two types of Lorentz spaces that will be used in the sequel. Suppose that $1 < p < \infty$. Then, for any measurable function f on \mathbb{R}^n , we define

$$\|f\|_{L^{p,1}(\mathbb{R}^n)} = \int_0^\infty f^*(t) t^{\frac{1}{p}-1} dt$$

and

$$\|f\|_{L^{p,\infty}(\mathbb{R}^n)} = \sup_{t>0} f^*(t) t^{\frac{1}{p}}.$$

It can be shown that

$$\|f\|_{L^{p,1}(\mathbb{R}^n)} = p \int_0^\infty |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|^{\frac{1}{p}} d\lambda$$

and

$$\|f\|_{L^{p,\infty}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|^{\frac{1}{p}}.$$

The space $L^{p',\infty}(\mathbb{R}^n)$, where $p' = \frac{p}{p-1}$, is a kind of a measure theoretic dual of the space $L^{p,1}(\mathbb{R}^n)$, in the sense that the following form of Hölder's inequality

$$\int_{\mathbb{R}^n} |fg| \leq \|f\|_{L^{p,1}(\mathbb{R}^n)} \|g\|_{L^{p',\infty}(\mathbb{R}^n)}$$

holds.

In what follows, $B(x, r)$ denotes the ball centered at point x and having the radius r . If a ball of radius r is centered at the origin, we shall denote it simply by B_r . Let $q \geq 1$

be a real number. We consider the centered maximal operator M_{L^q} defined by

$$M_{L^q}f(x) = \sup_{r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^q dy \right)^{\frac{1}{q}}.$$

Observe that

$$M_{L^q}f = (M|f|^q)^{\frac{1}{q}},$$

where M stands for the classical Hardy-Littlewood maximal operator.

The crucial step towards proving Theorem 2.2 is the following lemma, which can be understood as a sharp variant of [8, Theorem 2.1.10].

Lemma 2.1. *Assume that $n \in \mathbb{N}$, $s \in (0, n)$ and $q > \frac{n}{s}$. Then there is a positive constant C depending on n , s and q such that for any $j \in \mathbb{Z}$ and any measurable function f on \mathbb{R}^n ,*

$$(2.7) \quad \left\| \frac{f(x + 2^{-j}y)}{(1 + |y|)^s} \right\|_{L^{\frac{n}{s}, \infty}(\mathbb{R}^n)} \leq CM_{L^q}f(x), \quad x \in \mathbb{R}^n.$$

Proof. We may assume, without loss of generality, that $j = 0$ and $x = 0$. Indeed, setting $g(y) = f(x + 2^{-j}y)$, we obtain

$$(2.8) \quad \left\| \frac{f(x + 2^{-j}y)}{(1 + |y|)^s} \right\|_{L^{\frac{n}{s}, \infty}(\mathbb{R}^n)} = \left\| \frac{g(y)}{(1 + |y|)^s} \right\|_{L^{\frac{n}{s}, \infty}(\mathbb{R}^n)}$$

and

$$(2.9) \quad \begin{aligned} M_{L^q}f(x) &= \sup_{r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^q dy \right)^{\frac{1}{q}} \\ &= \sup_{r>0} \left(\frac{1}{2^{jn}|B(x,r)|} \int_{B(0,2^j r)} |f(x + 2^{-j}z)|^q dz \right)^{\frac{1}{q}} \\ &= \sup_{r'>0} \left(\frac{1}{|B(0,r')|} \int_{B(0,r')} |g(y)|^q dy \right)^{\frac{1}{q}} \\ &= M_{L^q}g(0). \end{aligned}$$

Hence, it suffices to show that for any measurable function g on \mathbb{R}^n ,

$$(2.10) \quad \left\| \frac{g(y)}{(1 + |y|)^s} \right\|_{L^{\frac{n}{s}, \infty}(\mathbb{R}^n)} \leq CM_{L^q}g(0).$$

If $M_{L^q}g(0) = \infty$, then inequality (2.10) holds trivially, so we can assume in what follows that $M_{L^q}g(0) < \infty$. Since the case $M_{L^q}g(0) = 0$ is trivial as well (as g needs to vanish a.e. in this case), dividing the function g by the positive constant $M_{L^q}g(0)$, we can in fact assume that $M_{L^q}g(0) = 1$.

Fix any $a > 0$ and $k \in \mathbb{N}_0$. Then

$$\begin{aligned} |\{y \in B_{2^{k+1}} \setminus B_{2^k} : |g(y)| > a\}| &\leq \frac{1}{a^q} \int_{B_{2^{k+1}} \setminus B_{2^k}} |g(y)|^q dy \\ &\leq \frac{|B_{2^{k+1}}|}{a^q} \cdot \frac{1}{|B_{2^{k+1}}|} \int_{B_{2^{k+1}}} |g(y)|^q dy \leq \frac{\omega_n 2^{(k+1)n}}{a^q}, \end{aligned}$$

where ω_n denotes the volume of the unit ball in \mathbb{R}^n . Combining this with the trivial estimate

$$|\{y \in B_{2^{k+1}} \setminus B_{2^k} : |g(y)| > a\}| \leq \omega_n 2^{(k+1)n},$$

we deduce that

$$\begin{aligned} &\left| \left\{ y \in \mathbb{R}^n : \frac{|g(y)|}{(1+|y|)^s} > a \right\} \right| \\ &= \left| \left\{ y \in B_1 : \frac{|g(y)|}{(1+|y|)^s} > a \right\} \right| + \sum_{k=0}^{\infty} \left| \left\{ y \in B_{2^{k+1}} \setminus B_{2^k} : \frac{|g(y)|}{(1+|y|)^s} > a \right\} \right| \\ &\leq |\{y \in B_1 : |g(y)| > a\}| + \sum_{k=0}^{\infty} |\{y \in B_{2^{k+1}} \setminus B_{2^k} : |g(y)| > 2^{ks} a\}| \\ &\leq |\{y \in B_1 : |g(y)| > a\}| + \sum_{k=0}^{\infty} \omega_n 2^{(k+1)n} \min \left\{ \frac{1}{2^{ksq} a^q}, 1 \right\} \\ &\leq |\{y \in B_1 : |g(y)| > a\}| + \sum_{k \in \mathbb{N}_0 : 2^k < \frac{1}{a^{1/s}}} \omega_n 2^n \cdot 2^{kn} + \sum_{k \in \mathbb{N}_0 : 2^k \geq \frac{1}{a^{1/s}}} \frac{\omega_n 2^n}{a^q} \cdot 2^{k(n-sq)} \\ &\leq |\{y \in B_1 : |g(y)| > a\}| + \frac{C}{a^{\frac{n}{s}}}. \end{aligned}$$

Notice that in the last inequality we have used the fact that $n - sq < 0$. Hence,

$$\begin{aligned} \left\| \frac{g(y)}{(1+|y|)^s} \right\|_{L^{\frac{n}{s}, \infty}(\mathbb{R}^n)} &= \sup_{a>0} a \left| \left\{ y \in \mathbb{R}^n : \frac{|g(y)|}{(1+|y|)^s} > a \right\} \right|^{\frac{s}{n}} \\ &\leq \sup_{a>0} a |\{y \in B_1 : |g(y)| > a\}|^{\frac{s}{n}} + C \\ &= \|g\|_{L^{\frac{n}{s}, \infty}(B_1)} + C \\ &\leq C' \|g\|_{L^q(B_1)} + C \\ &\leq C' \omega_n^{\frac{1}{q}} M_{L^q} g(0) + C \leq C'', \end{aligned}$$

where $C' > 0$ is the constant from the embedding $L^q(B_1) \hookrightarrow L^{\frac{n}{s}, \infty}(B_1)$. Since $M_{L^q} g(0) = 1$, this proves (2.10), and in turn (2.7) as well. \square

Theorem 2.2. *Let $p \in (1, \infty)$, $n \in \mathbb{N}$, $s \in (\frac{n}{2}, n)$. Let Ψ be as in Theorem 1.1. Then*

$$(2.11) \quad \|T_\sigma f\|_{L^p(\mathbb{R}^n)} \leq C \sup_{j \in \mathbb{Z}} \|(I - \Delta)^{\frac{s}{2}} [\widehat{\Psi}\sigma(2^j \cdot)]\|_{L^{\frac{n}{s}, 1}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. Let

$$K = \sup_{j \in \mathbb{Z}} \|(I - \Delta)^{\frac{s}{2}} [\widehat{\Psi}\sigma(2^j \cdot)]\|_{L^{\frac{n}{s}, 1}(\mathbb{R}^n)} < \infty.$$

Introduce the function Θ satisfying

$$\widehat{\Theta}(\xi) = \widehat{\Psi}(\xi/2) + \widehat{\Psi}(\xi) + \widehat{\Psi}(2\xi),$$

and observe that $\widehat{\Theta}$ is equal to 1 on the support of the function $\widehat{\Psi}$.

Let us denote by Δ_j and Δ_j^\ominus the Littlewood-Paley operators associated with Ψ and Θ , respectively. If f is a Schwartz function on \mathbb{R}^n , then standard manipulations yield

$$\begin{aligned} \Delta_j T_\sigma(f)(x) &= \int_{\mathbb{R}^n} \widehat{f}(\xi) \widehat{\Psi}(2^{-j}\xi) \sigma(\xi) e^{2\pi i x \cdot \xi} d\xi = \int_{\mathbb{R}^n} (\Delta_j^\ominus f)^\wedge(\xi) \widehat{\Psi}(2^{-j}\xi) \sigma(\xi) e^{2\pi i x \cdot \xi} d\xi \\ &= 2^{jn} \int_{\mathbb{R}^n} (\Delta_j^\ominus f)^\wedge(2^j \xi') \widehat{\Psi}(\xi') \sigma(2^j \xi') e^{2\pi i x \cdot 2^j \xi'} d\xi' \\ &= \int_{\mathbb{R}^n} (\Delta_j^\ominus f)(x + 2^{-j}y) [\widehat{\Psi}\sigma(2^j \cdot)]^\wedge(y) dy \\ &= \int_{\mathbb{R}^n} \frac{(\Delta_j^\ominus f)(x + 2^{-j}y)}{(1 + |y|)^s} (1 + |y|)^s [\widehat{\Psi}\sigma(2^j \cdot)]^\wedge(y) dy. \end{aligned}$$

By the Hölder inequality in Lorentz spaces, we therefore obtain

$$|\Delta_j T_\sigma(f)(x)| \leq \left\| \frac{(\Delta_j^\ominus f)(x + 2^{-j}y)}{(1 + |y|)^s} \right\|_{L^{\frac{n}{s}, \infty}(\mathbb{R}^n)} \left\| (1 + |y|)^s [\widehat{\Psi}\sigma(2^j \cdot)]^\wedge(y) \right\|_{L^{(\frac{n}{s})', 1}(\mathbb{R}^n)}.$$

Since $\frac{n}{s} < 2$, we can find a real number q such that $\frac{n}{s} < q < 2$. Lemma 2.1 now yields that

$$\left\| \frac{(\Delta_j^\ominus f)(x + 2^{-j}y)}{(1 + |y|)^s} \right\|_{L^{\frac{n}{s}, \infty}(\mathbb{R}^n)} \leq C M_{L^q}(\Delta_j^\ominus f)(x).$$

Using boundedness properties of the Fourier transform, we deduce that

$$\begin{aligned} \left\| (1 + |y|)^s [\widehat{\Psi}\sigma(2^j \cdot)]^\wedge(y) \right\|_{L^{(\frac{n}{s})', 1}(\mathbb{R}^n)} &\leq C \left\| (1 + |y|^2)^{\frac{s}{2}} [\widehat{\Psi}\sigma(2^j \cdot)]^\wedge(y) \right\|_{L^{(\frac{n}{s})', 1}(\mathbb{R}^n)} \\ &\leq C \|(I - \Delta)^{\frac{s}{2}} [\widehat{\Psi}\sigma(2^j \cdot)]\|_{L^{\frac{n}{s}, 1}(\mathbb{R}^n)} \leq CK. \end{aligned}$$

Altogether, we obtain the estimate

$$|\Delta_j T_\sigma(f)(x)| \leq CK M_{L^q}(\Delta_j^\ominus f)(x).$$

Assume that $p \geq 2$. Then we get, by applying the Littlewood-Paley theorem and the Fefferman-Stein inequality (notice that $\frac{p}{q} \geq \frac{2}{q} > 1$),

$$\begin{aligned} \|T_\sigma(f)\|_{L^p(\mathbb{R}^n)} &\leq C \left\| \left(\sum_{j \in \mathbb{Z}} |\Delta_j T_\sigma(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \leq CK \left\| \left(\sum_{j \in \mathbb{Z}} |M_{L^q}(\Delta_j^\ominus f)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \\ &= CK \left\| \left(\sum_{j \in \mathbb{Z}} (M(|\Delta_j^\ominus f|^q))^{\frac{q}{2}} \right)^{\frac{1}{q}} \right\|_{L^{\frac{p}{q}}(\mathbb{R}^n)} \leq CK \left\| \left(\sum_{j \in \mathbb{Z}} |\Delta_j^\ominus f|^{q \cdot \frac{2}{q}} \right)^{\frac{q}{2}} \right\|_{L^{\frac{p}{q}}(\mathbb{R}^n)}^{\frac{1}{q}} \\ &= CK \left\| \left(\sum_{j \in \mathbb{Z}} |\Delta_j^\ominus f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \leq CK \|f\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

If $p \in (1, 2)$ then the result follows by duality. \square

3. INTERPOLATION

Our main goal in this section will be to discuss the following result.

Theorem 3.1. *Suppose that $1 < p_1 < \infty$ and $0 < s_1 < n$. If*

$$(3.12) \quad \|T_\sigma f\|_{L^{p_1}(\mathbb{R}^n)} \leq C \sup_{j \in \mathbb{Z}} \|(I - \Delta)^{\frac{s_1}{2}} [\widehat{\Psi}\sigma(2^j \cdot)]\|_{L^{\frac{n}{s_1}, 1}(\mathbb{R}^n)} \|f\|_{L^{p_1}(\mathbb{R}^n)},$$

then

$$\|T_\sigma f\|_{L^p(\mathbb{R}^n)} \leq C \sup_{j \in \mathbb{Z}} \|(I - \Delta)^{\frac{s}{2}} [\widehat{\Psi}\sigma(2^j \cdot)]\|_{L^{\frac{n}{s}, 1}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}$$

for any $1 < p < \infty$ and $0 < s < s_1$ satisfying

$$(3.13) \quad \frac{1}{s} \left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{s_1} \left| \frac{1}{p_1} - \frac{1}{2} \right|.$$

Assuming Theorem 3.1, and using the estimate from Theorem 2.2 as the assumption (3.12), we finish the proof of our main result, Theorem 1.1, as follows.

Proof of Theorem 1.1. If $s \in (\frac{n}{2}, n)$, then inequality (1.4) follows from Theorem 2.2. If $s \leq \frac{n}{2}$, then we denote

$$\alpha = \frac{1}{s} \left| \frac{1}{p} - \frac{1}{2} \right|.$$

Since $\alpha \in (0, \frac{1}{n})$, we can find $p_1 \in (1, \infty)$ and $s_1 \in (\frac{n}{2}, n)$ such that

$$\alpha < \frac{1}{s_1} \left| \frac{1}{p_1} - \frac{1}{2} \right|.$$

A combination of Theorems 2.2 and 3.1 thus yields the desired assertion (1.4). \square

We now focus on Theorem 3.1. The main idea of its proof consists in applying a complex interpolation between estimate (3.12) and the usual L^2 estimate implied by Plancherel's theorem. To prove Theorem 3.1 we shall need a few auxiliary results:

Lemma 3.2 ([8, 10]). *Let F be analytic on the open strip $S = \{z \in \mathbb{C} : 0 < \Re(z) < 1\}$ and continuous on its closure. Assume that for every $0 \leq \tau \leq 1$ there exists a function A_τ on the real line such that*

$$|F(\tau + it)| \leq A_\tau(t) \quad \text{for all } t \in \mathbb{R},$$

and suppose that there exist constants $A > 0$ and $0 < a < \pi$ such that for all $t \in \mathbb{R}$ we have

$$0 < A_\tau(t) \leq \exp \{Ae^{a|t|}\}.$$

Then for $0 < \theta < 1$ we have

$$|F(\theta)| \leq \exp \left\{ \frac{\sin(\pi\theta)}{2} \int_{-\infty}^{\infty} \left[\frac{\log |A_0(t)|}{\cosh(\pi t) - \cos(\pi\theta)} + \frac{\log |A_1(t)|}{\cosh(\pi t) + \cos(\pi\theta)} \right] dt \right\}.$$

We also need the following lemma, whose standard proof is omitted.

Lemma 3.3. *Let $1 < p, p_1 < \infty$ be related as in $1/p = (1 - \theta)/2 + \theta/p_1$ for some $\theta \in (0, 1)$. Given $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $\varepsilon > 0$, there exist smooth functions h_j^ε , $j = 1, \dots, N_\varepsilon$, supported in cubes on \mathbb{R}^n with pairwise disjoint interiors, and nonzero complex constants c_j^ε such that*

$$(3.14) \quad f_z^\varepsilon = \sum_{j=1}^{N_\varepsilon} |c_j^\varepsilon|^{\frac{p}{2}(1-z) + \frac{p}{p_1}z} h_j^\varepsilon,$$

$$\|f_\theta^\varepsilon - f\|_{L^2(\mathbb{R}^n)} < \varepsilon, \|f_{it}^\varepsilon\|_{L^2(\mathbb{R}^n)} \leq (\|f\|_{L^p(\mathbb{R}^n)} + \varepsilon)^{\frac{p}{2}}, \text{ and } \|f_{1+it}^\varepsilon\|_{L^{p_1}(\mathbb{R}^n)} \leq (\|f\|_{L^p(\mathbb{R}^n)} + \varepsilon)^{\frac{p}{p_1}}.$$

The next three lemmas generalize results which are well known in the context of Lebesgue spaces to the setting of Lorentz spaces $L^{p,1}(\mathbb{R}^n)$.

Lemma 3.4. *Let $0 < s < n$. Then*

$$\|(I - \Delta)^{-\frac{s}{2}} f\|_{L^\infty(\mathbb{R}^n)} \leq C(n) \frac{s}{n-s} \|f\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)}.$$

Proof. Let G_s be the function defined for any $x \in \mathbb{R}^n$ by

$$G_s(x) = \frac{1}{(4\pi)^{\frac{s}{2}} \Gamma(\frac{s}{2})} \int_0^\infty e^{-\frac{\pi|x|^2}{\delta}} e^{-\frac{\delta}{4\pi}} \delta^{-\frac{n+s}{2}} \frac{d\delta}{\delta}.$$

It is not difficult to show that $G_s(x) \leq C(n) \frac{s}{n-s} |x|^{-n+s}$. Therefore,

$$|(I - \Delta)^{-\frac{s}{2}} f(x)| = |G_s * f(x)| \leq \int_{\mathbb{R}^n} G_s(y) |f(x-y)| dy \leq C(n) \frac{s}{n-s} \|f\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)}.$$

□

Lemma 3.5. *Let $1 < a < b < \infty$. Then, for any $p \in (a, b)$ and any $t \in \mathbb{R}$,*

$$\|(I - \Delta)^{-it} f\|_{L^{p,1}(\mathbb{R}^n)} \leq C(n, a, b)(1 + |t|)^{\frac{n}{2}+1} \|f\|_{L^{p,1}(\mathbb{R}^n)}.$$

Proof. Set $b_0 = 2b$. By the Hörmander multiplier theorem, one has

$$\|(I - \Delta)^{-it} f\|_{L^{1,\infty}(\mathbb{R}^n)} \leq C(n)(1 + |t|)^{\frac{n}{2}+1} \|f\|_{L^1(\mathbb{R}^n)}$$

and

$$\|(I - \Delta)^{-it} f\|_{L^{b_0}(\mathbb{R}^n)} \leq C(n, b)(1 + |t|)^{\frac{n}{2}+1} \|f\|_{L^{b_0}(\mathbb{R}^n)}.$$

Notice that the second estimate implies, in particular, the corresponding weak-type inequality. An interpolation between these two estimates using the Marcinkiewicz interpolation theorem [1, Chapter 4, Theorem 4.13] yields the required assertion. \square

Lemma 3.6. *Let $1 < p < \infty$ and $s > 0$, and let Ψ be as in Theorem 1.1. Then we have the a priori estimate*

$$(3.15) \quad \|(I - \Delta)^{\frac{s}{2}} [\widehat{\Psi} f]\|_{L^{p,1}(\mathbb{R}^n)} \leq C(n, s, p, \Psi) \|(I - \Delta)^{\frac{s}{2}} f\|_{L^{p,1}(\mathbb{R}^n)}.$$

Proof. Pick real numbers p_0, p_1 satisfying $1 < p_0 < p < p_1 < \infty$. Denote by T the linear operator defined by

$$Tf = (I - \Delta)^{\frac{s}{2}} [\widehat{\Psi} (I - \Delta)^{-\frac{s}{2}} f].$$

Thanks to the Kato-Ponce inequality, T is bounded on both $L^{p_0}(\mathbb{R}^n)$ and $L^{p_1}(\mathbb{R}^n)$, so, in particular, it is of weak type (p_0, p_0) and (p_1, p_1) . By the Marcinkiewicz interpolation theorem [1, Chapter 4, Theorem 4.13], T is bounded on $L^{p,1}(\mathbb{R}^n)$, which yields (3.15). \square

Lemma 3.7. *Let $0 < a < s < n$. Then*

$$(3.16) \quad \int_0^\infty (f^*(r)r^{\frac{s-a}{n}})^*(y)y^{\frac{a}{n}-1} dy \leq \frac{C(n)}{a} \int_0^\infty f^*(r)r^{\frac{s}{n}-1} dr.$$

Proof. Estimates of this type are known in the literature, see, e.g., [7]. For the convenience of the reader, we provide an elementary proof of inequality (3.16). The proof follows the ideas of [4, Section 9].

We may assume that

$$\int_0^\infty f^*(r)r^{\frac{s}{n}-1} dr < \infty.$$

Then $f^*(r)r^{\frac{s}{n}} \leq C$, and thus $\lim_{r \rightarrow \infty} f^*(r)r^{\frac{s-a}{n}} = 0$. Since the function f^* is left-continuous, $\sup_{y \leq r < \infty} f^*(r)r^{\frac{s-a}{n}}$ is attained for any $y > 0$ and the set

$$M = \{y \in (0, \infty) : \sup_{y \leq r < \infty} f^*(r)r^{\frac{s-a}{n}} > f^*(y)y^{\frac{s-a}{n}}\}$$

is open. Hence, M is a countable union of open intervals, namely, $M = \bigcup_{k \in S} (a_k, b_k)$, where S is a countable set of positive integers. Also, observe that if $y \in (a_k, b_k)$, then $\sup_{y \leq r < \infty} f^*(r) r^{\frac{s-a}{n}} = f^*(b_k) b_k^{\frac{s-a}{n}}$. We have

$$\begin{aligned} \int_0^\infty (f^*(r) r^{\frac{s-a}{n}})^*(y) y^{\frac{a}{n}-1} dy &\leq \int_0^\infty \sup_{y \leq r < \infty} f^*(r) r^{\frac{s-a}{n}} y^{\frac{a}{n}-1} dy \\ &= \int_{(0, \infty) \setminus \bigcup_{k \in S} (a_k, b_k)} f^*(y) y^{\frac{s}{n}-1} dy + \sum_{k \in S} f^*(b_k) b_k^{\frac{s-a}{n}} \int_{a_k}^{b_k} y^{\frac{a}{n}-1} dy. \end{aligned}$$

Furthermore, for every $k \in S$,

$$\begin{aligned} f^*(b_k) b_k^{\frac{s-a}{n}} \int_{a_k}^{b_k} y^{\frac{a}{n}-1} dy &\leq f^*(b_k) b_k^{\frac{s-a}{n}} \int_{\max(a_k, \frac{b_k}{2})}^{b_k} y^{\frac{a}{n}-1} dy \cdot \frac{\int_0^{b_k} y^{\frac{a}{n}-1} dy}{\int_{\frac{b_k}{2}}^{b_k} y^{\frac{a}{n}-1} dy} \\ &= \frac{1}{1 - (\frac{1}{2})^{\frac{a}{n}}} f^*(b_k) b_k^{\frac{s-a}{n}} \int_{\max(a_k, \frac{b_k}{2})}^{b_k} y^{\frac{a}{n}-1} dy \\ &\leq \frac{2^{\frac{s-a}{n}}}{1 - (\frac{1}{2})^{\frac{a}{n}}} \int_{a_k}^{b_k} f^*(y) y^{\frac{s}{n}-1} dy \\ &\leq \frac{C(n)}{a} \int_{a_k}^{b_k} f^*(y) y^{\frac{s}{n}-1} dy. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^\infty (f^*(r) r^{\frac{s-a}{n}})^*(y) y^{\frac{a}{n}-1} dy &\leq \int_0^\infty f^*(y) y^{\frac{s}{n}-1} dy + \frac{C(n)}{a} \sum_{k \in S} \int_{a_k}^{b_k} f^*(y) y^{\frac{s}{n}-1} dy \\ &\leq \frac{C(n)}{a} \int_0^\infty f^*(y) y^{\frac{s}{n}-1} dy. \end{aligned}$$

□

To prove Theorem 3.1 we will also need the notion of a measure preserving transformation. We say that a mapping $h : \mathbb{R}^n \rightarrow (0, \infty)$ is measure preserving if, whenever E is a measurable subset of $(0, \infty)$, the set $h^{-1}E = \{x \in \mathbb{R}^n : h(x) \in E\}$ is a measurable subset of \mathbb{R}^n and the n -dimensional Lebesgue measure of $h^{-1}E$ is equal to the one-dimensional Lebesgue measure of E . For more details on measure preserving transformations, see, e.g., [1, Chapter 2, Section 7].

Proof of Theorem 3.1. We first observe that, by (3.13), we have $p_1 \neq 2$. In fact, we can assume that $1 < p_1 < 2$ and $1 < p \leq 2$, otherwise the result will follow by duality. Further, if $p = 2$ then Theorem 3.1 is a consequence of Plancherel's theorem and of the Sobolev embedding from Lemma 3.4, so it is sufficient to focus on the case $p < 2$ in what

follows. Define

$$\theta = \frac{\frac{1}{p} - \frac{1}{2}}{\frac{1}{p_1} - \frac{1}{2}}.$$

The assumption (3.13) yields $\theta \in (0, \frac{s}{s_1})$, and therefore

$$\theta = \frac{s - s_0}{s_1 - s_0}$$

for some $s_0 \in (0, s)$. Fix a function σ satisfying

$$(3.17) \quad \sup_{j \in \mathbb{Z}} \|(I - \Delta)^{\frac{s}{2}} [\widehat{\Psi} \sigma(2^j \cdot)]\|_{L^{\frac{n}{s}, 1}(\mathbb{R}^n)} < \infty,$$

and denote $\varphi_j = (I - \Delta)^{\frac{s}{2}} [\widehat{\Psi} \sigma(2^j \cdot)]$, $j \in \mathbb{Z}$. Thanks to (3.17), we have $\lim_{r \rightarrow \infty} \varphi_j^*(r) = 0$. By [1, Chapter 2, Corollary 7.6], there is a measure preserving transformation $h_j : \mathbb{R}^n \rightarrow (0, \infty)$ such that $|\varphi_j| = \varphi_j^* \circ h_j$.

For a complex number z with $0 \leq \Re(z) \leq 1$, we define

$$(3.18) \quad \sigma_z(\xi) = \sum_{j \in \mathbb{Z}} (I - \Delta)^{-\frac{s_0(1-z) + s_1 z}{2}} [\varphi_j h_j^{\frac{s - (1-z)s_0 - z s_1}{n}}] (2^{-j} \xi) \widehat{\Phi}(2^{-j} \xi),$$

where $\widehat{\Phi}$ is a Schwartz function supported in the set $\{\xi \in \mathbb{R}^n : \frac{1}{4} \leq |\xi| \leq 4\}$ and $\widehat{\Phi} \equiv 1$ on the support of $\widehat{\Psi}$.

Fix $f, g \in \mathcal{C}_0^\infty$. Given $\varepsilon > 0$, let f_z^ε and g_z^ε be functions having the form (3.14), with f replaced by g and with p replaced by p' in the latter case, satisfying $\|f_\theta^\varepsilon - f\|_{L^2(\mathbb{R}^n)} < \varepsilon$, $\|g_\theta^\varepsilon - g\|_{L^2(\mathbb{R}^n)} < \varepsilon$, and

$$(3.19) \quad \begin{aligned} \|f_{it}^\varepsilon\|_{L^2(\mathbb{R}^n)} &\leq (\|f\|_{L^p(\mathbb{R}^n)} + \varepsilon)^{\frac{p}{2}}, & \|f_{1+it}^\varepsilon\|_{L^{p_1}(\mathbb{R}^n)} &\leq (\|f\|_{L^p(\mathbb{R}^n)} + \varepsilon)^{\frac{p}{p_1}}, \\ \|g_{it}^\varepsilon\|_{L^2(\mathbb{R}^n)} &\leq (\|g\|_{L^{p'}(\mathbb{R}^n)} + \varepsilon)^{\frac{p'}{2}}, & \|g_{1+it}^\varepsilon\|_{L^{p'_1}(\mathbb{R}^n)} &\leq (\|g\|_{L^{p'}(\mathbb{R}^n)} + \varepsilon)^{\frac{p'}{p'_1}}. \end{aligned}$$

For a complex number z with $0 \leq \Re(z) \leq 1$, define

$$F(z) = \int_{\mathbb{R}^n} T_{\sigma_z}(f_z^\varepsilon) g_z^\varepsilon dx = \int_{\mathbb{R}^n} \sigma_z(\xi) \widehat{f}_z^\varepsilon(\xi) \widehat{g}_z^\varepsilon(\xi) d\xi.$$

It is straightforward (but rather tedious) to verify that F is analytic on the strip $S = \{z \in \mathbb{C} : 0 < \Re(z) < 1\}$ and continuous on its closure.

Let us write $z = \tau + it$, $0 \leq \tau \leq 1$ and $t \in \mathbb{R}$, and denote $s_\tau = s_0(1 - \tau) + s_1\tau$. Then, applying Lemmas 3.4 and 3.5 and using the fact that h_j is measure preserving, we obtain

$$\begin{aligned} \|\sigma_z\|_{L^\infty(\mathbb{R}^n)} &\leq C(n) \sup_{j \in \mathbb{Z}} \|(I - \Delta)^{-\frac{s_0(1-z) + s_1 z}{2}} [\varphi_j h_j^{\frac{s - (1-z)s_0 - z s_1}{n}}]\|_{L^\infty(\mathbb{R}^n)} \\ &\leq C(n) \frac{s_\tau}{n - s_\tau} \sup_{j \in \mathbb{Z}} \|(I - \Delta)^{-\frac{s_0(-it) + s_1 it}{2}} [\varphi_j h_j^{\frac{s - (1-\tau-it)s_0 - (\tau+it)s_1}{n}}]\|_{L^{\frac{n}{s_\tau}, 1}(\mathbb{R}^n)} \end{aligned}$$

$$\begin{aligned}
&\leq C(n, s_0, s_1) \frac{s_\tau}{n - s_\tau} (1 + |t|)^{\frac{n}{2}+1} \sup_{j \in \mathbb{Z}} \|\varphi_j h_j\|_{L^{\frac{n}{s_\tau}, 1}(\mathbb{R}^n)}^{s - (1-\tau-it)s_0 - (\tau+it)s_1} \\
&\leq C(n, s_0, s_1) (1 + |t|)^{\frac{n}{2}+1} \sup_{j \in \mathbb{Z}} \|\varphi_j^*(r) r^{\frac{s-(1-\tau)s_0-\tau s_1}{n}}\|_{L^{\frac{n}{s_\tau}, 1}(0, \infty)} \\
&\leq C(n, s_0, s_1) (1 + |t|)^{\frac{n}{2}+1} \sup_{j \in \mathbb{Z}} \|\varphi_j^*\|_{L^{\frac{n}{s}, 1}(0, \infty)} \\
&\leq C(n, s_0, s_1) (1 + |t|)^{\frac{n}{2}+1} \sup_{j \in \mathbb{Z}} \|\varphi_j\|_{L^{\frac{n}{s}, 1}(\mathbb{R}^n)}.
\end{aligned}$$

Notice that if $\tau \in [0, \theta)$, then the penultimate inequality follows from Lemma 3.7. Thus,

$$\begin{aligned}
(3.20) \quad |F(z)| &\leq \|\sigma_z\|_{L^\infty(\mathbb{R}^n)} \|f_z^\varepsilon\|_{L^2(\mathbb{R}^n)} \|g_z^\varepsilon\|_{L^2(\mathbb{R}^n)} \\
&\leq C(n, s_0, s_1) (1 + |t|)^{\frac{n}{2}+1} \sup_{j \in \mathbb{Z}} \|\varphi_j\|_{L^{\frac{n}{s}, 1}(\mathbb{R}^n)} \|f_z^\varepsilon\|_{L^2(\mathbb{R}^n)} \|g_z^\varepsilon\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Since $\|f_z^\varepsilon\|_{L^2(\mathbb{R}^n)} \|g_z^\varepsilon\|_{L^2(\mathbb{R}^n)}$ can be bounded from above by a constant independent of z , the previous estimate yields

$$(3.21) \quad |F(z)| \leq C(n, s_0, s_1, p, p_1, \varepsilon, f, g) (1 + |t|)^{\frac{n}{2}+1} \sup_{j \in \mathbb{Z}} \|\varphi_j\|_{L^{\frac{n}{s}, 1}(\mathbb{R}^n)} \leq \exp\{Ae^{a|t|}\}$$

for a suitable choice of constants $A > 0$ and $a \in (0, \pi)$. Also, if $z = it$, $t \in \mathbb{R}$, then (3.20) combined with (3.19) yield

$$(3.22) \quad |F(it)| \leq C(n, s_0, s_1) (1 + |t|)^{\frac{n}{2}+1} (\|f\|_{L^p(\mathbb{R}^n)} + \varepsilon)^{\frac{p}{2}} (\|g\|_{L^{p'}(\mathbb{R}^n)} + \varepsilon)^{\frac{p'}{2}} \sup_{j \in \mathbb{Z}} \|\varphi_j\|_{L^{\frac{n}{s}, 1}(\mathbb{R}^n)}.$$

Finally, by Hölder's inequality and by (3.12),

$$\begin{aligned}
|F(1 + it)| &\leq \|T_{\sigma_{1+it}}(f_{1+it}^\varepsilon)\|_{L^{p_1}(\mathbb{R}^n)} \|g_{1+it}^\varepsilon\|_{L^{p'_1}(\mathbb{R}^n)} \\
&\leq C \sup_{j \in \mathbb{Z}} \|(I - \Delta)^{\frac{s_1}{2}} [\widehat{\Psi} \sigma_{1+it}(2^j \cdot)]\|_{L^{\frac{n}{s_1}, 1}(\mathbb{R}^n)} \|f_{1+it}^\varepsilon\|_{L^{p_1}(\mathbb{R}^n)} \|g_{1+it}^\varepsilon\|_{L^{p'_1}(\mathbb{R}^n)}.
\end{aligned}$$

Notice that $\widehat{\Psi} \sigma_{1+it}(2^k \cdot)$ picks up only those terms j of (3.18) which differ from k by at most two units. For simplicity, we may therefore take $j = k$ in the calculation below.

We have

$$\begin{aligned}
&\|(I - \Delta)^{\frac{s_1}{2}} [\widehat{\Psi}(I - \Delta)^{-\frac{s_1+it(s_1-s_0)}{2}} [\varphi_j h_j^{\frac{s-s_1+it(s_0-s_1)}{n}}]]\|_{L^{\frac{n}{s_1}, 1}(\mathbb{R}^n)} \\
&\leq C \|(I - \Delta)^{\frac{s_1}{2}} [(I - \Delta)^{-\frac{s_1+it(s_1-s_0)}{2}} [\varphi_j h_j^{\frac{s-s_1+it(s_0-s_1)}{n}}]]\|_{L^{\frac{n}{s_1}, 1}(\mathbb{R}^n)} \\
&\leq C \|(I - \Delta)^{-\frac{it(s_1-s_0)}{2}} [\varphi_j h_j^{\frac{s-s_1+it(s_0-s_1)}{n}}]\|_{L^{\frac{n}{s_1}, 1}(\mathbb{R}^n)} \\
&\leq C (1 + |t|)^{\frac{n}{2}+1} \|\varphi_j h_j^{\frac{s-s_1}{n}}\|_{L^{\frac{n}{s_1}, 1}(\mathbb{R}^n)} = C (1 + |t|)^{\frac{n}{2}+1} \|\varphi_j^*(r) r^{\frac{s-s_1}{n}}\|_{L^{\frac{n}{s_1}, 1}(0, \infty)} \\
&= C (1 + |t|)^{\frac{n}{2}+1} \|\varphi_j^*\|_{L^{\frac{n}{s}, 1}(0, \infty)} = C (1 + |t|)^{\frac{n}{2}+1} \|\varphi_j\|_{L^{\frac{n}{s}, 1}(\mathbb{R}^n)}.
\end{aligned}$$

Notice that in the previous estimate we consecutively used Lemmas 3.6 and 3.5 and the fact that h_j is measure preserving. Therefore,

$$(3.23) \quad |F(1+it)| \leq C(1+|t|)^{\frac{n}{2}+1} \sup_{j \in \mathbb{Z}} \|\varphi_j\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)} (\|f\|_{L^p(\mathbb{R}^n)} + \varepsilon)^{\frac{p}{p_1}} (\|g\|_{L^{p'}(\mathbb{R}^n)} + \varepsilon)^{\frac{p'}{p_1}}.$$

A combination of (3.21), (3.22), (3.23) and Lemma 3.2 yields

$$(3.24) \quad |F(\theta)| \leq C \sup_{j \in \mathbb{Z}} \|\varphi_j\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)} (\|f\|_{L^p(\mathbb{R}^n)} + \varepsilon) (\|g\|_{L^{p'}(\mathbb{R}^n)} + \varepsilon).$$

Observe that $F(\theta) = \int_{\mathbb{R}^n} \sigma(\xi) \widehat{f}_\theta^\varepsilon(\xi) \widehat{g}_\theta^\varepsilon(\xi) d\xi$ as for every $\xi \neq 0$,

$$\begin{aligned} \sigma_\theta(\xi) &= \sum_{j \in \mathbb{Z}} (I - \Delta)^{-\frac{s}{2}} [(I - \Delta)^{\frac{s}{2}} [\sigma(2^j \cdot) \widehat{\Psi}]] (2^{-j} \xi) \widehat{\Phi}(2^{-j} \xi) \\ &= \sum_{j \in \mathbb{Z}} \sigma(\xi) \widehat{\Psi}(2^{-j} \xi) \widehat{\Phi}(2^{-j} \xi) = \sum_{j \in \mathbb{Z}} \sigma(\xi) \widehat{\Psi}(2^{-j} \xi) = \sigma(\xi). \end{aligned}$$

Notice that

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \sigma(\xi) \widehat{f}_\theta^\varepsilon(\xi) \widehat{g}_\theta^\varepsilon(\xi) d\xi - \int_{\mathbb{R}^n} \sigma(\xi) \widehat{f}(\xi) \widehat{g}(\xi) d\xi \right| \\ &= \left| \int_{\mathbb{R}^n} \sigma(\xi) \left[\widehat{f}_\theta^\varepsilon(\xi) (\widehat{g}_\theta^\varepsilon(\xi) - \widehat{g}(\xi)) + \widehat{g}(\xi) (\widehat{f}_\theta^\varepsilon(\xi) - \widehat{f}(\xi)) \right] d\xi \right| \\ &\leq \|\sigma\|_{L^\infty(\mathbb{R}^n)} \left[\|\widehat{f}_\theta^\varepsilon\|_{L^2(\mathbb{R}^n)} \|\widehat{g}_\theta^\varepsilon - \widehat{g}\|_{L^2(\mathbb{R}^n)} + \|\widehat{g}\|_{L^2(\mathbb{R}^n)} \|\widehat{f}_\theta^\varepsilon - \widehat{f}\|_{L^2(\mathbb{R}^n)} \right] \\ &\leq C \sup_{j \in \mathbb{Z}} \|(I - \Delta)^{\frac{s}{2}} [\widehat{\Psi} \sigma(2^j \cdot)]\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)} \left[\|\widehat{f}_\theta^\varepsilon\|_{L^2(\mathbb{R}^n)} \|\widehat{g}_\theta^\varepsilon - \widehat{g}\|_{L^2(\mathbb{R}^n)} + \|\widehat{g}\|_{L^2(\mathbb{R}^n)} \|\widehat{f}_\theta^\varepsilon - \widehat{f}\|_{L^2(\mathbb{R}^n)} \right]. \end{aligned}$$

Recall that $\widehat{f}_\theta^\varepsilon - \widehat{f}$ and $\widehat{g}_\theta^\varepsilon - \widehat{g}$ converge to zero in $L^2(\mathbb{R}^n)$ as $\varepsilon \rightarrow 0$. Letting $\varepsilon \rightarrow 0$ in (3.24) yields

$$\left| \int_{\mathbb{R}^n} \sigma(\xi) \widehat{f}(\xi) \widehat{g}(\xi) d\xi \right| \leq C \sup_{j \in \mathbb{Z}} \|(I - \Delta)^{\frac{s}{2}} [\sigma(2^j \cdot) \widehat{\Psi}]\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)}.$$

Taking the supremum over all functions $g \in L^{p'}(\mathbb{R}^n)$ with $\|g\|_{L^{p'}(\mathbb{R}^n)} \leq 1$ we obtain

$$\|T_\sigma(f)\|_{L^p(\mathbb{R}^n)} \leq C \sup_{j \in \mathbb{Z}} \|(I - \Delta)^{\frac{s}{2}} [\sigma(2^j \cdot) \widehat{\Psi}]\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.$$

This completes the proof of Theorem 3.1. \square

We would like to thank the referee who pointed out that Theorem 3.1 in Section 3 could also be obtained by an argument using the idea in Connett and Schwartz [5], [6].

REFERENCES

- [1] C. Bennett and R. Sharpley, *Interpolation of operators*, Academic Press, Boston, 1988.
- [2] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution, II*, Adv. Math. **24** (1977), 101–171.

- [3] A. Cianchi and L. Pick, *Sobolev embeddings into BMO, VMO and L^∞* , Ark. Mat. **36** (1998), 317–340.
- [4] A. Cianchi, L. Pick and L. Slavíková, *Higher-order Sobolev embeddings and isoperimetric inequalities*, Adv. Math. **273** (2015), 568–650.
- [5] W. C. Connett and A. L. Schwartz, *The theory of ultraspherical multipliers*, Mem. Amer. Math. Soc. **9** (1977), no. 183, iv+92 pp.
- [6] W. C. Connett and A. L. Schwartz, *A remark about Calderón's upper s method of interpolation*, Interpolation spaces and allied topics in analysis (Lund, 1983), 48–53, Lecture Notes in Math., 1070, Springer, Berlin, 1984.
- [7] D. E. Edmunds and B. Opic, *Boundedness of fractional maximal operators between classical and weak type Lorentz spaces*, Dissertationes Math. (Rozprawy Mat.) **410** (2002), 50 pp.
- [8] L. Grafakos, *Classical Fourier Analysis*, 3rd edition, GTM 249, Springer-Verlag, NY 2014.
- [9] L. Grafakos, D. He, P. Honzík and H. V. Nguyen, *The Hörmander multiplier theorem I: The linear case revisited*, to appear in Illinois J. Math.
- [10] I. I. Jr. Hirschman, *A convexity theorem for certain groups of transformations*, J. Analyse Math. **2** (1953), 209–218.
- [11] I. I. Jr. Hirschman, *On multiplier transformations*, Duke Math. J. **26** (1959), 221–242.
- [12] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. **104** (1960), 93–139.
- [13] S. G. Mikhlin, *On the multipliers of Fourier integrals*. (Russian) Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 701–703.
- [14] A. Miyachi, *On some Fourier multipliers for $H^p(\mathbb{R}^n)$* , J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), 157–179.
- [15] A. Miyachi, N. Tomita, *Minimal smoothness conditions for bilinear Fourier multipliers*, Rev. Mat. Iberoam. **29** (2013), 495–530.
- [16] A. Seeger, *A limit case of the Hörmander multiplier theorem*, Monatsh. Math. **105** (1988), 151–160.
- [17] A. Seeger, *Estimates near L^1 for Fourier multipliers and maximal functions*, Arch. Math. (Basel) **53** (1989), 188–193.
- [18] A. Seeger, *Remarks on singular convolution operators*, Studia Math. **97** (1990), no. 2, 91–114.
- [19] E. M. Stein, *Singular integral and differentiability properties of functions*, Princeton Univ. Press, Princeton, NJ, 1970.
- [20] E. M. Stein, *Editor's note: The differentiability of functions in \mathbb{R}^n* , Ann. of Math. (2) **113** (1981), no. 2, 383–385.
- [21] S. Wainger, *Special trigonometric series in k -dimensions*, Mem. Amer. Math. Soc. **59** (1965), 1–102.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA MO 65211, USA
E-mail address: grafakosl@missouri.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA MO 65211, USA
E-mail address: slavikoval@missouri.edu