TWO COUNTEREXAMPLES IN THE THEORY OF SINGULAR INTEGRALS

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Abstract. In these lectures we discuss examples that are relevant to two questions in the theory of singular integrals. The first question is the L^p boundedness of the maximal operator formed by dilates of Mikhlin-Hörmander multipliers, while the second concerns the L^p boundedness of a well-known object, the classical L^2-bounded Calderón-Zygmund homogeneous singular integral associated with an integrable function on the sphere that is very rough.

1. Introduction

We denote the Fourier transform of a complex-valued function $f(t)$ on $\mathbf{R}^d$ by

$$\widehat{f}(\tau) = \int_{\mathbf{R}^d} f(t) e^{-2\pi i \tau \cdot t} dt$$

and its inverse Fourier transform by $f^\vee(\tau) = \widehat{f}(-\tau)$. Many linear operators can be expressed in terms of their action on the Fourier transform of the input function. In particular, convolution operators are identified by operators given by multiplication on the Fourier transform, i.e. operators of the form $T_m(f) = (\widehat{f} m)^\vee$. Here we will always be interested in L^2-bounded convolution operators for which the corresponding multiplying functions m (called the Fourier multipliers) must be essentially bounded functions. The Fourier multiplier associated in this way with an operator bounded on $L^p(\mathbf{R}^d)$ is called an L^p Fourier multiplier. The space of all L^p Fourier multipliers on $\mathbf{R}^d$ will be denoted by $M_p(\mathbf{R}^d)$. This is a Banach space (in fact algebra) with norm $\|m\|_{M_p} = \|T_m\|_{L^p \to L^p}$.

The classical Mikhlin multiplier theorem [13] states that if a function $m(\xi)$ on $\mathbf{R}^d$ satisfies

$$|\partial_\xi^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|} \tag{1.1}$$

for all multiindices α with $|\alpha| \le [\frac{d}{2}] + 1$, then it must be an L^p Fourier multiplier for all $1 < p < \infty$. This theorem was extended by Hörmander [12] to functions m satisfying the weaker condition

$$\sup_{k \in \mathbf{Z}} \|\varphi(\xi) m(2^k \xi)\|_{L^2_\beta(d\xi)} < \infty \tag{1.2}$$

for some $\beta > d/2$. Here φ is a smooth nonzero bump supported in the annulus $1 < |\xi| < 2$ not vanishing on a smaller annulus and L^2_β is the Sobolev space of functions with "β

Date: April 27, 2006.

1991 *Mathematics Subject Classification.* Primary 42B20, 42B25. Secondary 46B70, 47B38.

Key words and phrases. Calderón-Zygmund singular integrals, Mikhlin-Hörmander multiplier theorem, maximal multipliers.

Work supported by the NSF under grant DMS 0400387.

This article is an expansion of the lectures the author gave at the conference entitled "'Harmonic Analysis and its Applications at Sapporo" during the period August 22-24, 2005.

derivatives" in L^2. The space L^2_β is one of the Sobolev spaces L^p_γ with norm

$$\|f\|_{L^p_\gamma} = \big\|\big(\widehat{f}(\xi)(1+|\xi|^2)^{\gamma/2}\big)^\vee\big\|_{L^p(d\xi)},$$

where $1 \le p < \infty$ and $\gamma \in \mathbf{R}$. We remark that in Hörmader's version of this multiplier theorem the Sobolev space L^2_β in (1.2) can be replaced by L^r_γ, where $\gamma > d/r$ and $1 \le r \le 2$; however the least restrictive condition is when $r = 2$.

By duality an L^p Fourier multiplier must always be an $L^{p'}$ Fourier multiplier (where $p' = p/(p-1)$) and hence by interpolation it must be an L^q Fourier multiplier for all q between p and p'. Finding examples of functions that are L^q Fourier multipliers for some $q > 2$ but not L^s Fourier multipliers for some $s > q$ may not be an easy task. A question of this sort will be addressed in section 5.

In the next section we will discuss a problem concerning the L^p boundedness of the supremum of a family of Mikhlin-Hörmander Fourier multipliers.

2. Maximal Mikhlin-Hörmander Fourier multipliers

Suppose that we are given a bounded function on $\mathbf{R}^d$ that satisfies condition (1.1) (or even (1.2)). The question that we would like to address is whether the maximal operator

$$\mathcal{M}_m(f)(x) = \sup_{t>0} \big|(\widehat{f}(\xi)m(t\xi))^\vee(x)\big|$$

is bounded from $L^p(\mathbf{R}^d)$ into itself.

This question is motivated by the almost everywhere convergence questions

$$(\widehat{f}(\xi)m(t\xi))^\vee(x) \to m(0)\, f(x) \qquad \text{for almost all } x \text{ as} \quad t \to 0$$

$$(\widehat{f}(\xi)m(t\xi))^\vee(x) \to m(\infty)\, f(x) \qquad \text{for almost all } x \text{ as} \quad t \to \infty\,,$$

provided, of course, the quantities $m(0)$ and $m(\infty)$ exist.

We recall that in the usual proof of the Mikhlin-Hörmander multiplier theorem one obtains a weak type $(1,1)$ estimate using the trivial L^2 estimate and a smoothing condition on the kernel. Then the boundedness for the remaining p's follows by interpolation and duality.

By changing our point of view, we may consider $\mathcal{M}_m$ as a linear map from

$$L^p(\mathbf{R}^d) \to L^p(\mathbf{R}^d, L^\infty(\mathbf{R}^+) \tag{2.3}$$

and we may ask whether the classical scalar argument argument based on the weak type $(1,1)$ estimate holds in this setting. In the context of the vector-valued setting described in (2.3) the corresponding multiplier satisfies Mikhlin's condition but for the weak type $(1,1)$ argument to go through one needs to know an initial estimate at a single exponent. In the scalar case, one uses Plancherel's theorem to obtain the L^2 estimate for free but in the vector-valued case the $L^2(\mathbf{R}^d, L^\infty(\mathbf{R}^+))$ estimate cannot be obtained using Plancherel's theorem, in fact as we will shortly see, it may fail.

The underlying problem here is that the Banach space L^∞ is not a UMD space and for this reason many analogues of some of the scalar results in the theory of singular integrals do not hold in the Banach-valued setting.

Theorem 1. *(M. Christ, L. Grafakos, P. Honzík, and A. Seeger* [4]*) There exists a bounded function m such that for all multiindices α there are constants C_α such that*

$$\sup_\xi \sup_k |\partial^\alpha_\xi(\varphi(\xi)m(2^k\xi))| \le C_\alpha\,,$$

hence m is an L^p Fourier multiplier for all $1 < p < \infty$, but $\mathcal{M}_m$ is unbounded on $L^p(\mathbf{R}^d)$.

We discuss some of the ideas of the proof of this result.

Proof. Let $S = \{1, -1, i, -i\}$. Enumerate the set of all sequences of length N formed by elements of S as follows: $S^N = \{s_1, s_2, \dots, s_{4^N}\}$. Let Φ be a smooth function supported in $\frac{6}{8} \le |\xi| \le \frac{10}{8}$ satisfying $\Phi = 1$ on $\frac{7}{8} \le |\xi| \le \frac{9}{8}$. Define for $N \ge 10$

$$m_N(\xi) = \sum_{\ell=1}^{4^N} \sum_{\nu=1}^{N} s_\ell(\nu) \Phi(2^{-N\ell} 2^{-\nu}\, \xi)$$

and also

$$m(\xi) = \sum_{N=1}^{\infty} m_N(2^{-N8^N} \xi)\,.$$

It is straightforward that for all multiindices α there are constants C_α so that

$$|\partial^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|}$$

and it is also easy to check that for any $k_0 \in \mathbf{Z}$ we have

$$|\partial^\alpha (m(2^{k_0}\xi)\varphi(\xi))| \le C_\alpha\,.$$

Pick ψ with Fourier transform supported in $B(0, 1/8)$ with $\|\psi\|_{L^p} = 1$. Let

$$g_N(x) = \sum_{j=1}^{N} e^{2\pi i 2^j x_1} \psi(x)$$

and note that

$$\widehat{g_N}(\xi) = \sum_{j=1}^{N} \widehat{\psi}(\xi - 2^j(1, 0, 0, \dots, 0))\,.$$

Also let

$$f_{N,p}(x) = N^{-\frac{1}{2}} (2^{N8^N})^{\frac{d}{p}} g_N(2^{N8^N} x)$$

and notice that in view of the Littlewood-Paley Theorem (see [8]) we have that

$$\|g_N\|_{L^p} \approx N^{1/2}$$

while

$$\|f_{N,p}\|_{L^p} \approx C_p\,.$$

The main ingredient we need is the following lower estimate whose proof we postpone momentarily:

$$\Big\| \sup_{1 \le k \le N4^N} \big|(m_N(2^k\xi)\widehat{g_N}(\xi))^\vee\big\|_{L^p} \ge c\,N\,. \tag{2.4}$$

This implies that

$$\|\mathcal{M}_m(f_{N,p})\|_{L^p} \ge c\sqrt{N} = c\sqrt{N}\,\|f_{N,p}\|_{L^p}\,.$$

The reason for this is that $m(\xi) = \sum_{n=1}^{\infty} m_n(2^{-n8^n}\xi)$ and

$$m_n(2^{-n8^n} 2^k \xi)\widehat{f_{N,p}}(\xi) = m_n(2^{-n8^n} 2^k \xi)\widehat{g_N}(2^{-N8^N}\xi) = 0$$

for all $1 \le k \le N4^N$ unless $n \ne N$.

It remains to prove (2.4). We observe that

$$\sup_{c \in \{1, -, 1, i, -i\}} \mathrm{Re}(c\,z) \ge |z|/\sqrt{2}\,.$$

Thus for all $x \in \mathbf{R}^d$ and all $j \in \{1, 2, \dots, N\}$ there is a $c_j(x) \in \{1, -, 1, i, -i\}$ such that

$$\mathrm{Re}\big[c_j(x)\, e^{2\pi i 2^j x_1}\psi(x)\big] \geq |\psi(x)|/\sqrt{2}\,.$$

Therefore there is a $\kappa_x \in \{1, 2, \dots, 4^N\}$ such that

$$s_{\kappa_x} = (c_1(x), c_2(x), \dots, c_N(x))\,.$$

We then have

$$\begin{aligned}
&\sup_{1\leq k\leq N4^N} |(m_N(2^k\xi)\widehat{g_N}(\xi))^\vee(x)| \\
&\geq \mathrm{Re}\Big[\int_{\mathbf{R}^d}\sum_{\ell=1}^{4^N}\sum_{\nu=1}^{N} s_\ell(\nu)\Phi(2^{-N\ell-\nu}2^{N\kappa_x}\xi)\sum_{j=1}^{N}\widehat{\psi}(\xi-2^j e_1)e^{2\pi i x\cdot\xi}d\xi\Big]\,,
\end{aligned}$$

as easily follows by taking $k = N\kappa_x$.

Our choice of exponents makes the previous expression inside the the square brackets zero unless $\ell = \kappa_x$ and $j = \nu$. Also $\Phi = 1$ on support$(\widehat{\psi})$ and hence this expression is at least

$$\sum_{j=1}^{N}\mathrm{Re}\big[s_{k_x(j)}(\widehat{\psi}(\xi-2^j))^\vee(x)\big] \geq N|\psi(x)|/\sqrt{2}$$

which proves (2.4).

□

3. A positive result related to the previous counterexample

We recall the main observation in the previous section which can be rephrased as follows:

$$\Big\|\sup_{1\leq k\leq N4^N}\big|(m_N(2^k\xi)\widehat{g_N}(\xi))^\vee\big|\,\Big\|_{L^p} \geq c\sqrt{N}\,\|g_N\|_{L^p}\,. \tag{3.5}$$

Replacing $N4^N$ by $\mathcal{N}$ we see that the supremum of a family of $\mathcal{N}$ Mikhlin-Hörmander multipliers has operator norm on L^p at least as big as a constant multiple of $(\log\mathcal{N})^{\frac{1}{2}}$.

The question we would like to address is whether this lower estimate is sharp. We precisely formulate our question.

Question: *Suppose that m_j, $1 \leq j \leq N$, are Mikhlin multipliers satisfying*

$$|\partial^\alpha m_j(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}\,,$$

uniformly in j for all $|\alpha| \leq [\frac{d}{2}]+1$. What is the growth as $N \to \infty$ of the smallest constant $A(N)$ such that

$$\Big\|\sup_{1\leq j\leq N}\big|(m_j\widehat{f}\,)^\vee\big|\Big\|_{L^p(\mathbf{R}^d)} \leq A(N)\|f\|_{L^p(\mathbf{R}^d)}$$

holds for all f?

The counterexample in the previous section shows that for $N \geq 10$ we have

$$A(N) \geq c\sqrt{\log N}$$

and we would like to know if the converse inequality also holds for some other constant c'. The following theorem answers this question.

Theorem 2. *(L. Grafakos, P. Honzik, and A. Seeger [9]) Let $1 < r < 2$ and suppose*

$$\sup_{1\le j\le N} |\partial^\alpha m_j(\xi)|\,|\xi|^{|\alpha|} \le B$$

for all $|\alpha| \le [\frac{d}{2}]+1$. Then for any $1 < p < \infty$ there is a constant $C_{d,p}$ such that for all $N \ge 10$ we have

$$\Big\| \sup_{1\le j\le N} \big|(m_j \widehat{f}\,)^\vee\big| \Big\|_{L^p(\mathbf{R}^d)} \le C_{d,p} B \sqrt{\log N}\, \|f\|_{L^p(\mathbf{R}^d)}\,.$$

Therefore for $N \ge 10$ we have that

$$A(N) \le c' \sqrt{\log N}$$

and this shows that $A(N)$ grows indeed like the square root of the logarithm of N as $N \to \infty$.

We will outline a proof of this theorem in the next section, but before we do so, it will be illuminating to discuss a model case that contains the core idea and forms the basic outline of the proof in the general case. The model case comes from the theory of Rademacher multipliers. Let us recall the Rademacher functions defined on the interval $[0,1]$ as follows:

$$\begin{aligned} r_0(t) &= 1 \\ r_1(t) &= \chi_{[0,1/2]} - \chi_{[1/2,1]} \\ r_2(t) &= \chi_{[0,1/4]} - \chi_{[1/4,1/2]} + \chi_{[1/2,3/4]} - \chi_{[3/4,1]} \end{aligned}$$

etc. The inspiration comes by studying the growth in N of the L^p norms of simple-looking maximal functions of the form

$$\sup_{1\le k\le N} \Big|\sum_j a_j^k r_j\Big|\,,$$

where a_j^k is a fixed matrix and r_j is the j-th Rademacher function. Let us denote the sequence $(a_j^k)_j$ by a^k.

It turns out that

$$\Big\| \sup_{1\le i\le N} \Big|\sum_j a_j^k r_j\Big| \Big\|_{L^2([0,1])} \le C(N) \sup_{1\le k\le N} \|a^k\|_{\ell^2([0,1])} \tag{3.6}$$

where $C(N)$ grows like $\sqrt{\log N}$ as $N \to \infty$.

To see this we set $F_k = \sum_j a_j^k r_j$. One has the following exponential decay of sums of Rademacher functions (see [15], [8])

$$\big|\{s \in [0,1] :\ |F_k(s)| > \lambda\}\big| \le 2\, e^{-\lambda^2/4\|a^k\|_{\ell^2}^2}\,. \tag{3.7}$$

Then for $N \ge 10$ we have

$$\begin{aligned} \Big\| \sup_{1\le k\le N} |F_k| \Big\|_{L^2([0,1])}^2 &= \int_0^\infty \lambda \big|\{s \in [0,1] :\ \sup_k |F_k(s)| > \lambda\}\big|\, d\lambda \\ &= \int_0^{u_N} \dots\, d\lambda + \int_{u_N}^\infty \dots\, d\lambda \\ &\le \int_0^{u_N} \lambda\, d\lambda + \int_{u_N}^\infty \sum_{k=1}^N \lambda |\{s \in [0,1] :\ |F_k(s)| > \lambda\}|\, d\lambda\,. \end{aligned}$$

We now use (3.7) and calculate the integrals in question. We obtain

$$\begin{aligned}\Big\|\sup_{1\le k\le N}|F_k|\Big\|^2_{L^2([0,1])} &\le \frac{1}{2}u_N^2+\sum_{k=1}^{N}\int_{u_N}^{\infty}2\,\lambda\, e^{-\lambda^2/4\|a^k\|^2_{\ell^2}}d\lambda\\ &\le \frac{1}{2}u_N^2+2N\,e^{-u_N^2/4}\sup_{1\le k\le N}\|a^k\|^2_{\ell^2}\\ &\le c\log N\sup_{1\le k\le N}\|a^k\|^2_{\ell^2}\,,\end{aligned}$$

using the optimal choice of

$$u_N=\sqrt{4\log N}\sup_{1\le k\le N}\|a^k\|_{\ell^2}\,.$$

This proves (3.6).

4. THE GENERAL CASE

We now adapt the idea of the previous section to prove Theorem 2.

Proof. Let D_k be the dyadic cubes in $\mathbf{R}^d$ of sidelength 2^{-k}. We recall the dyadic averaging operator E_k, the martingale difference operator E_k, and the martingale square function $S(f)$ associated with the family of dyadic cubes:

$$\begin{aligned}E_k(f)&=\sum_{Q\in D_k}\chi_Q\frac{1}{|Q|}\int_Q f(t)\,dt\,,\\ D_k(f)&=E_{k+1}(f)-E_k(f)\,,\\ S(f)&=\Big(\sum_k|D_k(f)|^2\Big)^{1/2}\,,\end{aligned}$$

The key element in the proof is the Chang-Wilson-Wolff inequality [3] :

$$\big|\{x\in\mathbf{R}^d:\sup_{k\ge0}|E_k(g)-E_0(g)|>2\lambda\,,\,S(g)<\varepsilon\lambda\}\big|\le C_d\,e^{-\frac{c_d}{\varepsilon^2}}\big|\{x\in\mathbf{R}^d:\sup_k|E_k(g)|>\lambda\}\big|$$

which is valid for all functions g, all $\lambda>0$, $\varepsilon\in(0,1)$, and for some fixed constants C_d,c_d (both depending on d).

Recall that we denote by T_m the operator $f\to(\widehat{f}m)^\vee$. Start with

$$\Big\|\sup_{1\le k\le N}|T_{m_k}(f)|\Big\|_{L^p}=\Big(p\,4^p\int_0^\infty\lambda^{p-1}|\{\sup_k|T_{m_k}(f)|>4\lambda\}|\,d\lambda\Big)^{\frac{1}{p}}$$

and control the measure of the set that appears in the previous line by the sum of three terms:

$$|\{\sup_k|T_{m_k}(f)|>4\lambda\}|\quad\le\quad I_\lambda+II_\lambda+III_\lambda\,,$$

where

$$\begin{aligned}I_\lambda&=|\{\sup_k|T_{m_k}(f)-E_0(T_{m_k}(f))|>2\lambda\ ,\ G_p(f)<\varepsilon_N\lambda/(AB)\}|\,,\\ II_\lambda&=|\{G_p(f)>\varepsilon_N\lambda/(AB)\}|\,,\\ III_\lambda&=|\{\sup_k|E_0(T_{m_k}(f))|>2\lambda\}|\,.\end{aligned}$$

Here A is a constant and $f \to G_p(f)$ is an L^p bounded maximal operator which controls the square function applied to each T_{m_k}, precisely it satisfies:

$$S(T_{m_k}(f)) \leq A\left(\sup_{\xi}\sup_{|\alpha|\leq[\frac{d}{2}]+1}|\xi|^{|\alpha|}|\partial_\xi^\alpha m_k(\xi)|\right)G_p(f) \leq A\,B\,G_p(f)$$

for all Schwartz functions f. (For a precise definition of G_p, see [9].)

To estimate

$$\Big(p\,4^p\int_0^\infty \lambda^{p-1}\,I_\lambda\,d\lambda\Big)^{\frac{1}{p}} \tag{4.8}$$

we use the Chang Wilson Wolff theorem to write

$$\begin{aligned} I_\lambda &\leq |\{\sup_k |T_{m_k}(f)-E_0(T_{m_k}(f))|>2\lambda\ ,\ G_p(f)<\varepsilon_N\lambda/(AB)\}| \\ &\leq \sum_{k=1}^N |\{|T_{m_k}(f)-E_0(T_{m_k}(f))|>2\lambda\ ,\ G_p(f)<\varepsilon_N\lambda/(AB)\}| \\ &\leq \sum_{k=1}^N |\{|T_{m_k}(f)-E_0(T_{m_k}(f))|>2\lambda\ ,\ S(T_{m_k}(f))<\varepsilon_N\lambda\}| \\ &\leq \sum_{k=1}^N C_d\,e^{-c_d/\varepsilon_N^2}|\{\sup_l|E_l(T_{m_k}(f))|>\lambda\}|\,. \end{aligned}$$

Insert this estimate in (4.8) to obtain

$$\Big(p\,4^p\int_0^\infty \lambda^{p-1}\,I_\lambda\,d\lambda\Big)^{\frac{1}{p}} \leq C_d\bigg(\sum_{k=1}^N e^{-c_d/\varepsilon_N^2}\|\sup_l|E_l(T_{m_k}(f))|\|_{L^p}^p\bigg)^{\frac{1}{p}} \leq c'B\big[Ne^{-c_d/\varepsilon_N^2}\big]^{\frac{1}{p}} \approx B$$

provided we choose $\varepsilon_N = c''/\sqrt{\log N}$. Here we used that the maximal operator

$$g \to \sup_l |E_l(g)|$$

is controlled by the Hardy-Littlewood maximal function and is therefore L^p bounded for all $1<p<\infty$, while all T_{m_k} are L^p bounded with norm at most a multiple of B.

Next we turn our attention to the corresponding integral for term II_λ

$$\Big(p\,4^p\int_0^\infty \lambda^{p-1}\,II_\lambda d\lambda\Big)^{\frac{1}{p}}\,.$$

Using that

$$II_\lambda = |\{G_p(f)>(\varepsilon_N/AB)\lambda\}|,$$

where G_p is L^p bounded, we deduce that

$$\Big(p\,4^p\int_0^\infty \lambda^{p-1}\,II_\lambda\,d\lambda\Big)^{\frac{1}{p}} \leq C\frac{B\,A_r}{\varepsilon_N}\|G_p(f)\|_{L^p} \leq C_p\frac{B}{\varepsilon_N}\|f\|_{L^p}\,.$$

This last expression is equal to

$$C_r'B\sqrt{\log N}\,\|f\|_{L^p}$$

since ε_N was chosen to be $c''/\sqrt{\log N}$.

Finally we need to control

$$\Big(p\,4^p\int_0^\infty \lambda^{p-1}\,III_\lambda d\lambda\Big)^{\frac{1}{p}}\,. \tag{4.9}$$

It turns out that for any $1 < r < p$ one has an estimate (see [9])

$$|E_0(T_{m_k}(f))| \le C_r\, B\, 2^{-\frac{N}{r}} (MMM(|f|^r))^{\frac{1}{r}} \tag{4.10}$$

whenever $m_k(\xi) = 0$ on $|\xi| \le 2^N$ (M is the Hardy-Littlewood maximal operator). This assumption can be made on each multiplier m_k, $k = 1, \dots, N$ as follows: working with f such that $\widehat{f}$ is compactly supported, we may assume that the multipliers m_k are supported in a finite union of dyadic annuli, which, by changing scales, may assume that do not intersect the ball $|\xi| \le 2^N$.

Insert estimate (4.10) in (4.9) to obtain

$$\Big(p\, 4^p \int_0^\infty \lambda^{p-1}\, III_\lambda\, d\lambda\Big)^{\frac{1}{p}} \le \Big(\sum_{k=1}^N \big\|E_0(T_{m_k}(f))\big\|_{L^p}^p\Big)^{\frac{1}{p}} \le B\, N^{\frac{1}{p}}\, 2^{-\frac{N}{r}} \|f\|_{L^p}$$

which is trivially controlled by $B\sqrt{\log N}\, \|f\|_{L^p}$. □

5. A problem involving homogeneous singular integrals

Mihklin-Hörmander multipliers correspond to kernels $K(y)$ on $\mathbf{R}^d$ that are singular at the origin, satisfy an estimate $|K(y)| \le C|y|^{-d}$ for some $C < \infty$ and all $y \ne 0$, and possess a certain amount of smoothness. This smoothness suffices to guarantee the boundedness on all $L^p(\mathbf{R}^d)$ ($1 < p < \infty$) for the corresponding Fourier multiplier operator (given by convolution with K) as well as its weak type $(1,1)$ property.

In this section, we study a problem concerning Fourier multipliers given by convolution with kernels that are homogeneous of degree $-d$ on $\mathbf{R}^d$. Such kernels are determined by their restriction on the unit sphere $\mathbf{S}^{d-1}$. Let K be such a kernel and let Ω be its restriction on $\mathbf{S}^{d-1}$. One may check that the function $\Omega(y/|y|)|y|^{-d}$, $y \ne 0$ coincides with a principal value distribution on $\mathbf{R}^d$ if and only if Ω has mean value zero on the sphere. Only in this case one can make sense of convolution with K.

Let therefore Ω be an integrable function on $\mathbf{S}^{d-1}$ with mean value zero. We will be considering Calderón-Zygmund singular integrals of the form

$$T_\Omega(f)(x) = f * \text{p.v.}\frac{\Omega(x/|x|)}{|x|^d} = \lim_{\varepsilon\to 0} \int\limits_{|y|>\varepsilon} f(x-y)\frac{\Omega(y/|y|)}{|y|^d}\, dy\,, \tag{5.11}$$

where f is a Schwartz function on $\mathbf{R}^d$. This type of singular integrals were introduced by Calderón and Zygmund in [1].

If Ω is odd then the method of rotations (see [2]) gives

$$T_\Omega(f)(x) = \frac{\pi}{2}\int_{\mathbf{S}^{d-1}} H_\theta(f)(x)\, \Omega(\theta)\, d\theta\,, \tag{5.12}$$

where H_θ is the *directional Hilbert transform*

$$H_\theta(f)(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int\limits_{|t|>\varepsilon} f(x-t\theta)\frac{dt}{t}\,.$$

A simple argument using change of variables yields that the operator H_θ is bounded on L^p exactly when $H_{(1,0,\dots,0)}$ is; the latter is the Hilbert transform in the first variable and the identity operator in the remaining variables and hence it is trivially bounded on $L^p(\mathbf{R}^d)$ (and is of weak type $(1,1)$.)

Thus the boundedness of T_Ω on $L^p(\mathbf{R}^d)$ for Ω odd is an easy consequence of (5.12) and of the boundedness of H_θ on $L^p(\mathbf{R}^d)$ (which is uniform in θ). We point out that, as of this writing, the weak type $(1,1)$ boundedness of T_Ω for Ω odd, remains an open question.

The problem of the L^p boundedness of T_Ω is therefore interesting for Ω even. We begin our discussion by recalling the results of Calderón and Zygmund [2] who showed that if Ω lies in the space $L\log L(\mathbf{S}^{d-1})$, then T_Ω is bounded on $L^p(\mathbf{R}^d)$ for all $1 < p < \infty$. The more delicate issue of the weak type $(1,1)$ property of T_Ω was shown much later by Christ and Rubio de Francia [5] (for $d \le 7$, published only the case $d = 2$) and Seeger [14] for all d.

We note that M. Weiss and A. Zygmund [16] have constructed examples of even functions Ω in $L^1(\mathbf{S}^{d-1})$ such that T_Ω is unbounded on L^2 (even when restricted to continuous functions) and therefore on all other L^p. For an operator to be bounded on $L^2(\mathbf{R}^d)$ a certain condition on Ω is required. A calculation using the Fourier transform gives that the multiplier corresponding to the kernel $\Omega(y/|y|)|y|^{-d}$ is the function

$$\xi \to \int_{\mathbf{S}^{d-1}} \Omega(\theta)\, \log \frac{1}{|\xi\cdot\theta|}\, d\theta\,.$$

Therefore we have the equivalence

$$\operatorname*{essup}_{|\xi|=1} \left| \int_{\mathbf{S}^{d-1}} \Omega(\theta)\, \log \frac{1}{|\xi\cdot\theta|}\, d\theta \right| < +\infty \iff T_\Omega : L^2 \to L^2$$

and hence condition

$$\operatorname*{essup}_{|\xi|=1} \int_{\mathbf{S}^{d-1}} |\,\Omega(\theta)|\, \log \frac{1}{|\xi\cdot\theta|}\, d\theta < +\infty \tag{5.13}$$

implies the L^2 boundedness of T_Ω.

Since condition (5.13) arises naturally, it is reasonable to ask whether it implies the boundedness of T_Ω on L^p for some (or all) $p \neq 2$, The underlying question here is whether the p-independence boundedness property in Calderón-Zygmund theory holds for rough kernels.

This question was answered in the negative by P. Honzík and D. Ryabogin, in collaboration with the author, who constructed an example of an even function Ω on $\mathbf{S}^{d-1}$ such that the corresponding operator T_Ω is bounded on L^p exactly when $p = 2$.

In fact these authors have obtained the following sharper result:

Theorem 3. *(L. Grafakos, P. Honzík, D. Ryabogin* [10]*): Let $0 \le \alpha < \frac{1}{2}$. Then there exists $\Omega \in L^1(\mathbf{S}^{d-1})$ with mean value zero such that*

$$\operatorname*{essup}_{|\xi|=1} \int_{\mathbf{S}^{d-1}} |\,\Omega(\theta)|\, \log^{1+\alpha} \frac{1}{|\xi\cdot\theta|}\, d\theta < +\infty$$

but T_Ω is unbounded on $L^p(\mathbf{R}^d)$ for all

$$\left|\frac{1}{p} - \frac{1}{2}\right| > \alpha\,.$$

Taking $\alpha = 0$ yields the previous case.

6. The second counterexample

In this section we discuss the counterexample of Theorem 3. In the proof we restrict our attention to the case $d = 2$ and we note that higher dimensional examples can be constructed using the two-dimensional example.

Let $M_p(\mathbf{Z})$ be the space of multipliers on $L^p([0,1])$, i.e. the space of all bounded sequences $(b_m)_{m\in\mathbf{Z}}$ such that the linear operator

$$h(x) \to \sum_{m\in\mathbf{Z}} b_m \bigg(\int_0^1 h(t)e^{-2\pi imt}dt \bigg) e^{2\pi imx} \tag{6.14}$$

maps 1-periodic functions h in $L^p([0,1])$ to functions in $L^p([0,1])$. The $M_p(\mathbf{Z})$ norm of the sequence $(b_m)_m$ is then the norm of the operator in (6.14) on $L^p([0,1])$.

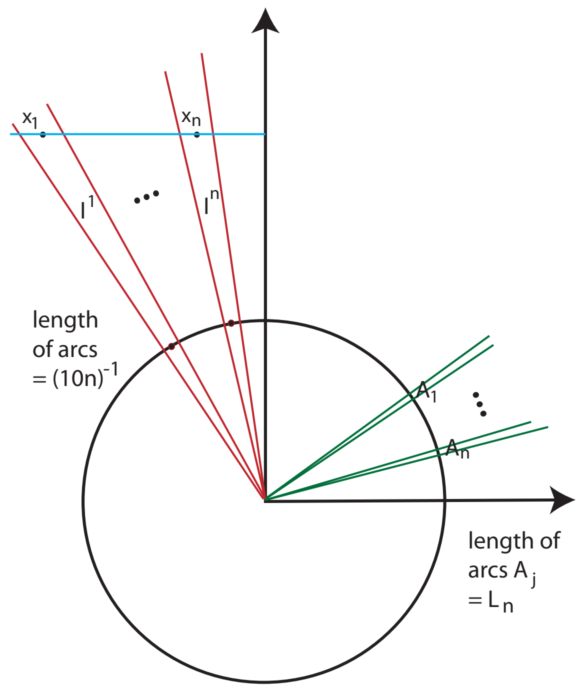

FIGURE 1. The points $x_1, \dots, x_n$ lie on a straight line perpendicular to the vertical coordinate axis. The arcs A_j lie in the first quadrant of the unit circle and have length L_n, a quantity to be determined. The cones I^j lie in the second quadrant and they meet the unit circle on arcs of length $(10n)^{-1}$ centered at the points $x_j/|x_j|$. The centers of the arcs A_j and the points $x_j/|x_j|$ form an angle of size $\pi/2$.

The basis $\{e^{2\pi ikx}\}_{k=-\infty}^{\infty}$ of $L^p([0,1])$, $p \neq 2$, is not unconditional. This means that for all $n = 1, 2, \dots$ there exist complex sequences a_k^n and $|\varepsilon_k^n| \le 1$ such that

$$\Big\| \sum_{k=1}^{n} \varepsilon_k^n a_k^n e^{2\pi ikx} \Big\|_{L^p[0,1]} \ge c_p\, n^{|\frac{1}{2}-\frac{1}{p}|} \Big\| \sum_{k=1}^{n} a_k^n e^{2\pi ikx} \Big\|_{L^p[0,1]}, \tag{6.15}$$

for some constant c_p. To see this we consider the sequence of $a_k^n = 1$ for all k for which the L^p norm if calculated explicitly and gives $\approx n^{1-1/p}$ and a random sequence of ± 1's, which by the Khintchine's inequality gives the constant $\sqrt{n}$.

Rephrased in the language of multipliers, estimate (6.15) is saying that for some constant c'_p we have

$$\|(\dots, 0, \dots, 0, \varepsilon_1^n, \varepsilon_2^n, \dots, \varepsilon_n^n, 0, \dots)\|_{M_p(\mathbf{Z})} \ge c'_p n^{|\frac{1}{2}-\frac{1}{p}|}\,.$$

We may choose the sequence $\{\varepsilon_k^n\}_{k=1}^n$ to be "maximal" in the sense that its M_p norm is the supremum of the M_p norms of all other sequences of ℓ^∞ norm 1 that satisfy (6.15) for the choice of a_k^n.

We now define

$$m(\omega)(\xi) = \int_{\mathbf{S}^{d-1}} \omega(\theta) \log \frac{1}{|\xi \cdot \theta|}\, d\theta$$

and we also define a similar quantity

$$m_\alpha(\omega)(\xi) = \int_{\mathbf{S}^{d-1}} |\omega(\theta)| \log^{1+\alpha} \frac{1}{|\xi \cdot \theta|}\, d\theta\,,$$

while for each integer n we define a even function

$$\Omega_n = \sum_{k=1}^n \varepsilon_k^n \underbrace{C(n) \sum_{j=0}^3 (-1)^j \chi_{A_k \text{ rotated by } \frac{j\pi}{2}}}_{\omega_k^n}\,,$$

where A_k are the arcs of Figure 1. Here $C(n)$ is a constant chosen so that

$$m_\alpha(\omega_k^n)(x_k/|x_k|) = 1/2$$

for all k. Finally we denote by $D(n)$ the constant

$$D(n) = m(\omega_k^n)(x_k/|x_k|)\,.$$

It is not difficult to check that

$$\begin{aligned} C(n) &\approx L_n^{-1} |\log L_n|^{-1-\alpha} \\ D(n) &\approx |\log L_n|^{-\alpha}. \end{aligned}$$

while for all $x \notin \bigcup_{j=0}^3 (I^k \text{ rotated by } \frac{j\pi}{2}) \cap \mathbf{S}^1$ we have

$$\begin{aligned} |m(\omega_k^n)(x)| &\lesssim (\log n) |\log L_n|^{-1-\alpha} \\ m_\alpha(\omega_k^n)(x) &\lesssim (\log n)^{1+\alpha} |\log L_n|^{-1-\alpha}\,. \end{aligned}$$

It follows from these estimates that

$$\big\|\Omega_n\big\|_{L^1(\mathbf{S}^1)} \lesssim n\,(\log n)\,|\log L_n|^{-1}\,. \tag{6.16}$$

On the other hand we have

$$m(\Omega_n)(x_k) = D(n)\,\varepsilon_k^n + \sum_{1 \le i \ne k \le n} \varepsilon_i^n m(\omega_i^{\epsilon_n})(x_k) = D(n)\varepsilon_k^n + o_k^n,$$

where o_k^n is an error term which satisfies

$$|o_k^n| \le D(n)/4$$

provided

$$n^{4n} \lesssim L_n^{-1}\,.$$

We now take a look at certain multiplier norms. Using a deLeeuw type argument [6] we can restrict the $M_p(\mathbf{R}^2)$ norm of a multiplier to its values at the points $(x_k)_k$ which lie on a line parallel to the horizontal coordinate axis and we can thus estimate from below the

$M_p(\mathbf{R}^2)$ norm of the multiplier by the $M_p(\mathbf{Z})$ norm of the sequence of the first coordinates of its values at the points $(x_k)_k$. This way we obtain

$$\begin{aligned}\|m(\Omega_n)\|_{M_p(\mathbf{R}^2)} &\geq c_p\|(..,0,m(\Omega_n)(x_1),...,m(\Omega_n)(x_n),0,...)\|_{M_p(\mathbf{Z})}\\ &\geq c_p\,D(n)\Big[\|(...,0,\varepsilon_1^n,...,\varepsilon_n^n,0,...)\|_{M_p(\mathbf{Z})}-\\ &\qquad\qquad \frac{1}{D(n)}\|(...,0,o_1^n,...,o_n^n,0,...)\|_{M_p(\mathbf{Z})}\Big]\\ &\geq \frac{1}{2}\,c_p\,D(n)\,\|(...,0,\varepsilon_1^n,...,\varepsilon_n^n,0,...)\|_{M^p(\mathbf{Z})}\,,\end{aligned}$$

since the inequality

$$\frac{1}{D(n)}\Big\|(...,0,o_1^n,...,o_n^n,0,...)\Big\|_{M_p(\mathbf{Z})} > \frac{1}{2}\,\Big\|(...,0,\varepsilon_1^n,...,\varepsilon_n^n,0,...)\Big\|_{M_p(\mathbf{Z})}$$

would contradict the "maximal" choice of $(\varepsilon_k^n)_{k=1}^n$.

We now recall that

$$\Big\|(...,0,\varepsilon_1^n,...,\varepsilon_n^n,0,...)\Big\|_{M_p(\mathbf{Z})} \geq c_p'\,n^{|\frac{1}{2}-\frac{1}{p}|}\,,$$

which implies that

$$\|m(\Omega_n)\|_{M_p(\mathbf{R}^2)} \geq c'\,D(n)\,n^{|\frac{1}{2}-\frac{1}{p}|} \approx |\log L_n|^{-\alpha}n^{|\frac{1}{2}-\frac{1}{p}|}\,.$$

We finally choose the L_n's. We had the restriction

$$n^{4n} \leq L_n^{-1}$$

while the need to make the expression on the right in equation (6.16) equal to a constant forces us to choose

$$|\log L_n| \approx n\log n\,.$$

With this choice of L_n and all the facts we have accumulated so far we have

$$\begin{aligned}\|T_{\Omega_n}\|_{L^p\to L^p} &= \|m(\Omega_n)\|_{M_p(\mathbf{R}^2)}\\ &\geq c'|\log L_n|^{-\alpha}n^{|\frac{1}{2}-\frac{1}{p}|}\\ &\approx (\log n)^{-\alpha}n^{|\frac{1}{2}-\frac{1}{p}|-\alpha}\,.\end{aligned}$$

We have now constructed a sequence of even integrable functions Ω_n with L^1 norm at most a constant such that

$$\|T_{\Omega_n}\|_{L^p\to L^p} \to \infty$$

when $|\frac{1}{2}-\frac{1}{p}| > \alpha$.

To complete the proof we need some functional analysis. Let $\mathcal{B}_\alpha$ the Banach space of all even integrable functions Ω on $\mathbf{S}^1$ with mean value zero with norm

$$\|\Omega\|_{\mathcal{B}_\alpha} \equiv \|\Omega\|_{L^1(\mathbf{S}^1)} + \|m_\alpha(\Omega)\|_{L^\infty(\mathbf{S}^1)} < \infty.$$

Consider the family of linear maps

$$\Omega \to T_\Omega(f) \quad : \quad \mathcal{B}_\alpha \to L^p(\mathbf{R}^2)$$

indexed by the set

$$U = \{f \in L^p(\mathbf{R}^2):\ \|f\|_{L^p} = 1\}.$$

If no claimed Ω existed, then for all $\Omega \in \mathcal{B}_\alpha$ we would have

$$\sup_{f\in U}\|T_\Omega(f)\|_{L^p} \leq C(\Omega) < \infty\,.$$

The uniform boundedness principle implies the existence of a constant $K < \infty$ such that

$$\|T_\Omega\|_{L^p \to L^p} = \sup_{f \in U} \|T_\Omega(f)\|_{L^p} \leq K \, \|\Omega\|_{\mathcal{B}_\alpha}$$

for all $\Omega \in \mathcal{B}_\alpha$. But this contradicts the construction of Ω_n's for $|\frac{1}{2} - \frac{1}{p}| > \alpha$.

7. CONDITIONS THAT DISTINGUISH BETWEEN p'S

Theorem 3 suggests that one should look for conditions on Ω that distinguish boundedness on $L^p(\mathbf{R}^d)$ for different values of p's.

A natural condition that one should introduce in the study of this problem is the following:

$$CL(\alpha) \qquad \operatorname*{essup}_{|\xi|=1} \int_{\mathbf{S}^{d-1}} |\,\Omega(\theta)|\, \log^{1+\alpha} \frac{1}{|\xi \cdot \theta|}\, d\theta < +\infty$$

A result of Stefanov and the author [11] says that $CL(\alpha)$ implies the L^p boundedness of T_Ω whenever

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{\alpha}{2(2+\alpha)}\,.$$

This restriction was weakened by Fan, Guo, Pan [7] to

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{\alpha}{2(1+\alpha)}\,.$$

Since $\alpha > \alpha/(2(\alpha+1))$, it remains an open question to find out if L^p boundedness holds for values of p in between. We pose therefore the following question:

(a) Assume that $CL(\alpha)$ holds for some $\alpha < 1/2$. Does it follow that

$$T_\Omega : L^p \to L^p$$

whenever

$$\alpha \geq \left|\frac{1}{p} - \frac{1}{2}\right| \geq \frac{\alpha}{2(\alpha+1)} \qquad ?$$

For $\alpha \geq 1/2$ the counterexample does not work and one may guess that in this case T_Ω is bounded on L^p for the whole range

(b) Assume that $CL(\alpha)$ holds for some $\alpha \geq 1/2$. Does it follow that

$$T_\Omega : L^p \to L^p$$

for all $1 < p < \infty$?

The author would like to thank the organizers of the meeting in Sapporo for their hospitality and for the inspiring mathematical atmosphere they created during the conference.

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