MULTILINEAR HARMONIC ANALYSIS

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In memory of Nigel Kalton

1. Introduction

An operator acting on function spaces may not only depend on a main variable but also on several other function-variables that are often treated as parameters. Examples of such operators are ubiquitous in harmonic analysis: multiplier operators, homogeneous singular integrals associated with functions on the sphere, Littlewood-Paley operators, the Calderón commutators, and the Cauchy integral along Lipschitz curves.

Of the aforementioned examples, we discuss the latter: The Cauchy integral along a Lipschitz curve Γ is given by

$$C_\Gamma(h)(z) = \frac{1}{2\pi i}\,\text{p.v.}\int_\Gamma \frac{h(\zeta)}{\zeta - z}\,d\zeta\,,$$

where h is a function on Γ, which is taken to be the graph of a Lipschitz function $A:\ \mathbf{R}\to\mathbf{R}$, and z is a point on the curve Γ. A. Calderón wrote

$$C_\Gamma(h)(z) = \frac{1}{2\pi i}\sum_{m=0}^{\infty}(-i)^m \mathcal{C}_m(f;A)(x)\,, \tag{1}$$

where $z = x + iA(x)$, $f(y) = h(y + iA(y))(1 + iA'(y))$, and

$$\mathcal{C}_m(f;A)(x) = \text{p.v.}\int_{\mathbf{R}}\left(\frac{A(x)-A(y)}{x-y}\right)^m \frac{f(y)}{x-y}\,dy\,.$$

The operators $\mathcal{C}_m(f;A)$ are called the *mth Calderón commutators* and they provide examples of singular integrals whose action on the function 1 has inspired the fundamental work on the $T1$ theorem [11].

Identity (1) reduces the boundedness of $C_\Gamma(h)$ to that of the operators $\mathcal{C}_m(f;A)$ (recall $f(y) = h(y+iA(y))(1+iA'(y))$); certainly for this approach to bear fruit, one

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would also need to know that the operators $\mathcal{C}_m(f;A)$ are bounded with norms having moderate growth in m. At this point, it seems that we reduced the boundedness of a linear operator to another operator that contains powers of the function A and thus it is nonlinear in it. To adopt a truly multilinear point of view, we introduce the $(m+1)$-linear operator

$$\mathcal{E}_{m+1}(f;A_1,\dots,A_m)(x) = \text{p.v.} \int_{\mathbf{R}} \left(\frac{A_1(x)-A_1(y)}{x-y}\right)\dots\left(\frac{A_m(x)-A_m(y)}{x-y}\right)\frac{f(y)}{x-y}\,dy$$

and seek estimates for it. Any estimate for $\mathcal{E}_{m+1}$ from a product of function spaces $Z_1\times Z_2\times\cdots\times Z_{m+1}$, where $Z_2=\cdots=Z_{m+1}$ gives yield to an estimate for $\mathcal{C}_m(f;A)$ in terms of f and A. This point of view leads to the following result:

Theorem 1. *([15]) Let $0<1/p=\sum_{j=1}^{m+1}1/p_j$. Then the $(m+1)$-linear operator $\mathcal{E}_{m+1}$ maps $L^{p_1}(\mathbf{R})\times\cdots\times L^{p_{m+1}}(\mathbf{R})$ to $L^{p,\infty}(\mathbf{R})$ whenever $1\le p_1,\dots,p_{m+1}\le\infty$ and it also maps $L^{p_1}(\mathbf{R})\times\cdots\times L^{p_{m+1}}(\mathbf{R})$ to $L^p(\mathbf{R})$ when $1<p_j<\infty$ for all j. In particular it maps $L^1(\mathbf{R})\times\cdots\times L^1(\mathbf{R})$ to $L^{1/(m+1),\infty}(\mathbf{R})$.*

The endpoint conclusion $L^1(\mathbf{R})\times\cdots\times L^1(\mathbf{R})$ to $L^{1/(m+1),\infty}(\mathbf{R})$ of Theorem 1 has been obtained by C. Calderón [2] when $m=1$ and Coifman and Meyer [6] when $m=1,2$ while the case $m\ge 3$ was completed by Duong, Grafakos, and Yan [15].

The underlying idea in the proof of Theorem 1 is the simultaneous Calderón-Zymgund decomposition on all functions that $\mathcal{E}_{m+1}$ acts on. This decomposition resembles the classical *linear* Calderón-Zymgund decomposition, but is more complicated due to the presence of several tuples of combinations of good and bad functions. This decomposition is discussed in Section 3. However, the proof contained in Section 3 does not directly apply to Theorem 1; the latter requires a more flexible version of the Calderón-Zymgund decomposition, since the kernel of $\mathcal{E}_{m+1}$ does not obey the standard multilinear Calderón-Zymgund kernel conditions, see Section 3. Indeed the kernel of $\mathcal{E}_{m+1}$ is the function of $(m+1)$-variables

$$K(y_0,\dots,y_{m+1}) = \frac{(-1)^{e(y_{m+1}-y_0)m}}{(y_0-y_{m+1})^{m+1}}\prod_{\ell=1}^{m}\chi_{\left(\min(y_0,y_{m+1}),\max(y_0,y_{m+1})\right)}(y_\ell)$$

which contains characteristic functions. The proof of Theorem 1 is achieved in [15] and is modeled after the approach devised by Duong and A. McIntosh [16] for linear operators.

Another class of operators closely related to the commutators of Calderón is the family

$$H_{\alpha_1,\alpha_2}(f_1,f_2)(x) = \text{p.v.}\int_{\mathbf{R}} f_1(x-\alpha_1 t)f_2(x-\alpha_2 t)\,\frac{dt}{t}, \quad \alpha_1,\alpha_2,x\in\mathbf{R},$$

called today the *bilinear Hilbert transforms.* These were also introduced by A. Calderón in an attempt to show that the commutator $\mathcal{C}_1(f;A)$ is bounded on $L^2(\mathbf{R})$ when $A(t)$ is a function on the line with bounded derivative. The idea of this approach is that the linear operator $f\to\mathcal{C}_1(f;A)$ can be expressed as the average

$$\mathcal{C}_1(f;A)(x) = \int_0^1 H_{1,\alpha}(f,A')(x)\,d\alpha\,, \tag{2}$$

and thus the boundedness of $\mathcal{C}_1(f;A)$ can be reduced to the (uniform in α) boundedness of $H_{1,\alpha}$. Naturally, the estimates for $H_{1,\alpha}$ should depend (linearly) on both functions f and A'. This operator is discussed in Section 7

The previous discussion leads to the conclusion that treating the function A as a frozen parameter provides limited results in terms of its smoothness. If we have estimates in terms of a few function space norms of both f and A, we may use the power of *multilinear interpolation*, to deduce boundedness of $\mathcal{C}_1(f;A)$ on various function spaces, of ranging degree of regularity. Certainly this fact is not only pertinent to Calderón's first commutator $\mathcal{C}_1$, but all multilinear operators.

In summary, we advocate the following point of view in the study of multivariable operators: unfreeze the functions serving the roles of a parameter and treat them as input variables. This approach often yields sharper results in terms of the regularity of the input functions. In these notes we pursue this idea in a systematic way. We present certain fundamental results concerning linear (or sublinear) operators of several variables, henceforth called multilinear (or multisublinear), that contain challenges that appear in their study, despite the great resemblances with their linear analogues. The proofs given in the next sections contain most necessary details but references are provided for the sake of completeness in the exposition.

2. Examples of multivariable operators

We embark on the study of multilinear harmonic analysis with the class of operators that extends the concept of Calderón-Zygmund operators in the multilinear setting. These operators have kernels that satisfy standard estimates and possess boundedness properties analogous to those of the classical linear ones. This class of operators has been previously studied by Coifman and Meyer [6], [7], [8], [9], [34], assuming sufficient smoothness on their symbols and kernels.

If an m-linear operator T commutes with translations in the sense that

$$T(f_1,\dots,f_m)(x+t) = T(f_1(\cdot+t),\dots,f_m(\cdot+t))(x) \tag{3}$$

for all $t,x\in\mathbf{R}^n$, then it incorporates a certain amount of homogeneity. Indeed, if it maps $L^{p_1}\times\cdots\times L^{p_m}$ to L^p, then one must necessarily have $1/p_1+\cdots+1/p_m\ge 1/p$; this was proved in [25] for compactly supported kernels but extended for general kernels in [13].

We use the following definition for the Fourier transform in n-dimensional Euclidean space

$$\widehat{f}(\xi)=\int_{\mathbf{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx,$$

while $f^\vee(\xi)=\widehat{f}(-\xi)$ denotes the inverse Fourier transform. Multilinear operators that commute with translations as in (3) are exactly the multilinear multiplier operators that have the form

$$T(f_1,\dots,f_m)(x)=\int_{(\mathbf{R}^n)^m}\sigma(\xi_1,\dots,\xi_m)\widehat{f_1}(\xi_1)\cdots\widehat{f_m}(\xi_m)e^{2\pi i x\cdot(\xi_1+\cdots+\xi_m)}d\xi_1\dots d\xi_m \tag{4}$$

for some bounded function σ.

Endpoint estimates for linear singular integrals are usually estimates of the form $L^1 \to L^1$ or $L^1 \to L^{1,\infty}$. The analogous m-linear estimates are $L^1 \times \cdots \times L^1 \to L^{1/m,\infty}$. Although one expects some similarities with the linear case, there exist some differences as well. For example, if a linear translation-invariant operator has a positive kernel and it maps $L^1 \to L^{1,\infty}$, then it must have an integrable kernel and thus it actually maps L^1 to L^1. In the multilinear case, it is still true that if a multilinear translation-invariant operator has a positive kernel and maps $L^1 \times \cdots \times L^1$ to $L^{1/m,\infty}$, then it must have an integrable kernel, but having an integrable positive kernel does not necessarily imply that the corresponding operator maps $L^1 \times \cdots \times L^1$ to $L^{1/m}$. Results of this type have been obtained in [23].

We provide a few examples of mutlivariable (multilinear and multisublinear) operators:

Example 1: The *identity operator* in the m-linear setting is the product operator

$$T_1(f_1, \dots, f_m)(x) = f_1(x) \cdots f_m(x).$$

By Hölder's inequality T_1 maps $L^{p_1} \times \cdots \times L^{p_m} \to L^p$ whenever $1/p_1 + \cdots + 1/p_m = 1/p$.

Example 2: The action of a linear operator L on the product $f_1 \cdots f_m$ gives rise to a more general *degenerate m-linear operator*

$$T_2(f_1, \dots, f_m)(x) = L(f_1 \cdots f_m)(x).$$

that still maps $L^{p_1} \times \cdots \times L^{p_m} \to L^p$ whenever $1/p_1 + \cdots + 1/p_m = 1/p$, provided L is a bounded operator on L^p.

Example 3: The previous example captures "the majority of interesting" m-linear operators. Let L_0 be a linear operator acting on functions defined on $\mathbf{R}^{mn}$. We define

$$T_3(f_1, \dots, f_m)(x) = L_0(f_1 \otimes \cdots \otimes f_m)(x).$$

Here $f_1 \otimes \cdots \otimes f_m$ is the tensor product of these functions, defined as a function on $\mathbf{R}^{mn}$ as follows: $(f_1 \otimes \cdots \otimes f_m)(x_1, \dots, x_m) = f_1(x_1) \dots f_m(x_m)$. In particular, L_0 could be a singular integral acting on functions on $\mathbf{R}^{mn}$. The boundedness of T_3 from $L^{p_1} \times \cdots \times L^{p_m} \to L^p$ whenever $1/p_1 + \cdots + 1/p_m = 1/p$ may not always be an easy task. It often requires a delicate study aspects of which are investigated in this article for certain classes of linear (and also sublinear) operators L_0.

The situation where $1/p_1 + \cdots + 1/p_m = 1/p$ will be referred to as the *singular integral case*. This is because, it needs to be distinguished from the *fractional integral case* in which $1/p < 1/p_1 + \cdots + 1/p_m$. This name is due to the fact that most examples of multilinear operators bounded in this case have fractional integral homogeneity, such as these:

$$(f_1, \dots, f_m) \to \int_{\mathbf{R}^{mn}} f_1(x-y_1) \dots f_m(x-y_m)(|x-y_1| + \cdots + |x-y_m|)^{-mn+\alpha} dy_1 \dots dy_m \,.$$

Example 4: Taking L_0 to be a linear multiplier operator on $(\mathbf{R}^n)^m$ with symbol σ, we obtain a *multilinear multiplier operator* of the form (4) Then σ is called the *symbol* or *multiplier* of the m-linear multiplier.

Multilinear multipliers operators arise in many situations. For instance, to prove the *Kato-Ponce inequality* [28] (Leibniz rule for fractional derivatives D^α, $\alpha > 0$)

$$\big\|D^\alpha(fg)\big\|_{L^r} \le C_{p,q,r}\Big[\big\|D^\alpha f\big\|_{L^p}\big\|g\big\|_{L^q} + \big\|f\big\|_{L^p}\big\|D^\alpha g\big\|_{L^q}\Big]$$

where $1/p + 1/q = 1/r$, one would have to study bilinear multiplier operators with symbols

$$|\xi+\eta|^\alpha \sum_j \Big[\Psi(2^{-j}\xi)\Phi(2^{-j}\eta) + \Phi(2^{-j}\xi)\Psi(2^{-j}\eta) + \sum_{|j-k|\le 2} \Psi(2^{-j}\xi)\Psi(2^{-k}\eta)\Big]. \tag{5}$$

Here Ψ and Φ are smooth functions supported in an annulus and in small disjoint ball, both centered at the origin, respectively. The main idea is in the first term in (5) we have $|\xi+\eta| \approx |\xi|$ and so $|\xi+\eta|^\alpha$ could be replaced by $|\xi|^\alpha$ via some multiplier theorem. This would yield the term $\|D^\alpha f\|_{L^p}\|g\ \|_{L^q}$ in L^r norm. An analogous estimate with the roles of f and g interchanged holds for the second term in (5), while the third term is easier. Such a study requires a multiplier theory for multilinear operators. The topic of multilinear multipliers will be addressed in Section 5.

Example 5: The maximal function

$$\mathcal{M}(f_1,\dots,f_m)(x) = \sup_{Q\ni x}\left(\frac{1}{|Q|^m}\int_Q \cdots \int_Q |f_1(y_1)|\dots|f_m(y_m)|\,dy_1\dots dy_m\right)$$

where the supremum is taken over all cubes in $\mathbf{R}^n$ with sides parallel to the axes. This was introduced in the work of Lerner, Ombrosi, Pérez, Torres and Trujillo-González [32] and plays an important role in the characterization of the class of multiple A_p weights.

Example 6: A larger operator is the *strong multilinear maximal function.* It is defined for $x \in \mathbf{R}^n$ as

$$\mathcal{M}_{\mathcal{R}}(f_1,\dots,f_m)(x) = \sup_{R\ni x}\left(\frac{1}{|R|^m}\int_R \cdots \int_R |f_1(y_1)|\dots|f_m(y_m)|\,dy_1\dots dy_m\right),$$

where the supremum is taken over all rectangles R in $\mathbf{R}^n$ with sides parallel to the axes. When $m = 1$, this operator reduces to the strong maximal function on $\mathbf{R}^n$.

3. Multilinear Calderón-Zygmund operators

In this section we set up the background of the theory of multilinear Calderón-Zygmund operators. We will be working on n-dimensional space $\mathbf{R}^n$. We denote by $\mathscr{S}(\mathbf{R}^n)$ the space of all Schwartz functions on $\mathbf{R}^n$ and by $\mathscr{S}'(\mathbf{R}^n)$ its dual space, the set of all tempered distributions on $\mathbf{R}^n$.

An m-linear operator $T: \mathscr{S}(\mathbf{R}^n)\times\cdots\times\mathscr{S}(\mathbf{R}^n) \to \mathscr{S}'(\mathbf{R}^n)$ is linear in every entry and consequently it has m formal transposes. The first transpose T^{*1} of T is defined via

$$\langle T^{*1}(f_1,f_2,\dots,f_m)\,,\,h\rangle = \langle T(h,f_2,\dots,f_m)\,,\,f_1\rangle,$$

for all $f_1, f_2, \dots, f_m, h$ in $\mathscr{S}(\mathbf{R}^n)$. Analogously one defines T^{*j}, for $j \ge 2$ and we also set $T^{*0} = T$.

Let $K(x, y_1, \dots, y_m)$ be a locally integrable function defined away from the *diagonal* $x = y_1 = \dots = y_m$ in $(\mathbf{R}^n)^{m+1}$, which satisfies the *size estimate*

$$|K(x, y_1, \dots, y_m)| \le \frac{A}{(|x - y_1| + \dots + |x - y_m|)^{mn}} \tag{6}$$

for some $A > 0$ and all $(x, y_1, \dots, y_m) \in (\mathbf{R}^n)^{m+1}$ with $x \ne y_j$ for some j. Furthermore, assume that for some $\varepsilon > 0$ we have the smoothness estimates

$$|K(x, y_1, \dots, y_m) - K(x', y_1, \dots, y_m)| \le \frac{A\,|x - x'|^\varepsilon}{(|x - y_1| + \dots + |x - y_m|)^{mn+\varepsilon}} \tag{7}$$

whenever $|x - x'| \le \frac{1}{2} \max\big(|x - y_1|, \dots, |x - y_m|\big)$ and also that

$$|K(x, y_1, y_2, \dots, y_m) - K(x, y_1', y_2, \dots, y_m)| \le \frac{A\,|y_j - y_j'|^\varepsilon}{(|x - y_1| + \dots + |x - y_m|)^{mn+\varepsilon}} \tag{8}$$

whenever $|y_1 - y_1'| \le \frac{1}{2} \max\big(|x - y_1|, \dots, |x - y_m|\big)$ as well as a similar estimate with the roles of y_1 and y_j reversed. Kernels satisfying these conditions are called multilinear Calderón-Zygmund kernels and are denoted by $m\text{-}CZK(A, \varepsilon)$. A multilinear operator T is said to be associated with K if

$$T(f_1, \dots, f_m)(x) = \int_{(\mathbf{R}^n)^m} K(x, y_1, \dots, y_m) f_1(y_1) \dots f_m(y_m)\, dy_1 \dots dy_m\,,$$

whenever $f_1, \dots, f_m$ are smooth functions with compact support and x does not lie in the intersection of the support of f_j.

Certain homogeneous distributions of order $-mn$ are examples of kernels in the class $m\text{-}CZK(A, \varepsilon)$. For this reason, boundedness properties of operators T with kernels in $m\text{-}CZK(A, \varepsilon)$ from a product $L^{p_1} \times \dots \times L^{p_m}$ into another L^p space can only hold when

$$\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p}$$

as dictated by homogeneity. If such boundedness holds for a certain triple of Lebesgue spaces, then the corresponding operator is called multilinear Calderón-Zygmund.

A fundamental result concerning these operators is the multilinear extension of the classical Calderón-Zygmund [3]; the linear result states that if an operator with smooth enough kernel is bounded on a certain L^r space, then it is of weak type $(1, 1)$ and is also bounded on all L^p spaces for $1 < p < \infty$. A version of this theorem for operators with kernels in the class $m\text{-}CZK(A, \varepsilon)$ has been obtained by Grafakos and Torres [25]. A special case of this result was also obtained by Kenig and Stein [29]; both approaches build on previous work by Coifman and Meyer [6].

Theorem 2. *([25]) Let T be a multilinear operator with kernel K in $m\text{-}CZK(A, \varepsilon)$. Assume that for some $1 \le q_1, \dots, q_m \le \infty$ and some $0 < q < \infty$ with*

$$\frac{1}{q_1} + \dots + \frac{1}{q_m} = \frac{1}{q}\,,$$

T maps $L^{q_1} \times \dots \times L^{q_m} \to L^{q,\infty}$. Then T can be extended to a bounded operator from $L^1 \times \dots \times L^1$ into $L^{1/m,\infty}$. Moreover, for some constant C_n (that depends only on the

parameters indicated) we have that

$$\|T\|_{L^1\times\cdots\times L^1\to L^{1/m,\infty}} \le C_n\big(A+\|T\|_{L^{q_1}\times\cdots\times L^{q_m}\to L^{q,\infty}}\big). \tag{9}$$

Proof. Set $B=\|T\|_{L^{q_1}\times\cdots\times L^{q_m}\to L^{q,\infty}}$. Fix an $\alpha>0$ and consider functions $f_j\in L^1$ for $1\le j\le m$. Without loss of generality we may assume that $\|f_1\|_{L^1}=\cdots=\|f_m\|_{L^1}=1$. Set $E_\alpha=\{x:\ |T(f_1,\dots,f_m)(x)|>\alpha\}$. We need to show that there is a constant $C=C_{m,n}$ such that

$$|E_\alpha|\le C(A+B)^{1/m}\alpha^{-1/m}. \tag{10}$$

Once (10) has been established for f_j's with norm one, the general case follows by replacing each f_j by $f_j/\|f_j\|_{L^1}$. Let γ be a positive real number to be determined later. Apply the Calderón-Zygmund decomposition to the function f_j at height $(\alpha\gamma)^{1/m}$ to obtain 'good' and 'bad' functions g_j and b_j, and families of cubes $\{Q_{j,k}\}_k$ with disjoint interiors such that

$$f_j=g_j+b_j$$

and

$$b_j=\sum_k b_{j,k}$$

where

$$\begin{gathered}\text{support}(b_{j,k})\subset Q_{j,k}\\ \int b_{j,k}(x)dx=0\\ \int |b_{j,k}(x)|dx\le C(\alpha\gamma)^{1/m}|Q_{j,k}|\\ |\cup_k Q_{j,k}|\le C(\alpha\gamma)^{-1/m}\\ \|b_j\|_{L^1}\le C\\ \|g_j\|_{L^s}\le C(\alpha\gamma)^{1/ms'}\end{gathered}$$

for all $j=1,2,\dots,m$ and any $1\le s\le\infty$. Define the sets

$$\begin{aligned}E_1=&\{x:\ |T(g_1,g_2,\dots,g_m)(x)|>\alpha/2^m\}\\ E_2=&\{x:\ |T(b_1,g_2,\dots,g_m)(x)|>\alpha/2^m\}\\ E_3=&\{x:\ |T(g_1,b_2,\dots,g_m)(x)|>\alpha/2^m\}\\ &\dots\\ E_{2^m}=&\{x:\ |T(b_1,b_2,\dots,b_m)(x)|>\alpha/2^m\},\end{aligned}$$

where each set E_s has the form $\{x:\ |T(h_1,h_2,\dots,h_m)(x)|>\alpha/2^m\}$ with $h_j\in\{g_j,b_j\}$ and all the sets E_s are distinct. Since $|\{x:\ |T(f_1,\dots,f_m)(x)|>\alpha\}|\le\sum_{s=1}^{2^m}|E_s|$, it will suffice to prove estimate (10) for each one of the 2^m sets E_s.

Chebychev's inequality and the $L^{q_1} \times \cdots \times L^{q_m} \to L^{q,\infty}$ boundedness give

$$
\begin{aligned}
|E_1| \leq & \frac{(2^m B)^q}{\alpha^q} \|g_1\|_{L^{q_1}}^q \dots \|g_m\|_{L^{q_m}}^q \leq \frac{CB^q}{\alpha^q} \prod_{j=1}^m (\alpha\gamma)^{\frac{q}{mq_j'}} \\
= & \frac{C'B^q}{\alpha^q} (\alpha\gamma)^{(m-\frac{1}{q})\frac{q}{m}} = C' B^q \alpha^{-\frac{1}{m}} \gamma^{q-\frac{1}{m}}.
\end{aligned} \tag{11}
$$

Consider now a set E_s defined above with $2 \leq s \leq 2^m$. Suppose that for some $1 \leq l \leq m$ we have l bad functions and $m-l$ good functions appearing in $T(h_1, \dots, h_m)$, where $h_j \in \{g_j, b_j\}$ and assume that the bad functions appear at the entries $j_1, \dots, j_l$. We will show that

$$
|E_s| \leq C\alpha^{-1/m}\big(\gamma^{-1/m} + \gamma^{-1/m}(A\gamma)^{1/l}\big). \tag{12}
$$

Let $l(Q)$ denote the side-length of a cube Q and let Q^* be a certain dimensional dilate of Q with the same center. Fix an $x \notin \cup_{j=1}^m \cup_k (Q_{j,k})^*$. Also fix for the moment the cubes $Q_{j_1,k_1}, \dots, Q_{j_l,k_l}$ and without loss of generality suppose that Q_{j_1,k_1} has the smallest size among them. Let c_{j_1,k_1} be the center of Q_{j_1,k_1}. For fixed $y_{j_2}, \dots, y_{j_l} \in \mathbf{R}^n$, the mean value property of the function b_{j_1,k_1} gives

$$
\begin{aligned}
& \left| \int_{Q_{j_1,k_1}} K(x, y_1, \dots, y_{j_1}, \dots, y_m) b_{j_1,k_1}(y_{j_1})\, dy_{j_1} \right| \\
= & \left| \int_{Q_{j_1,k_1}} \big(K(x, y_1, \dots, y_{j_1}, \dots, y_m) - K(x, y_1, \dots, c_{j_1,k_1}, \dots, y_m) \big) b_{j_1,k_1}(y_{j_1})\, dy_{j_1} \right| \\
\leq & \int_{Q_{j_1,k_1}} |b_{j_1,k_1}(y_{j_1})| \frac{A\,|y_{j_1} - c_{j_1,k_1}|^\varepsilon}{(|x-y_1| + \cdots + |x-y_m|)^{mn+\varepsilon}}\, dy_{j_1} \\
\leq & \int_{Q_{j_1,k_1}} |b_{j_1,k_1}(y_{j_1})| \frac{C\,A\,l(Q_{j_1,k_1})^\varepsilon}{(|x-y_1| + \cdots + |x-y_m|)^{mn+\varepsilon}}\, dy_{j_1},
\end{aligned}
$$

where the previous to last inequality above is due to the fact that

$$
|y_{j_1} - c_{j_1,k_1}| \leq c_n\, l(Q_{j_1,k_1}) \leq \frac{1}{2}|x - y_{j_1}| \leq \frac{1}{2} \max_{1\leq j \leq m} |x - y_j|.
$$

Multiplying the just derived inequality

$$
\left| \int_{Q_{j_1,k_1}} K(x, \vec{y}) b_{j_1,k_1}(y_{j_1})\, dy_{j_1} \right| \leq \int_{Q_{j_1,k_1}} \frac{C\,A\,|b_{j_1,k_1}(y_{j_1})|\, l(Q_{j_1,k_1})^\varepsilon}{(|x-y_1| + \cdots + |x-y_m|)^{mn+\varepsilon}}\, dy_{j_1}
$$

by $\prod_{i\notin\{j_1,\dots,j_l\}} |g_i(y_i)|$ and integrating over all y_i with $i \notin \{j_1,\dots,j_l\}$, we obtain the estimate

$$
\begin{aligned}
&\int_{(\mathbf{R}^n)^{m-l}} \prod_{i\notin\{j_1,\dots,j_l\}} |g_i(y_i)| \left| \int_{Q_{j_1,k_1}} K(x,\vec{y}) b_{j_1,k_1}(y_{j_1})\, dy_{j_1} \right| \prod_{i\notin\{j_1,\dots,j_l\}} dy_i \\
&\le \prod_{i\notin\{j_1,\dots,j_l\}} \|g_i\|_{L^\infty} \int_{Q_{j_1,k_1}} |b_{j_1,k_1}(y_{j_1})| \frac{A\,C\,l(Q_{j_1,k_1})^{\varepsilon}}{(\sum_{j=1}^{l}|x-y_j|)^{mn-(m-l)n+\varepsilon}}\, dy_{j_1} \\
&\le C\,A \prod_{i\notin\{j_1,\dots,j_l\}} \|g_i\|_{L^\infty} \|b_{j_1,k_1}\|_{L^1} \frac{l(Q_{j_1,k_1})^{\varepsilon}}{\big(\sum_{j=1}^{l}(l(Q_{i,k_i})+|x-c_{i,k_i}|)\big)^{nl+\varepsilon}} \\
&\le C\,A \prod_{i\notin\{j_1,\dots,j_l\}} \|g_i\|_{L^\infty} \|b_{j_1,k_1}\|_{L^1} \prod_{i=1}^{l} \frac{l(Q_{j_i,k_i})^{\frac{\varepsilon}{l}}}{(l(Q_{i,k_i})+|x-c_{i,k_i}|)^{n+\frac{\varepsilon}{l}}}.
\end{aligned}
\tag{13}
$$

The inequality before the last one is due to the fact that for $x \notin \cup_{j=1}^{m} \cup_k (Q_{j,k})^*$ and $y_j \in Q_{j,k}$ we have that $|x-y_j| \approx l(Q_{j,k_j}) + |x-c_{j,k_j}|$, while the last inequality is due to our assumption that the cube Q_{j_1,k_1} has the smallest side length. It is now a simple consequence of (13) that for $x \notin \cup_{j=1}^{m} \cup_k (Q_{j,k})^*$ we have

$$
\begin{aligned}
&|T(h_1,\dots,h_m)(x)| \\
\le & CA\int_{(\mathbf{R}^n)^{m-1}} \prod_{i\notin\{j_1,\dots,j_l\}} |g_i(y_i)| \prod_{i=2}^{l} \Big(\sum_{k_i} |b_{j_i,k_i}(y_{j_i})|\Big) \left| \int_{Q_{j_1,k_1}} K(x,\vec{y})\, b_{j_1,k_1}(y_{j_1})\, dy_{j_1} \right| \prod_{i\ne j_1} dy_i \\
\le & CA \prod_{i\notin\{j_1,\dots,j_l\}} \|g_i\|_{L^\infty} \prod_{i=1}^{l} \frac{l(Q_{j_i,k_i})^{\frac{\varepsilon}{l}}}{(l(Q_{i,k_i})+|x-c_{i,k_i}|)^{n+\frac{\varepsilon}{l}}} \int_{(\mathbf{R}^n)^{l-1}} \prod_{i=2}^{l} \Big(\sum_{k_i} |b_{j_i,k_i}(y_{j_i})|\Big) dy_{i_2}\dots dy_{i_l} \\
\le & CA \prod_{i\notin\{j_1,\dots,j_l\}} \|g_i\|_{L^\infty} \prod_{i=2}^{l} \left(\sum_{k_i} \frac{\|b_{j_i,k_i}\|_{L^1}\, l(Q_{j_i,k_i})^{\frac{\varepsilon}{l}}}{(l(Q_{i,k_i})+|x-c_{i,k_i}|)^{n+\frac{\varepsilon}{l}}} \right) \\
\le & C'A(\alpha\gamma)^{\frac{m-l}{m}} \prod_{i=1}^{l} \left(\sum_{k_i} \frac{(\alpha\gamma)^{1/m}\, l(Q_{j_i,k_i})^{n+\frac{\varepsilon}{l}}}{(l(Q_{i,k_i})+|x-c_{i,k_i}|)^{n+\frac{\varepsilon}{l}}} \right) = C''A\,\alpha\,\gamma \prod_{i=1}^{l} M_{i,\varepsilon/l}(x),
\end{aligned}
$$

where

$$
M_{i,\varepsilon/l}(x) = \sum_{k_i} \frac{l(Q_{j_i,k_i})^{n+\frac{\varepsilon}{l}}}{(l(Q_{i,k_i})+|x-c_{i,k_i}|)^{n+\frac{\varepsilon}{l}}}
$$

is the Marcinkiewicz function associated with the union of the cubes $\{Q_{i,k_i}\}_k$. It is a known fact (see for instance [36]) that

$$
\int_{\mathbf{R}^n} M_{i,\varepsilon/l}(x)\, dx \le C|\cup_{k_i} Q_{i,k_i}| \le C'(\alpha\gamma)^{-1/m}.
$$

Now, since

$$
|\cup_{j=1}^{m} \cup_k (Q_{j,k})^*| \le C(\alpha\gamma)^{-1/m},
$$

inequality (12) will be a consequence of the estimate

$$|\{x \notin \cup_{j=1}^m \cup_k (Q_{j,k})^* : \ |T(h_1,\dots,h_m)(x)| > \alpha/2^m\}| \le C(\alpha\gamma)^{-1/m}(A\gamma)^{1/l}. \tag{14}$$

We prove (14) using an $L^{1/l}$ estimate outside $\cup_{j=1}^m \cup_k (Q_{j,k})^*$; recall here that we are considering the situation where l is not zero. Using the size estimate derived above for $|T(h_1,\dots,h_m)(x)|$ outside the exceptional set, we obtain

$$\begin{aligned}
&|\{x \notin \cup_{j=1}^m \cup_k (Q_{j,k})^* : \ |T(h_1,\dots,h_m)(x)| > \alpha/2^m\}| \\
\le & C\alpha^{-1/l} \int_{\mathbf{R}^n \setminus \cup_{j=1}^m \cup_k (Q_{j,k})^*} \big(\alpha\gamma A\, M_{1,\varepsilon/l}(x)\dots M_{l,\varepsilon/l}(x)\big)^{1/l}\, dx \\
\le & C(\gamma A)^{1/l} \bigg(\int_{\mathbf{R}^n} M_{1,\varepsilon/l}(x)dx\bigg)^{1/l} \dots \bigg(\int_{\mathbf{R}^n} M_{l,\varepsilon/l}(x)dx\bigg)^{1/l} \\
\le & C'(\gamma A)^{1/l}\big((\alpha\gamma)^{-1/m}\dots(\alpha\gamma)^{-1/m}\big)^{1/l} = C'\alpha^{-1/m}(A\gamma)^{1/l}\gamma^{-1/m},
\end{aligned}$$

which proves (14) and thus (12).

We have now proved (12) for any $\gamma > 0$. Selecting $\gamma = (A+B)^{-1}$ in both (11) and (12) we obtain that all the sets E_s satisfy (10). Summing over all $1 \le s \le 2^m$ we obtain the conclusion of the theorem. □

Example Let R_1 be *the bilinear Riesz transform in the first variable*

$$R_1(f_1,f_2)(x) = \text{p.v.} \int_{\mathbf{R}}\int_{\mathbf{R}} \frac{x-y_1}{|(x-y_1,x-y_2)|^3} f_1(y_1)f_2(y_2)\, dy_1 dy_2.$$

Using an m-linear $T1$ theorem, it was shown in [25]) that R_1 maps $L^{p_1}(\mathbf{R}) \times L^{p_2}(\mathbf{R})$ to $L^p(\mathbf{R})$ for $1/p_1 + 1/p_2 = 1/p$, $1 < p_1, p_1 < \infty$, $1/2 < p < \infty$. Thus by Theorem 2 it also maps $L^1 \times L^1$ to $L^{1/2,\infty}$. However, it does not map $L^1 \times L^1$ to any Lorentz space $L^{1/2,q}$ for $q < \infty$. In fact, letting $f_1 = f_2 = \chi_{[0,1]}$, an easy computation shows that $R_1(f_1,f_2)(x)$ behaves at infinity like $|x|^{-2}$. This fact indicates that in Theorem 2 the space $L^{1/2,\infty}$ is best possible and cannot be replaced by any smaller space, in particular, it cannot be replaced by $L^{1/2}$.

4. Endpoint estimates and interpolation for multilinear Calderón-Zygmund operators

The theory of multilinear interpolation according to the real method is significantly more complicated than the linear one. Early versions appeared in the work of Janson [27] and Strichartz [37]. In this exposition we will use a version of real multilinear interpolation appearing in [18]. This makes use of the notion of *multilinear restricted weak type* $(p_1,\dots p_m,p)$ *estimates.* These are estimates of the form

$$\sup_{\lambda>0} \lambda\big|\{x : |T(\chi_{A_1},\dots,\chi_{A_m})(x)| > \lambda\}\big|^{1/p} \le M\,|A_1|^{1/p_1}\dots|A_m|^{1/p_m}$$

and have a wonderful interpolation property: if an operator T satisfies restricted weak type $(p_1,\dots,p_m,p)$ and $(q_1,\dots,q_m,q)$ estimates with constants M_0 and M_1, respectively, then it also satisfies a restricted weak type $(r_1, \dots, r_m, r)$ estimate with constant $M_0^{1-\theta} M_1^{\theta}$, where

$$\left(\frac{1}{r_1}, \dots, \frac{1}{r_m}, \frac{1}{r}\right) = (1-\theta)\left(\frac{1}{p_1}, \dots, \frac{1}{p_m}, \frac{1}{p}\right) + \theta\left(\frac{1}{q_1}, \dots, \frac{1}{q_m}, \frac{1}{q}\right).$$

More refined ideas can be employed to obtain the following multilinear interpolation result; for a precise formulation and a proof see [18].

Theorem 3. *Let $0 < p_{ij}, p_i \le \infty$, $i = 1, \dots, m+1$, $j = 1, \dots, m$, and suppose that the points $(1/p_{11}, \dots, 1/p_{1m})$, $(1/p_{21}, \dots, 1/p_{m2})$, $(1/p_{(m+1)1}, \dots, 1/p_{(m+1)m})$ satisfy a certain nondegeneracy condition. Let $(1/q_1, \dots, 1/q_m)$ be in the interior of the convex hull of these $m+1$ points. Suppose that a multilinear operator T satisfies restricted weak type $(p_{i1}, \dots, p_{im}, p_i)$ estimates for $i = 1, \dots, m+1$. Then T has a bounded extension from $L^{q_1} \times \cdots \times L^{q_m} \to L^q$ whenever $1/q \le 1/q_1 + \cdots + 1/q_m$.*

There is also an interpolation theorem saying that if a linear operator (that satisfies a mild assumption) and its transpose are of restricted weak type $(1,1)$, then the operator is L^2 bounded. We prove here a multilinear analogue of this result due to Grafakos and Tao [24]:

Theorem 4. *([24]) Let $1 < p_1, \dots, p_m < \infty$ be such that $1/p_1 + \cdots + 1/p_m = 1/p < 1$. Suppose that an m-linear operator has the property that*

$$\sup_{A_0, A_1, \dots, A_m} |A_0|^{-1/p'} |A_1|^{-1/p_1} \dots |A_m|^{-1/p_m} \left| \int_{A_0} T(\chi_{A_1}, \dots, \chi_{A_m})\, dx \right| < \infty \tag{15}$$

where the supremum is taken over all subsets $A_0, A_1, \dots, A_m$ of finite measure. Also suppose that T^{*j}, $j = 0, 1, \dots, m$ are of restricted weak type $(1, 1, \dots, 1/m)$; this means that these operators map $L^1 \times \cdots \times L^1$ to $L^{1/m,\infty}$ when restricted to characteristic functions with constants $B_0, B_1, \dots, B_m$, respectively. Then there is a constant $C_{p_1,\dots,p_m}$ such that T maps the product of Lorentz spaces $L^{p_1,1} \times \cdots \times L^{p_m,1}$ to weak L^p when restricted to characteristic functions with norm at most

$$C_{p_1,\dots,p_m} B_0^{1/(2p)} B_1^{1/(2p_1')} \dots B_m^{1/(2p_m')}.$$

Proof. We will make use of the following characterization of weak L^p (due to Tao):

$$\big\|g\big\|_{L^{p,\infty}} \approx \sup_{|E|<\infty} \inf_{\substack{E' \subset E \\ |E'| \ge \frac{1}{2}|E|}} |E|^{\frac{1}{p}-1} \left| \int_{E'} g(t)\, dt \right| \tag{16}$$

The easy proof of (16) is omitted.

Let M be the supremum in (15). We consider the following two cases:

Case 1: Suppose that $\frac{|A_0|}{\sqrt{B_0}} \ge \max\left(\frac{|A_1|}{\sqrt{B_1}}, \dots, \frac{|A_m|}{\sqrt{B_m}}\right)$. Since T maps $L^1 \times \cdots \times L^1$ to weak $L^{1/m}$ when restricted to characteristic functions, there exists a subset A_0' of A_0 of measure $|A_0'| \ge \frac{1}{2}|A_0|$ such that

$$\left| \int_{A_0'} T(\chi_{A_1}, \dots, \chi_{A_m}) dx \right| \le C\, B_0\, |A_1|\, \dots\, |A_m|\, |A_0|^{1-\frac{1}{1/m}}$$

for some constant C. Then

$$\begin{aligned}\Big|\int_{A_0} T(\chi_{A_1},\dots,\chi_{A_m})dx\Big| \le & \Big|\int_{A_0'} T(\chi_{A_1},\dots,\chi_{A_m})dx\Big| + \Big|\int_{A_0\setminus A_0'} T(\chi_{A_1},\dots,\chi_{A_m})dx\Big| \\ \le & C\, B_0\, |A_1|\,\dots|A_m|\,|A_0|^{-m+1} + M|A_1|^{\frac{1}{p_1}}\dots|A_2|^{\frac{1}{p_2}}\Big(\frac{1}{2}|A_0|\Big)^{\frac{1}{p'}} \\ \le & CB_0|A_1|^{\frac{1}{p_1}}\Big(\frac{\sqrt{B_1}}{\sqrt{B_0}}\Big)^{\frac{1}{p_1'}}\dots|A_m|^{\frac{1}{p_m}}\Big(\frac{\sqrt{B_m}}{\sqrt{B_0}}\Big)^{\frac{1}{p_m'}}|A_0|^{\sum_{s=1}^m\frac{1}{p_s'}-m+1} \\ & + M\, 2^{-\frac{1}{p'}}|A_1|^{\frac{1}{p_1}}\dots|A_m|^{\frac{1}{p_m}}|A_0|^{\frac{1}{p'}}\,.\end{aligned}$$

It follows that M has to be less than or equal to

$$C\, B_0\Big(\frac{\sqrt{B_1}}{\sqrt{B_0}}\Big)^{1/p_1'}\dots\Big(\frac{\sqrt{B_m}}{\sqrt{B_0}}\Big)^{1/p_m'} + M\, 2^{-1/p'}$$

and consequently

$$M \le \frac{C}{1-2^{-1/p'}}B_0^{1/(2p)}B_1^{1/(2p_1')}\dots B_m^{1/(2p_m')}\,.$$

Case 2: Suppose that $\frac{|A_j|}{\sqrt{B_j}} \ge \max\big(\frac{|A_0|}{\sqrt{B_0}},\dots,\frac{|A_m|}{\sqrt{B_m}}\big)$ for some $j\ge 1$. To simplify notation, let us take $j=1$. Here we use that T^{*1} maps $L^1\times\dots\times L^1$ to weak $L^{1/m}$ when restricted to characteristic functions. Then there exists a subset A_1' of A_1 of measure $|A_1'|\ge\frac{1}{2}|A_1|$ such that

$$\Big|\int_{A_1'} T^{*1}(\chi_{A_0},\dots,\chi_{A_m})dx\Big| \le C\, B_1\, |A_0|\,|A_2|\dots|A_m|\,|A_1|^{-m+1}$$

for some constant C. Equivalently, we write this statement as

$$\Big|\int_{A_0} T(\chi_{A_1'},\chi_{A_2}\dots,\chi_{A_m})dx\Big| \le C\, B_1\, |A_0|\,|A_2|\dots|A_m|\,|A_1|^{-m+1}.$$

by the definition of the first dual operator T^{*1}. Therefore we obtain

$$\begin{aligned}\Big|\int_{A_0} T(\chi_{A_1},\dots)dx\Big| \le & \Big|\int_{A_0} T(\chi_{A_1'},\chi_{A_2},\dots)dx\Big| + \Big|\int_{A_0} T(\chi_{A_1\setminus A_1'},\chi_{A_2},\dots)dx\Big| \\ \le & CB_1|A_0|\Big(\prod_{s=2}^m|A_s|\Big)|A_1|^{-m+1} + M|A_0|^{\frac{1}{p'}}\prod_{s=2}^m|A_s|^{\frac{1}{p_s}}\Big(\frac{1}{2}|A_1|\Big)^{\frac{1}{p_1}} \\ \le & CB_1|A_1|^{-m+1+\frac{1}{p}+\sum_{s=2}^m\frac{1}{p_s'}}\Big(\frac{\sqrt{B_0}}{\sqrt{B_1}}\Big)^{\frac{1}{p}}|A_0|^{\frac{1}{p'}}\prod_{s=2}^m|A_s|^{\frac{1}{p_s}}\Big(\frac{\sqrt{B_s}}{\sqrt{B_1}}\Big)^{\frac{1}{p_s'}} \\ & + M\, 2^{-1/p_1}|A_1|^{\frac{1}{p_1}}|A_2|^{\frac{1}{p_2}}\dots|A_m|^{\frac{1}{p_m}}|A_0|^{\frac{1}{p'}}\,.\end{aligned}$$

By the definition of M, it follows that

$$M \le \frac{C}{1-2^{-1/p_1}}B_0^{1/(2p)}B_1^{1/(2p_1')}\dots B_m^{1/(2p_m')}\,.$$

Then the statement of the theorem follows with

$$C_{p_1,\dots,p_m} = C \max\left(\frac{1}{1-2^{-1/p_1}}, \dots, \frac{1}{1-2^{-1/p_m}}, \frac{1}{1-2^{-1/p'}}\right).$$

□

Assumption (15) is not as restrictive as it looks. To apply this theorem for m-linear Calderón-Zygmund operators, one needs to consider the family of operators whose kernels are truncated near the origin, i.e.,

$$K_\delta(x, y_1, \dots, y_m) = K(x, y_1, \dots, y_m)\zeta\big((|x-y_1| + \dots + |x-y_m|)/\delta\big),$$

where ζ is a smooth function that is equal to 1 on $[2,\infty)$ and vanishes on $[0,1]$. The kernels K_δ are essentially in the same Calderón-Zygmund kernel class as K, that is if K lies in m-$CZK(A,\varepsilon)$, then K_ε lie in m-$CZK(A',\varepsilon)$, where A' is a multiple of A. Using Hölder's inequality with exponents $p_1, \dots, p_m, p'$, it is easy to see that for the operators T_δ with kernels K_δ, assumption (15) holds with constants depending on δ.

Theorem 4 provides an interpolation machinery needed to pass from bounds at one point to bounds at every point for multilinear Calderón-Zygmund operators. (An alternative interpolation technique was described in [25].) We have:

Theorem 5. *Suppose that an operator T with kernel in m-$CZK(A,\delta)$ and all of its truncations T_δ map $L^{r_1} \times \dots \times L^{r_m} \to L^r$ for a single tuple of indices $r_1, \dots, r_m, r$ satisfying $1/r_1 + \dots + 1/r_m = 1/r$ and $1 < r_1, \dots, r_m, r < \infty$ uniformly in δ. Then T is bounded from $L^{p_1} \times \dots \times L^{p_m} \to L^p$ for all indices $p_1, \dots, p_m, p$ satisfying $1/p_1 + \dots + 1/p_m = 1/p$ and $1 < p_1, \dots, p_m < \infty$, $1/m < p < \infty$.*

Proof. Since T_δ maps $L^{r_1} \times \dots \times L^{r_m} \to L^r$ and $r > 1$, duality gives that T_δ^{*1} maps $L^{r'} \times L^{r_2} \times \dots \times L^{r_m} \to L^{r_1'}$ and likewise for the remaining adjoints (uniformly in δ). It follows from Theorems 4 and 3 that T_δ are bounded from $L^{p_1} \times \dots \times L^{p_m} \to L^p$ for all indices $p_1, \dots, p_m, p$ satisfying $1/p_1 + \dots + 1/p_m = 1/p$ and $1 < p_1, \dots, p_m < \infty$, $1/m < p < \infty$. Passing to the limit, using Fatou's lemma, the same conclusion may be obtained for the non truncated operator T. □

This approach has the drawback that it uses the redundant assumption that if T is bounded from $L^{r_1} \times L^{r_2} \to L^r$, then so are all its truncations T_δ (uniformly in $\delta > 0$). This is hardly a problem in concrete applications since the kernels of T and T_δ satisfy equivalent estimates (uniformly in $\delta > 0$) and the method used in the proof of the boundedness of the former almost always applies for the latter.

5. The m-linear Mikhlin-Hörmander multiplier theorem

In this section, we focus on an analogue of a classical linear multiplier theorem in the multilinear case. We first note that the Marcinkiewicz multiplier theorem fails for multilinear operators, see [19]. However, the Mikhlin -Hörmander multiplier theorem (see [35], [26]) has a multilinear extension, which we discuss below.

The multilinear Fourier multiplier operator T_σ associated with a symbol σ is defined by

$$T_\sigma(f_1,\dots,f_m)(x)=\int_{(\mathbf{R}^n)^m} e^{2\pi i x\cdot(\xi_1+\cdots+\xi_m)}\sigma(\xi_1,\dots,\xi_m)\widehat{f_1}(\xi_1)\cdots\widehat{f_m}(\xi_m)\,d\xi_1\cdots d\xi_m$$

for $f_i\in\mathscr{S}(\mathbf{R}^n)$, $i=1,\cdots,m$.

Coifman and Meyer [8] proved that if σ is a bounded function on $\mathbf{R}^{mn}\setminus\{0\}$ that satisfies

$$|\partial_{\xi_1}^{\alpha_1}\cdots\partial_{\xi_m}^{\alpha_m}\sigma(\xi_1,\dots,\xi_m)|\le C_\alpha(|\xi_1|+\cdots+|\xi_m|)^{-(|\alpha_1|+\cdots+|\alpha_m|)} \tag{17}$$

away from the origin for all sufficiently large multiindices α_j, then T_σ is bounded from the product $L^{p_1}(\mathbf{R}^n)\times\cdots\times L^{p_m}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ for all $1<p_1,\dots,p_m,p<\infty$ satisfying $\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p}$. Their proof is based on the idea of writing the Fourier multiplier σ as a rapidly convergent sum of products of functions of the variables ξ_j. The multiplier theorem of Coifman and Meyer was extended to indices $p<1$ (and larger than $1/m$ by Grafakos and Torres [25] and Kenig and Stein [29] (when $m=2$).

A different approach was taken by Tomita [38] who extended the proof of the Hörmander multiplier theorem in [14] to obtain the following result in the m-linear case:

Theorem A. [38] *Let $\sigma\in L^\infty(\mathbf{R}^{mn})$. Let Ψ be a Schwartz function whose Fourier transform is supported in the set $\{\vec{\xi}\in(\mathbf{R}^n)^m: 1/2\le|\vec{\xi}|\le 2\}$ and satisfies*

$$\sum_{j\in\mathbf{Z}}\widehat{\Psi}(\vec{\xi}/2^j)=1 \tag{18}$$

for all $\vec{\xi}\in(\mathbf{R}^n)^m\setminus\{0\}$. Suppose that for some $s>mn/2$, the function $\sigma\in L^\infty(\mathbf{R}^{mn})$ satisfies

$$\sup_{k\in\mathbf{Z}}\|\sigma^k\,\widehat{\Psi}\|_{L^2_s}<\infty.$$

where for $k\in\mathbf{Z}$, σ^k is defined as

$$\sigma^k(\xi_1,\dots,\xi_m)=\sigma(2^k\xi_1,\dots,2^k\xi_m)\,. \tag{19}$$

Then T_σ is bounded from $L^{p_1}(\mathbf{R}^n)\times\cdots\times L^{p_m}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$, whenever $1<p_1,p_2,\dots,p_m,p<\infty$ and $1/p_1+\cdots+1/p_m=1/p$.

Let $\mathscr{S}_1(\mathbf{R}^d)$ be the set of all Schwartz functions Ψ on $\mathbf{R}^d$, whose Fourier transform is supported in an annulus of the form $\{\xi:\ c_1<|\xi|<c_2\}$, is nonvanishing in a smaller annulus $\{\xi:\ c_1'\le|\xi|\le c_2'\}$ (for some choice of constants $0<c_1<c_1'<c_2'<c_2<\infty$), and satisfies

$$\sum_{j\in\mathbf{Z}}\widehat{\Psi}(2^{-j}\xi)=\text{constant},\qquad \xi\in\mathbf{R}^{\mathbf{d}}\setminus\{0\}. \tag{20}$$

Theorem A has an extension to the case where the target space is L^p for $p\le 1$:

Theorem 6. *([22]) Let $1 < r \le 2$. Suppose that σ is a function on $\mathbf{R}^{nm}$ and Ψ is a function in $\mathscr{S}_1(\mathbf{R}^{nm})$ that satisfies for some $\gamma > \frac{mn}{r}$*

$$\sup_{k\in\mathbf{Z}} \|\sigma^k\,\widehat{\Psi}\|_{L^r_\gamma(\mathbf{R}^{mn})} = K < \infty, \tag{21}$$

where σ^k is defined in (19). Then there is a number $\delta = \delta(mn,\gamma,r)$ satisfying $0 < \delta \le r-1$, such that the m-linear operator T_σ, associated with the multiplier σ, is bounded from $L^{p_1}(\mathbf{R}^n) \times \cdots \times L^{p_m}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$, whenever $r-\delta < p_j < \infty$ for all $j = 1,\dots,m$, and p is given by

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}\,. \tag{22}$$

In the rest of this section, we prove Theorem 6.

5.1. **Preliminary material.** We develop some preliminary material needed in the proof of Theorem 6. For $s \in \mathbf{R}$ we denote by w_s the weight

$$w_s(x) = (1 + 4\pi^2|x|^2)^{s/2}\,.$$

Definition 1. *For $1 \le p < \infty$, the weighted Lebesgue space $L^p(w_s)$ is defined as the set of all measurable functions f on $\mathbf{R}^d$ such that*

$$\|f\|_{L^p(w_s)} = \left(\int_{\mathbf{R}^d} |f(x)|^p\, w_s(x)\, dx\right)^{1/p} < \infty.$$

We note that for $1 < r \le 2$ one has

$$\begin{aligned}
\|\widehat{g}\,\|_{L^{r'}(w_s)} &= \left(\int_{\mathbf{R}^d} |\widehat{g}\,|^{r'} w_s\, d\xi\right)^{\frac{1}{r'}} \\
&= \left(\int_{\mathbf{R}^d} |\widehat{g}\, w_{s/r'}|^{r'}\, d\xi\right)^{\frac{1}{r'}} \\
&= \left(\int_{\mathbf{R}^d} \big|\big[(I-\Delta)^{\frac{s}{2r'}} g\big]\widehat{\ }\big|^{r'}\, d\xi\right)^{\frac{1}{r'}} \\
&\le \left(\int_{\mathbf{R}^d} \big|(I-\Delta)^{\frac{s}{2r'}} g\big|^{r}\, dx\right)^{\frac{1}{r}} \\
&= \|g\,\|_{L^r_{s/r'}}\,,
\end{aligned} \tag{23}$$

via the Hausdorff-Young inequality.

Lemma 1. *Let $1 \le p < q < \infty$. Then for every $s \ge 0$ there exists a constant $C = C(p,q,s,d) > 0$ such that for all functions g supported in a ball of a fixed finite radius in $\mathbf{R}^d$ we have*

$$\|g\|_{L^p_s(\mathbf{R}^d)} \le C\,\|g\|_{L^q_s(\mathbf{R}^d)}\,.$$

Proof. Since g is supported in a ball of finite fixed radius, then $g = g\,\varphi$ for some compactly supported smooth function φ that is equal to one on the support of g. Pick r such that

$$1/p = 1/q + 1/r\,.$$

The Kato-Ponce rule [28] gives the estimate

$$\begin{aligned}\|g\|_{L^p_s(\mathbf{R}^d)} &= \big\|(I-\Delta)^{s/2}(g\,\varphi)\big\|_{L^p}\\ &\le C\big[\big\|(I-\Delta)^{s/2}g\big\|_{L^q}\|\varphi\|_{L^r}+\|g\|_{L^q}\big\|(I-\Delta)^{s/2}\varphi\big\|_{L^r}\big]\\ &= C_\varphi\big[\big\|(I-\Delta)^{s/2}g\big\|_{L^q}+\|g\|_{L^q}\big]\,.\end{aligned}$$

Now the Bessel potential operator $J_s=(I-\Delta)^{-s/2}$ is bounded from L^q to itself for all $s>0$. This implies that

$$\|g\|_{L^q}\le C'\big\|(I-\Delta)^{s/2}g\big\|_{L^q}$$

Combining this estimate with the one previously obtained, we deduce that

$$\|g\|_{L^p_s(\mathbf{R}^d)}\le 2\,C_\varphi\,C'\big\|(I-\Delta)^{s/2}g\big\|_{L^q(\mathbf{R}^d)}=C\,\|g\|_{L^q_s(\mathbf{R}^d)}\,.$$

□

Lemma 2. *Suppose that $s\ge 0$ and $1<r<\infty$. Assume that φ lies in $\mathscr{S}(\mathbf{R}^d)$. Then there is a constant c_φ such that for all $g\in L^r_s(\mathbf{R}^d)$ we have*

$$\|g\,\varphi\|_{L^r_s}\le c_\varphi\,\|g\|_{L^r_s}\,.$$

Proof. We write

$$(I-\Delta)^{s/2}(g\,\varphi)=\int_{\mathbf{R}^d}\widehat{\varphi}(\tau)(I-\Delta)^{s/2}(g\,e^{2\pi i\tau\cdot(\cdot)})\,d\tau\,.$$

It will suffice to show that the L^r norm of $(I-\Delta)^{s/2}(g\,e^{2\pi i\tau\cdot(\cdot)})$ is controlled by $C_M\,(1+|\tau|)^M$ times the L^r norm of $(I-\Delta)^{s/2}g$, for some $M>0$. This statement is equivalent to showing that the function

$$\left(\frac{1+|\xi-\tau|^2}{1+|\xi|^2}\right)^{\frac{s}{2}}$$

is an L^r Fourier multiplier with norm at most a multiple of $(1+|\tau|)^M$. But this is an easy consequence of the Mihlin multiplier theorem. □

Lemma 3. *Let Δ_k be the Littlewood-Paley operator given by $\Delta_k(g)\widehat{\ }(\xi)=\widehat{g}(\xi)\widehat{\Psi}(2^{-k}\xi)$, $k\in\mathbf{Z}$, where Ψ is a Schwartz function whose Fourier transform is supported in the annulus $\{\xi:2^{-b}<|\xi|<2^b\}$, for some $b\in\mathbf{Z}^+$ and satisfies $\sum_{k\in\mathbf{Z}}\widehat{\Psi}(2^{-k}\xi)=c_0$, for some constant c_0. Let $0<p<\infty$. Then there is a constant $c=c(n,p,c_0,\Psi)$, such that for L^p functions f we have*

$$\|f\|_{L^p}\le c\left\|\Big(\sum_{k\in\mathbf{Z}}|\Delta_k(f)|^2\Big)^{1/2}\right\|_{L^p}.$$

Proof. Let Φ be a Schwartz function with integral one. Then the following quantity provides a characterization of the H^p norm:

$$\|f\|_{H^p}\approx\big\|\sup_{t>0}|f*\Phi_t|\,\big\|_{L^p}\,.$$

It follows that for f in $H^p\cap L^2$, which is a dense subclass of H^p, one has the estimate

$$|f|\le\sup_{t>0}|f*\Phi_t|\,,$$

since the family $\{\Phi_t\}_{t>0}$ is an approximate identity. Thus

$$\|f\|_{L^p} \le c\,\|f\|_{H^p}$$

whenever f is a function in H^p.

Keeping this observation in mind we can write:

$$\begin{aligned}\|f\|_{L^p} \le&\ c\,\|f\|_{H^p}\\ \le&\ \Big\|\Big(\sum_{j\in\mathbf{Z}}|\Delta_j(f)|^2\Big)^{1/2}\Big\|_{L^p}\\ =&\ c\Big\|\Big(\sum_{j\in\mathbf{Z}}\Big|\Delta_j\Big(\sum_{k\in\mathbf{Z}}\Delta_k(f)\Big)\Big|^2\Big)^{1/2}\Big\|_{L^p}\\ \le&\ c'\Big\|\Big(\sum_{k\in\mathbf{Z}}|\Delta_k(f)|^2\Big)^{1/2}\Big\|_{L^p}\end{aligned}$$

in view of the fact that $\Delta_j\Delta_k = 0$ unless $|j-k|\le b$. □

5.2. **The proof of Theorem 6.** Having disposed of the preliminary material, we now prove Theorem 6.

Proof. For each $j = 1,\dots,m$, we let R_j be the set of points $(\xi_1,\dots,\xi_m)$ in $(\mathbf{R}^n)^m$ such that $|\xi_j| = \max\{|\xi_1|,\dots,|\xi_m|\}$. For $j = 1,\dots,m$, we introduce nonnegative smooth functions ϕ_j on $[0,\infty)^{m-1}$ that are supported in $[0,\frac{11}{10}]^{m-1}$ such that

$$1 = \sum_{j=1}^{m}\phi_j\Big(\frac{|\xi_1|}{|\xi_j|},\dots,\widetilde{\frac{|\xi_j|}{|\xi_j|}},\dots,\frac{|\xi_m|}{|\xi_j|}\Big)$$

for all $(\xi_1,\dots,\xi_m)\neq 0$, with the understanding that the variable with the tilde is missing. These functions introduce a partition of unity of $(\mathbf{R}^n)^m\setminus\{0\}$ subordinate to a conical neighborhood of the region R_j.

Each region R_j can be written as the union of sets

$$R_{j,k} = \big\{(\xi_1,\dots,\xi_m)\in R_j:\ |\xi_k|\ge|\xi_s|\quad\text{for all } s\neq j\big\}$$

over $k = 1,\dots,m$. We need to work with a finer partition of unity, subordinate to each $R_{j,k}$. To achieve this, for each j, we introduce smooth functions $\phi_{j,k}$ on $[0,\infty)^{m-2}$ supported in $[0,\frac{11}{10}]^{m-2}$ such that

$$1 = \sum_{\substack{k=1\\k\neq j}}^{m}\phi_{j,k}\Big(\frac{|\xi_1|}{|\xi_k|},\dots,\widetilde{\frac{|\xi_k|}{|\xi_k|}},\dots,\widetilde{\frac{|\xi_j|}{|\xi_k|}},\dots,\frac{|\xi_m|}{|\xi_k|}\Big)$$

for all $(\xi_1,\dots,\xi_m)$ in the support of ϕ_j with $\xi_k\neq 0$ (with missing kth and jth entries).

We now have obtained the following partition of unity of $(\mathbf{R}^n)^m\setminus\{0\}$:

$$1 = \sum_{j=1}^{m}\sum_{\substack{k=1\\k\neq j}}^{m}\phi_j(\dots)\,\phi_{j,k}(\dots)\,,$$

where the dots indicate the variables of each function.

We introduce a nonnegative smooth bump ψ supported in $[(10m)^{-1}, 2]$ and equal to 1 on the interval $[(5m)^{-1}, \frac{12}{10}]$, and we decompose the identity on $(\mathbf{R}^n)^m \setminus \{0\}$ as follows

$$1 = \sum_{j=1}^{m} \sum_{\substack{k=1 \\ k\neq j}}^{m} \left[\Phi_{j,k} + \Psi_{j,k} \right],$$

where

$$\Phi_{j,k}(\xi_1, \dots, \xi_m) = \phi_j(\dots)\, \phi_{j,k}(\dots) \bigg(1 - \psi\Big(\frac{|\xi_k|}{|\xi_j|} \Big) \bigg)$$

and

$$\Psi_{j,k}(\xi_1, \dots, \xi_m) = \phi_j(\dots)\, \phi_{j,k}(\dots) \psi\Big(\frac{|\xi_k|}{|\xi_j|} \Big).$$

This partition of unity induces the following decomposition of σ:

$$\sigma = \sum_{j=1}^{m} \sum_{\substack{k=1 \\ k\neq j}}^{m} \left[\sigma\, \Phi_{j,k} + \sigma\, \Psi_{j,k} \right].$$

We will prove the required assertion for each piece of this decomposition, i.e., for the multipliers $\sigma\, \Phi_{j,k}$ and $\sigma\, \Psi_{j,k}$ for each pair (j, k) in the previous sum. In view of the symmetry of the decomposition, it suffices to consider the case of a *fixed* pair (j, k) in the previous sum. To simplify notation, we fix the pair $(m, m-1)$, thus, for the rest of the proof we fix $j = m$ and $k = m-1$ and we prove boundedness for the m-linear operators whose symbols are $\sigma_1 = \sigma\, \Phi_{m,m-1}$ and $\sigma_2 = \sigma\, \Psi_{m,m-1}$. These correspond to the m-linear operators T_{σ_1} and T_{σ_2}, respectively. The important thing to keep in mind is that σ_1 is supported in the set where

$$\max(|\xi_1|, \dots, |\xi_{m-2}|) \le \tfrac{11}{10}\, |\xi_{m-1}| \le \tfrac{11}{10} \cdot \tfrac{1}{5m}\, |\xi_m|$$

and σ_2 is supported in the set where

$$\max(|\xi_1|, \dots, |\xi_{m-2}|) \le \tfrac{11}{10}\, |\xi_{m-1}|$$

and

$$\tfrac{1}{10m} \le \tfrac{|\xi_{m-1}|}{|\xi_m|} \le 2\,.$$

We first consider $T_{\sigma_1}(f_1, \dots, f_m)$, where f_j are fixed Schwartz functions. We fix a Schwartz radial function η whose Fourier transform is supported in the annulus $1 - \frac{1}{25} \le |\xi| \le 2$ and satisfies

$$\sum_{j\in\mathbf{Z}} \widehat{\eta}(2^{-j}\xi) = 1, \qquad \xi \in \mathbf{R}^n \setminus \{0\}.$$

Associated with η we define the Littlewood-Paley operator $\Delta_j(f) = f * \eta_{2^{-j}}$, where $\eta_t(x) = t^{-n}\eta(t^{-1}x)$ for $t > 0$. We decompose the function f_m as $\sum_{j\in\mathbf{Z}} \Delta_j(f_m)$ and we note that the spectrum (i.e. the Fourier transform) of $T_{\sigma_1}(f_1, \dots, f_{m-1}, \Delta_j(f_m))$ is contained in the set

$$\left\{ \xi_1 : |\xi_1| \le \tfrac{3\cdot 2^j}{5m} \right\} + \cdots + \left\{ \xi_{m-1} : |\xi_{m-1}| \le \tfrac{3\cdot 2^j}{5m} \right\} + \left\{ \xi_m : \tfrac{24}{25} \cdot 2^j \le |\xi_m| \le 2 \cdot 2^j \right\}$$

This algebraic sum of these sets is contained in the annulus

$$\{z \in \mathbf{R}^n : \ \tfrac{9}{25} \cdot 2^j \le |z| \le \tfrac{65}{25} \cdot 2^j\}.$$

We now introduce another bump that is equal to 1 on the annulus $\{z \in \mathbf{R}^n : \ \frac{9}{25} \le |z| \le \frac{65}{25}\}$ and vanishes in the complement of the larger annulus $\{z \in \mathbf{R}^n : \ \frac{8}{25} < |z| < \frac{66}{25}\}$. We call $\widetilde{\Delta}_j$ the Littlewood-Paley operators associated with this bump and we note that

$$\widetilde{\Delta}_j(T_{\sigma_1}(f_1, \dots, \Delta_j(f_m))) = T_{\sigma_1}(f_1, \dots, \Delta_j(f_m)).$$

Finally, we define an operator S_j by setting

$$S_j(g) = g * \zeta_{2^{-j}} ,$$

where ζ is a smooth function whose Fourier transform is equal to 1 on the ball $|z| < 3/5m$ and vanishes outside the double of this ball. Using this notation, we may write

$$\begin{aligned} T_{\sigma_1}(f_1, \dots, f_{m-1}, f_m) &= \sum_j T_{\sigma_1}\big(f_1, \dots, f_{m-1}, \Delta_j(f_m)\big) \\ &= \sum_j T_{\sigma_1}\big(S_j(f_1), \dots, S_j(f_{m-1}), \Delta_j(f_m)\big) \\ &= \sum_j \widetilde{\Delta}_j\big(T_{\sigma_1}(S_j(f_1), \dots, S_j(f_{m-1}), \Delta_j(f_m))\big) . \end{aligned}$$

Since the Fourier transforms of $\widetilde{\Delta}_j\big(T_{\sigma_1}(S_j(f_1), \dots, S_j(f_{m-1}), \Delta_j(f_m))\big)$ have bounded overlap, Lemma 3 yields that

$$\|T_{\sigma_1}(f_1, \dots, , f_m)\|_{L^p} \le C \bigg\| \Big[\sum_j \big|T_{\sigma_1}\big(S_j(f_1), \dots, S_j(f_{m-1}), \Delta_j(f_m)\big)\big|^2 \Big]^{\frac{1}{2}} \bigg\|_{L^p}$$

Obviously, we have

$$\begin{aligned} &T_{\sigma_1}\big(S_j(f_1), \dots, S_j(f_{m-1}), \Delta_j(f_m)\big)(x) \\ &= \int_{(\mathbf{R}^n)^m} e^{2\pi i x\cdot(\xi_1+\cdots+\xi_m)} \sigma_1(\xi_1, \dots, \xi_m) \prod_{k=1}^{m-1} \widehat{S_j(f_k)}(\xi_k)\ \widehat{\Delta_j(f_m)}(\xi_m)\, d\xi_1 \cdots d\xi_m . \end{aligned}$$

A simple calculation yields that the support of the integrand in the previous integral is contained in the annulus

$$\Big\{(\xi_1, \dots, \xi_m) \in (\mathbf{R}^n)^m : \ \tfrac{7}{10} \cdot 2^j < |(\xi_1, \dots, \xi_m)| < \tfrac{21}{10} \cdot 2^j \Big\},$$

so one may introduce in the previous integral the factor $\widehat{\Psi}(2^{-j}\xi_1, \dots, 2^{-j}\xi_m)$, where Ψ is a radial function in $\mathscr{S}_1((\mathbf{R}^n)^m)$ whose Fourier transform is supported in some annulus and is equal to 1 on the annulus

$$\Big\{(z_1, \dots, z_m) \in (\mathbf{R}^n)^m : \ \tfrac{7}{10} \le |(z_1, \dots, z_m)| \le \tfrac{21}{10} \Big\}.$$

Inserting this factor and taking the inverse Fourier transform, we obtain that

$$T_{\sigma_1}\big(S_j(f_1),\dots,S_j(f_{m-1}),\Delta_j(f_m)\big)(x)$$

is equal to

$$\int_{(\mathbf{R}^n)^m} 2^{mnj}(\sigma_1^j\,\widehat{\Psi})^\vee(2^j(x-y_1),\dots,2^j(x-y_m))\prod_{i=1}^{m-1}S_j(f_i)(y_i)\,\Delta_j(f_m)(y_m)\,d\vec{y},$$

where $d\vec{y}=dy_1\dots dy_m$, the check indicates the inverse Fourier transform in all variables, and

$$\sigma_1^j(\xi_1,\xi_2,\dots,\xi_m)=\sigma_1(2^j\xi_1,\dots,2^j\xi_m)\,.$$

We pick a ρ such that $1<\rho<r\le 2$ and $\gamma>mn/\rho$. This is possible since $\gamma>mn/r$; for instance $\rho=\frac{mn}{\gamma}+\frac{1}{1000}(r-\frac{mn}{\gamma})$ is a good choice if this number is bigger than 1, otherwise we set $\rho=\frac{1+r}{2}$. We define $\delta=r-\rho$. We now have:

$$\begin{aligned}
&|T_{\sigma_1}(S_j(f_1),\dots,S_j(f_{m-1}),\Delta_j(f_m))(x)|\\
\le&\int_{(\mathbf{R}^n)^m} w_\gamma\big(2^j(x-y_1),\dots,2^j(x-y_m)\big)\;|(\sigma_1^j\,\widehat{\Psi})^\vee(2^j(x-y_1),\dots,2^j(x-y_m))|\\
&\qquad\qquad\times\frac{2^{mnj}|S_j(f_1)(y_1)\cdots S_j(f_{m-1})(y_{m-1})\Delta_j(f_m)(y_m)|}{w_\gamma\big(2^j(x-y_1),\dots,2^j(x-y_m)\big)}\,d\vec{y}\\
\le&\bigg[\int_{(\mathbf{R}^n)^m}\Big|\big(w_\gamma\,(\sigma_1^j\,\widehat{\Psi})^\vee\big)(2^j(x-y_1),\dots,2^j(x-y_m))\Big|^{\rho'}d\vec{y}\bigg]^{\frac{1}{\rho'}}\\
&\qquad\times 2^{mnj}\left(\int_{(\mathbf{R}^n)^m}\frac{|S_j(f_1)(y_1)\cdots S_j(f_{m-1})(y_{m-1})\Delta_j(f_m)(y_m)|^\rho}{w_{\gamma\rho}\big(2^j(x-y_1),\dots 2^j(x-y_m)\big)}\,d\vec{y}\right)^{\frac{1}{\rho}}\\
\le&C\left(\int_{(\mathbf{R}^n)^m} w_{\gamma\rho'}(y_1,\dots,y_m)|(\sigma_1^j\,\widehat{\Psi})^\vee(y_1,\dots,y_m)|^{\rho'}d\vec{y}\right)^{\frac{1}{\rho'}}\\
&\qquad\times\left(\int_{(\mathbf{R}^n)^m}\frac{2^{mnj}|S_j(f_1)(y_1)\cdots S_j(f_{m-1})(y_{m-1})\Delta_j(f_m)(y_m)|^\rho}{(1+2^j|x-y_1|)^{\gamma\rho/m}\cdots(1+2^j|x-y_m|)^{\gamma\rho/m}}\,d\vec{y}\right)^{\frac{1}{\rho}}\\
\le&\|(\sigma_1^j\,\widehat{\Psi})^\vee\|_{L^{\rho'}(w_{\gamma\rho'})}\prod_{i=1}^{m-1}\left(\int_{\mathbf{R}^n}\frac{2^{jn}|S_j(f_i)(y_i)|^\rho}{(1+2^j|x-y_i|)^{\gamma\rho/m}}\,dy_i\right)^{\frac{1}{\rho}}\\
&\qquad\qquad\times\left(\int_{\mathbf{R}^n}\frac{2^{jn}|\Delta_j(f_m)(y_m)|^\rho}{(1+2^j|x-y_m|)^{\gamma\rho/m}}\,dy_m\right)^{\frac{1}{\rho}}\\
\le&\|(\sigma_1^j\,\widehat{\Psi})^\vee\|_{L^{\rho'}(w_{\gamma\rho'})}c^{m/\rho}\prod_{i=1}^{m-1}\left(\mathcal{M}(\mathcal{M}(f_i)^\rho)(x)\right)^{\frac{1}{\rho}}\left(\mathcal{M}(|\Delta_j(f_m)|^\rho)(x)\right)^{\frac{1}{\rho}}\ ,
\end{aligned}$$

where we used that

$$\int_{\mathbf{R}^n}\frac{2^{jn}|h(y)|}{(1+2^j|x-y|)^{\gamma\rho/m}}\,dy\le c\,\mathcal{M}(h)(x)\,,$$

a consequence of the fact that $\gamma\rho/m>n$.

We now have the sequence of inequalities:

$$\|(\sigma_1^j\,\widehat{\Psi})^\vee\|_{L^{\rho'}(w_{\gamma\rho'})} \le \|\sigma_1^j\,\widehat{\Psi}\|_{L^\rho_\gamma} \le C''\,\|\sigma_1^j\,\widehat{\Psi}\|_{L^r_\gamma} \le C'\,\|\sigma^j\,\widehat{\Psi}\|_{L^r_\gamma} < C\,K\,,$$

justified by the result in the calculation (23) for the first, Lemma 1 together with the facts that $1 < \rho < r$ and σ_1^j is supported in a ball of a fixed radius for the second inequality, Lemma 2 for the third, and the hypothesis of Theorem 6 for the last inequality.

Thus we have obtained the estimate:

$$\begin{aligned}&|T_{\sigma_1}(S_j(f_1),\dots,S_j(f_{m-1}),\Delta_j(f_m))|\\ &\qquad\le C\,K\prod_{i=1}^{m-1}\big(\mathcal{M}(\mathcal{M}(f_i)^\rho)\big)^{\frac{1}{\rho}}\,(\mathcal{M}(|\Delta_j(f_m)|^\rho))^{\frac{1}{\rho}}\ .\end{aligned}$$

We now square the previous expression, we sum over $j\in\mathbf{Z}$ and we take square roots. Since $r-\delta=\rho$, the hypothesis $p_j > r-\delta$ implies $p_j > \rho$, and thus each term $(\mathcal{M}(\mathcal{M}(f_i)^\rho))^{\frac{1}{\rho}}$ is bounded on $L^{p_j}(\mathbf{R}^n)$. We obtain

$$\begin{aligned}&\big\|T_{\sigma_1}(f_1,\dots,f_{m-1},f_m)\big\|_{L^p(\mathbf{R}^n)}\\
&\le C\,K\Big\|\Big\{\sum_j|T_{\sigma_1}(S_j(f_1),\dots,S_j(f_{m-1}),\Delta_j(f_m))|^2\Big\}^{\frac12}\Big\|_{L^p(\mathbf{R}^n)}\\
&\le C'\,K\Big\|\Big\{\sum_j\mathcal{M}(|\Delta_j(f_m)|^\rho)^{\frac{2}{\rho}}\Big\}^{\frac12}\Big\|_{L^{p_m}(\mathbf{R}^n)}\prod_{i=1}^{m-1}\big\|\big(\mathcal{M}\big(\mathcal{M}(f_i)^\rho\big)\big)^{\frac{1}{\rho}}\big\|_{L^{p_i}(\mathbf{R}^n)}\\
&\le C''\,K\Big\|\Big\{\sum_j\mathcal{M}(|\Delta_j(f_m)|^\rho)^{\frac{2}{\rho}}\Big\}^{\frac{\rho}{2}}\Big\|^{\frac{1}{\rho}}_{L^{p_m/\rho}(\mathbf{R}^n)}\prod_{i=1}^{m-1}\|f_i\|_{L^{p_i}(\mathbf{R}^n)}\\
&\le C''\,K\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(\mathbf{R}^n)}\end{aligned}$$

in view of the Fefferman-Stein vector-valued inequality for the Hary-Littlewood maximal function [17] and the Littlewood-Paley theorem.

Next we deal with σ_2. Using the notation introduced earlier, we write

$$T_{\sigma_2}(f_1,\dots,f_{m-1},f_m)=\sum_{j\in\mathbf{Z}}T_{\sigma_2}(f_1,\dots,f_{m-1},\Delta_j(f_m))\,.$$

The key observation in this case is that

$$T_{\sigma_2}(f_1,\dots,f_{m-1},\Delta_j(f_m))=T_{\sigma_2}\big(S_j'(f_1),\dots,S_j'(f_{m-2}),\Delta_j'(f_{m-1}),\Delta_j(f_m)\big)$$

for some other Littlewood-Paley operator Δ_j' which is given on the Fourier transform by multiplication with a bump $\widehat{\Theta}(2^{-j}\xi)$, where $\widehat{\Theta}$ is equal to one on the annulus $\{\xi\in\mathbf{R}^n:\ \frac{24}{25}\cdot\frac{1}{10m}\le|\xi|\le 4\}$ and vanishes on a larger annulus. Also, S_j' is given by convolution with $\zeta'_{2^{-j}}$, where ζ' is a smooth function whose Fourier transform is equal to 1 on the ball $|z|<\frac{22}{10}$ and vanishes outside the double of this ball.

As in the previous case, one has that in the support of the integral

$$
\begin{aligned}
&T_{\sigma_2}\big(S'_j(f_1),\dots,S'_j(f_{m-2}),\Delta'_j(f_{m-1}),\Delta_j(f_m)\big)(x)\\
&=\int_{(\mathbf{R}^n)^m} e^{2\pi i x\cdot(\xi_1+\cdots+\xi_m)}\sigma_2(\vec{\xi}\,)\prod_{t=1}^{m-2}\widehat{S'_j(f_t)}(\xi_t)\ \widehat{\Delta'_j(f_{m-1})}(\xi_{m-1})\widehat{\Delta_j(f_m)}(\xi_m)\,d\vec{\xi}
\end{aligned}
$$

we have that

$$
|\xi_1|+\cdots+|\xi_m|\approx 2^j\,,
$$

thus one may insert in the integrand the factor $\widehat{\Psi}(2^{-j}\xi_1,\dots,2^{-j}\xi_m)$, for some Ψ in $\mathscr{S}_1((\mathbf{R}^n)^m)$ that is equal to one on a sufficiently wide annulus.

A calculation similar to the one in the case for σ_1 yields the estimate

$$
\begin{aligned}
&|T_{\sigma_2}(S'_j(f_1),\dots,S'_j(f_{m-2}),\Delta'_j(f_{m-1}),\Delta_j(f_m))|\\
&\le C\,K\prod_{i=1}^{m-2}\left(\mathcal{M}(\mathcal{M}(f_i)^\rho)\right)^{\frac{1}{\rho}}\big(\mathcal{M}(|\Delta'_j(f_{m-1})|^\rho)\big)^{\frac{1}{\rho}}\left(\mathcal{M}(|\Delta_j(f_m)|^\rho)\right)^{\frac{1}{\rho}}\,.
\end{aligned}
$$

Summing over j and taking L^p norms yields

$$
\begin{aligned}
&\|T_{\sigma_2}(f_1,\dots,,f_{m-1},f_m)\|_{L^p(\mathbf{R}^n)}\\
\le& C\,K\Big\|\prod_{i=1}^{m-2}\left(\mathcal{M}(\mathcal{M}(f_i)^\rho)\right)^{\frac{1}{\rho}}\sum_{j\in\mathbf{Z}}\big(\mathcal{M}\left(|\Delta'_j(f_{m-1})|^\rho\right)\big)^{\frac{1}{\rho}}\left(\mathcal{M}\left(|\Delta_j(f_m)|^\rho\right)\right)^{\frac{1}{\rho}}\Big\|_{L^p}\\
\le& C\,K\Big\|\prod_{i=1}^{m-2}\left(\mathcal{M}(\mathcal{M}(f_i)^\rho)\right)^{\frac{1}{\rho}}\Big\{\prod_{i=m-1}^{m}\sum_{j\in\mathbf{Z}}|\mathcal{M}\left(|\Delta_j(f_i)|^\rho\right)|^{\frac{2}{\rho}}\Big\}^{\frac{1}{2}}\Big\|_{L^p(\mathbf{R}^n)}
\end{aligned}
$$

where the last step follows by the Cauchy-Schwarz inequality and we omitted the prime from the term with $i=m-1$ for matters of simplicity. Applying Hölder's inequality and using that $\rho<2$ and Lemma B we obtain the conclusion that the expression above is bounded by

$$
C'\,K\,\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\cdots\|f_m\|_{L^{p_m}(\mathbf{R}^n)}\,.
$$

This conclues the proof of the theorem. □

6. The multilinear strong maximal function

In this section we study the maximal function $\mathcal{M}_{\mathcal{R}}$ introduced in Example 6 of Section 2. It turns out that this operator can be used to characterize the class of multiple A_p weights introduced in [32] suitably modified for rectangles, see [21]. Here, we will be concerned with endpoint boundedness properties of $\mathcal{M}_{\mathcal{R}}$. This will require a quick review of some facts from the theory of Orlicz spaces.

A *Young function* is a continuous, convex, increasing function $\Phi:[0,\infty)\to[0,\infty)$ with $\Phi(0)=0$ and such that $\Phi(t)\to\infty$ as $t\to\infty$. The properties of Φ easily imply that for $0<\epsilon<1$ and $t\ge 0$

$$
\Phi(\epsilon\,t)\le\epsilon\,\Phi(t)\,. \tag{24}
$$

The Φ-*norm* of a function f over a set E with finite measure is defined by

$$\|f\|_{\Phi,E} = \inf\left\{\lambda > 0 \,:\, \frac{1}{|E|}\int_E \Phi\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\}. \tag{25}$$

It follows from this definition that

$$\|f\|_{\Phi,E} > 1 \quad \text{if and only if} \quad \frac{1}{|E|}\int_E \Phi\left(|f(x)|\right)dx > 1. \tag{26}$$

Associated with each Young function Φ, there is its *complementary Young function*

$$\bar{\Phi}(s) = \sup_{t>0}\{st - \Phi(t)\} \tag{27}$$

for $s \ge 0$. Such $\bar{\Phi}$ is also a Young function and has the property that

$$st \le C\left[\Phi(t) + \bar{\Phi}(s)\right] \tag{28}$$

for all $s, t \ge 0$. Also the $\bar{\Phi}$-norms are related to the L_Φ-norms via the *the generalized Hölder inequality*, namely

$$\frac{1}{|E|}\int_E |f(x)\,g(x)|\,dx \le 2\,\|f\|_{\Phi,E}\,\|g\|_{\bar{\Phi},E}. \tag{29}$$

In this section we will work with the pair of Young functions

$$\Phi_n(t) := t[\log(e+t)]^{n-1} \quad \text{and} \quad \bar{\Phi}_n(t) \approx \Psi_n(t) := \exp(t^{\frac{1}{n-1}}) - 1, \quad t \ge 0.$$

It is the case that the pair Φ_n, Ψ_n satisfies (28), see the article by Bagby [1], page 887. Observe that the above function Φ_n is submultiplicative, that is, for $s, t > 0$

$$\Phi_n(st) \le c\,\Phi_n(s)\,\Phi_n(t).$$

We introduce the function $\Phi^{(m)} := \overbrace{\Phi \circ \Phi \circ \cdots \circ \Phi}^{m \text{ times}}$ which is increasing with respect to the input variable and also with respect to $m \in \mathbb{N}$.

6.1. **Some Lemmas.** We begin by proving some useful general lemmas about averaging functions and Orlicz spaces.

Lemma 4. *Let Φ be any Young function, then for any $f \ge 0$ and any measurable set E*

$$1 < \|f\|_{\Phi,E} \quad \Rightarrow \quad \|f\|_{\Phi,E} \le \frac{1}{|E|}\int_E \Phi(f(x))\,dx\,.$$

Proof. Indeed, by homogeneity this is equivalent to

$$\left\|\frac{f}{\lambda_{f,E}}\right\|_{\Phi,E} \le 1\,,$$

where

$$\lambda_{f,E} = \frac{1}{|E|}\int_E \Phi(f(x))\,dx\,.$$

But this is the same as

$$\frac{1}{|E|}\int_E \Phi\Big(\frac{f(x)}{\lambda_{f,E}}\Big)\,dx \le 1$$

by definition of the norm (25). In view of Property (24), it would be enough to show that

$$\lambda_{f,E} = \frac{1}{|E|} \int_E \Phi(f(x))\, dx \geq 1.$$

But this is exactly the case in view of Property (26). □

Lemma 5. *Let Φ be a submultiplicative Young function, let $m \in \mathbb{N}$ and let E be any set. Then there is a constant c such that whenever*

$$1 < \prod_{i=1}^{m} \|f_i\|_{\Phi,E} \tag{30}$$

holds, then

$$\prod_{i=1}^{m} \|f_i\|_{\Phi,E} \leq c \prod_{i=1}^{m} \frac{1}{|E|} \int_E \Phi^{(m)}(f_i(x))\, dx\,. \tag{31}$$

Proof. a) *The case $m = 1$.* This is the content of Lemma 4.

b) *The case $m = 2$.* Fix functions for which (30) holds:

$$1 < \prod_{i=1}^{2} \|f_i\|_{\Phi,E}.$$

Without loss of generality we may assume that

$$\|f_1\|_{\Phi,E} \leq \|f_2\|_{\Phi,E}\,.$$

Observe that by (30) we must have $\|f_2\|_{\Phi,E} > 1$.

Suppose first that $1 \leq \|f_1\|_{\Phi,E}$, then (31) follows from Lemma 4:

$$1 < \prod_{i=1}^{2} \|f_i\|_{\Phi,E} \leq \prod_{i=1}^{2} \frac{1}{|E|} \int_E \Phi(f_i(x))\, dx$$

with $m = 1$ and $c = 1$.

Assume now

$$\|f_1\|_{\Phi,E} \leq 1 \leq \|f_2\|_{\Phi,E}\,.$$

Then we have by Lemma 4, submultiplicativity and Jensen's inequality

$$\begin{aligned}
1 &< \prod_{i=1}^{2} \|f_i\|_{\Phi,E} \\
&= \|f_1\|_{\Phi,E}\, \|f_2\|_{\Phi,E} \\
&= \Big\| f_1\, \|f_2\|_{\Phi,E} \Big\|_{\Phi,E} \\
&\leq c\, \frac{1}{|E|} \int_E \Phi(f_1(x) \|f_2\|_{\Phi,E})\, dx \\
&\leq c\, \frac{1}{|E|} \int_E \Phi(f_1(x))\, dx\, \Phi(\|f_2\|_{\Phi,E}) \\
&\leq c\, \frac{1}{|E|} \int_E \Phi(f_1(x))\, dx\, \Phi(c\, \frac{1}{|E|} \int_E \Phi(f_2(x))\, dx\,)
\end{aligned}$$

$$\leq c\,\frac{1}{|E|}\int_E \Phi(f_1(x))\,dx\,\frac{1}{|E|}\int_E \Phi^{(2)}(f_2(x))\,dx$$
$$\leq c\,\prod_{i=1}^{2}\frac{1}{|E|}\int_E \Phi^{(2)}(f_i(x))\,dx\,,$$

which is exactly (31).

c) *The case* $m \geq 3$. By induction, assuming that the result holds for the integer $m-1 \geq 2$, we will prove it for m. Fix functions for which (30) holds:

$$1 < \prod_{i=1}^{m} \|f_i\|_{\Phi,E},$$

and without loss of generality assume that

$$\|f_1\|_{\Phi,E} \leq \|f_2\|_{\Phi,E} \leq \cdots \leq \|f_m\|_{\Phi,E}\,.$$

Observe that we must have $\|f_m\|_{\Phi,E} > 1$.

If we suppose that $1 \leq \|f_1\|_{\Phi,E}$, then (31) follows directly from Lemma 4:

$$1 < \prod_{i=1}^{m} \|f_i\|_{\Phi,E} \leq \prod_{i=1}^{m}\frac{1}{|E|}\int_E \Phi(f_i(x))\,dx$$

with $c = 1$ and Φ instead of $\Phi^{(2)}$.

Assume now that for some integer $k \in \{1, 2, \ldots, m-1\}$ we have

$$\|f_1\|_{\Phi,E} \leq \|f_2\|_{\Phi,E} \leq \cdots \leq \|f_k\|_{\Phi,E} \leq 1 \leq \|f_{k+1}\|_{\Phi,E} \leq \cdots \leq \|f_m\|_{\Phi,E}\,.$$

Since

$$1 < \prod_{i=1}^{m} \|f_i\|_{\Phi,E} = \|f_1\|_{\Phi,E}\prod_{i=2}^{m} \|f_i\|_{\Phi,E},$$

we must have $\prod_{i=2}^{m} \|f_i\|_{\Phi,E} > 1$. Using the induction hypothesis we have

$$\|f_1\|_{\Phi,E}\prod_{i=2}^{m} \|f_i\|_{\Phi,E} \leq c\,\|f_1\|_{\Phi,E}\prod_{i=2}^{m}\frac{1}{|E|}\int_E \Phi^{(m-1)}(f_i(x))\,dx = \|f_1\,R\|_{\Phi,E}\,, \tag{32}$$

where $R = \prod_{i=2}^{m}\frac{1}{|E|}\int_E \Phi^{(m-1)}(f_i(x))\,dx$. Applying Lemma 4 to the function $f_1\,R$ we obtain by submultiplicativity and Jensen's inequality

$$\|f_1\,R\|_{\Phi,E} \leq c\,\frac{1}{|E|}\int_E \Phi(f_1(x)\,R)\,dx$$
$$\leq c\,\frac{1}{|E|}\int_E \Phi(f_1(x))\,dx\;\Phi(R)$$
$$\leq c\,\frac{1}{|E|}\int_E \Phi(f_1(x))\,dx\prod_{i=2}^{m}\Phi\left(\frac{1}{|E|}\int_E \Phi^{(m-1)}(f_i(x))\,dx\right)$$
$$\leq c\,\frac{1}{|E|}\int_E \Phi(f_1(x))\,dx\prod_{i=2}^{m}\frac{1}{|E|}\int_E \Phi^{(m)}(f_i(x))\,dx.$$

Combining this result with (32) we deduce

$$\prod_{i=1}^{m} \|f_i\|_{\Phi,E} \leq c \prod_{i=1}^{m} \frac{1}{|E|} \int_E \Phi^{(m)}(f_i(x))\, dx\,,$$

thus proving (31). □

6.2. **The main result.** The previous lemmas are used in the proof of the following result due to Grafakos, Liu, Pérez, and Torres.

Theorem 7. [21] *There exists a positive constant C depending only on m and n such that for all $\lambda > 0$,*

$$\left|\left\{x \in \mathbf{R}^n : \mathcal{M}_{\mathcal{R}}(\vec{f})(x) > \lambda^m\right\}\right| \leq C \left\{\prod_{i=1}^{m} \int_{\mathbf{R}^n} \Phi_n^{(m)}\left(\frac{|f_i(x)|}{\lambda}\right) dx\right\}^{1/m} \tag{33}$$

for all f_i on $\mathbf{R}^n$ and for all $i = 1, \ldots, m$. Furthermore, the theorem is sharp in the sense that we cannot replace $\Phi_n^{(m)}$ by $\Phi_n^{(k)}$ for $k \leq m-1$.

Proof. By homogeneity, positivity of the operator, and the doubling property of Φ_n, it is enough to prove

$$\left|\left\{x \in \mathbf{R}^n : \mathcal{M}_{\mathcal{R}}(\vec{f})(x) > 1\right\}\right| \leq C \left\{\prod_{j=1}^{m} \int_{\mathbf{R}^n} \Phi_n^{(m)}\left(f_j(x)\right) dx\right\}^{1/m}, \tag{34}$$

for some constant C independent of the nonnegative functions $\vec{f} = (f_1, \cdots, f_m)$.

Let $E = \{x \in \mathbf{R}^n : \mathcal{M}_{\mathcal{R}}(\vec{f})(x) > 1\}$, then by the continuity property of the Lebesgue measure we can find a compact set K such that $K \subset E$ and

$$|K| \leq |E| \leq 2|K|.$$

Such a compact set K can be covered with a finite collection of rectangles $\{R_j\}_{j=1}^N$ such that

$$\prod_{i=1}^{m} \frac{1}{|R_j|} \int_{R_j} f_i(y)\, dy > 1, \quad j = 1, \cdots, N. \tag{35}$$

We will use the following version of the Córdoba-Fefferman rectangle covering lemma [10] due to Bagby ([1] Theorem 4.1 (C)): there are dimensional positive constants δ, c and a subfamily $\{\widetilde{R}_j\}_{j=1}^{\ell}$ of $\{R_j\}_{j=1}^N$ satisfying

$$\left|\bigcup_{j=1}^{N} R_j\right| \leq c \left|\bigcup_{j=1}^{\ell} \widetilde{R}_j\right|,$$

and

$$\int_{\bigcup_{j=1}^{\ell} \widetilde{R}_j} \exp\left(\delta \sum_{j=1}^{\ell} \chi_{\widetilde{R}_j}(x)\right)^{\frac{1}{n-1}} dx \leq 2 \left|\bigcup_{j=1}^{\ell} \widetilde{R}_j\right|.$$

Setting $\widetilde{E} = \bigcup_{j=1}^{\ell} \widetilde{R}_j$ and recalling that $\Psi_n(t) = \exp(t^{\frac{1}{n-1}}) - 1$ the latter inequality is

$$\frac{1}{|\widetilde{E}|} \int_{\widetilde{E}} \Psi_n\Big(\delta \sum_{j=1}^{\ell} \chi_{\widetilde{R}_j}(x)\Big)\, dx \le 1$$

which is equivalent to

$$\Big\| \sum_{j=1}^{\ell} \chi_{\widetilde{R}_j} \Big\|_{\Psi_n, \widetilde{E}} \le \frac{1}{\delta} \tag{36}$$

by the definition of the norm. Now, since

$$|E| \le 2|K| \le C|\widetilde{E}|$$

we can use (35) and Hölder's inequality as follows

$$\begin{aligned}
|\widetilde{E}| &= \Big| \bigcup_{j=1}^{\ell} \widetilde{R}_j \Big| \\
&\le \sum_{j=1}^{\ell} |\widetilde{R}_j| \\
&\le \sum_{j=1}^{\ell} \Big(\prod_{i=1}^{m} \int_{\widetilde{R}_j} f_i(y)\, dy \Big)^{\frac{1}{m}} \\
&\le \Big(\prod_{i=1}^{m} \sum_{j=1}^{\ell} \int_{\widetilde{R}_j} f_i(y)\, dy \Big)^{\frac{1}{m}} \\
&\le \Big(\prod_{i=1}^{m} \int_{\bigcup_{j=1}^{\ell} \widetilde{R}_j} \sum_{j=1}^{\ell} \chi_{\widetilde{R}_j}(y) f_i(y)\, dy \Big)^{\frac{1}{m}} \\
&= \Big(\prod_{i=1}^{m} \int_{\widetilde{E}} \sum_{j=1}^{\ell} \chi_{\widetilde{R}_j}(y) f_i(y)\, dy \Big)^{\frac{1}{m}}.
\end{aligned}$$

By this inequality and (29), we deduce

$$\begin{aligned}
1 &\le \prod_{i=1}^{m} \frac{1}{|\widetilde{E}|} \int_{\widetilde{E}} \sum_{j=1}^{\ell} \chi_{\widetilde{R}_j}(y) f_i(y)\, dy \\
&\le \prod_{i=1}^{m} \Big\| \sum_{j=1}^{\ell} \chi_{\widetilde{R}_j} \Big\|_{\Psi_n, \widetilde{E}} \|f_i\|_{\Phi_n, \widetilde{E}} \\
&\le \prod_{i=1}^{m} \frac{1}{\delta} \|f_i\|_{\Phi_n, \widetilde{E}} \\
&= \prod_{i=1}^{m} \Big\| \frac{f_i}{\delta} \Big\|_{\Phi_n, \widetilde{E}}.
\end{aligned}$$

Finally, it is enough to apply Lemma 5 and that $\Phi_n^{(m)}$ is submultiplicative to conclude the proof of (34).

Finally, we turn to the claimed sharpness that one cannot replace $\Phi_n^{(m)}$ by $\Phi_n^{(k)}$ for $k \le m-1$ in (33). In the case $m = n = 2$, we show that the estimate (E)

$$\left|\left\{x \in \mathbf{R}^2 : \mathcal{M}_{\mathcal{R}}(f,g)(x) > \alpha^2\right\}\right| \le C\left\{\int_{\mathbf{R}^2} \Phi_2\left(\frac{|f(x)|}{\alpha}\right) dx \int_{\mathbf{R}^2} \Phi_2\left(\frac{|g(x)|}{\alpha}\right) dx\right\}^{\frac{1}{2}}$$

cannot hold for $\alpha > 0$ and functions f, g with a constant C independent of these parameters.

For $N = 1, 2, \dots$, consider the functions

$$f = \chi_{[0,1]^2} \quad \text{and} \quad g_N = N\chi_{[0,1]^2}$$

and the parameter $\alpha = \frac{1}{10}$. Then the left hand side of (E) reduces to

$$\begin{aligned}\left|\left\{x \in \mathbb{R}^2 : \mathcal{M}_{\mathcal{R}}(f, g_N)(x) > \frac{1}{100}\right\}\right| &= \left|\left\{x \in \mathbb{R}^2 : M_{\mathcal{R}}(\chi_{[0,1]^2})(x) > \frac{1}{10\sqrt{N}}\right\}\right| \\ &\approx \sqrt{N}\,(\log N),\end{aligned}$$

where the last estimate is a simple calculation concerning the strong maximal function. However, the right hand side of (E) is equal to

$$C(\Phi_2(1/\alpha))^{1/2}\,(\Phi_2(N/\alpha))^{1/2} = C(\Phi_2(10))^{1/2}\,(\Phi_2(10N))^{1/2} \approx \sqrt{N\log N}$$

and obviously it cannot control the left hand side of (E) for N large.

For general m, the vector $\vec{f} = (f_1, \dots, f_m)$ with

$$f_1 = f_2 = \cdots = f_{m-1} = \chi_{[0,1]^2} \quad \text{and} \quad f_m = N\chi_{[0,1]^2}$$

also provides a counterexample. □

7. The bilinear Hilbert transform and the method of rotations

It is a classical result obtained by Calderón and Zygmund [4] using the method of rotations, that homogeneous linear singular integrals with odd kernels are always L^p bounded for $1 < p < \infty$. We indicate what happens if the method of rotations is used in the multilinear setting. For an integrable function Ω on $\mathbf{S}^{2n-1}$ with vanishing integral, we consider the bilinear operator

$$T_\Omega(f_1, f_2)(x) = \iint_{\mathbf{R}^{2n}} \frac{\Omega((y_1,y_2)/|(y_1,y_2)|)}{|(y_1,y_2)|^{2n}} f_1(x-y_1) f_2(x-y_2)\, dy_1 dy_2\,. \tag{37}$$

Suppose that Ω is an odd function on $\mathbf{S}^{2n-1}$. Using polar coordinates in $\mathbf{R}^{2n}$ we express

$$T_\Omega(f_1, f_2)(x) = \int_{\mathbf{S}^{2n-1}} \Omega(\theta_1,\theta_2)\left\{\int_0^{+\infty} f_1(x - t\theta_1) f_2(x - t\theta_2)\frac{dt}{t}\right\} d(\theta_1,\theta_2).$$

Replacing (θ_1, θ_2) by $-(\theta_1, \theta_2)$, changing variables, and using that Ω is odd we obtain

$$T_\Omega(f_1, f_2)(x) = \int_{\mathbf{S}^{2n-1}} \Omega(\theta_1,\theta_2)\left\{\int_0^{+\infty} f_1(x + t\theta_1) f_2(x + t\theta_2)\frac{dt}{t}\right\} d(\theta_1,\theta_2)$$

and averaging these identities we deduce that

$$T_\Omega(f_1,f_2)(x) = \frac{1}{2}\int_{\mathbf{S}^{2n-1}} \Omega(\theta_1,\theta_2)\left\{\int_{-\infty}^{+\infty} f_1(x-t\theta_1)f_2(x-t\theta_2)\frac{dt}{t}\right\} d(\theta_1,\theta_2).$$

The method of rotations gives rise to the operator inside the curly brackets above and one would like to know that this operator is bounded from a product of two Lebesgue spaces into another Lebesgue space (and preferably) uniformly bounded in θ_1,θ_2. Motivated by this calculation, for vectors $u,v\in\mathbf{R}^n$ we introduce the family of operators

$$\mathcal{H}_{u,v}(f_1,f_2)(x) = \text{p.v.}\int_{-\infty}^{+\infty} f_1(x-tu)f_2(x-tv)\frac{dt}{t}$$

We call this operator the *directional bilinear Hilbert transform* (in the direction indicated by the vector (u,v) in $\mathbf{R}^{2n}$). In the special case $n=1$, we use the notation

$$H_{\alpha,\beta}(f,g)(x) = \text{p.v.}\int_{-\infty}^{+\infty} f(x-\alpha t)g(x-\beta t)\frac{dt}{t}$$

for the bilinear Hilbert transform defined for functions f,g on the line and $x,\alpha,\beta\in\mathbf{R}$.

We mention results concerning the boundedness of these operators. The operator $H_{\alpha,\beta}$ was first shown to be bounded by Lacey and Thiele [30], [31] in the range

$$1 < p,q \le \infty\,, \qquad 2/3 < r < \infty\,, \qquad 1/p+1/q = 1/r\,. \tag{38}$$

Uniform L^r bounds (in α,β) for $H_{\alpha,\beta}$ were obtained by Grafakos and Li [20] in the local L^2 case, (i.e the case when $2 < p,q,r' < \infty$) and extended by Li [33] in the hexagonal region

$$1 < p,q,r < \infty\,, \qquad \left|\frac{1}{p}-\frac{1}{q}\right| < \frac{1}{2}\,, \qquad \left|\frac{1}{p}-\frac{1}{r'}\right| < \frac{1}{2}\,, \qquad \left|\frac{1}{q}-\frac{1}{r'}\right| < \frac{1}{2}\,. \tag{39}$$

We use an idea similar to that Calderón used to express the first commutator $\mathcal{C}_1$ as an average of the bilinear Hilbert transforms as in (2), to obtain new bounds for a higher dimensional commutator introduced by Christ and Journé [5]. The n-dimensional commutator is defined as

$$\mathcal{C}_1^{(n)}(f,a)(x) = \text{p.v.}\int_{\mathbf{R}^n} K(x-y)\int_0^1 f(y)a((1-t)x+ty)\,dt\,dy \tag{40}$$

where $K(x)$ is a Calderón-Zygmund kernel in dimension n and f, a are functions on $\mathbf{R}^n$. Christ and Journé [5] proved that $\mathcal{C}_1^{(n)}$ is bounded from $L^p(\mathbf{R}^n)\times L^\infty(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ for $1<p<\infty$. Here we discuss some off-diagonal bounds $L^p\times L^q\to L^r$, whenever $1/p+1/q=1/r$ and $1<p,q,r<\infty$.

As the operator $\mathcal{C}_1^{(n)}(f,a)$ is n-dimensional, we will need to "transfer" $H_{\alpha,\beta}$ in higher dimensions. To achieve this we use rotations. We have the following lemma:

Lemma 6. *Suppose that K is kernel in $\mathbf{R}^{2n}$ (which may be a distribution) and let T_K be the bilinear singular integral operator associated with K*

$$T_K(f,g)(x) = \iint K(x-y,x-z)f(y)g(z)\,dy\,dz\,.$$

Assume that T_K is bounded from $L^p(\mathbf{R}^n) \times L^q(\mathbf{R}^n) \to L^r(\mathbf{R}^n)$ with norm $\|T\|$ when $1/p + 1/q = 1/r$. Let M be a $n \times n$ invertible matrix. Define a $2n \times 2n$ invertible matrix

$$\widetilde{M} = \begin{pmatrix} M & O \\ O & M \end{pmatrix},$$

where O is the zero $n \times n$ matrix. Then the operator $T_{K\circ\widetilde{M}}$ is also bounded from $L^p(\mathbf{R}^n) \times L^q(\mathbf{R}^n) \to L^r(\mathbf{R}^n)$ with norm at most $\|T\|$.

Proof. To prove the lemma we note that

$$T_{K\circ\widetilde{M}}(f,g)(x) = T_K(f\circ M^{-1}, g\circ M^{-1})(Mx)$$

from which it follows that

$$\begin{aligned}\|T_{K\circ\widetilde{M}}(f,g)\|_{L^r} &= (\det M)^{-1/r}\|T_K(f\circ M^{-1}, g\circ M^{-1})\|_{L^r}\\ &\le (\det M)^{-1/r}\|T\|\,\|f\circ M^{-1}\|_{L^p}\|g\circ M^{-1}\|_{L^q}\\ &= \|T\|\,(\det M)^{-1/r}\|T\|\,\|f\|_{L^p}(\det M)^{1/p}\|g\|_{L^q}(\det M)^{1/p}\\ &= \|T\|\,\|f\|_{L^p}\|g\|_{L^q}\,.\end{aligned}$$

□

We apply Lemma 6 to the bilinear Hilbert transform. Let $e_1 = (1, 0, \ldots, 0)$ be the standard coordinate vector on $\mathbf{R}^n$. We begin with the observation that the operator $\mathcal{H}_{\alpha e_1,\beta e_1}(f,g)$ defined for functions f, g on $\mathbf{R}^n$ is bounded from $L^p(\mathbf{R}^n) \times L^q(\mathbf{R}^n)$ to $L^r(\mathbf{R}^n)$ for the same range of indices as the bilinear Hilbert transform. Indeed, the operator $\mathcal{H}_{\alpha e_1,\beta e_1}$ can be viewed as the classical one-dimensional bilinear Hilbert transform in the coordinate x_1 followed by the identity operator in the remaining coordinates $x_2, \ldots, x_n$, where $x = (x_1, \ldots, x_n)$. By Lemma 6, for an invertible $n \times n$ matrix M and $x \in \mathbf{R}^n$ we have

$$\mathcal{H}_{\alpha e_1,\beta e_1}(f\circ M^{-1}, g\circ M^{-1})(Mx) = \text{p.v.}\int_{-\infty}^{+\infty} f(x-\alpha t M^{-1}e_1)g(x-\beta t M^{-1}e_1)\,\frac{dt}{t}$$

maps $L^p(\mathbf{R}^n)\times L^q(\mathbf{R}^n) \to L^r(\mathbf{R}^n)$ with norm the same as the one-dimensional bilinear Hilbert transform $H_{\alpha,\beta}$ whenever the indices p, q, r satisfy (38). If M is a rotation (i.e. an orthogonal matrix), then $M^{-1}e_1$ can be any unit vector in $\mathbf{S}^{n-1}$. We conclude that the family of operators

$$\mathcal{H}_{\alpha\theta,\beta\theta}(f,g)(x) = \text{p.v.}\int_{-\infty}^{+\infty} f(x-\alpha t\,\theta)g(x-\beta t\,\theta)\,\frac{dt}{t} \qquad x \in \mathbf{R}^n$$

is bounded from $L^p(\mathbf{R}^n) \times L^q(\mathbf{R}^n)$ to $L^r(\mathbf{R}^n)$ with a bound independent of $\theta \in \mathbf{S}^{n-1}$ whenever the indices p, q, r satisfy (38). This bound is also independent of α, β whenever the indices p, q, r satisfy (39).

It remains to express the higher dimensional commutator $\mathcal{C}_1^{(n)}$ in terms of the operators $\mathcal{H}_{\alpha\theta,\beta\theta}$. Here we make the assumption that K is an odd homogeneous singular integral operator on $\mathbf{R}^n$, such as a Riesz transform. For a fixed $x \in \mathbf{R}^n$ we apply polar coordinates centered at x by writing $y = x - r\theta$. Then we can express the higher dimensional commutator in (40) as

$$\int_{\mathbf{S}^{n-1}} \int_0^\infty \frac{K(\theta)}{r^n} \int_0^1 f(x-r\theta)a(x-tr\theta)\,dt\,r^{n-1}\,dr\,d\theta\,. \tag{41}$$

Changing variables from $\theta \to -\theta$, $r \to -r$ and using that $K(\theta)$ is odd we write this expression as

$$\int_{\mathbf{S}^{n-1}} \int_{-\infty}^0 K(\theta) \int_0^1 f(x-r\theta)a(x-tr\theta)\,dt\,\frac{dr}{r}\,d\theta\,. \tag{42}$$

Averaging the (41) and (41) we arrive at the identity

$$\mathcal{C}_1^{(n)}(f,a)(x) = \frac{1}{2}\int_{\mathbf{S}^{n-1}} K(\theta) \int_0^1 \mathcal{H}_{\theta,t\theta}(f,a)(x)\,dt\,d\theta\,.$$

This identity implies the boundedness of $\mathcal{C}_1^{(n)}$ from $L^p(\mathbf{R}^n)\times L^q(\mathbf{R}^n)$ to $L^r(\mathbf{R}^n)$ whenever the indices p,q,r satisfy (39). Interpolation with the known $L^p \times L^\infty \to L^p$ bounds yield the following result due to Duong, Grafakos, and Yan [15]:

Theorem 8. *([15]) Let K be an odd homogeneous singular integral on $\mathbf{R}^n$. Then the n-dimensional commutator $\mathcal{C}_1^{(n)}$ associated with K maps $L^p(\mathbf{R}^n)\times L^q(\mathbf{R}^n) \to L^r(\mathbf{R}^n)$ whenever $1/p+1/q=1/r$ and $(1/p,1/q,1/r)$ lies in the open convex hull of the pentagon with vertices $(0,1/2,1/2)$, $(0,0,0)$, $(1,0,1)$, $(1/2,1/2,1)$, and $(1/6,4/6,5/6)$.*

Finally, we briefly discuss the situation in which the function Ω in (37) is even. The following result was recently obtained by Diestel, Grafakos, Honzík, Si, and Terwilleger [12]:

Theorem 9. *Let $\Omega \in L\log L(\mathbf{S}^1)$ be an even function with mean zero. Then the bilinear operator T_Ω defined in (37) is bounded from $L^p(\mathbf{R})\times L^q(\mathbf{R}) \to L^r(\mathbf{R})$ for all $2<p,q,r'<\infty$ satisfying $1/p+1/q=1/r$.*

The proof of this result also relies on expressing the operator T_Ω as an average of the operators $H_{\alpha,\beta}$. The details are omitted.

8. Closing Remarks

The topics discussed in these lectures by no means exhaust the full richness and broadness of multilinear harmonic analysis. However, they provide representative results of current research interests in this rapidly developing subject. I hope that investigators will find inspiration in these results to pursue further research in the area. The author would also like to thank the organizers of School on Nonlinear Analysis, Function Spaces and Applications 9 for their invitation to deliver these lectures and for providing an inspiring environment for mathematical interaction and research during the meeting.

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