An improvement of the Marcinkiewicz multiplier theorem

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Abstract

We provide an improvement of the Marcinkiewicz multiplier theorem on $\mathbb{R}^n$ by relaxing the integrability hypothesis on the multiplier. In particular, when $r > 1$ and $\min(s_1, \dots, s_n) > 1/r$, we replace the product-type Sobolev norm $\big\|(I-\partial_1^2)^{s_1/2}\cdots(I-\partial_n^2)^{s_n/2}[\Phi\, g]\big\|_{L^r}$ by the larger Sobolev norm $\big\|(I-\partial_1^2)^{s_1/2}\cdots(I-\partial_n^2)^{s_n/2}[\Phi\, g]\big\|_{L^{q,1}}$ built upon the Lorentz space with first indices $q = 1/\min(s_1, \dots, s_n)$ and 1. Here $s_i > 0$ are distinct numbers and Φ is a function with compact support away from the origin.

1 Introduction

Given any function σ in $L^\infty(\mathbb{R}^n)$ we consider the multiplier operator

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)\sigma(\xi)e^{2\pi i x\cdot\xi}d\xi$$

initially defined for functions f in the Schwartz class on $\mathbb{R}^n$. Here $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}$ is the Fourier transform of f. A classical problem in harmonic analysis is to find optimal conditions on σ such that T_σ, initially defined on Schwartz functions, admits a bounded extension on $L^p(\mathbb{R}^n)$ for some $p \in (1, \infty)$.

There are a few known conditions on σ that ensure the $L^p(\mathbb{R}^n)$ boundedness of T_σ. Most prominent results in the area are the classical versions of Mikhlin's theorem [18], Hörmander's multiplier theorem [14], and the Marcinkiewicz multiplier theorem [17]. Recently, Grafakos and Slavíková [13], provided a version of the Hörmander multiplier theorem in [12] in which the classical assumption

$$\sup_{k\in\mathbb{Z}} \Big\|(I-\Delta)^{\frac{s}{2}}[\widehat{\Psi}(\cdot)\sigma(2^k\cdot)]\Big\|_{L^r(\mathbb{R}^n)} < \infty, \quad 1 \le r \le 2, \quad r > n/s$$

is replaced by the weaker condition

$$\sup_{k\in\mathbb{Z}} \Big\|(I-\Delta)^{\frac{s}{2}}[\widehat{\Psi}(\cdot)\sigma(2^k\cdot)]\Big\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)} < \infty,$$

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and this implies boundneness for T_σ from $L^p(\mathbb{R}^n)$ to itself for $|\frac{1}{p}-\frac{1}{2}| < s$. Here Δ is the Laplacian, $(I-\Delta)^{s/2}$ is the operator given by multiplication by $(1+4\pi^2|\xi|^2)^{s/2}$ on the Fourier transform side, and Ψ is a Schwartz function whose Fourier transform is supported in the annulus $\{\xi : 1/2 < \xi < 2\}$ and satisfies $\sum_{j\in\mathbb{Z}} \widehat{\Psi}(2^{-j}\xi) = 1$ for all $\xi \neq 0$. Also $L^{\frac{n}{s},1}$ is the Lorentz space with indices n/s and 1. Moreover, Slavíková [19] obtained an example indicating that $L^p(\mathbb{R}^n)$ boundedness fails on the line segments $|\frac{1}{p}-\frac{1}{2}| = \frac{s}{n}$.

It is natural to consider an analogous version for the Marcinkiewicz multiplier theorem. The classical version of this theorem appears for instance in Stein [20, Page 109]. As of this writing, the strongest version of this theorem in terms of the weakest smoothness of the multiplier says that if

$$\sup_{j_1,\dots,j_n\in\mathbb{Z}} \Big\| \prod_{\rho=1}^{n} (I-\partial_\rho^2)^{\frac{s_\rho}{2}} \Big[\prod_{k=1}^{n} \widehat{\psi}(\xi_k)\sigma(2^{j_1}\xi_1,\dots,2^{j_n}\xi_n) \Big] \Big\|_{L^r(\mathbb{R}^n,d\xi)} < \infty, \quad (1.1)$$

where $1 \le r < \infty$, $s_\rho > 0$, $\min(s_1,\dots,s_n) > 1/r$, then T_σ maps $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $|\frac{1}{p}-\frac{1}{2}| < \min(s_1,\dots,s_n)$. Here ψ is a Schwartz function on the line whose Fourier transform is supported in $[-2,-1/2]\cup[1/2,2]$ and which satisfies $\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1$ for all $\xi \neq 0$. A proof of this result can be found in Grafakos and Slavíková [12]. Earlier versions were given by Carbery [2], who considered the case in which the multiplier lies in a product-type L^2-based Sobolev space, and Carbery and Seeger [3, Remark after Prop. 6.1], who considered the case $s_1 = \cdots = s_n > |\frac{1}{p}-\frac{1}{2}| = \frac{1}{r}$. The positive direction of Carbery and Seeger's result in the range $|\frac{1}{p}-\frac{1}{2}| < \frac{1}{r}$ also appeared in [4, Condition (1.4)]; notice that the range is expressed in terms of the integrability of the multiplier and not in terms of its smoothness. Another extension of the Marcinkiewicz multiplier theorem in a different direction was obtained by Hytönen [15].

Our objective in this paper is to relax the condition on the multiplier to a Lorentz type inequality. In this work we will be restricting attention to case, in which the smoothness indices are different for each variable. The more difficult case in which two smoothness indices are equal, say $s_1 = s_2$ in $\mathbb{R}^2$, and the significantly more complicated higher-dimensional case in which some, but not all, indices are equal will be considered in a subsequent publication.

The Lorentz space $L^{\frac{1}{s},1}(\mathbb{R}^n)$ is defined in terms of the norm

$$\|f\|_{L^{\frac{1}{s},1}(\mathbb{R}^n)} = \int_0^\infty f^*(r) r^{s-1}\, dr,$$

where f^* is the nonincreasing rearrangement of the function f, namely, the unique nonincreasing left-continuous function on $(0,\infty)$ that is equimeasurable with f and is given explicitly by

$$f^*(t) = \inf\left\{ r \ge 0 : \ |\{y \in \mathbb{R}^n : \ |f(y)| > r\}| < t \right\}.$$

We now introduce some notation. We denote by ∂_j differentiation in the jth variable in $\mathbb{R}^n$. For complex numbers s_j with nonnegative real part we define the differential operator

$$\Gamma(s_1,\dots,s_n) := (I-\partial_1^2)^{s_1/2}\cdots(I-\partial_n^2)^{s_n/2},$$

We also define the dilation operator

$$D_{k_1,\dots,k_n} f(x_1,\dots,x_n) = f(2^{k_1}x_1,\dots,2^{k_n}x_n)$$

with respect to the anisotropic set of dilations given by powers of 2 (here $k_j \in \mathbb{Z}$).

We now state the main result of this paper. In dimension $n=2$, Theorem 1.1 was announced in [9]. In this work we undertake the more technically-involved n-dimensional case, providing all details of the proof.

Theorem 1.1. *Let n be a natural number, $1<p<\infty$, and $s_1,\dots,s_n$ be distinct numbers in $(0,1)$. Let ψ be a Schwartz function on real line whose Fourier transform is supported in $[-2,-1/2]\cup[1/2,2]$ and satisfies $\sum_{j\in\mathbb{Z}}\widehat{\psi}\left(2^{-j}y\right)=1$ for all $y\in\mathbb{R}\setminus\{0\}$, and let $\widehat{\Psi}(\xi_1,\dots,\xi_n)=\prod_{k=1}^n\widehat{\psi}(\xi_k)$. If a bounded function σ on $\mathbb{R}^n$ satisfies*

$$\sup_{j_1,\dots,j_n\in\mathbb{Z}} \left\|\Gamma\left(s_1,\dots,s_n\right)\left[\widehat{\Psi}\left(D_{j_1,\dots,j_n}\sigma\right)\right]\right\|_{L^{\frac{1}{\min(s_1,\dots,s_n)},1}(\mathbb{R}^n)} < \infty, \tag{1.2}$$

then T_σ extends to a bounded operator from $L^p(\mathbb{R}^n)$ to itself for all $1<p<\infty$ satisfying

$$\left|\frac{1}{p}-\frac{1}{2}\right| < \min(s_1,\dots,s_n).$$

Moreover, this condition is sharp.

For the sharpness of this result we refer to the example [19], taken in dimension $n=1$, indicating the unboundedness holds on the line $|\frac{1}{p}-\frac{1}{2}|=s_1$. For higher dimensions one may use a product type example of a function of one variable times another function of the remaining variables. Permuting the variables, we may assume that $0<s_1<s_2<\cdots<s_n<1$, and we will do so in the rest of this article.

We begin by proving a version of this theorem when $s_1\in(\frac{1}{2},1)$. Then we use interpolation to extend this result to values of $s_1\le\frac{1}{2}$. In Section 3 we provide a proof of a reduced version of the main theorem. In Section 4 we prove certain lemmas and the interpolation needed to complete the argument.

Throughout this paper, C denotes a constant depending only on inessential parameters.

2 Preliminary material

It is well known that the Hardy-Littlewood maximal operator pointwise controls maximal convolutions. An improved version of this result in the context of

Lorentz spaces is given in [13]. In this paper we work with the strong maximal operator

$$\mathcal{M}(f)(x) = \sup_{R \ni x} \frac{1}{|R|} \int_R |f(y)|dy,$$

where the supremum is taken over all rectangles R in $\mathbb{R}^n$ with sides parallel to the axes containing the point x, instead of balls used for the classical Hardy-Littlewood maximal operator. This maximal function is of weak type (p,p) for $p > 1$ and satisfies an endpoint estimate which resembles a weak type $(1,1)$ with a logarithmic term; see [16] and also [5, Theorem 2]. We also define the following L^q version of $\mathcal{M}$ by:

$$\mathcal{M}_{L^q}(f) = \mathcal{M}(|f|^q)^{\frac{1}{q}}.$$

As mentioned in the introduction we first prove a restriction of Theorem 1.1 when $s_1 \in \left(\frac{1}{2}, 1\right)$. This is as follows:

Proposition 2.1. *Let $n \in \mathbb{N}$ and $\frac{1}{2} < s_1 < \cdots < s_n < 1$. Let ψ be a Schwartz function on real line whose Fourier transform is supported in $1/2 < |\xi| < 2$ and satisfies $\sum_{j\in\mathbb{Z}} \widehat{\psi}\left(2^{-j}\xi\right) = 1$ for $\xi \neq 0$. Define a function on $\mathbb{R}^n$ minus the coordinate planes by setting $\widehat{\Psi}(\xi_1, \dots, \xi_n) = \prod_{k=1}^n \widehat{\psi}(\xi_k)$. If σ is a bounded function on $\mathbb{R}^n$ that satisfies*

$$K = \sup_{j_1,\dots,j_n\in\mathbb{Z}} \left\| \Gamma(s_1, \dots, s_n) \left[\widehat{\Psi}\left(D_{j_1,\dots,j_n}\sigma\right)\right] \right\|_{L^{\frac{1}{s_1},1}(\mathbb{R}^n)} < \infty,$$

then there is a constant C such that for every Schwartz function f on $\mathbb{R}^n$ we have

$$\|T_\sigma(f)\|_{L^p(\mathbb{R}^n)} \leq CK \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.1}$$

Thus T_σ admits a bounded extension on $L^p(\mathbb{R}^n)$ with the same bound for any $1 < p < \infty$.

The proof of this proposition relies on the following estimate.

Lemma 2.2. *Let $0 < 1/q < s_1 < \cdots < s_n < 1$. Then there is a constant C depending on these parameters such that for all measurable functions f on $\mathbb{R}^n$, all for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, and all $j_1, \dots, j_n \in \mathbb{Z}$ we have*

$$\left\| \frac{f(x_1 + 2^{-j_1}y_1, \dots, x_n + 2^{-j_n}y_n)}{(1+|y_1|)^{s_1} \cdots (1+|y_n|)^{s_n}} \right\|_{L^{\frac{1}{s_1},\infty}(dy_1\cdots dy_n)} \leq C\mathcal{M}_{L^q}(f)(x). \tag{2.2}$$

Proof. In proving (2.2) we may assume that $j_1 = \cdots = j_n = 0$ and $(x_1, \dots, x_n) = 0$ as the general case follows by applying the special case to $g(y_1, \dots, y_n) = f(x_1 + 2^{-j_1}y_1, \dots, x_n + 2^{-j_n}y_n)$.

So we only need to prove the following inequality.

$$\left\| \frac{g(y_1, \dots, y_n)}{(1+|y_1|)^{s_1} \cdots (1+|y_n|)^{s_n}} \right\|_{L^{\frac{1}{s_1},\infty}(\mathbb{R}^n)} \leq C\mathcal{M}_{L^q}(g)(0). \tag{2.3}$$

The cases where $\mathcal{M}_{L^q}(g)(0) = \infty$ and 0 can be easily verified, so we can divide the function g by the positive constant $\mathcal{M}_{L^q}(g)(0)$ and can therefore assume that $\mathcal{M}_{L^q}(g)(0) = 1$. For $j_1, \dots, j_n$ nonnegative integers define

$$R_{j_1,\dots,j_n} = \left\{ (y_1, \dots, y_n) \in \mathbb{R}^n : \begin{cases} 2^{j_i} < |y_i| \le 2^{j_i+1} & \text{if } j_i \ge 1 \\ |y_i| \le 1 & \text{if } j_i = 0, \end{cases} \quad i = 1, \dots, n \right\}$$

and notice that the family of rectangles $R_{j_1,\dots,j_n}$ is a tiling of $\mathbb{R}^n$ when $j_1, \dots, j_n$ run over all nonnegative integers.

In the sequel we denote by y the vector $(y_1, \dots, y_n)$. For $a > 0$ and $j_1, \dots, j_n$ nonnegative integers we have

$$|\{ y \in R_{j_1,\dots,j_n} : \ |g(y)| > a \}| \le \frac{1}{a^q} \int_{R_{j_1,\dots,j_n}} |g(y)|^q dy \le \frac{2^{j_1+\dots+j_n+2n}}{a^q}$$

since we are assuming that $\mathcal{M}_{L^q}(g)(0) = 1$. So, in view of the trivial estimate $|R_{j_1,\dots,j_n}| \le 2^{j_1+\dots+j_n+2n}$ we obtain

$$|\{ y \in R_{j_1,\dots,j_n} : \ |g(y)| > a \}| \le 2^{2n} 2^{j_1+\dots+j_n} \min\big(1, a^{-q}\big). \tag{2.4}$$

It follows from (2.4) that for all $j_1, \dots, j_n \ge 0$ we have

$$\begin{aligned} \left| \left\{ y \in R_{j_1,\dots,j_n} : \ \frac{|g(y)|}{(1+|y_1|)^{s_1} \cdots (1+|y_n|)^{s_n}} > a \right\} \right| & \\ \le \ 2^{2n} 2^{j_1+\dots+j_n} \min\left(1, \frac{1}{(a 2^{j_1 s_1 + \cdots j_n s_n})^q}\right). & \end{aligned} \tag{2.5}$$

We split $g = g_0 + g_1$, where $g_0 = g \chi_{R_{0,\dots,0}}$. It will suffice to obtain (2.3) for each one of g_0 and g_1. We begin with g_0. We have

$$\begin{aligned} & \left\| \frac{g_0(y)}{(1+|y_1|)^{s_1} \cdots (1+|y_n|)^{s_n}} \right\|_{L^{\frac{1}{s_1},\infty}} \\ & = \sup_{a>0} a \left| \left\{ y \in R_{0,\dots,0} : \ \frac{g(y)}{(1+|y_1|)^{s_1} \dots (1+|y_n|)^{s_n}} > a \right\} \right|^{s_1} \\ & \le \sup_{a>0} a |\{ y \in R_{0,\dots,0} : \ |g(y)| > a \}|^{s_1} \\ & = \|g\|_{L^{1/s_1,\infty}(R_{0,\dots,0})} \\ & \le C \, \|g\|_{L^q(R_{0,\dots,0})} \\ & \le C' \mathcal{M}_{L^q}(g)(0) \\ & = C', \end{aligned}$$

as $L^q(R_{0,\dots,0})$ embeds in $L^{1/s_1,\infty}(R_{0,\dots,0})$ when $q > 1/s_1$. This proves (2.3) for g_0 in place of g.

We now turn our attention to g_1. Using (2.5) we write

$$\left| \left\{ y \in \mathbb{R}^n : \ \frac{|g_1(y)|}{(1+|y_1|)^{s_1} \cdots (1+|y_n|)^{s_n}} > a \right\} \right|$$

$$\leq \sum_{\substack{j_1,\dots,j_n=0\\ j_1+\dots+j_n>0}}^{\infty} \left|\left\{y\in R_{j_1,\dots,j_n}:\ \frac{|g(y)|}{(1+|y_1|)^{s_1}\cdots(1+|y_n|)^{s_n}}>a\right\}\right|$$
$$\leq \sum_{\substack{j_1,\dots,j_n=0\\ j_1+\dots+j_n>0}}^{\infty} 2^{j_1+\dots+j_n+2n}\min\left(1,\frac{1}{a^q(2^{j_1s_1+\dots+j_ns_n})^q}\right)$$
$$\leq 2^{2n+s_1q+\dots+s_nq}\int\cdots\int_{[0,\infty)^n\setminus[0,1]^n}\min\left[1,\frac{1}{a^q(\max(1,t_1)^{s_1}\cdots\max(1,t_n)^{s_n})^q}\right]dt_1\cdots dt_n$$
$$=: 2^{2n+s_1q+\dots+s_nq}\, I(a,n),$$

where the last inequality follows from the monotonicity of the integrand. Let S be the set of all $(t_1,\dots t_n)\in[0,\infty)^n\setminus[0,1]^n$ such that

$$1\leq \frac{1}{a\max(1,t_1)^{s_1}\cdots\max(1,t_n)^{s_n}}. \tag{2.6}$$

If S is nonempty, then we must have $a\leq 1$. Let us fix a two-set partition $I=\{i_1,\dots,i_m\}$ and $J=\{j_1,\dots,j_k\}$ of $\{1,2,\dots,n\}$. We split S as a union of sets $S_{I,J}$ (ranging over all such pairs of partitions) for which

$$(t_1,\dots,t_n)\in S_{I,J} \iff t_i\leq 1 \text{ for all } i\in I \text{ and } t_j>1 \text{ for all } j\in J. \tag{2.7}$$

Then the n-dimensional measure $|S_{I,J}|$ of $S_{I,J}$ is at most the k-th dimensional measure of the subset of $[1,\infty)^k$

$$S^k_{I,J}=\left\{(t_{j_1},\dots,t_{j_k}):\ t_{j_1}^{s_{j_1}}\cdots t_{j_k}^{s_{j_k}}\leq\frac{1}{a}\right\}$$

as the vector of the remaining m coordinates is contained in the cube $[0,1]^m$ which has m-th dimensional measure equal to 1. Let us assume, without loss of generality, that $s_{j_1}=\min(s_{j_1},\dots,s_{j_k})$. Then

$$\begin{aligned}|S^k_{I,J}| &\leq \int_{t_{j_2}=1}^{\infty}\cdots\int_{t_{j_k}=1}^{\infty}\left|\left\{t_{j_1}:\ 1\leq t_{j_1}\leq a^{-\frac{1}{s_{j_1}}}t_{j_2}^{-\frac{s_{j_2}}{s_{j_1}}}\cdots t_{j_k}^{-\frac{s_{j_k}}{s_{j_1}}}\right\}\right|dt_{j_k}\cdots dt_{j_2}\\ &\leq a^{-\frac{1}{s_{j_1}}}\chi_{a\leq 1}\int_{t_{j_2}=1}^{\infty}\cdots\int_{t_{j_k}=1}^{\infty}t_{j_2}^{-\frac{s_{j_2}}{s_{j_1}}}\cdots t_{j_k}^{-\frac{s_{j_k}}{s_{j_1}}}\,dt_{j_k}\cdots dt_{j_2}\\ &\leq Ca^{-\frac{1}{s_{j_1}}}\chi_{a\leq 1}\\ &\leq Ca^{-\frac{1}{s_1}}\chi_{a\leq 1},\end{aligned}$$

recalling that $s_1=\min\{s_1,s_2,\dots,s_n\}$ and that $\frac{s_{j_2}}{s_{j_1}}>1,\dots,\frac{s_{j_k}}{s_{j_1}}>1$. Summing over all partitions (I,J) of $\{1,2,\dots,n\}$ yields the required estimate for $I(a,n)$ whenever (2.6) holds.

Now let S' be the set of all $(t_1,\dots t_n)\in[0,\infty)^n\setminus[0,1]^n$ such that

$$a>\frac{1}{\max(1,t_1)^{s_1}\cdots\max(1,t_n)^{s_n}}. \tag{2.8}$$

Then S' is complementary to S in $[0,\infty)^n \setminus [0,1]^n$. Writing S' as a union of sets $S'_{I,J}$ over all partitions (I,J) of $\{1,2,\dots,n\}$ as in (2.7), matters reduce to estimating the integral

$$\frac{1}{a^q}\int\cdots\int_{\substack{t_{j_1},\dots,t_{j_k}\geq 1\\ t_{j_1}^{s_{j_1}}\cdots t_{j_k}^{s_{j_k}}>\frac{1}{a}}} \frac{dt_{j_1}\cdots dt_{j_k}}{(t_{j_1}^{s_{j_1}}\cdots t_{j_k}^{s_{j_k}})^q} \tag{2.9}$$

for each subset $\{j_1,\dots,j_k\}$ of $\{1,\dots,n\}$. Now if $a>1$, the integral in (2.9) is over the set $[1,\infty)^k$ and, as $s_{j_1}q>1,\dots,s_{j_k}q>1$, the expression in (2.9) is bounded by

$$C\,a^{-q}\chi_{a>1}\leq C\,a^{-\frac{1}{s_1}},$$

since $q>1/s_1$. So we focus attention to the case $a\leq 1$ in (2.9). Let us again assume, without loss of generality, that $s_{j_1}=\min(s_{j_1},\dots,s_{j_k})$. Then

$$\frac{1}{a^q}\int\cdots\int_{\substack{t_{j_1},\dots,t_{j_k}\geq 1\\ t_{j_1}^{s_{j_1}}\cdots t_{j_k}^{s_{j_k}}>\frac{1}{a}}} \frac{dt_{j_1}\cdots dt_{j_k}}{(t_{j_1}^{s_{j_1}}\cdots t_{j_k}^{s_{j_k}})^q}\leq \frac{1}{a^q}\int_{t_{j_2}=1}^{\infty}\cdots\int_{t_{j_k}=1}^{\infty}\frac{\int_{t_{j_1}=L}^{\infty}\frac{dt_{j_1}}{t_{j_1}^{s_{j_1}q}}}{t_{j_2}^{s_{j_2}q}\cdots t_{j_k}^{s_{j_k}q}}\,dt_{j_k}\cdots dt_{j_2},$$

where $L=(at_{j_2}^{s_{j_2}}\cdots t_{j_k}^{s_{j_k}})^{-1/s_{j_1}}$. As $s_{j_1}q>1$, the t_{j_1} integral is convergent and produces the term

$$c\,a^{q-\frac{1}{s_{j_1}}}t_{j_2}^{s_{j_2}q-\frac{s_{j_2}}{s_{j_1}}}\cdots t_{j_k}^{s_{j_k}q-\frac{s_{j_k}}{s_{j_1}}},$$

which plugged in the $t_{j_2},\dots,t_{j_n}$ integrals yields the expression

$$C'\frac{1}{a^q}\,a^{q-\frac{1}{s_{j_1}}}\chi_{a\leq 1}\leq C'a^{-\frac{1}{s_1}},$$

since $\frac{1}{s_{j_1}}<\frac{1}{s_1}$. The convergence of the $t_{j_2},\dots,t_{j_n}$ integrals relies on the fact that $s_{j_i}>s_{j_1}$ for all $i=1,\dots,k$. (Recall our assumption that all s_i are distinct.) Summing over all partitions (I,J) of $\{1,2,\dots,n\}$ yields the required estimate for $I(a,n)$ whenever (2.8) holds.

These estimates provide the required conclusion for g_1, hence (2.3) holds for both g_0 and g_1, thus it holds for g. □

3 The proof of Proposition 2.1

Proof. Let ψ be as in the statement of Proposition 2.1. Consider a Schwartz function θ on $\mathbb{R}$ with

$$\widehat{\theta}(\xi)=\widehat{\psi}(\xi/2)+\widehat{\psi}(\xi)+\widehat{\psi}(2\xi). \tag{3.1}$$

Then $\widehat{\theta}$ is supported in the annulus $1/4 < |\xi| < 4$ and $\widehat{\theta} = 1$ on the support of ψ. For $j \in \mathbb{Z}$ we define the Littlewood-Paley operators in each variable (associated with the bump ψ) by

$$\Delta_j^{\psi,i}(f)(x) = \int_{\mathbf{R}} f\left(x_1,\ldots,x_{i-1},x_i - y, x_{i+1},\ldots,x_n\right) 2^j \psi\left(2^j y\right) dy$$

with obvious modifications when $i = 1$ or $i = n$. Analogously, we define $\Delta_j^{\theta,i}(f)$ with θ in place of ψ. Since $\widehat{\theta} = 1$ on support of $\widehat{\psi}$, the function $\widehat{\Theta}(\xi_1,\ldots,\xi_n) = \prod_{\rho=1}^n \widehat{\theta}(2^{-j_\rho}\xi_\rho)$ equals 1 on the support of $\widehat{\Psi}(\xi_1,\ldots,\xi_n) = \prod_{\rho=1}^n \widehat{\psi}(2^{-j_\rho}\xi_\rho)$ for any $j_1,\ldots,j_n \in \mathbb{Z}$ and so we can write:

$$\begin{aligned}
&\Delta_{j_1}^{\psi,1}\cdots\Delta_{j_n}^{\psi,n} T_\sigma(f)\left(x_1,\ldots,x_n\right)\\
&= \int_{\mathbb{R}^n} \widehat{f}(\xi)\big(D_{-j_1,\ldots,-j_n}\widehat{\Theta}\big)(\xi)\big(D_{-j_1,\ldots,-j_n}\widehat{\Psi}\big)(\xi)\sigma(\xi)e^{2\pi i(x_1\xi_1+\cdots+x_n\xi_n)}d\xi\\
&= \int_{\mathbb{R}^n} (\Delta_{j_1}^{\theta,1}\cdots\Delta_{j_n}^{\theta,n} f)\widehat{\ }(\xi)\big(D_{-j_1,\ldots,-j_n}\widehat{\Psi}\big)(\xi)\sigma(\xi)e^{2\pi i(x_1\xi_1+\cdots+x_n\xi_n)}d\xi\\
&= \int_{\mathbb{R}^n} 2^{j_1+\cdots+j_n}(\Delta_{j_1}^{\theta,1}\cdots\Delta_{j_n}^{\theta,n} f)\widehat{\ }(2^{j_1}\xi_1',\ldots,2^{j_n}\xi_n')\\
&\qquad\qquad \widehat{\Psi}(\xi')\big(D_{j_1,\ldots,j_n}\sigma\big)(\xi')\, e^{2\pi i(2^{j_1}x_1\xi_1'+\cdots+2^{j_n}x_n\xi_n')}d\xi'\\
&= \int_{\mathbb{R}^n} (\Delta_{j_1}^{\theta,1}\cdots\Delta_{j_n}^{\theta,n} f)(2^{-j_1}y_1',\ldots,2^{-j_n}y_n')\\
&\qquad\qquad \Big[\widehat{\Psi}\big(D_{j_1,\ldots,j_n}\sigma\big)\Big]\widehat{\ }(y_1' - 2^{j_1}x_1,\ldots,y_n' - 2^{j_n}x_n)dy_1'\cdots dy_n'\\
&= \int_{\mathbb{R}^n} (\Delta_{j_1}^{\theta,1}\cdots\Delta_{j_n}^{\theta,n} f)(2^{-j_1}y_1 + x_1,\ldots,2^{-j_n}y_n + x_n)\Big[\widehat{\Psi}\big(D_{j_1,\ldots,j_n}\sigma\big)\Big](y)dy\\
&= \int_{\mathbb{R}^n} \frac{\big(\Delta_{j_1}^{\theta,1}\cdots\Delta_{j_n}^{\theta,n} f\big)(2^{-j_1}y_1 + x_1,\ldots,2^{-j_n}y_n + x_n)}{(1+|y_1|)^{s_1}\cdots(1+|y_n|)^{s_n}}\\
&\qquad\qquad (1+|y_1|)^{s_1}\cdots(1+|y_n|)^{s_n}\Big[\widehat{\Psi}\big(D_{j_1,\ldots,j_n}\sigma\big)\Big]\widehat{\ }(y)dy.
\end{aligned}$$

Here $y = (y_1,\ldots,y_n)$. In view of Hölder's inequality for Lorentz spaces, the last displayed expression is bounded by

$$\left\|\frac{\big(\Delta_{j_1}^{\theta,1}\cdots\Delta_{j_n}^{\theta,n} f\big)(2^{-j_1}y_1 + x_1,\ldots,2^{-j_n}y_n + x_n)}{(1+|y_1|)^{s_1}\cdots(1+|y_n|)^{s_n}}\right\|_{L^{\frac{1}{s_1},\infty}(\mathbb{R}^n,dy)} \times$$
$$\left\|(1+|y_1|)^{s_1}\cdots(1+|y_n|)^{s_n}\Big[\widehat{\Psi}\big(D_{j_1,\ldots,j_n}\sigma\big)\Big]\widehat{\ }(y)\right\|_{L^{(\frac{1}{s_1})',1}(\mathbb{R}^n,dy)}.$$

We estimate the second term by the Hausdorff-Young inequality for Lorentz spaces[1]

$$\left\|(1+|y_1|)^{s_1}\cdots(1+|y_n|)^{s_n}\Big[\widehat{\Psi}\big(D_{j_1,\ldots,j_n}\sigma\big)\Big]\widehat{\ }(y)\right\|_{L^{(\frac{1}{s_1})',1}(\mathbb{R}^n)}$$

[1] this can be proved using the off-diagonal Marcinkiewicz interpolation theorem [7] using the classical Hausdorff-Young inequality $\|\widehat{f}\|_{L^{p_i'}} \le \|f\|_{L^{p_i}}$ with $1 < p_1 < p < p_2 < 2$, $i = 1, 2$.

$$
\begin{aligned}
&\le C \left\| \prod_{\rho=1}^{n} (1+4\pi^2|y_\rho|^2)^{\frac{s_\rho}{2}} \left[\widehat{\Psi}\big(D_{j_1,\dots,j_n}\sigma\big)\right]\widehat{\ }(y) \right\|_{L^{(\frac{1}{s_1})',1}(\mathbb{R}^n)} \\
&\le C \left\| \Gamma\left(s_1,\dots,s_n\right) \left[\widehat{\Psi}\big(D_{j_1,\dots,j_n}\sigma\big)\right] \right\|_{L^{\frac{1}{s_1},1}(\mathbb{R}^n)} \\
&\le CK, \qquad (3.2)
\end{aligned}
$$

in view of hypothesis (1.2). Pick a number q such that $\frac{1}{s_1} < q < 2$. Then Lemma 2.2 yields

$$
\begin{aligned}
&\left\| \frac{\big(\Delta_{j_1}^{\theta,1}\cdots\Delta_{j_n}^{\theta,n} f\big)(2^{-j_1}y_1+x_1,\dots,2^{-j_n}y_n+x_n)}{(1+|y_1|)^{s_1}\cdots(1+|y_n|)^{s_n}} \right\|_{L^{\frac{1}{s_1},\infty}(\mathbb{R}^n,dy)} \qquad (3.3)\\
&\qquad\qquad \le C\mathcal{M}_{L^q}\big(|\Delta_{j_1}^{\theta,1}\cdots\Delta_{j_n}^{\theta,2n} f|\big)(x_1,\dots,x_n).
\end{aligned}
$$

Combining estimates (3.2) and (3.3), for any $x\in\mathbb{R}^n$ we obtain

$$
|\Delta_{j_1}^{\psi,1}\cdots\Delta_{j_n}^{\psi,n} T_\sigma(f)\,(x)| \le CK\mathcal{M}_{L^q}\big(|\Delta_{j_1}^{\theta,1}\cdots\Delta_{j_n}^{\theta,n} f|\big)(x).
$$

We may assume that $p\ge 2$ as the case $1<p<2$ follows by duality. For $p\ge 2$, by the Littlewood-Paley theorem and the Fefferman-Stein [6] inequality we write:

$$
\begin{aligned}
\|T_\sigma(f)\|_{L^p(\mathbb{R}^n)} &\le C \left\| \Big(\sum_{j_1,\dots,j_n\in\mathbb{Z}} \Big| \Delta_{j_1}^{\psi,1}\cdots\Delta_{j_n}^{\psi,n} T_\sigma(f)\Big|^2 \Big)^{\frac12} \right\|_{L^p(\mathbb{R}^n)} \\
&\le CK \left\| \Big(\sum_{j_1,\dots,j_n\in\mathbb{Z}} \Big| \mathcal{M}_{L^q}\left(|\Delta_{j_1}^{\theta,1}\cdots\Delta_{j_n}^{\theta,n} f|\right)\Big|^2 \Big)^{\frac12} \right\|_{L^p(\mathbb{R}^n)} \\
&\le CK \left\| \Big(\sum_{j_1,\dots,j_n\in\mathbb{Z}} \left(\mathcal{M}|\Delta_{j_1}^{\theta,1}\cdots\Delta_{j_n}^{\theta,n} f|^q\right)^{\frac{2}{q}} \Big)^{\frac12} \right\|_{L^p(\mathbb{R}^n)} \\
&\le CK \left\| \Big(\sum_{j_1,\dots,j_n\in\mathbb{Z}} \left(\mathcal{M}|\Delta_{j_1}^{\theta,1}\cdots\Delta_{j_n}^{\theta,n} f|^q\right)^{\frac{2}{q}} \Big)^{\frac{q}{2}} \right\|_{L^{\frac{p}{q}}(\mathbb{R}^n)}^{\frac1q} \\
&\le C'K \left\| \Big(\sum_{j_1,\dots,j_n\in\mathbb{Z}} |\Delta_{j_1}^{\theta,1}\cdots\Delta_{j_n}^{\theta,n} f|^{q\cdot\frac{2}{q}} \Big)^{\frac{q}{2}} \right\|_{L^{\frac{p}{q}}(\mathbb{R}^n)}^{\frac1q} \\
&\le C''K \left\| \Big(\sum_{j_1,\dots,j_n\in\mathbb{Z}} |\Delta_{j_1}^{\theta,1}\cdots\Delta_{j_n}^{\theta,n} f|^2 \Big)^{\frac12} \right\|_{L^p(\mathbb{R}^n)} \\
&\le C'''K\,\|f\|_{L^p(\mathbb{R}^n)}\,.
\end{aligned}
$$

The Fefferman-Stein inequality can be used as $1<p/q<\infty$ and $1<2/q<\infty$. This completes the proof of Proposition 2.1. □

4 Interpolation

Next we state the interpolation result needed to complete our proof. We denote by $\mathscr{C}_0^\infty(\mathbb{R}^n)$ the space of smooth functions with compact support on $\mathbb{R}^n$.

Proposition 4.1. *Suppose that $1 < p_0, p_1 < \infty$, $0 < s_1^0, \dots, s_n^0, s_1^1, \dots, s_n^1 < \infty$ and $s_1^k < \cdots < s_n^k$ for $k = 0, 1$. Let ψ be a Schwartz function whose Fourier transform is supported in the cube $[-2, -1/2] \cup [1/2, 2]$ which satisfies $\sum_{j\in\mathbb{Z}} \widehat{\psi}\left(2^{-j}y\right) = 1$ for $y \neq 0$, and define $\widehat{\Psi}(\xi_1, \dots, \xi_n) = \widehat{\psi}(\xi_1) \cdots \widehat{\psi}(\xi_n)$. Suppose that for all functions f in $\mathscr{C}_0^\infty(\mathbb{R}^n)$ we have*

$$\|T_\sigma(f)\|_{L^{p_0}} \le K_0 \sup_{j_1,\dots,j_n\in\mathbb{Z}} \left\|\Gamma\left(s_1^0, \dots, s_n^0\right)\left[\widehat{\Psi}\, D_{j_1,\dots,j_n}\sigma\right]\right\|_{L^{\frac{1}{s_1^0},1}} \|f\|_{L^{p_0}}$$

and

$$\|T_\sigma(f)\|_{L^{p_1}} \le K_1 \sup_{j_1,\dots,j_n\in\mathbb{Z}} \left\|\Gamma\left(s_1^1, \dots, s_n^1\right)\left[\widehat{\Psi}\, D_{j_1,\dots,j_n}\sigma\right]\right\|_{L^{\frac{1}{s_1^1},1}} \|f\|_{L^{p_1}}.$$

Then for each $0 < \theta < 1$ there exists a constant

$$C_* = C(\psi, \theta, p_0, p_1, s_1^0, \dots, s_n^0, s_1^1, \dots, s_n^1)$$

such that for all $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ we have

$$\|T_\sigma(f)\|_{L^p} \le C_* K_0^{1-\theta} K_1^\theta \sup_{j_1,\dots,j_n\in\mathbb{Z}} \left\|\Gamma\left(s_1, \dots, s_n\right)\left[\widehat{\Psi}\, D_{j_1,\dots,j_n}\sigma\right]\right\|_{L^{\frac{1}{s_1},1}} \|f\|_{L^p},$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad s_1 = (1-\theta)s_1^0 + \theta s_1^1, \ \dots \ , s_n = (1-\theta)s_n^0 + \theta s_n^1.$$

This proposition will be proved in the next section but its proof is based on a few lemmas proved in this section.

Lemma 4.2. *For $0 < s_1 < \cdots < s_n < 1$ there is a constant $C_{s_1,\dots,s_n}$ such that for all functions f in $L^{\frac{1}{s_1},1}(\mathbb{R}^n)$ we have*

$$\|\Gamma(-s_1, \dots, -s_n) f\|_{L^\infty(\mathbb{R}^n)} \le C_{s_1,\dots,s_n} \|f\|_{L^{\frac{1}{s_1},1}(\mathbb{R}^n)}. \tag{4.1}$$

Proof of Lemma 4.2. Recall that for $0 < s < 1$ the one-dimensional kernel G_s of $(I - \partial^2)^{-s/2}$ (called the Bessel potential) satisfies

$$0 < G_s(x) \le C_s \begin{cases} e^{-\frac{|x|}{2}} & \text{if } |x| > 2 \\ |x|^{-1+s} & \text{if } |x| \le 2 \end{cases}$$

(see [8]). It follows that G_s lies in $L^{(1/s)',\infty}(\mathbb{R})$.

Set $G_{s_1}\otimes\cdots\otimes G_{s_n}(y_1,\dots,y_n)=G_{s_1}(y_1)\cdots G_{s_n}(y_n)$. We have

$$\begin{aligned}|\Gamma(-s_1,\dots,-s_n)\,f(x)| &\le \int_{\mathbb{R}^n}(G_{s_1}\otimes\cdots\otimes G_{s_n})(y)\,|f(x-y)|dy\\ &\le C\,\|G_{s_1}\otimes\cdots\otimes G_{s_n}\|_{L^{(\frac{1}{s_1})',\infty}(\mathbb{R}^n)}\,\|f\|_{L^{\frac{1}{s_1},1}(\mathbb{R}^n)},\end{aligned}$$

where in the last step we have used Hölder's inequality for Lorentz spaces. It will then suffice to show that

$$\sup_{\lambda>0}\lambda\,|\{y\in\mathbb{R}^n:\ G_{s_1}(y_1)\cdots G_{s_n}(y_n)>\lambda\}|^{1-s_1}\le C'_{s_1,\dots,s_n}<\infty.$$

To prove this we write

$$\begin{aligned}&|\{y\in\mathbb{R}^n:\ G_{s_1}(y_1)\cdots G_{s_n}(y_n)>\lambda\}|\\ &=\int_{y_2\in\mathbb{R}}\cdots\int_{y_n\in\mathbb{R}}\left|\left\{y_1\in\mathbb{R}:\ G_{s_1}(y_1)>\frac{\lambda}{G_{s_2}(y_2)\cdots G_{s_n}(y_n)}\right\}\right|dy_n\cdots dy_2\\ &\le\frac{C}{\lambda^{\frac{1}{1-s_1}}}\int_{y_2\in\mathbb{R}}G_{s_2}(y_2)^{\frac{1}{1-s_1}}dy_2\cdots\int_{y_n\in\mathbb{R}}G_{s_n}(y_n)^{\frac{1}{1-s_1}}dy_n\\ &=\frac{C'}{\lambda^{\frac{1}{1-s_1}}},\end{aligned}$$

since $G_{s_i}(y_i)\le C_{s_i}|y_i|^{-1+s_i}$ for $|y_i|\le 2$ and have an exponential decay at infinity, and so the integrals above converge as $s_1<s_i$ for all $i\ge 2$. □

Lemma 4.3. *Let $1<p<\infty$. Then for any $t_1,\dots,t_n\in\mathbb{R}$ we have*

$$\|\Gamma(it_1,\dots,it_n)\,f\|_{L^{p,1}(\mathbb{R}^n)}\le C(p,n)(1+|t_1|)\cdots(1+|t_n|)\,\|f\|_{L^{p,1}(\mathbb{R}^n)}.$$

Proof. We pick p_0 and p_1 such that $p_0<p<p_1$. Then the claimed inequality holds with L^{p_0} and L^{p_1} in place of $L^{p,1}$ in view of the classical version of the Marcinkiewicz multiplier theorem. We then appeal to the off-diagonal version of the Marcinkiewicz interpolation theorem (see [7, Theorem 1.4.19]) to conclude the proof. □

Lemma 4.4. *Let $1<p<\infty$, $s_1,s_2>0$ and let Ψ be as in the Proposition 2.1. Then we have the following estimate*

$$\left\|\Gamma(s_1,\dots,s_n)\left[\widehat{\Psi}f\right]\right\|_{L^{p,1}(\mathbb{R}^n)}\le C(s_1,s_2,p,\Psi)\,\|\Gamma(s_1,\dots,s_n)\,f\|_{L^{p,1}(\mathbb{R}^n)}$$

Proof. We start by picking numbers p_0,p_1 satisfying $1<p_0<p<p_1<\infty$. Then we define the linear operator

$$T(f)=\Gamma(s_1,\dots,s_n)\left[\widehat{\Psi}\,\Gamma(-s_1,\dots,-s_n)\,f\right].$$

By the product-type version of the Kato-Ponce inequality (see [11, Theorem 4]), T is bounded on $L^{p_0}(\mathbb{R}^n)$ and $L^{p_1}(\mathbb{R}^n)$. Now by the off diagonal Marcinkiewicz interpolation we get the desired result. □

Lemma 4.5. *Let $0 < a, s < 1$. Then for any measurable function f on $\mathbb{R}^n$ we have*

$$\big\|f^*(r)r^{s-a}\big\|_{L^{1/a,1}(0,\infty)} \le \frac{C}{a}\big\|f^*\big\|_{L^{1/s,1}(0,\infty)}.$$

Proof. If $a < s$, this is Lemma 3.7 in [13] with $n = 1$. For $a \ge s$ the inequality becomes equality with constant 1 as $(f^*(r)r^{s-a})^* = f^*(r)r^{s-a}$. □

The proof of the following lemma can be found in [10] (Lemma 2.1).

Lemma 4.6. *Let $0 < p_0 \le p_1 < \infty$ and define p via $1/p = (1-\theta)/p_0 + \theta/p_1$, where $0 < \theta < 1$. Given $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ and $\varepsilon > 0$, there exist smooth functions h_j^ε, $j = 1, \dots, N_\varepsilon$, supported in cubes with disjoint interiors, and there exist nonzero complex constants c_j^ε such that the functions*

$$f_z^\varepsilon = \sum_{j=1}^{N_\varepsilon} |c_j^\varepsilon|^{\frac{p}{p_0}(1-z)+\frac{p}{p_1}z}\, h_j^\varepsilon \tag{4.2}$$

satisfy

$$\big\|f_\theta^\varepsilon - f\big\|_{L^{p_0}} + \big\|f_\theta^\varepsilon - f\big\|_{L^{p_1}} + \big\|f_\theta^\varepsilon - f\big\|_{L^2} < \varepsilon \tag{4.3}$$

and

$$\|f_{it}^\varepsilon\|_{L^{p_0}}^{p_0} \le \|f\|_{L^p}^p + \varepsilon', \quad \|f_{1+it}^\varepsilon\|_{L^{p_1}}^{p_1} \le \|f\|_{L^p}^p + \varepsilon', \tag{4.4}$$

where ε' depends on $\varepsilon, p, \|f\|_{L^p}$ and tends to zero as $\varepsilon \to 0$.

Proof of Proposition 4.1. Let us fix a function σ such that

$$\sup_{j_1,\dots,j_n\in\mathbb{Z}} \Big\|\Gamma(s_1,\dots,s_n)\Big[\widehat{\Psi}\, D_{j_1,\dots,j_n}\sigma\Big]\Big\|_{L^{\frac{1}{s_1},1}(\mathbb{R}^n)} < \infty \tag{4.5}$$

and for $j_1, \dots, j_n \in \mathbb{Z}$ define

$$\varphi_{j_1,\dots,j_n} = \Gamma(s_1,\dots,s_n)\Big[\widehat{\Psi}\, D_{j_1,\dots,j_n}\sigma\Big].$$

As $\varphi_{j_1,\dots,j_n} \in L^{\frac{1}{s_1},1}(\mathbb{R}^n)$, we have $\sup_{\lambda>0} \lambda\left(\varphi^*_{j_1,\dots,j_n}(\lambda)\right)^{s_1} < \infty$ and thus $\varphi^*_{j_1,\dots,j_n}(\lambda) \to 0$ as $\lambda \to \infty$. Now by [1, Corollary 7.6 in Chapter 2], there is a measure preserving transformation $h_{j_1,\dots,j_n} : \mathbb{R}^n \to (0,\infty)$ such that

$$|\varphi_{j_1,\dots,j_n}| = \varphi^*_{j_1,\dots,j_n} \circ h_{j_1,\dots,j_n}. \tag{4.6}$$

Recall that $s_1^0 < \cdots < s_n^0$ and $s_1^1 < \cdots < s_n^1$. For $z \in \mathbb{C}$ with $0 \le \mathrm{Re}(z) \le 1$, we define complex polynomials

$$P_\rho(z) = s_\rho^0(1-z) + s_\rho^1 z$$

for $\rho = 1, 2, \dots, n$. We also define the family of multipliers

$$\sigma_z = \sum_{k_1,\dots,k_n\in\mathbb{Z}} D_{-k_1,\dots,-k_n}\left[\widehat{\Phi}\,\Gamma\big(-P_1(z),\dots,-P_n(z)\big)\Big[\varphi_{k_1,\dots,k_n} h_{k_1,\dots,k_n}^{s_1-P_1(z)}\Big]\right],$$

where $\widehat{\Phi}(\xi_1,\dots,\xi_n)=\widehat{\phi}(\xi_1)\cdots\widehat{\phi}(\xi_n)$ and ϕ is smooth with Fourier transform supported in $[-4,-1/4]\cup[1/4,4]$ and equal to 1 on $[-2,-1/2]\cup[1/2,2]$ so that $\widehat{\Phi}\equiv 1$ on the support of $\widehat{\Psi}$. Notice that $\sigma_\theta=\sigma$ a.e. as $P_j(\theta)=s_j$ for $1\le j\le m$, and as $\sum_{k_1,\dots,k_n\in\mathbb{Z}}\widehat{\Psi}\left(2^{-k_1}\xi_1,\dots,2^{-k_n}\xi_n\right)=1$ when all $\xi_k\neq 0$.

Fix $f,g\in\mathscr{C}_0^\infty(\mathbb{R}^n)$. Given $\epsilon>0$ find f_z^ϵ and g_z^ϵ as in the Lemma 4.6. So we have $\|f_\theta^\epsilon-f\|_{L^p}+\|f_\theta^\epsilon-f\|_{L^2}<\epsilon$, $\|g_\theta^\epsilon-g\|_{L^{p'}}+\|g_\theta^\epsilon-g\|_{L^2}\le\epsilon$

$$\|f_{it}^\epsilon\|_{L^{p_0}(\mathbb{R}^n)}^{p_0}\le\|f\|_{L^p(\mathbb{R}^n)}^p+\epsilon',\quad \left\|f_{1+it}^\epsilon\right\|_{L^{p_1}(\mathbb{R}^n)}^{p_1}\le\|f\|_{L^p(\mathbb{R}^n)}^p+\epsilon',$$
$$\|g_{it}^\epsilon\|_{L^{p_0'}(\mathbb{R}^n)}^{p_0'}\le\|g\|_{L^{p'}(\mathbb{R}^n)}^{p'}+\epsilon',\quad \left\|g_{1+it}^\epsilon\right\|_{L^{p_1'}(\mathbb{R}^n)}^{p_1'}\le\|g\|_{L^{p'}(\mathbb{R}^n)}^{p'}+\epsilon'.$$

Now define on the unit strip $\{z\in\mathbb{C}:0\le\operatorname{Re}(z)\le 1\}$ the following function

$$F(z)=\int_{\mathbb{R}^n}\sigma_z(\xi)\widehat{f_z^\epsilon}(\xi)\widehat{g_z^\epsilon}(\xi)d\xi=\int_{\mathbb{R}^n}T_{\sigma_z}(f_z^\epsilon)(x)\,g_z^\epsilon(-x)\,dx$$

which is analytic in the interior of this strip and is continuous on its closure. Hölder's inequality and one hypothesis of Proposition 4.1 give

$$\begin{aligned}&|F(it)|\le\|T_{\sigma_{it}}(f_{it}^\epsilon)\|_{L^{p_0}}\,\|g_{it}^\epsilon\|_{L^{p_0'}}\\&\le K_0\sup_{j_1,\dots,j_n\in\mathbb{Z}}\left\|\Gamma\left(s_1^0,\dots,s_n^0\right)\left[\widehat{\Psi}\,D_{j_1,\dots,j_n}\sigma_{it}\right]\right\|_{L^{\frac{1}{s_1^0},1}}\|f_{it}^\epsilon\|_{L^{p_0}}\,\|g_{it}^\epsilon\|_{L^{p_0'}}\,.\end{aligned}\tag{4.7}$$

Using the definition of σ_z with $z=it$ we write

$$\begin{aligned}&\widehat{\Psi}\left(D_{j_1,\dots,j_n}\sigma_{it}\right)\\&=\sum_{k_1,\dots,k_n\in\mathbb{Z}}\widehat{\Psi}\,D_{j_1-k_1,\dots,j_n-k_n}\left[\widehat{\Phi}\,\Gamma\big(-P_1(it),\dots,-P_n(it)\big)\left[\varphi_{k_1,\dots,k_n}h_{k_1,\dots,k_n}^{s_1-P_1(it)}\right]\right].\end{aligned}$$

In view of the support properties of the bumps $\widehat{\Psi}$ and $\widehat{\Phi}$, all terms in the sum above are zero if $k_i\notin\{j_i-2,j_i-1,j_i,j_i+1,j_i+2\}$ for some $i\in\{1,\dots,n\}$. Using this observation and Lemma 4.4 we write

$$\begin{aligned}&\left\|\Gamma\left(s_1^0,\dots,s_n^0\right)\left[\widehat{\Psi}\,D_{j_1,\dots,j_n}\sigma_{it}\right]\right\|_{L^{\frac{1}{s_1^0},1}(\mathbb{R}^n)}\\&\le\sum_{\substack{1\le i\le n\\-2\le a_i\le 2}}\Bigg\|\Gamma\left(s_1^0,\dots,s_n^0\right)\Bigg[\widehat{\Psi}\big(D_{a_1,\dots,a_n}\widehat{\Phi}\big)\\&\quad D_{a_1,\dots,a_n}\Big\{\Gamma\big(-P_1(it),\dots,-P_n(it)\big)\big(\varphi_{j_1+a_1,\dots,j_n+a_n}h_{j_1+a_1,\dots,j_n+a_n}^{s_1-P_1(it)}\big)\Big\}\Bigg]\Bigg\|_{L^{\frac{1}{s_1^0},1}(\mathbb{R}^n)}\\&\le\sum_{\substack{1\le i\le n\\-2\le a_i\le 2}}\Bigg\|\Gamma\left(s_1^0,\dots,s_n^0\right)D_{a_1,\dots,a_n}\Big\{\Gamma\left(-s_1^0,\dots,-s_n^0\right)\\&\quad\Gamma\big(it(s_1^0-s_1^1),\dots,it(s_n^0-s_n^1)\big)\big(\varphi_{j_1+a_1,\dots,j_n+a_n}h_{j_1+a_1,\dots,j_n+a_n}^{s_1-P_1(it)}\big)\Big\}\Bigg]\Bigg\|_{L^{\frac{1}{s_1^0},1}(\mathbb{R}^n)}\end{aligned}$$

$$\leq C \sum_{\substack{1\leq i\leq n\\ -2\leq a_i\leq 2}} \Big\| \Gamma\big(it(s_1^0 - s_1^1), \dots, it(s_n^0 - s_n^1)\big) \Big[\varphi_{j_1+a_1,\dots,j_n+a_n} h_{j_1+a_1,\dots,j_n+a_n}^{s_1 - P_1(it)}\Big] \Big\|_{L^{\frac{1}{s_1^0},1}(\mathbb{R}^n)},$$

as $-P_j(it) = -s_j^0 + it(s_j^0 - s_j^1)$. In the last inequality we made use of the fact that the function

$$\prod_{i=1}^n \left(\frac{1 + 4\pi^2 |\xi_i|^2}{1 + 4\pi^2 |\xi_i/2^{a_i}|^2} \right)^{s_i^0/2}$$

is an L^{1/s_1^0} Fourier multiplier, and thus an $L^{1/s_1^0,1}$ Fourier multiplier (this is a consequence of the classical version of the Marcinkiewicz multiplier theorem and of interpolation [7, Theorem 1.4.19]). Using successively Lemma 4.3, the fact that Re $P_1(it) = s_1^0$, identity (4.6) together with the fact that $h_{j_1,\dots,j_n}$ is measure-preserving, and Lemma 4.5, we estimate the last term in the alignment as follows:

$$\begin{aligned}
C &\sum_{\substack{1\leq i\leq n\\ -2\leq a_i\leq 2}} \Big\| \Gamma\big(it(s_1^0 - s_1^1), \dots, it(s_n^0 - s_n^1)\big) \Big[\varphi_{j_1+a_1,\dots,j_n+a_n} h_{j_1+a_1,\dots,j_n+a_n}^{s_1 - P_1(it)}\Big] \Big\|_{L^{\frac{1}{s_1^0},1}(\mathbb{R}^n)} \\
&\leq C \sum_{\substack{1\leq i\leq n\\ -2\leq a_i\leq 2}} (1+|t|)^n \Big\| \varphi_{j_1+a_1,\dots,j_n+a_n} h_{j_1+a_1,\dots,j_n+a_n}^{s_1 - s_1^0} \Big\|_{L^{\frac{1}{s_1^0},1}(\mathbb{R}^n)} \\
&= C \sum_{\substack{1\leq i\leq n\\ -2\leq a_i\leq 2}} (1+|t|)^n \Big\| \varphi^*_{j_1+a_1,\dots,j_n+a_n}(r) r^{s_1 - s_1^0} \Big\|_{L^{\frac{1}{s_1^0},1}((0,\infty),dr)} \\
&\leq C(1+|t|)^n \sum_{\substack{1\leq i\leq n\\ -2\leq a_i\leq 2}} \big\| \varphi^*_{j_1+a_1,\dots,j_n+a_n} \big\|_{L^{\frac{1}{s_1},1}(0,\infty)} \\
&\leq C(1+|t|)^n \sup_{j_1,\dots,j_n\in\mathbb{Z}} \|\varphi_{j_1,\dots,j_n}\|_{L^{\frac{1}{s_1},1}(\mathbb{R}^n)} .
\end{aligned}$$

Inserting this estimate in (4.7) and using Lemma 4.6 we obtain

$$|F(it)| \leq CK_0 (1+|t|)^n \sup_{j_1,\dots,j_n\in\mathbb{Z}} \|\varphi_{j_1,\dots,j_n}\|_{L^{\frac{1}{s_1},1}} \big(\|f\|_{L^p}^p + \epsilon'\big)^{\frac{1}{p_0}} \big(\|g\|_{L^{p'}}^{p'} + \epsilon'\big)^{\frac{1}{p_0'}}.$$

A similar argument yields

$$|F(1+it)| \leq CK_1 (1+|t|)^n \sup_{j_1,\dots,j_n\in\mathbb{Z}} \|\varphi_{j_1,\dots,j_n}\|_{L^{\frac{1}{s_1},1}} \big(\|f\|_{L^p}^p + \epsilon'\big)^{\frac{1}{p_1}} \big(\|g\|_{L^{p'}}^{p'} + \epsilon'\big)^{\frac{1}{p_1'}}.$$

Moreover, for $\tau \in [0,1]$, we claim that $|F(\tau + it)| \leq A_\tau(t)$ where $A_\tau(t)$ has at most polynomial growth as $|t| \to \infty$; we prove this assertion at the end. Thus we can apply Hirschman's lemma ([7, Lemma 1.3.8]). Using the estimates for $|F(it)|$ and $|F(1+it)|$ we obtain for $\theta \in (0,1)$

$$|F(\theta)| \leq C_* K_0^{1-\theta} K_1^\theta \sup_{j_1,\dots,j_n\in\mathbb{Z}} \|\varphi_{j_1,\dots,j_n}\|_{L^{\frac{1}{s_1},1}} \big(\|f\|_{L^p}^p + \epsilon'\big)^{\frac{1}{p}} \big(\|g\|_{L^{p'}}^{p'} + \epsilon'\big)^{\frac{1}{p'}}.$$

Adding and subtracting a term we write

$$\left|F(\theta)-\int_{\mathbb{R}^n}\widehat{T_\sigma(f)}\,\widehat{g}(\xi)\,d\xi\right|=\left|\int_{\mathbb{R}^n}\sigma(\xi)\widehat{f_\theta^\epsilon}(\xi)\widehat{g_\theta^\epsilon}(\xi)\,d\xi-\int_{\mathbb{R}^n}\sigma(\xi)\widehat{f}(\xi)\widehat{g}(\xi)\,d\xi\right|$$
$$\leq\big\|\sigma\big\|_{L^\infty}\Big[\big\|f_\theta^\epsilon-f\big\|_{L^2}\big\|g\big\|_{L^2}+\big\|g_\theta^\epsilon-g\big\|_{L^2}\big\|f\big\|_{L^2}\Big],$$

which tends to zero as $\epsilon\to 0$ (which implies $\epsilon'\to 0$). Thus

$$\left|\int_{\mathbb{R}^n}\widehat{T_\sigma(f)}\,\widehat{g}(\xi)\,d\xi\right|\leq C_*K_0^{1-\theta}K_1^\theta\sup_{j_1,\dots,j_n\in\mathbb{Z}}\big\|\varphi_{j_1,\dots,j_n}\big\|_{L^{\frac{1}{s_1},1}}\,\|f\|_{L^p}\,\|g\|_{L^{p'}}\,.$$

But the integral on the left is equal to $\int_{\mathbb{R}^n}T_\sigma(f)(x)g(-x)dx$. Taking the supremum over all functions $g\in\mathscr{C}_0^\infty(\mathbb{R}^n)$ with $\|g\|_{L^{p'}}\leq 1$ we deduce for $f\in\mathscr{C}_0^\infty(\mathbb{R}^n)$:

$$\|T_\sigma(f)\|_{L^p(\mathbb{R}^n)}\leq C_*K_0^{1-\theta}K_1^\theta\sup_{j_1,\dots,j_n\in\mathbb{Z}}\big\|\varphi_{j_1,\dots,j_n}\big\|_{L^{\frac{1}{s_1},1}(\mathbb{R}^n)}\,\|f\|_{L^p(\mathbb{R}^n)}\,.$$

Notice that the constant C_* depends on the parameters indicated in the statement.

We now return to the assertion that $|F(\tau+it)|\leq A_\tau(t)$, where $A_\tau(t)$ has at most polynomial growth in $|t|$. (This was one of the hypotheses in Hirschman's lemma.) Let $z=\tau+it$ where $t\in\mathbb{R}$ and $0\leq\tau\leq 1$. We use that

$$|F(\tau+it)|\leq\|\sigma_{\tau+it}\|_{L^\infty}\|f_{\tau+it}\|_{L^2}\|g_{\tau+it}\|_{L^2},$$

and we notice that in view of (4.2), the L^2 norms of $f_{\tau+it}$ and $g_{\tau+it}$ are bounded by constants independent of t. We now estimate $\|\sigma_z\|_{L^\infty}$. Let E be the set of all $(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$ with some $\xi_i=0$. Then for all $(\xi_1,\dots,\xi_n)\in\mathbb{R}^n\setminus E$ there are only finitely many indices k_i in the summation defining $\sigma_z(\xi_1,\dots,\xi_n)$ that produce a nonzero term, in fact the indices with $|\xi_i|/4\leq 2^{k_i}\leq 4|\xi_i|$ for all $i\in\{1,\dots,n\}$. Also, $P_\rho(\tau+it)=P_\rho(\tau)+(s_1^1-s_1^0)(it)$ which implies that

$$\begin{aligned}&\Gamma\big(-P_1(\tau+it),\dots,-P_n(\tau+it)\big)\\&\qquad=\Gamma\big(-P_1(\tau),\dots,-P_n(\tau)\big)\ \Gamma\big(it(s_1^0-s_1^1),\dots,it(s_n^0-s_n^1)\big).\end{aligned}\tag{4.8}$$

Applying identity (4.8), and using successively Lemma 4.2, Lemma 4.3, the fact that Re $P_1(\tau+it)=P_1(\tau)$, identity (4.6) together with the fact that $h_{j_1,\dots,j_n}$ is measure-preserving, and Lemma 4.5, we estimate $\|\sigma_{\tau+it}\|_{L^\infty}$ by

$$\sup_{\xi\in\mathbb{R}^n\setminus E}\sum_{\substack{1\leq i\leq n\\ \frac{|\xi_i|}{4}\leq 2^{k_i}\leq 4|\xi_i|}}\left\|\Gamma\big(-P_1(\tau+it),\dots,-P_n(\tau+it)\big)\Big[\varphi_{k_1,\dots,k_n}h_{k_1,\dots,k_n}^{s_1-P_1(\tau+it)}\Big]\right\|_{L^\infty}$$
$$\leq C\,(1+|t|)^n\sup_{\xi\in\mathbb{R}^n\setminus E}\sum_{\substack{1\leq i\leq n\\ \frac{|\xi_i|}{4}\leq 2^{k_i}\leq 4|\xi_i|}}\left\|\varphi_{k_1,\dots,k_n}h_{k_1,\dots,k_n}^{s_1-P_1(\tau)}\right\|_{L^{\frac{1}{P_1(\tau)},1}(\mathbb{R}^n)}$$
$$\leq C\,(1+|t|)^n\sup_{\xi\in\mathbb{R}^n\setminus E}\sum_{\substack{1\leq i\leq n\\ \frac{|\xi_i|}{4}\leq 2^{k_i}\leq 4|\xi_i|}}\left\|\varphi_{k_1,\dots,k_n}^*(r)r^{s_1-P_1(\tau)}\right\|_{L^{\frac{1}{P_1(\tau)},1}(0,\infty)}$$

$$
\begin{aligned}
&\leq C(1+|t|)^n \sup_{\xi\in\mathbb{R}^n\setminus E} \sum_{\substack{1\leq i\leq n\\ \frac{|\xi_i|}{4}\leq 2^{k_i}\leq 4|\xi_i|}} \left\|\varphi^*_{k_1,\dots,k_n}\right\|_{L^{\frac{1}{s_1},1}(0,\infty)}\\
&\leq C\,(1+|t|)^n 5^n \sup_{k_1,\dots,k_n\in\mathbb{Z}} \left\|\varphi_{k_1,\dots,k_n}\right\|_{L^{\frac{1}{s_1},1}(\mathbb{R}^n)},
\end{aligned}
$$

and the last expression is finite in view of assumption (4.5). This proves that $|F(\tau+it)|\leq A_\tau(t)$, where $A_\tau(t)\leq C'\,(1+|t|)^n$. □

5 The conclusion of the proof, final comments and remarks

We now conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. We begin by noticing that the statement is true by Proposition 2.1 when $s_1\in(\frac{1}{2},1)$, so to complete the proof we only need to consider the case when $s_1\leq\frac{1}{2}$. We will rely on Proposition 4.1 (interpolation) to complete this argument, using $p=2$ as one interpolation endpoint, with the second endpoint coming from the Proposition 2.1. Additionally, we note that it is actually enough to consider $p\in(1,2)$, as the case $p\in(2,\infty)$ follows by duality. Let ψ and Ψ be as in Theorem 1.1. When $p=2$, by Plancherel's theorem, we have

$$
\begin{aligned}
\|T_\sigma(f)\|_{L^2(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n}\left|\sum_{j_1,\dots,j_n\in\mathbb{Z}} \widehat{\psi}(2^{-j_1}\xi_1)\cdots\widehat{\psi}(2^{-j_n}\xi_n)\sigma(\xi)\widehat{f}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}}\\
&\leq C\sup_{j_1,\dots,j_n\in\mathbb{Z}}\left\|\big(D_{-j_1,\dots,-j_n}\widehat{\Psi}\big)\,\sigma\right\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^2}\\
&= C\sup_{j_1,\dots,j_n\in\mathbb{Z}}\left\|\widehat{\Psi}\big(D_{j_1,\dots,j_n}\sigma\big)\right\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^2}\\
&\leq C\sup_{j_1,\dots,j_n\in\mathbb{Z}}\left\|\Gamma\left(s_1^0,\dots,s_n^0\right)\left[\widehat{\Psi}\big(D_{j_1,\dots,j_n}\sigma\big)\right]\right\|_{L^{\frac{1}{s},1}}\|f\|_{L^2},
\end{aligned}
$$

using that the sum contains only 3^n nonzero terms and Lemma 4.2 in the last step, for some choice of exponents $0<s_1^0<s_2^0<\cdots<s_n^0$. Now for $p\in(1,2)$ with $\frac{1}{p}-\frac{1}{2}=|\frac{1}{p}-\frac{1}{2}|<s_1$, there exists a $\tau\in(0,1)$ such that

$$
\frac{1}{p}-\frac{1}{2}<\tau s_1. \tag{5.1}
$$

Set $p_1=\frac{2}{\tau+1}$, $s_1^1=\frac{1}{2}+\epsilon_1<\cdots<s_n^1=\frac{1}{2}+\epsilon_n$, $0<\epsilon_1<\cdots<\epsilon_n$ to be specified later. Since $p_1>1$ and $s_1^1,\dots,s_n^1\in(\frac{1}{2},1)$ with $s_1^1<s_1^2<\cdots<s_1^n$, by Proposition 2.1 we get

$$
\|T_\sigma(f)\|_{L^{p_1}}\leq C\sup_{j_1,\dots,j_n\in\mathbb{Z}}\left\|\Gamma\left(s_1^1,\dots,s_n^1\right)\left[\widehat{\Psi}\big(D_{j_1,\dots,j_n}\sigma\big)\right]\right\|_{L^{\frac{1}{s_1^1},1}}\|f\|_{L^{p_1}}.
$$

Now set $p_0 = 2$ and pick θ such that $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{p_1}$, i.e., $\theta = \frac{2}{\tau}\left(\frac{1}{p} - \frac{1}{2}\right)$. Moreover by (5.1), $\theta < 2s_1 \le 1$. Pick $s_1^0 < \cdots < s_n^0$ such that

$$s_1 = (1-\theta)s_1^0 + \theta s_1^1, \; \ldots \; , s_n = (1-\theta)s_n^0 + \theta s_n^1.$$

Notice that if ϵ_1 is small enough then $s_1^0 = \frac{s_1 - \theta s_1^1}{1-\theta} < s_1^1 < 1$ and similarly for s_2^0 if ϵ_2 is small enough and so on. An application of Proposition 4.1 yields the desired result. □

We end indicating why hypothesis (1.2) is indeed weaker than (1.1). Picking a smooth and compactly supported function $\widehat{\Theta}$ that equals 1 on the support of $\widehat{\Psi}$, matters reduce to proving the inequality

$$\big\|\Gamma(s_1,\ldots,s_n)[\widehat{\Theta}g]\big\|_{L^{q,1}} \le C_{q,r}\big\|\Gamma(s_1,\ldots,s_n)g\big\|_{L^r}, \tag{5.2}$$

whenever $s_1,\ldots,s_n > 0$ and $1 < q < r < \infty$. Note that (5.2) is quite easy if all s_j are even integers as $L^r(K)$ embeds in $L^{q,1}(K)$ when K is has compact support. For other values of s_j we write (5.2) in the equivalent form

$$\Big\|\Gamma(s_1,\ldots,s_n)\big[\widehat{\Theta}\Gamma(-s_1,\ldots,-s_n)g\big]\Big\|_{L^{q,1}} \le C_{q,r}\big\|g\big\|_{L^r}. \tag{5.3}$$

We obtain (5.3) via complex interpolation. Let N be an even integer larger than $\max(s_1,\ldots,s_n)$. Using Lemma 4.3 we obtain the validity of (5.3) when $\operatorname{Re} s_j \in \{0, N\}$ for all j and then we deduce (5.3) by an n-fold application of the interpolation theorem for analytic families of operators.

In this article we considered an improvement of the Marcinkiewicz interpolation theorem in which the indices $s_1,\ldots,s_n$, measuring the Sobolev smoothness of the multiplier $\sigma(\xi_1,\ldots,\xi_n)$ in each variable ξ_j, are different. In a forthcoming paper, we address the situation where the exponents could be equal. In this particular case, the multiplier needs to be locally in a Marcinkiewicz space.

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