A REMARK ON AN ENDPOINT KATO-PONCE INEQUALITY

LOUKAS GRAFAKOS, DIEGO MALDONADO, AND VIRGINIA NAIBO

Abstract. This note introduces bilinear estimates intended as a step towards an L^∞-endpoint Kato-Ponce inequality. In particular, a bilinear version of the classical Gagliardo-Nirenberg interpolation inequalities for a product of functions is proved.

1. Introduction and main result

The following inequality appears to be missing from the vast literature on a class of inequalities known as Kato-Ponce inequalities or fractional Leibniz rules: For every $s > 0$ there exists $C > 0$, depending only on s and dimension n, such that

$$\|D^s(fg)\|_{L^\infty} \leq C\left(\|D^s f\|_{L^\infty}\|g\|_{L^\infty} + \|D^s g\|_{L^\infty}\|f\|_{L^\infty}\right), \quad \text{for all } f, g \in \mathcal{S}(\mathbb{R}^n), \tag{1.1}$$

where D^s is the s-derivative operator* defined for $h \in \mathcal{S}(\mathbb{R}^n)$ as

$$\widehat{D^s h}(\xi) := |\xi|^s \hat{h}(\xi), \quad \forall \xi \in \mathbb{R}^n.$$

Inequality (1.1) represents an endpoint case of inequalities of Kato-Ponce type (see [1, 3, 4, 5, 6, 8, 9, 10, 11] and references therein) and we do not know whether it holds true or not. Moreover, the fact that for any $s > 0$ and any $f, g \in \mathcal{S}(\mathbb{R}^n)$, both sides of (1.1) are finite, makes it quite difficult to find a counter-example to (1.1). Such counter-example should violate the structure of the right-hand side of (1.1), but not the fact that the left-hand side is finite. As a step towards (1.1) the purpose of this note is to prove the following results

Theorem 1. *Let $0 \leq r < s < t$ and set*

$$\alpha := \frac{t-s}{t-r} \quad \text{and} \quad \beta := \frac{s-r}{t-r}. \tag{1.2}$$

Then, for every $f, g \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\|D^s(fg)\|_{L^\infty} \lesssim \|D^r f\|^\alpha_{\dot{B}^{0,\infty}_\infty}\left\|D^t f\right\|^\beta_{\dot{B}^{0,\infty}_\infty}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|D^r g\|^\alpha_{\dot{B}^{0,\infty}_\infty}\left\|D^t g\right\|^\beta_{\dot{B}^{0,\infty}_\infty}, \tag{1.3}$$

where the implicit constant depends only on r, s, t, and dimension n. In particular,

$$\|D^s(fg)\|_{L^\infty} \lesssim \|D^r f\|^\alpha_{L^\infty}\left\|D^t f\right\|^\beta_{L^\infty}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|D^r g\|^\alpha_{L^\infty}\left\|D^t g\right\|^\beta_{L^\infty}. \tag{1.4}$$

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*The notation D^s seems to be standard for this operator although other notations include $|D|^s$, $|\nabla|^s$ and $(-\Delta)^{\frac{s}{2}}$.

Remark 1. Inequality (1.4) can be regarded as a combination of Leibniz-rule and interpolation (or bilinear Gagliardo-Nirenberg) inequalities. Notice that (1.4) is weaker than (1.1). Indeed, given $0 \leq r < s < t$, by the linear Gagliardo-Nirenberg inequality (see, for instance, Theorem 2.44 in [2]), we have

$$\|D^s f\|_{L^\infty} \lesssim \|D^r f\|_{L^\infty}^{\frac{t-s}{t-r}} \left\|D^t f\right\|_{L^\infty}^{\frac{s-r}{t-r}}, \quad \forall f \in \mathcal{S}(\mathbb{R}^n). \tag{1.5}$$

Then, it follows that (1.1), if true, would imply (1.4).

Theorem 2. *Suppose $s > 2n+1$. Let $1 < p_1, p_2 < \infty$ and $\varepsilon > 0$ with $n/p := (1/p_1 + 1/p_2)n < \varepsilon < 1$. Then for every $f, g \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$\begin{aligned} \|D^s(fg)\|_{L^\infty} \lesssim\; & \|D^s f\|_{L^{p_1}}^{1-\frac{n}{p\varepsilon}} \left\|D^{s+\varepsilon} f\right\|_{L^{p_1}}^{\frac{n}{p\varepsilon}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|D^s g\|_{L^{p_2}}^{1-\frac{n}{p\varepsilon}} \left\|D^{s+\varepsilon} g\right\|_{L^{p_2}}^{\frac{n}{p\varepsilon}} \\ & + \|D^s f\|_{L^\infty} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^s g\|_{L^\infty}, \end{aligned}$$

where the implicit constant depends only on s, n, ε, p_1, and p_2.

Remark 2. In the case $s > 2n+1$, the proof of Theorem 2 will be based on a connection between Kato-Ponce inequalities and the bilinear Calderón-Zygmund theory, see Section 4. Notice that the inequality in Theorem 2 involves no derivatives lower than D^s. Also, $\varepsilon > 0$ can be arbitrarily small and $p_1, p_2 \in (1, \infty)$ arbitrarily large, as long as $(1/p_1 + 1/p_2)n < \varepsilon$.

2. Preliminaries

Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a smooth, non-negative, radial function supported in $\{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$ with $\Phi \equiv 1$ in $\{\xi \in \mathbb{R}^n : |\xi| \leq 1\}$. Define $\Psi : \mathbb{R}^n \to \mathbb{R}$ supported in $1/2 \leq |\xi| \leq 2$ as $\Psi(\xi) := \Phi(\xi) - \Phi(2\xi)$ for $\xi \in \mathbb{R}^n$, so that

$$\sum_{j \in \mathbb{Z}} \Delta_j h = h \text{ in } \mathcal{S}'(\mathbb{R}^n) \quad \forall h \in \mathcal{S}(\mathbb{R}^n), \tag{2.1}$$

where, as usual, $\Delta_j h$ is defined for $h \in \mathcal{S}(\mathbb{R}^n)$ as

$$\widehat{\Delta_j h}(\xi) := \Psi(2^{-j}\xi)\widehat{h}(\xi) \quad \forall \xi \in \mathbb{R}^n.$$

We recall that the Besov $\dot{B}^{0,\infty}_\infty$-norm is given by

$$\|h\|_{\dot{B}^{0,\infty}_\infty} := \sup_{j \in \mathbb{Z}} \|\Delta_j h\|_{L^\infty} \leq \left\|\widehat{\Psi}\right\|_{L^1} \|h\|_{L^\infty}. \tag{2.2}$$

For $f \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda > 0$ set $f_\lambda(x) := f(\lambda x)$ for every $x \in \mathbb{R}^n$. For $s \geq 0$ we have

$$\|D^s(f_\lambda)\|_{\dot{B}^{0,\infty}_\infty} = \lambda^s \|D^s f\|_{\dot{B}^{0,\infty}_\infty} \qquad \text{for all } \lambda = 2^{j_0},\ j_0 \in \mathbb{Z}. \tag{2.3}$$

We note tha $\tilde{\Phi}(\xi + \eta)\Phi(\xi)\Psi(\eta) = \Phi(\xi)\Psi(\eta)$ for every $\xi, \eta \in \mathbb{R}^n$, where $\tilde{\Phi}(\cdot) := \Phi(4^{-1}\cdot)$, and write $\Phi_{(s)}(\cdot) := |\cdot|^s \tilde{\Phi}(\cdot)$. Reasoning as in [6], the absolutely convergent Fourier series for $\Phi_{(s)}(t)\chi_{[-8,8]^n}(t)$,

$$\Phi_{(s)}(t) = \sum_{m \in \mathbb{Z}^n} c_{s,m} e^{\frac{2\pi i}{16} m \cdot t} \chi_{[-8,8]^n}(t), \tag{2.4}$$

has coefficients $c_{s,m}$ satisfying

$$c_{s,m} = O(1+|m|^{-n-s}). \tag{2.5}$$

3. Proof of Theorem 1

Proof. Fix $0 \le r < s < t$. By (2.1), we have

$$D^s(fg)(x) = \int_{\mathbb{R}^{2n}} |\xi+\eta|^s \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i(\xi+\eta)\cdot x} d\xi d\eta =: \Pi(f,g)(x) + \tilde{\Pi}(f,g)(x),$$

with

$$\Pi(f,g)(x) := \int_{\mathbb{R}^{2n}} \sum_{j\in\mathbb{Z}} \sum_{k\le j} |\xi+\eta|^s \Psi(2^{-j}\xi)\Psi(2^{-k}\eta)\widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i(\xi+\eta)\cdot x} d\xi d\eta$$

and

$$\tilde{\Pi}(f,g)(x) := \int_{\mathbb{R}^{2n}} \sum_{j\in\mathbb{Z}} \sum_{j<k} |\xi+\eta|^s \Psi(2^{-j}\xi)\Psi(2^{-k}\eta)\widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i(\xi+\eta)\cdot x} d\xi d\eta.$$

Now, we split $\Pi(f,g)$ (and then, similarly, $\tilde{\Pi}$) as follows

$$\begin{aligned}
\Pi(f,g)(x) &= \int_{\mathbb{R}^{2n}} \sum_{j\in\mathbb{Z}} |\xi+\eta|^s \Psi(2^{-j}\xi)\Phi(2^{-j}\eta)\widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i(\xi+\eta)\cdot x} d\xi d\eta \\
&= \int_{\mathbb{R}^{2n}} \sum_{j\le 0} |\xi+\eta|^s \Psi(2^{-j}\xi)\Phi(2^{-j}\eta)\widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i(\xi+\eta)\cdot x} d\xi d\eta \\
&+ \int_{\mathbb{R}^{2n}} \sum_{j> 0} |\xi+\eta|^s \Psi(2^{-j}\xi)\Phi(2^{-j}\eta)\widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i(\xi+\eta)\cdot x} d\xi d\eta \\
&= \int_{\mathbb{R}^{2n}} \sum_{j\le 0} \frac{|\xi+\eta|^s}{|\xi|^r} \Psi(2^{-j}\xi)\Phi(2^{-j}\eta)\widehat{D^r f}(\xi)\widehat{g}(\eta) e^{2\pi i(\xi+\eta)\cdot x} d\xi d\eta \\
&+ \int_{\mathbb{R}^{2n}} \sum_{j> 0} \frac{|\xi+\eta|^s}{|\xi|^t} \Psi(2^{-j}\xi)\Phi(2^{-j}\eta)\widehat{D^t f}(\xi)\widehat{g}(\eta) e^{2\pi i(\xi+\eta)\cdot x} d\xi d\eta \\
&=: \Pi_1(D^r f, g) + \Pi_2(D^t f, g).
\end{aligned}$$

We now look at the bilinear kernel of Π_1 (the kernel for Π_2 will be dealt with in a similar way).

$$\Pi_1(f,g)(x) = \int_{\mathbb{R}^{2n}} K_1(x-y, x-z) f(y) g(z) dy dz, \tag{3.1}$$

where, after putting $\Psi_{(-r)}(\cdot) := |\cdot|^{-r}\Psi(\cdot)$ and using that $\tilde{\Phi}(\xi+\eta)\Phi(\xi)\Psi(\eta) = \Phi(\xi)\Psi(\eta)$ for every $\xi, \eta \in \mathbb{R}^n$, K_1 is given by

$$\begin{aligned}
K_1(y,z) &= \int_{\mathbb{R}^{2n}} \sum_{j\le 0} \frac{|\xi+\eta|^s}{|\xi|^r} \Psi(2^{-j}\xi)\Phi(2^{-j}\eta) e^{2\pi i(\xi\cdot y+\eta\cdot z)} d\xi d\eta \\
&= \int_{\mathbb{R}^{2n}} \sum_{j\le 0} \frac{2^{js}}{2^{jr}} \Phi_{(s)}(2^{-j}(\xi+\eta))\Psi_{(-r)}(2^{-j}\xi)\Phi(2^{-j}\eta) e^{2\pi i(\xi\cdot y+\eta\cdot z)} d\xi d\eta.
\end{aligned}$$

Hence, using the Fourier expansion in (2.4) and noting that the support of $\psi_{(-r)}(\xi)\phi(\eta)$ is contained in $\{(\xi,\eta) : |\xi+\eta| \le 4\}$, we get

$$\begin{aligned} K_1(y,z) &= \int_{\mathbb{R}^{2n}} \sum_{j\le 0} \sum_{m\in\mathbb{Z}^n} c_{s,m} 2^{j(s-r)} e^{\frac{2\pi i}{16} m\cdot 2^{-j}(\xi+\eta)} \Psi_{(-r)}(2^{-j}\xi)\Phi(2^{-j}\eta) e^{2\pi i(\xi\cdot y+\eta\cdot z)} d\xi d\eta \\ &= \sum_{j\le 0} \sum_{m\in\mathbb{Z}^n} c_{s,m} 2^{j(s-r)} \int_{\mathbb{R}^{2n}} e^{\frac{2\pi i}{16} m\cdot 2^{-j}(\xi+\eta)} \Psi_{(-r)}(2^{-j}\xi)\Phi(2^{-j}\eta) e^{2\pi i(\xi\cdot y+\eta\cdot z)} d\xi d\eta \\ &= \sum_{j\le 0} \sum_{m\in\mathbb{Z}^n} c_{s,m} 2^{j(s-r)} 2^{2jn} \int_{\mathbb{R}^{2n}} e^{\frac{2\pi i}{16} m\cdot (\xi+\eta)} \Psi_{(-r)}(\xi)\Phi(\eta) e^{2\pi i 2^j(\xi\cdot y+\eta\cdot z)} d\xi d\eta \\ &= \sum_{j\le 0} \sum_{m\in\mathbb{Z}^n} c_{s,m} 2^{j(s-r)} 2^{2jn} \widehat{\Psi_{(-r)}}(\tfrac{m}{16}+2^j y)\widehat{\Phi}(\tfrac{m}{16}+2^j z). \end{aligned}$$

Now,

$$\begin{aligned} \Pi_1(f,g)(x) &= \sum_{l\in\mathbb{Z}} \int_{\mathbb{R}^{2n}} K_1(x-y,x-z)(\Delta_l f)(y) g(z) dydz \\ &\le \sum_{j\le 0} \sum_{m\in\mathbb{Z}^n} \sum_{l\in\mathbb{Z}} c_{s,m} 2^{j(s-r)} \\ &\quad \times \int_{\mathbb{R}^{2n}} 2^{2jn} \widehat{\Psi_{(-r)}}(\tfrac{m}{16}+2^j(x-y))\widehat{\Phi}(\tfrac{m}{16}+2^j(x-z))\Delta_l f(y) g(z) dydz. \end{aligned}$$

For a fixed $j\in\mathbb{Z}$ we look at the integral in y

$$\int_{\mathbb{R}^n} \widehat{\Psi_{(-r)}}(\tfrac{m}{16}+2^j(x-y))\Delta_l f(y) dy = \int_{\mathbb{R}^n} \frac{e^{2\pi i\xi\cdot(2^{-j}\frac{m}{16}+x)}}{2^{jn}} \Psi(2^{-j}\xi)\Psi(2^{-l}\xi)\hat{f}(\xi) d\xi,$$

which, due to the support conditions on Ψ, vanishes for every $l\in\mathbb{Z}\setminus\{j-1,j,j+1\}$. Consequently,

$$\begin{aligned} |\Pi_1(f,g)(x)| &\le \sum_{j\le 0} \sum_{m\in\mathbb{Z}^n} \sum_{l=j-1,j,j+1} |c_{s,m}| 2^{j(s-r)} \\ &\times \int_{\mathbb{R}^{2n}} 2^{2jn} |\widehat{\Psi_{(-r)}}(\tfrac{m}{16}+2^j(x-y))||\widehat{\Phi}(\tfrac{m}{16}+2^j(x-z))||\Delta_l f(y)||g(z)| dydz \\ &\le 3\left(\sum_{j\le 0} 2^{j(s-r)}\right)\left(\sum_{m\in\mathbb{Z}^n} |c_{s,m}|\right) \left\|\widehat{\Psi_{(-r)}}\right\|_{L^1(\mathbb{R}^n)} \left\|\widehat{\Phi}\right\|_{L^1(\mathbb{R}^n)} \|f\|_{\dot{B}^{0,\infty}_\infty} \|g\|_{L^\infty}. \end{aligned}$$

Since $s-r>0$ we have $\sum_{j\le 0} 2^{j(s-r)} < \infty$ and, from (2.5), $\sum_{m\in\mathbb{Z}^n} |c_{s,m}| < \infty$. Hence,

$$|\Pi_1(f,g)(x)| \le C \left\|\widehat{\Psi_{(-r)}}\right\|_{L^1(\mathbb{R}^n)} \left\|\widehat{\Phi}\right\|_{L^1(\mathbb{R}^n)} \|f\|_{\dot{B}^{0,\infty}_\infty} \|g\|_{L^\infty} \quad \forall x\in\mathbb{R}^n,$$

where $C>0$ depends only on r, s, and n.

Along the same lines, now for $s<t$ one gets the bound for $\Pi_2(f,g)$,

$$|\Pi_2(f,g)(x)| \le c\left(\sum_{j>0} 2^{j(s-t)}\right)\left(\sum_{m\in\mathbb{Z}^n} |c_{s,m}|\right) \left\|\widehat{\Psi_{(-t)}}\right\|_{L^1(\mathbb{R}^n)} \left\|\widehat{\Phi}\right\|_{L^1(\mathbb{R}^n)} \|f\|_{\dot{B}^{0,\infty}_\infty} \|g\|_{L^\infty},$$

with $s-t<0$. Then

$$\|\Pi(f,g)\|_{L^\infty} \le C(\|D^r f\|_{\dot B^{0,\infty}_\infty} + \left\|D^t f\right\|_{\dot B^{0,\infty}_\infty})\,\|g\|_{L^\infty}\,. \tag{3.2}$$

Interchanging the roles of f and g to deal with $\tilde\Pi$ yields

$$\left\|\tilde\Pi(f,g)\right\|_{L^\infty} \le C(\|D^r g\|_{\dot B^{0,\infty}_\infty} + \left\|D^t g\right\|_{\dot B^{0,\infty}_\infty})\,\|f\|_{L^\infty}\,. \tag{3.3}$$

Given a positive dyadic number μ, plugging in f_μ and g_μ into (3.2) and (3.3), using the scaling property (2.3) and the fact that $\Pi(f_\mu,g_\mu)=\mu^s\,\Pi(f,g)_\mu$ and $\tilde\Pi(f_\mu,g_\mu)=\mu^s\,\tilde\Pi(f,g)_\mu$, we get

$$\|\Pi(f,g)\|_{L^\infty} \lesssim (\lambda^{r-s}\,\|D^r f\|_{\dot B^{0,\infty}_\infty} + \lambda^{t-s}\left\|D^t f\right\|_{\dot B^{0,\infty}_\infty})\,\|g\|_{L^\infty}\,,$$

$$\left\|\tilde\Pi(f,g)\right\|_{L^\infty} \lesssim (\lambda^{r-s}\,\|D^r g\|_{\dot B^{0,\infty}_\infty} + \lambda^{t-s}\left\|D^t g\right\|_{\dot B^{0,\infty}_\infty})\,\|f\|_{L^\infty}\,,$$

for every positive number λ. Minimizing in λ each of the above inequalities leads to

$$\|\Pi(f,g)\|_{L^\infty} \lesssim \|D^r f\|^{\alpha}_{\dot B^{0,\infty}_\infty}\left\|D^t f\right\|^{\beta}_{\dot B^{0,\infty}_\infty}\|g\|_{L^\infty}\,,$$

$$\left\|\tilde\Pi(f,g)\right\|_{L^\infty} \lesssim \|f\|_{L^\infty}\,\|D^r g\|^{\alpha}_{\dot B^{0,\infty}_\infty}\left\|D^t g\right\|^{\beta}_{\dot B^{0,\infty}_\infty}\,,$$

from which (1.3) follows. □

4. The case $s > 2n+1$

A smooth function $\sigma:\mathbb{R}^{2n}\setminus\{(0,0)\}\to\mathbb{C}$ is said to belong to the class of bilinear Coifman-Meyer multipliers if for all multi-indices $\alpha,\beta\in\mathbb{N}_0^n$ with $|\alpha|+|\beta|\le 2n+1$ there exist constants $c_{\alpha,\beta}>0$ such that

$$|\partial_\xi^\alpha\partial_\eta^\beta\sigma(\xi,\eta)| \le c_{\alpha,\beta}(|\xi|+|\eta|)^{-|\alpha|-|\beta|},\quad \forall(\xi,\eta)\in\mathbb{R}^{2n}\setminus\{(0,0)\}. \tag{4.1}$$

In [6], the bilinear mapping $(f,g)\mapsto D^s(fg)$ was decomposed into the sum of three bilinear multipliers as follows

$$D^s(fg) = T_{1,s}(D^s f,g) + T_{2,s}(f,D^s g) + T_{3,s}(f,D^s g), \tag{4.2}$$

where, keeping with the notation in Section 3, for $(\xi,\eta)\in\mathbb{R}^{2n}\setminus\{(0,0)\}$ the bilinear multipliers for $T_{1,s}$ and $T_{2,s}$ are given by

$$\sigma_{1,s}(\xi,\eta) := \sum_{j\in\mathbb{Z}}\Psi(2^{-j}\xi)\Phi(2^{-j+3}\eta)\frac{|\xi+\eta|^s}{|\xi|^s}\quad\text{and}\quad \sigma_{2,s}(\xi,\eta) := \sigma_{1,s}(\eta,\xi), \tag{4.3}$$

respectively, which belong to the Coifman-Meyer class for every $s>0$. On the other hand, the multiplier for $T_{3,s}$, denoted by $\sigma_{3,s}$, can be expressed as

$$\sigma_{3,s}(\xi,\eta) := \sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^n} c_{s,m}e^{\frac{2\pi i}{16}2^{-k}(\xi+\eta)\cdot m}\Psi(2^{-k}\xi)\Psi_{(-s)}(2^{-k}\eta). \tag{4.4}$$

For fixed $\xi,\eta\in\mathbb{R}^{2n}\setminus\{(0,0)\}$ the condition on the support of Ψ implies that the sum in k has only finitely many terms; namely, those with $2^k\sim|\xi|\sim|\eta|$. When derivatives in ξ and η of the product $e^{\frac{2\pi i}{16}2^{-k}(\xi+\eta)\cdot m}\Psi(2^{-k}\xi)\Psi_{(-s)}(2^{-k}\eta)$ are taken, after each derivative a factor 2^{-k} $(\sim |\xi|^{-1} \sim |\eta|^{-1} \sim (|\xi|+|\eta|)^{-1})$ appears, producing the right-hand side of (4.1). However, when the derivatives fall on the factor $e^{\frac{2\pi i}{16}2^{-k}(\xi+\eta)\cdot m}$ also components of $m \in \mathbb{Z}^n$ appear. Since the definition of a Coifman-Meyer multiplier requires at most $2n+1$ derivatives, the worst case scenario for the sum over $m \in \mathbb{Z}^n$ (i.e., the case in which all $2n+1$ derivatives fall on $e^{\frac{2\pi i}{16}2^{-k}(\xi+\eta)\cdot m}$) leads to the sum

$$\sum_{m\in\mathbb{Z}^n} |c_{s,m}||m|^{2n+1}.$$

By (2.5), the sum above will be finite provided that $s > 2n+1$. That is, whenever $s > 2n+1$ all three bilinear operators in (4.2), and therefore the mapping $(f,g) \mapsto D^s(fg)$, can be realized as bilinear Coifman-Meyer multipliers. Since the class of Coifman-Meyer multipliers is included in the family of bilinear Calderón-Zygmund operators (see, [7, Section 6]) all the mapping properties of the type

$$\|T(f,g)\|_Z \lesssim \|f\|_X \|g\|_Y , \tag{4.5}$$

that apply to bilinear C-Z operators T on function spaces X, Y, and Z will also apply to $(f,g) \mapsto D^s(fg)$. For example, for a bilinear C-Z operator T, given $1 < p_1, p_2 < \infty$ and $1/p := 1/p_1 + 1/p_2$, it holds that

$$\|T(f,g)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \tag{4.6}$$

and (see [7, Proposition 1]) that,

$$\|T(f,g)\|_{BMO} \lesssim \|f\|_{L^\infty} \|g\|_{L^\infty} , \tag{4.7}$$

as well as other end-point estimates such as

$$\|T(f,g)\|_{L^{1,\infty}} \lesssim \|D^s f\|_{L^\infty} \|g\|_{L^1} + \|f\|_{L^1} \|D^s g\|_{L^\infty} . \tag{4.8}$$

As a consequence of the results above, we have

Theorem 3. *If $s > 2n+1$, then for every $f, g \in \mathcal{S}(\mathbb{R}^n)$ we have the endpoint inequalities*

$$\|D^s(fg)\|_{BMO} \lesssim \|D^s f\|_{L^\infty} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^s g\|_{L^\infty} . \tag{4.9}$$

and

$$\|D^s(fg)\|_{L^{1,\infty}} \lesssim \|D^s f\|_{L^\infty} \|g\|_{L^1} + \|f\|_{L^1} \|D^s g\|_{L^\infty} . \tag{4.10}$$

Remark 3. We note that the conditions (4.1) being satisfied with up to $n+1$ derivatives (instead of $2n+1$) are sufficient for the corresponding multiplier operator to be bounded from $L^{p_1} \times L^{p_2}$ into L^p for $1 < p_1, p_2, p < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, as shown in Tomita [13]. The endpoint boundedness $L^\infty \times L^\infty$ into BMO for Coifman-Meyer multipliers, with only up to $n+1$ derivatives in (4.1), is unknown to us. To pass through the bilinear C-Z theory, as done above, it suffices that the conditions (4.1) be satisfied with up to $2n+1$ derivatives.

Proof of Theorem 2. By hypothesis, $1/p := 1/p_1 + 1/p_2$, so that $n/p < \varepsilon < 1$. It was proved in [12, pp.193–198] that a function F with $\|D^\varepsilon F\|_{L^p} + \|F\|_{BMO} + \|F\|_{L^p} < \infty$ can be written as $F = F_0 + G + F_1$ where

$$\|F_0\|_{L^\infty} \lesssim \|D^\varepsilon F\|_{L^p} , \quad \|G\|_{L^\infty} \lesssim \|F\|_{BMO} , \quad \text{and} \quad \|F_1\|_{L^\infty} \lesssim \|F\|_{L^p} . \tag{4.11}$$

Now, with $T_{1,s}$ as in the decomposition (4.2), let us first choose $F := T_{1,s}(D^s f, g)$, so that from (4.11) we get

$$\|T_{1,s}(D^s f,g)\|_{L^\infty} \lesssim \|T_{1,s}(D^s f,g)\|_{L^p} + \|T_{1,s}(D^s f,g)\|_{BMO} + \|D^\varepsilon(T_{1,s}(D^s f,g))\|_{L^p}.$$

The fact that $T_{1,s}$ is a bilinear C-Z operator and (4.6) yield

$$\|T_{1,s}(D^s f,g)\|_{L^p} \lesssim \|D^s f\|_{L^{p_1}} \|g\|_{L^{p_2}}.$$

Also, from (4.7), it follows that

$$\|T_{1,s}(D^s f,g)\|_{BMO} \lesssim \|D^s f\|_{L^\infty} \|g\|_{L^\infty}.$$

On the other hand, notice that

$$D^\varepsilon(T_{1,s}(D^s f,g)) =: T_{1,s+\varepsilon}(D^{s+\varepsilon} f, g),$$

where the bilinear symbol for the operator $T_{1,s+\varepsilon}$ equals $\sigma_{1,s+\varepsilon}(\xi,\eta)$ (using the notation in (4.3)), also a Coifman-Meyer multiplier. Hence, (4.6) gives

$$\|D^\varepsilon(T_{1,s}(D^s f,g))\|_{L^p} \lesssim \left\|D^{s+\varepsilon} f\right\|_{L^{p_1}} \|g\|_{L^{p_2}}.$$

Putting all together, for $T_{1,s}(f,g)$ we have

$$\|T_{1,s}(D^s f,g)\|_{L^\infty} \lesssim (\|D^s f\|_{L^{p_1}} + \left\|D^{s+\varepsilon} f\right\|_{L^{p_1}}) \|g\|_{L^{p_2}} + \|D^s f\|_{L^\infty} \|g\|_{L^\infty}. \tag{4.12}$$

Given a positive dyadic number μ, by replacing f and g in (4.12) with f_μ and g_μ and using the facts that

$$\|D^s(f_\mu)\|_{L^q} = \mu^{s-\frac{n}{q}} \|D^s f\|_{L^q}, \quad \forall q \in [1,\infty],$$

that $1/p = 1/p_1 + 1/p_2$, and that $T_{1,s}(D^s f_\mu, g_\mu) = \mu^s\, T_{1,s}(D^s f, g)_\mu$, we obtain

$$\|T_{1,s}(D^s f,g)\|_{L^\infty} \lesssim (\lambda^{-\frac{n}{p}} \|D^s f\|_{L^{p_1}} + \lambda^{\varepsilon-\frac{n}{p}} \left\|D^{s+\varepsilon} f\right\|_{L^{p_1}}) \|g\|_{L^{p_2}} + \|D^s f\|_{L^\infty} \|g\|_{L^\infty},$$

for every positive number λ. Minimization over λ then implies

$$\|T_{1,s}(D^s f,g)\|_{L^\infty} \lesssim \|D^s f\|_{L^{p_1}}^{1-\frac{n}{p\varepsilon}} \left\|D^{s+\varepsilon} f\right\|_{L^{p_1}}^{\frac{n}{p\varepsilon}} \|g\|_{L^{p_2}} + \|D^s f\|_{L^\infty} \|g\|_{L^\infty}. \tag{4.13}$$

And, by an analogous argument based on $T_{2,s}$,

$$\|T_{2,s}(f,D^s g)\|_{L^\infty} \lesssim \|f\|_{L^{p_1}} \|D^s g\|_{L^{p_2}}^{1-\frac{n}{p\varepsilon}} \left\|D^{s+\varepsilon} g\right\|_{L^{p_2}}^{\frac{n}{p\varepsilon}} + \|f\|_{L^\infty} \|D^s g\|_{L^\infty}. \tag{4.14}$$

It only remains to consider $T_{3,s}$. Since $s > 2n+1$, again from (4.6) and (4.7), we have

$$\|T_{3,s}(D^s f,g)\|_{L^p} + \|T_{3,s}(D^s f,g)\|_{BMO} \lesssim \|D^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|D^s f\|_{L^\infty} \|g\|_{L^\infty}.$$

Now,

$$D^\varepsilon(T_{3,s}(D^s f,g)) =: T_{3,s+\varepsilon}(D^{s+\varepsilon} f, g)$$

where the bilinear symbol for $T_{3,s+\varepsilon}$ is similar to $\sigma_{3,s}$ in (4.4) but with $c_{s,m}$ replaced by $c_{s+\varepsilon,m}$, the Fourier coefficients for $\Phi_{(s+\varepsilon)}$ which will satisfy $c_{s+\varepsilon,m} = O(1+|m|^{-n-s-\varepsilon})$. Consequently,

$$\|D^\varepsilon(T_{3,s}(D^s f,g))\|_{L^p} \lesssim \left\|D^{s+\varepsilon} f\right\|_{L^{p_1}} \|g\|_{L^{p_2}}$$

and, proceeding as before, after scaling we get

$$\|T_{3,s}(D^s f,g)\|_{L^\infty} \lesssim \|D^s f\|_{L^{p_1}}^{1-\frac{n}{p\varepsilon}} \left\|D^{s+\varepsilon} f\right\|_{L^{p_1}}^{\frac{n}{p\varepsilon}} \|g\|_{L^{p_2}} + \|D^s f\|_{L^\infty} \|g\|_{L^\infty}. \tag{4.15}$$

Finally, Theorem 2 follows from (4.2), (4.13), (4.14), and (4.15). □

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Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: grafakosl@missouri.edu

Department of Mathematics, Kansas State University, Manhattan, KS 66506, USA
E-mail address: dmaldona@math.ksu.edu

Department of Mathematics, Kansas State University, Manhattan, KS 66506, USA
E-mail address: vnaibo@math.ksu.edu