ON THE NORM OF THE OPERATOR $aI + bH$ ON $L^p(\mathbb{R})$

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ABSTRACT. We provide a direct proof of the following theorem of Kalton, Hollenbeck, and Verbitsky [7]: let H be the Hilbert transform and let a, b be real constants. Then for $1 < p < \infty$ the norm of the operator $aI + bH$ from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ is equal to

$$\left(\max_{x\in\mathbb{R}} \frac{|ax - b + (bx + a)\tan\frac{\pi}{2p}|^p + |ax - b - (bx + a)\tan\frac{\pi}{2p}|^p}{|x + \tan\frac{\pi}{2p}|^p + |x - \tan\frac{\pi}{2p}|^p} \right)^{\frac{1}{p}}.$$

Our proof avoids passing through the analogous result for the conjugate function on the circle, as in [7], and is given directly on the line. We also provide new approximate extremals for $aI + bH$ in the case $p > 2$.

1. Introduction

In this note we revisit the celebrated result of Kalton, Hollenbeck, and Verbitsky [7] concerning the value of the norm of the operator $aI + bH$ from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ for $1 < p < \infty$ and a, b real constants. We provide a self-contained direct proof of this result on the real line. The original proof in [7] was given for the conjugate function on the circle in lieu of the Hilbert transform and the corresponding result for the line was obtained from the periodic case via a transference-type argument due to Zygmund [13, Ch XVI, Th. 3.8] known as "blowing up the circle". Here we work directly with the Hilbert transform on the line, using an idea contained in [4] and [6], which is based on applying subharmonicity on the boundary of a suitable family of discs that fill up the upper half space as their radii tend to infinity. The main estimates needed for our proof (Lemmas 3.1 and 3.2) are as in [7] but are included in this note for the sake of completeness (with a minor adjustment). The new contributions of this article are contained in Sections 4 and 5. In Section 4 we use a limiting argument and subharmonicity to prove the claimed bound for $aI + bH$. We obtain the approximate extremals for the operators $aI + bH$ in Section 5; recall that the approximate extremals for the Hilbert transform first appeared in Gohberg and Krupnik [5] for $1 < p < 2$ and were also used by Pichorides [12]. We find new approximate extremals for the Hilbert transform for $2 < p < \infty$ in Section 5 and we use them to construct corresponding approximate extremals for $aI + bH$ for this range of p's. We note that the case $a = 0, b = 1$ of this result was proved by Pichorides [12] and B. Cole (unpublished, see [2]), while the case $a = 0$, $b = 1$, $p = 2^m$, $m = 1, 2, \dots$, was obtained four years earlier by Gohberg and Krupnik [5]. For a short history on this topic we refer to Laeng [9]. It is

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noteworthy that the operator norm of the Hilbert transform on L^p is also the norm of other operators, for instance of the segment multiplier; on this see De Carli and Laeng [1].

2. The norm of $aI + bH$

Denote the identity operator by I. The Hilbert transform on the real line is defined by

$$Hf(x) = p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)}{x-t}\,dt$$

for a smooth function with compact support. For $a, b \in \mathbb{R}$, define

$$B_p = \max_{x\in\mathbb{R}} \frac{|ax - b + (bx+a)\tan\gamma|^p + |ax - b - (bx+a)\tan\gamma|^p}{|x+\tan\gamma|^p + |x - \tan\gamma|^p}, \tag{1}$$

where $\gamma = \frac{\pi}{2p}$. B_p can be defined equivalently by

$$B_p = (a^2+b^2)^{p/2} \max_{0\le\theta\le 2\pi} \frac{|\cos(\theta+\theta_0)|^p + |\cos(\theta+\theta_0+\frac{\pi}{p})|^p}{|\cos\theta|^p + |\cos(\theta+\frac{\pi}{p})|^p}, \tag{2}$$

where $\tan\theta_0 = b/a$. By letting $\theta = -\vartheta - \pi/p$,

$$B_p = (a^2+b^2)^{p/2} \max_{0\le\vartheta\le 2\pi} \frac{|\cos(\vartheta-\theta_0)|^p + |\cos(\vartheta-\theta_0+\frac{\pi}{p})|^p}{|\cos\vartheta|^p + |\cos(\vartheta+\frac{\pi}{p})|^p}. \tag{3}$$

Our goal is to provide a proof of the following result in [7]:

Theorem 1. [7] *Let $1 < p < \infty$ and $a, b \in \mathbb{R}$. Then for all smooth functions with compact support f on the line we have*

$$\|(aI + bH)f\|^p_{L^p(\mathbb{R})} \le B_p \|f\|^p_{L^p(\mathbb{R})}$$

where the constant B_p is sharp. In other words,

$$\|aI + bH\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})} = B_p^{\frac{1}{p}}.$$

Without loss of generality, we assume that $a = \cos\theta_0, b = \sin\theta_0$, so that $a^2 + b^2 = 1$. As $aI + bH$ maps real-valued functions to real-valued functions, in view of the Marcinkiewicz and Zygmund theorem [11] (see also [3, Theorem 5.5.1]), the norms of $aI + bH$ on real and complex L^p spaces are equal.[1] Thus we may work with a nice real-valued function f in the proof of Theorem 1.

3. Some Lemmas

In this section we provide two auxiliary results that are crucial in the proof of the main theorem.

Lemma 3.1. [7] *Suppose $p > 1/2$, $p \ne 1$, and F is a p-homogeneous continuous function on $\mathbb{C}$. Suppose there is a sector S so that F is subharmonic on S and superharmonic on the complementary sector S'. Suppose further there is no nontrivial sector on which F is harmonic. Suppose that $F(z) + F(e^{i\pi/p}z) \ge 0$ for all z, and there*

[1] for operators that do not map real-valued functions to real-valued functions, these norms may not be equal; for instance this is the case for the Riesz projections, see [8].

exists $z_0 \neq 0$ so that $F(z_0)+F(e^{i\pi/p}z_0) = 0$. Then there is a continuous p-homogeneous subharmonic function G with $G(z) \leq F(z)$ for all $z \in \mathbb{C}$.

Proof. Lemma 3.1 is a restatement of Theorem 3.5 in [7]. We only provide a sketch below making a minor modification in the proof in [7] (i.e., definition of h in (4)).

We can suppose there exists z_0 with $|z_0| = 1$ so that $F(z_0) + F(e^{i\pi/p}z_0) = 0$. Let $z_0 = e^{it_0}$, $z_1 = e^{i(t_0+\pi/p)}$, since $p > 1/2$, there exists $\epsilon > 0$ such that $t_0 - \epsilon < t_0 < t_0 + \pi/p < t_0 + 2\pi - \epsilon$. Write $F(re^{it}) = r^p f(pt)$, where f is a $2p\pi$-periodic function on $\mathbb{R}$. By Proposition 3.3 in [7], if I is any interval so that $e^{ix/p} \in S$ for $x \in I$, then f is trigonometrically convex on I, and if $e^{ix/p} \in S'$ for $x \in I$, then f is trigonometrically concave on I. At least one of z_0, z_1 is contained in S; let us suppose that $z_0 \in S$. The function $f(x) + f(x + \pi)$ has minimum at pt_0, hence $f'_-(pt_0) + f'_-(pt_0 + \pi) \leq 0$, $f'_+(pt_0) + f'_+(pt_0 + \pi) \geq 0$. This implies that there exist a and b such that $a + b = 0$ and $f'_-(pt_0) \leq a \leq f'_+(pt_0)$ and $f'_-(pt_0 + \pi) \leq b \leq f'_+(pt_0 + \pi)$. Now define

$$h(x) = f(pt_0)\cos(x - pt_0) + a\sin(x - pt_0). \tag{4}$$

Then by Lemma 3.1 in [7], $h \leq f$ on a neighborhood of pt_0. Lemma 3.2 in [7] implies that $h(x) \leq f(x)$ for $p\alpha + 2p\pi \leq x < pt_0 + \pi + \delta$ and for $pt_0 \leq x \leq p\beta$. By the Phragmén-Lindelöf theorem ([10]) we obtain that $h \leq f$ in a neighborhood of $[pt_0, pt_0 + \pi]$.

Let $T = \{re^{i\theta} : r > 0, t_0 < \theta < t_0 + \frac{\pi}{p}\}$ and define $H(re^{i\theta}) = r^p h(p\theta)$ for $t_0 < \theta < t_0 + \frac{\pi}{p}$ and

$$G(z) = \begin{cases} H(z) & \text{if } z \in T, \\ F(z) & \text{if } z \notin T. \end{cases}$$

Then $G(z) \leq F(z)$ for all $z \in \mathbb{C} = \left\{re^{it} : r > 0, t_0 - \epsilon \leq t < t_0 + 2\pi - \epsilon\right\}$ and G is subharmonic on both T and its complementary sector T'. It is easy to see G is then subharmonic on $\mathbb{C}\backslash\{0\}$ since $h \leq f$ in a neighborhood of pt_0 and $pt_0 + \pi$. Finally $h(x) + h(x + \pi) = 0$ and Lemma 3.1 in [7] imply that $G(z) + G(e^{i\pi/p}z) \geq 0$ for all z. Integrating over a circle around 0 yields the subharmonicity of G at 0. □

Next we have a version of Lemma 4.2 in [7] in which we provide an explicit formula for the subharmonic function G.

Lemma 3.2. *Let $1 < p < \infty$, B_p be given by (1), $T = \{re^{it} : r > 0, t_0 < t < t_0 + \frac{\pi}{p}\}$, where t_0 is the value that makes right part of (3) attain its maximum, and there exists $\varepsilon > 0$ such that $t_0 - \varepsilon < t_0 < t_0 + \pi/p < t_0 + \pi - \varepsilon$. Let $z = re^{it}, z_0 = re^{it_0}, G(z) = G(re^{it})$ be π-periodic of t and when $t_0 - \varepsilon < t < t_0 + \pi - \varepsilon$:*

$$G(z) = \begin{cases} B_p|\mathrm{Re}z_0|^{p-1}\mathrm{sgn}(\mathrm{Re}z_0)\mathrm{Re}[(\frac{z}{z_0})^p z_0] - |a\mathrm{Re}z_0 + b\mathrm{Im}z_0|^{p-1} & \\ \qquad \times\mathrm{sgn}(a\mathrm{Re}z_0 + b\mathrm{Im}z_0)(a\mathrm{Re}[(\frac{z}{z_0})^p z_0] + b\mathrm{Im}[(\frac{z}{z_0})^p z_0]), & \text{if} \quad z \in T \\ B_p|\mathrm{Re}z|^p - |a\mathrm{Re}z + b\mathrm{Im}z|^p, & \text{if} \quad z \notin T. \end{cases}$$

Then G is subharmonic on $\mathbb{C}$ and satisfies

$$|a\mathrm{Re}z + b\mathrm{Im}z|^p \leq B_p|\mathrm{Re}z|^p - G(z). \tag{5}$$

for all $z \in \mathbb{C}$.

Proof. The case $b = 0$ is trivial, so we assume $b \neq 0$, and we may further assume that $a^2 + b^2 = 1$. Let $F(z) = B_p|\mathrm{Re}z|^p - |a\mathrm{Re}z + b\mathrm{Im}z|^p$. Then $F(re^{it}) = r^p f(t)$, where $f(t) = B_p|\cos t|^p - |a\cos t + b\sin t|^p$ is π-periodic and continuously differentiable. The definition in (2) implies that

$$\min_{0\le t\le 2\pi}[f(t) + f(t + \pi/p)] = 0.$$

We observe that $\Delta F \geq 0$ is equivalent to

$$B_p|\mathrm{Re}z|^{p-2} \geq |a\mathrm{Re}z + b\mathrm{Im}z|^{p-2}.$$

In order for $F(z)$ to be subharmonic, the following must be true:

$$|a + b\tan t|^{p-2} \leq B_p.$$

We can see that for $p \neq 2$ there will be two separate "double sectors" where $F(z)$ is subharmonic, and superharmonic in their complement. So let $\widetilde{p} = p/2, \widetilde{t_0} = 2t_0$, define $\widetilde{F}(z) = F(z^{1/2})$, then $\widetilde{F}$ is $\widetilde{p}$-homogeneous and satisfies the hypotheses of Lemma 3.1 with $\widetilde{p}$ and $\widetilde{t_0}$. Write $\widetilde{F}(re^{it}) = r^{\widetilde{p}}\widetilde{f}(\widetilde{p}t)$, where

$$\widetilde{f}(t) = B_p|\cos(t/p)|^p - |a\cos(t/p) + b\sin(t/p)|^p.$$

We can get

$$\widetilde{f}(\widetilde{p}\widetilde{t_0}) = B_p|\cos t_0|^p - |\cos(t_0 - \theta_0)|^p, \tag{6}$$

$$\widetilde{f}'_-(\widetilde{p}\widetilde{t_0}) = \widetilde{f}'_+(\widetilde{p}\widetilde{t_0}) = -B_p\frac{|\cos t_0|^p}{\cos t_0}\sin t_0 + \frac{|\cos(t_0-\theta_0)|^p}{\cos(t_0-\theta_0)}\sin(t_0-\theta_0), \tag{7}$$

where $\tan\theta_0 = b/a$. By the proof of Lemma 3.1, let

$$h(x) = \widetilde{f}(\widetilde{p}\widetilde{t_0})\cos(x - \widetilde{p}\widetilde{t_0}) + \widetilde{f}'_+(\widetilde{p}\widetilde{t_0})\sin(x - \widetilde{p}\widetilde{t_0}),$$

then $h(x) \leq \widetilde{f}(x)$ for all x in a neighborhood of $[\widetilde{p}\widetilde{t_0}, \widetilde{p}\widetilde{t_0} + \pi]$.

Let $\widetilde{T} = \{re^{it} : r > 0, \widetilde{t_0} < t < \widetilde{t_0} + \frac{\pi}{\widetilde{p}}\}$, and $H(re^{it}) = r^{\widetilde{p}}h(\widetilde{p}t)$ for $\widetilde{t_0} < t < \widetilde{t_0} + \frac{\pi}{\widetilde{p}}$, let

$$\widetilde{G}(z) = \begin{cases} H(z) = H(re^{it}) = r^{\widetilde{p}}[\widetilde{f}(\widetilde{p}\widetilde{t_0})\cos(\widetilde{p}t - \widetilde{p}\widetilde{t_0}) + \widetilde{f}'_+(\widetilde{p}\widetilde{t_0})\sin(\widetilde{p}t - \widetilde{p}\widetilde{t_0})] & \text{if } z \in \widetilde{T}, \\ \widetilde{F}(z) = r^{\widetilde{p}}(B_p|\cos\frac{t}{2}|^p - |a\cos\frac{t}{2} + b\sin\frac{t}{2}|^p) & \text{if } z \notin \widetilde{T}. \end{cases}$$

So let $\epsilon = 2\varepsilon$, $\widetilde{G}$ is subharmonic and $\widetilde{G}(z) \leq \widetilde{F}(z)$ on $\{re^{it} : r > 0, \widetilde{t_0} - \epsilon \leq t < \widetilde{t_0} + 2\pi - \epsilon\}$ by Lemma 3.1. Now let $G(z) = \widetilde{G}(z^2)$, clearly G is p-homogeneous and satisfies $G(z) \leq F(z)$ for $\{re^{it} : r > 0, t_0 - \varepsilon < t < t_0 + \pi - \varepsilon\}$. Since z^2 is holomorphic, $G(z)$ is also subharmonic on $\{re^{it} : r > 0, t_0 - \varepsilon < t < t_0 + \pi - \varepsilon\}$. Now let function $G(z) = G(re^{it})$ be π-periodic. For $t_0 - \varepsilon \leq t < t_0 + \pi - \varepsilon$, by (6), (7) and $G(z) = \widetilde{G}(z^2)$ we have:

$$G(z) = \begin{cases} r^p[B_p\dfrac{|\cos t_0|^p}{\cos t_0}\cos(p(t-t_0)+t_0) - \dfrac{|\cos(t_0-\theta_0)|^p}{\cos(t_0-\theta_0)} & \cos(p(t-t_0)+t_0-\theta_0)], \\ & \text{if } z \in T, \\ r^p(B_p|\cos t|^p - |\cos(t-\theta_0)|^p), & \text{if } z \notin T, \end{cases}$$

where $\tan\theta_0 = b/a$. It is easy to see $G(z) \leq F(z)$ for all $z \in \mathbb{C}$, by this we mean $\{re^{it} : r > 0, t_0 - \varepsilon \leq t < t_0 + 2\pi - \varepsilon\}$, so we get (5). Using similar proof as Lemma 3.1

and the periodicity of G, we can get $G(z)$ is also subharmonic on $\mathbb{C}$. Since $z_0 = re^{it_0}$, the above formula is equivalent to

$$G(z) = \begin{cases} B_p|\mathrm{Re}z_0|^{p-1}\mathrm{sgn}(\mathrm{Re}z_0)\mathrm{Re}[(\frac{z}{z_0})^p z_0] - |a\mathrm{Re}z_0 + b\mathrm{Im}z_0|^{p-1} & \\ \quad \times\mathrm{sgn}(a\mathrm{Re}z_0 + b\mathrm{Im}z_0)(a\mathrm{Re}[(\frac{z}{z_0})^p z_0] + b\mathrm{Im}[(\frac{z}{z_0})^p z_0]), & \text{if} \quad z \in T, \\ B_p|\mathrm{Re}z|^p - |a\mathrm{Re}z + b\mathrm{Im}z|^p, & \text{if} \quad z \notin T. \end{cases}$$

This completes the proof of the lemma. □

4. Proof of Theorem 1

If $p = 2$, then obviously

$$\|aI + bH\|^2_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = a^2 + b^2 = B_2,$$

so we can assume $p \neq 2$. Consider the holomorphic extension of $f(x) + iH(f)(x)$ on the upper half space given by

$$u(z) + iv(z) = \frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{f(t)}{z - t}dt, \quad u, v \text{ real-valued.}$$

Let $G(z)$ be given by Lemma 3.2, our next step is to use Lemma 3.2 and replace z with $h(z) = u(z) + iv(z)$. Since $h(z)$ is holomorphic and G is subharmonic, it follows that $G(h(z))$ is subharmonic on the upper half space. We note that ([4])

$$|u(x + iy)| + |v(x + iy)| \leq \frac{C_f}{1 + |x| + |y|}.$$

By Lemma 3.2, we have that $|G(z)| \leq C|z|^p$, hence

$$|G(h(z))| \leq C|h(z)|^p \leq C(|u(z)| + |v(z)|)^p.$$

So

$$|G(h(z))| \leq \frac{C_f^p}{(1 + |x| + |y|)^p}, \tag{8}$$

where $z = x + iy$. The boundary values of $G(h(z))$ are $G(h(x + i0))$.

The following part of the argument is based on [6]. For $R > 100$, consider the circle with center $(0, R)$ and radius $R' = R - R^{-1}$, denote by

$$C_R^U = \{iR + R'e^{i\phi} : -\pi/4 \leq \phi \leq 5\pi/4\}$$

and

$$C_R^L = \{iR + R'e^{i\phi} : 5\pi/4 \leq \phi \leq 7\pi/4\}.$$

It follows from the subharmonicity of $G(h(z))$ that

$$\int_{C_R^U} G(h(z))ds + \int_{C_R^L} G(h(z))ds \geq 2\pi R' G(h(iR)). \tag{9}$$

Clearly (8) implies that

$$|R'G(h(iR))| \leq R'\frac{C}{(1 + R)^p} \to 0 \qquad \text{as } R \to \infty, \tag{10}$$

and that

$$\left|\int_{C_R^U} G(h(z))ds\right| \leq R'\frac{C}{(1 + R)^p} \to 0 \qquad \text{as } R \to \infty. \tag{11}$$

Letting $R\to\infty$ in (9), and using (10), (11), we obtain

$$\int_{\mathbb{R}} G(h(x))dx \geq 0 \tag{12}$$

provided

$$\int_{C_R^L} G(h(z))ds \to \int_{\mathbb{R}} G(h(x))dx \qquad \text{as } R\to\infty. \tag{13}$$

To show (13), using parametric equations, the integral $\int_{C_R^L} G(h(z))ds$ is equal to

$$\int_{-R'\sqrt{2}/2}^{R'\sqrt{2}/2} G\bigg(h\bigg(x+iR-iR'\sqrt{1-\frac{x^2}{R'^2}}\bigg)\bigg)\frac{dx}{\sqrt{1-\frac{x^2}{R'^2}}}. \tag{14}$$

In view of (8), for all $R>100$, the integrand in (14) is bounded by the integrable function $C_f(1+|x|)^{-p}$ since $\sqrt{1-\frac{x^2}{R'^2}}$ is bounded from below by $\sqrt{1/2}$ in the range of integration. Then the Lebesgue dominated convergence theorem gives that (14) converges to

$$\int_{\mathbb{R}} G(h(x))dx \tag{15}$$

as $R\to\infty$.

Then replace z with $h(x)=f(x)+iH(f)(x)$ in (5) and integrate (5) with respect to x, we get

$$\int_{\mathbb{R}} |af(x)+bH(f)(x)|^p dx \leq B_p\int_{\mathbb{R}} |f(x)|^p dx - \int_{\mathbb{R}} G(h(x))dx. \tag{16}$$

So by (12) we obtain

$$\|(aI+bH)f\|_{L^p(\mathbb{R})}^p \leq B_p\|f\|_{L^p(\mathbb{R})}^p. \tag{17}$$

5. The sharpness of the constant B_p

To deduce that the constant B_p is sharp, we need to show

$$\|aI+bH\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})}^p \geq B_p. \tag{18}$$

The proof of (18) relies on finding suitable analytic functions in H^p of the upper half space that will serve as approximate extremals. Unlike the case of the circle, where the functions $\big((1+z)/(1-z)\big)^{1/p-\epsilon}$ in H^p of the unit disc serve this purpose for all $1<p<\infty$ (see [7]) as $\epsilon\downarrow 0$, we need to consider the cases $p<2$ and $p>2$ separately.

Case 1: $1<p<2$. Recall the analytic function used in [5] (also used in [12]),

$$F(z)=(z+1)^{-1}\bigg(i\frac{z+1}{z-1}\bigg)^{2\gamma/\pi}$$

on the upper half plane. If $1<p<2$ and $\pi/2p'<\gamma<\pi/2p$, where $p'=p/(p-1)$, then $F(z)$ belongs to H^p (the Hardy Spaces) in the upper half plane. Let

$$f_\gamma(x)=\frac{1}{x+1}\bigg(\frac{|x+1|}{|x-1|}\bigg)^{2\gamma/\pi}\cos\gamma,$$

then we have

$$F(x+i0) = f_\gamma(x) + i \begin{cases} \frac{1}{x+1}\left(\frac{|x+1|}{|x-1|}\right)^{2\gamma/\pi}\sin\gamma & \text{when } |x|>1, \\ -\frac{1}{x+1}\left(\frac{|x+1|}{|x-1|}\right)^{2\gamma/\pi}\sin\gamma & \text{when } |x|<1, \end{cases}$$

and since this is equal to the boundary values of a holomorphic function on the upper half plane, it follows that

$$H(f_\gamma)(x) = \begin{cases} (\tan\gamma) f_\gamma(x) & \text{when } |x|>1, \\ -(\tan\gamma) f_\gamma(x) & \text{when } |x|<1, \end{cases}$$

So consider a function of the form $g_\gamma = \alpha f_\gamma + \beta H(f_\gamma)$, where $\alpha, \beta \in \mathbb{R}$. Notice that $H(g_\gamma) = \alpha H(f_\gamma) - \beta f_\gamma$, and the function $(|x-1|^{-\frac{2\gamma}{\pi}}|x+1|^{\frac{2\gamma}{\pi}-1})^p$ is integrable over the entire line since $\pi/2p' < \gamma < \pi/2p$, so for fixed α, β we have

$$\begin{aligned} \frac{\|(aI+bH)g_\gamma\|^p_{L^p(\mathbb{R})}}{\|g_\gamma\|^p_{L^p(\mathbb{R})}} &= \frac{\int_{\mathbb{R}} |(a\alpha - b\beta)f_\gamma + (a\beta + b\alpha)H(f_\gamma)|^p dx}{\int_{\mathbb{R}} |\alpha f_\gamma + \beta H(f_\gamma)|^p dx} \\ &= \frac{|(a\alpha - b\beta) + (a\beta + b\alpha)\tan\gamma|^p A_\gamma + |(a\alpha - b\beta) - (a\beta + b\alpha)\tan\gamma|^p B_\gamma}{|\alpha + \beta\tan\gamma|^p A_\gamma + |\alpha - \beta\tan\gamma|^p B_\gamma} \end{aligned}$$

where $A_\gamma = \int_{|x|>1} |f_\gamma(x)|^p dx$, $B_\gamma = \int_{|x|<1} |f_\gamma(x)|^p dx$. It is easy to get $A_\gamma \geq B_\gamma$, so

$$\begin{aligned} &\frac{\|(aI+bH)g_\gamma\|^p_{L^p(\mathbb{R})}}{\|g_\gamma\|^p_{L^p(\mathbb{R})}} \\ \geq\ & \frac{B_\gamma}{A_\gamma}\frac{|(a\alpha - b\beta) + (a\beta + b\alpha)\tan\gamma|^p + |(a\alpha - b\beta) - (a\beta + b\alpha)\tan\gamma|^p}{|\alpha + \beta\tan\gamma|^p + |\alpha - \beta\tan\gamma|^p}, \end{aligned} \tag{19}$$

and

$$\begin{aligned} &\frac{\|(aI+bH)g_\gamma\|^p_{L^p(\mathbb{R})}}{\|g_\gamma\|^p_{L^p(\mathbb{R})}} \\ \leq\ & \frac{A_\gamma}{B_\gamma}\frac{|(a\alpha - b\beta) + (a\beta + b\alpha)\tan\gamma|^p + |(a\alpha - b\beta) - (a\beta + b\alpha)\tan\gamma|^p}{|\alpha + \beta\tan\gamma|^p + |\alpha - \beta\tan\gamma|^p}. \end{aligned} \tag{20}$$

Now we argue that

$$\lim_{\gamma \to \frac{\pi}{2p}} \frac{A_\gamma}{B_\gamma} = 1. \tag{21}$$

In fact, by the second mean value theorem for definite integrals, there exists $\varepsilon \in (\delta, 1)$ where $0 < \delta < 1$ so that

$$\frac{\displaystyle\int_\delta^1 |x|^{p-2}\frac{|x+1|^{\frac{2\gamma p}{\pi}-p}}{|x-1|^{\frac{2\gamma p}{\pi}}}dx}{\displaystyle\int_\delta^1 \frac{|x+1|^{\frac{2\gamma p}{\pi}-p}}{|x-1|^{\frac{2\gamma p}{\pi}}}dx} = \frac{\frac{1}{\delta^{2-p}}\displaystyle\int_\delta^\varepsilon \frac{|x+1|^{\frac{2\gamma p}{\pi}-p}}{|x-1|^{\frac{2\gamma p}{\pi}}}dx + \int_\varepsilon^1 \frac{|x+1|^{\frac{2\gamma p}{\pi}-p}}{|x-1|^{\frac{2\gamma p}{\pi}}}dx}{\displaystyle\int_\delta^\varepsilon \frac{|x+1|^{\frac{2\gamma p}{\pi}-p}}{|x-1|^{\frac{2\gamma p}{\pi}}}dx + \int_\varepsilon^1 \frac{|x+1|^{\frac{2\gamma p}{\pi}-p}}{|x-1|^{\frac{2\gamma p}{\pi}}}dx}.$$

Since $\int_\varepsilon^1 |x+1|^{\frac{2\gamma p}{\pi}-p}|x-1|^{-\frac{2\gamma p}{\pi}}dx \to \infty$ as $\gamma \to \frac{\pi}{2p}$, we get

$$\lim_{\gamma\to\frac{\pi}{2p}} \frac{\displaystyle\int_\delta^1 |x|^{p-2}|x+1|^{\frac{2\gamma p}{\pi}-p}|x-1|^{-\frac{2\gamma p}{\pi}}dx}{\displaystyle\int_\delta^1 |x+1|^{\frac{2\gamma p}{\pi}-p}|x-1|^{-\frac{2\gamma p}{\pi}}dx} = 1.$$

Clearly this implies (21). Combining (19) with (20) we obtain

$$\begin{aligned} &\|aI+bH\|^p_{L^p(\mathbb{R})\to L^p(\mathbb{R})} \\ \geq\ & \max_{\alpha,\beta\in\mathbb{R}} \left(\frac{|(a\alpha-b\beta)+(a\beta+b\alpha)\tan\gamma'|^p+|(a\alpha-b\beta)-(a\beta+b\alpha)\tan\gamma'|^p}{|\alpha+\beta\tan\gamma'|^p+|\alpha-\beta\tan\gamma'|^p}\right)^{\frac{1}{p}}, \end{aligned}$$

where $\gamma' = \frac{\pi}{2p}$. Letting $x=\alpha/\beta$ in (1), we see that (18) holds, therefore the constant B_p is sharp for $1<p<2$.

Case 2: $2<p<\infty$. In this case, the function $(|x-1|^{-\frac{2\gamma}{\pi}}|x+1|^{\frac{2\gamma}{\pi}-1})^p$ used in Case 1 fails to be integrable over the entire line. So we consider the following analytic function:

$$F(z) = \big(i(z^2-1)\big)^{-\frac{2\gamma}{\pi}},$$

which belongs to H^p in the upper half plane when $2<p<\infty$ and $\pi/4p<\gamma<\pi/2p$. Let

$$f_\gamma(x) = |x+1|^{-\frac{2\gamma}{\pi}}|x-1|^{-\frac{2\gamma}{\pi}}\cos\gamma,$$

then we have

$$F(x+i0) = f_\gamma(x) + i\begin{cases} -|x+1|^{-\frac{2\gamma}{\pi}}|x-1|^{-\frac{2\gamma}{\pi}}\sin\gamma & \text{when } |x|>1, \\ \ \ |x+1|^{-\frac{2\gamma}{\pi}}|x-1|^{-\frac{2\gamma}{\pi}}\sin\gamma & \text{when } |x|<1. \end{cases}$$

It follows that

$$H(f_\gamma)(x) = \begin{cases} (\tan\gamma)f_\gamma(x) & \text{when } |x|<1, \\ -(\tan\gamma)f_\gamma(x) & \text{when } |x|>1. \end{cases}$$

Consider the function $g_\gamma = \alpha f_\gamma + \beta H(f_\gamma)$, where $\alpha,\beta\in\mathbb{R}$. Notice that the function $(|x-1|^{-\frac{2\gamma}{\pi}}|x+1|^{-\frac{2\gamma}{\pi}})^p$ is integrable over the entire line since $\pi/4p<\gamma<\pi/2p$, so for fixed α,β we have

$$\begin{aligned} &\frac{\|(aI+bH)g_\gamma\|^p_{L^p(\mathbb{R})}}{\|g_\gamma\|^p_{L^p(\mathbb{R})}} \\ =\ & \frac{|(a\alpha-b\beta)+(a\beta+b\alpha)\tan\gamma|^pA_\gamma+|(a\alpha-b\beta)-(a\beta+b\alpha)\tan\gamma|^pB_\gamma}{|\alpha+\beta\tan\gamma|^pA_\gamma+|\alpha-\beta\tan\gamma|^pB_\gamma}, \end{aligned}$$

where $A_\gamma = \int_{|x|<1}|f_\gamma(x)|^pdx$, $B_\gamma=\int_{|x|>1}|f_\gamma(x)|^pdx$. It is easy to see $A_\gamma\leq B_\gamma$, so

$$\begin{aligned} &\frac{\|(aI+bH)g_\gamma\|^p_{L^p(\mathbb{R})}}{\|g_\gamma\|^p_{L^p(\mathbb{R})}} \\ \leq\ & \frac{B_\gamma}{A_\gamma}\frac{|(a\alpha-b\beta)+(a\beta+b\alpha)\tan\gamma|^p+|(a\alpha-b\beta)-(a\beta+b\alpha)\tan\gamma|^p}{|\alpha+\beta\tan\gamma|^p+|\alpha-\beta\tan\gamma|^p}, \end{aligned} \tag{22}$$

and

$$\begin{aligned}(23)\qquad &\frac{\|(aI+bH)g_\gamma\|^p_{L^p(\mathbb{R})}}{\|g_\gamma\|^p_{L^p(\mathbb{R})}}\\ \geq\ &\frac{A_\gamma}{B_\gamma}\frac{|(a\alpha-b\beta)+(a\beta+b\alpha)\tan\gamma|^p+|(a\alpha-b\beta)-(a\beta+b\alpha)\tan\gamma|^p}{|\alpha+\beta\tan\gamma|^p+|\alpha-\beta\tan\gamma|^p}.\end{aligned}$$

By the second mean value theorem for definite integrals, there exists $\varepsilon\in(\delta,1)$ where $0<\delta<1$ so that

$$\begin{aligned}&\frac{\displaystyle\int_\delta^1 |x|^{\frac{4\gamma p}{\pi}-2}|x+1|^{-\frac{2\gamma p}{\pi}}|x-1|^{-\frac{2\gamma p}{\pi}}\,dx}{\displaystyle\int_\delta^1 |x+1|^{-\frac{2\gamma p}{\pi}}|x-1|^{-\frac{2\gamma p}{\pi}}\,dx}\\ =\ &\frac{\delta^{\frac{4\gamma p}{\pi}-2}\displaystyle\int_\delta^\varepsilon |x+1|^{-\frac{2\gamma p}{\pi}}|x-1|^{-\frac{2\gamma p}{\pi}}\,dx+\int_\varepsilon^1 |x+1|^{-\frac{2\gamma p}{\pi}}|x-1|^{-\frac{2\gamma p}{\pi}}\,dx}{\displaystyle\int_\delta^\varepsilon |x+1|^{-\frac{2\gamma p}{\pi}}|x-1|^{-\frac{2\gamma p}{\pi}}\,dx+\int_\varepsilon^1 |x+1|^{-\frac{2\gamma p}{\pi}}|x-1|^{-\frac{2\gamma p}{\pi}}\,dx}.\end{aligned}$$

Since $\int_\varepsilon^1 |x+1|^{-\frac{2\gamma p}{\pi}}|x-1|^{-\frac{2\gamma p}{\pi}}\,dx\to\infty$ as $\gamma\to\frac{\pi}{2p}$, we have

$$\lim_{\gamma\to\frac{\pi}{2p}}\frac{\displaystyle\int_\delta^1 |x|^{\frac{4\gamma p}{\pi}-2}|x+1|^{-\frac{2\gamma p}{\pi}}|x-1|^{-\frac{2\gamma p}{\pi}}\,dx}{\displaystyle\int_\delta^1 |x+1|^{-\frac{2\gamma p}{\pi}}|x-1|^{-\frac{2\gamma p}{\pi}}\,dx}=1.$$

This implies

$$\lim_{\gamma\to\frac{\pi}{2p}}\frac{B_\gamma}{A_\gamma}=1.$$

Combining (22) and (23) we obtain

$$\begin{aligned}&\|aI+bH\|^p_{L^p(\mathbb{R})\to L^p(\mathbb{R})}\\ \geq\ &\max_{\alpha,\beta\in\mathbb{R}}\left(\frac{|(a\alpha-b\beta)+(a\beta+b\alpha)\tan\frac{\pi}{2p}|^p+|(a\alpha-b\beta)-(a\beta+b\alpha)\tan\frac{\pi}{2p}|^p}{|\alpha+\beta\tan\frac{\pi}{2p}|^p+|\alpha-\beta\tan\frac{\pi}{2p}|^p}\right)^{\frac{1}{p}}.\end{aligned}$$

Letting $x=\alpha/\beta$ in (1), so (18) holds, therefore the constant B_p is sharp for $2<p<\infty$. □

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