OFF-DIAGONAL MULTILINEAR INTERPOLATION BETWEEN ADJOINT OPERATORS

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Abstract. We extend a theorem by Grafakos and Tao [5] on multilinear interpolation between adjoint operators to an off-diagonal situation. We provide an application.

1. Introduction and the Main Result

Multilinear interpolation has proved to be a powerful and indispensable tool in analysis. The two main linear interpolation theorems, the Marcinkiewicz and Riesz-Thorin theorems, have well-established multilinear analogs. The works [10], [6], [3], [4] provide multilinear extensions of the Marcinkiewicz interpolation theorem. The Riesz-Thorin theorem is easily adapted in the multilinear case in [12, 21, Chapter XII, (3.3)] and [1, Theorem 4.4.2]; related versions of this result have appeared in [11], [8], [9].

A different type of interpolation is that between adjoint operators. In the linear case a typical result would be as follows: if an operator and its adjoint are of weak type $(1, 1)$, then the operator is L^p bounded for all $p \in (1, \infty)$. A multilinear version of this result was obtained in [5]. This theorem says that, under an initial condition similar to (1.1) below, if an m-linear operator and all of its m adjoints are of restricted weak type $(1, 1, \dots, 1, 1/m)$, then the operator is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ to L^p for all $1 < p_1, \dots, p_m < \infty$ and $1/m < p < \infty$. In this context, an m-linear operator is called of restricted weak type $(p_1, \dots, p_m, p)$ if it maps $L^{p_1} \times \cdots \times L^{p_m}$ to $L^{p,\infty}$ when restricted to characteristic functions of sets of finite measure. The jth adjoint of an m-linear operator T (defined on products of simple functions on measure spaces (X_j, μ_j), $j \in \{1, \dots, m\}$, and taking values in another measure space (X_0, μ_0)) is another operator T^{*j} such that

$$\int_{A_0} T(\chi_{A_1}, \dots, \chi_{A_{j-1}}, \chi_{A_j}, \chi_{A_{j+1}}, \dots, \chi_{A_m}) d\mu_0 = \int_{A_j} T^{*j}(\chi_{A_1}, \dots, \chi_{A_{j-1}}, \chi_{A_0}, \chi_{A_{j+1}}, \chi_{A_m}) d\mu_j$$

for all measurable subsets A_i of X_i with nonzero finite measure. When T^{*0} is written, it is understood to be T itself. For $0 < q < \infty$, q' denotes the number $q/(q-1)$ and $1' = \infty$.

In this note we obtain the following off-diagonal version of the main result in [5] in which the diagonal case $t = 1$ and $s = 1/m$ was considered.

Theorem 1.1. *Let $1 \le t < \infty$, $0 < s \le 1$, $1 < p < t'$, and $t < p_1, \dots, p_m < \infty$ be such that $1/p_1 + \cdots + 1/p_m - 1/p = m/t - 1/s$. Let $(X_0, \mu_0), (X_1, \mu_1), \dots, (X_m, \mu_m)$ be σ-finite measure spaces. Suppose that an m-linear operator T is defined on the space of simple functions on $X_1 \times \cdots \times X_m$ and takes values in the space of measurable functions on X_0. We assume that T satisfies*

$$\sup_{A_0, A_1, \dots, A_m} \frac{1}{\mu_0(A_0)^{\frac{1}{p'}} \mu_1(A_1)^{\frac{1}{p_1}} \cdots \mu_m(A_m)^{\frac{1}{p_m}}} \left| \int_{A_0} T(\chi_{A_1}, \dots, \chi_{A_m})\, d\mu_0 \right| < \infty, \tag{1.1}$$

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where the supremum is taken over all measurable subsets A_i of X_i with nonzero finite measure. Suppose that for each $j \in \{0, 1, \ldots, m\}$, T^{*j} is of restricted weak type $(t, t, \ldots, t, s)$ with constant B_j. Then there is a constant $C = C(p_1, \ldots, p_m, p, t, s)$ such that T is of restricted weak type $(p_1, \ldots, p_m, p)$ with norm at most

$$CB_0^{\theta\left(\frac{1}{t}-\frac{1}{p'}\right)} B_1^{\theta\left(\frac{1}{t}-\frac{1}{p_1}\right)} \cdots B_m^{\theta\left(\frac{1}{t}-\frac{1}{p_m}\right)}. \tag{1.2}$$

where $\theta = (1/t + 1/s - 1)^{-1}$.

The following well-known characterization of weak L^p will be used in the proof; see for instance Proposition 7.2.12 in [2].

Proposition 1.2. *Let $0 < p < \infty$, $A, B > 0$, and let f be a measurable function on a σ-finite measure space (X, μ).*
(i) Suppose that $\|f\|_{L^{p,\infty}} \leq A$. Then for every measurable set E of finite measure there exists a measurable subset E' of E with $\mu(E') \geq \mu(E)/2$ such that f is bounded on E' and

$$\left| \int_{E'} f \, d\mu \right| \leq 2^{\frac{1}{p}} A \, \mu(E)^{1-\frac{1}{p}} .$$

(ii) Suppose that a measurable function f on X has the property that for any measurable subset E of X with $\mu(E) < \infty$ there is a measurable subset E' of E with $\mu(E') \geq \mu(E)/2$ such that f is integrable on E' and

$$\left| \int_{E'} f \, d\mu \right| \leq B \, \mu(E)^{1-\frac{1}{p}} .$$

Then we have that

$$\|f\|_{L^{p,\infty}} \leq B \, 2^{\frac{2}{p}+\frac{3}{2}}.$$

2. The Proof of Theorem 1.1

Proof. First consider the case where

$$\frac{\mu_0(A_0)}{B_0^\theta} \geq \max\left(\frac{\mu_1(A_1)}{B_1^\theta}, \ldots, \frac{\mu_m(A_m)}{B_m^\theta}\right). \tag{2.1}$$

Let M be the supremum given in (1.1). It will be enough to show that M is bounded above by the constant in (1.2), from which Proposition 1.2(ii) gives the desired boundedness.

Since T is restricted weak type $(t, \ldots, t, s)$, Proposition 1.2(i) gives a subset A_0' of A_0 with measure $\mu_0(A_0') \geq \mu_0(A_0)/2$ so that

$$\left| \int_{A_0'} T(\chi_{A_1}, \ldots, \chi_{A_m}) d\mu_0 \right| \leq KB_0 \, \mu_1(A_1)^{\frac{1}{t}} \cdots \mu_m(A_m)^{\frac{1}{t}} \mu_0(A_0)^{1-\frac{1}{s}}$$

for some constant $K = K(s)$. It then follows that

$$\begin{aligned}\left| \int_{A_0} T(\chi_{A_1}, \ldots, \chi_{A_m}) d\mu_0 \right| &\leq \left| \int_{A_0'} T(\chi_{A_1}, \ldots, \chi_{A_m}) d\mu_0 \right| + \left| \int_{A_0 \setminus A_0'} T(\chi_{A_1}, \ldots, \chi_{A_m}) d\mu_0 \right| \\ &:= I + II \,.\end{aligned}$$

We have that

$$I \leq KB_0 \mu_1(A_1)^{\frac{1}{t}} \cdots \mu_m(A_m)^{\frac{1}{t}} \mu_0(A_0)^{1-\frac{1}{s}}$$

$$\leq \mu_0(A_0)^{\frac{1}{p'}}\mu_1(A_1)^{\frac{1}{p_1}}\cdots\mu_m(A_m)^{\frac{1}{p_m}}\left(KB_0\left(\frac{B_1^\theta}{B_0^\theta}\right)^{\frac{1}{t}-\frac{1}{p_1}}\cdots\left(\frac{B_m^\theta}{B_0^\theta}\right)^{\frac{1}{t}-\frac{1}{p_m}}\right)$$

in view of (2.1) and

$$\begin{aligned} II &\leq M\mu_1(A_1)^{\frac{1}{p_1}}\cdots\mu_m(A_m)^{\frac{1}{p_m}}\left(\frac{1}{2}\mu_0(A_0)\right)^{\frac{1}{p'}} \\ &= \mu_0(A_0)^{\frac{1}{p'}}\mu_1(A_1)^{\frac{1}{p_1}}\cdots\mu_m(A_m)^{\frac{1}{p_m}}\left(M\,2^{-\frac{1}{p'}}\right) \end{aligned}$$

in view of (1.1). Consequently,

$$M \leq KB_0\left(\frac{B_1^\theta}{B_0^\theta}\right)^{\frac{1}{t}-\frac{1}{p_1}}\cdots\left(\frac{B_m^\theta}{B_0^\theta}\right)^{\frac{1}{t}-\frac{1}{p_m}} + M\,2^{-\frac{1}{p'}}$$

and since $M<\infty$ by assumption (1.1), we have

$$M \leq \frac{K}{1-2^{-\frac{1}{p'}}} B_0^{\theta\left(\frac{1}{t}-\frac{1}{p'}\right)} B_1^{\theta\left(\frac{1}{t}-\frac{1}{p_1}\right)}\cdots B_m^{\theta\left(\frac{1}{t}-\frac{1}{p_m}\right)}.$$

The preceding inequality is an implication of the fact that

$$1-\theta\left(\frac{m}{t}-\frac{1}{p_1}-\cdots-\frac{1}{p_m}\right) = 1-\theta\left(\frac{1}{s}-\frac{1}{p}\right) = \theta\left(\frac{1}{\theta}-\frac{1}{s}+\frac{1}{p}\right) = \theta\left(\frac{1}{t}-\frac{1}{p'}\right).$$

There are m more cases left in each of which $\mu_j(A_j)/B_j^\theta$ is interchanged with $\mu_0(A_0)/B_0^\theta$ in (2.1) for some $j\in\{1,\dots,m\}$. Fixing such a j we recall the assumption that the jth adjoint T^{*j} of T is also of restricted weak type $(t,\dots,t,s)$. Setting $p_0=p'$, we notice that (1.1) can be written as

$$\sup_{A_0,A_1,\dots,A_m}\frac{1}{\mu_j(A_j)^{\frac{1}{(p_j')'}}\mu_0(A_0)^{\frac{1}{p_0}}\prod_{i\neq j}\mu_i(A_i)^{\frac{1}{p_i}}}\left|\int_{A_j}T^{*j}(\chi_{A_1},\dots,\chi_{A_0},\dots,\chi_{A_m})d\mu_j\right|<\infty,$$

in which the $(m{+}1)$-tuple $(p_1,\dots,p_{j-1},p_0,p_{j+1},p_j')$ replaces $(p_1,\dots,p_m,p)$, and the identity $(\frac{1}{p_0}+\sum_{i\neq j}\frac{1}{p_i})-\frac{1}{p_j'}=\frac{m}{t}-\frac{1}{s}$ replaces $\frac{1}{p_1}+\cdots+\frac{1}{p_m}-\frac{1}{p}=\frac{m}{t}-\frac{1}{s}$. The argument in this case follows by an identical repetition of the argument in the preceding case, under this change of notation and concludes the proof in all cases. □

Remark 1. One may wonder whether hypothesis (1.1) weakens the statement of the main theorem. As in [5], it is an essential element of the proof, but in most applications it does not present any significant restriction. In fact, in most cases, one may work with truncated versions of an operator T for which (1.1) holds with constants depending on the truncation. Then boundedness is obtained for truncated operators with bounds independent of the truncation and a limiting argument implies the same conclusion for the original operator T.

Remark 2. It is also worth noting that if (1.1) holds for every point $(p_1,\dots,p_m)$ with $1<p<t'$, $t<p_1,\dots,p_m<\infty$ and $1/p_1+\cdots+1/p_m-1/p=m/t-1/s$, then one obtains restricted weak type estimates at every point in the open convex hull H of these points combined with the point $(\frac{1}{t},\dots,\frac{1}{t},\frac{1}{s})$. Then by the multilinear Marcinkiewicz interpolation theorem (see for instance [4]), it follows that T satisfies strong type bounds in H.

3. An Application

Let $0 < \alpha < n$. Consider the bilinear fractional integral

$$I_\alpha(f,g)(x) = \int_{\mathbf{R}^n} f(x+y)g(x-y)|y|^{\alpha-n}dy, \tag{3.1}$$

defined for positive functions f, g on $\mathbf{R}^n$. It was shown in [3] and [7] that I_α maps the product $L^{p_1}(\mathbf{R}^n)\times L^{p_2}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ whenever $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}+\frac{\alpha}{n}$ and $(\frac{1}{p_1},\frac{1}{p_2},\frac{1}{p})$ lies in the open convex hull of the points $(\frac{n}{\alpha},\infty,\infty)$, $(\infty,\frac{n}{\alpha},\infty)$, $(1,\infty,\frac{n}{n-\alpha})$, $(\infty,1,\frac{n}{n-\alpha})$, and $(1,1,\frac{n}{2n-\alpha})$. The proof is achieved in two steps: (a) restricted weak type estimates are proven at the aforementioned five points first; (b) then multilinear interpolation is used to obtain boundedness in the open convex hull H of these five points.

In this note we provide a simpler proof of the boundedness of I_α in H by reducing it to a restricted weak type estimate at *only the point* $(1,1,\frac{n}{2n-\alpha})$ for I_α and its two adjoints. We will use Theorem 1.1 with $t=1$ and $s=\frac{n}{2n-\alpha}$ for which $\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p}=\frac{2}{t}-\frac{1}{s}$. To satisfy condition (1.1) we introduce the following truncated version of I_α:

$$I_\alpha^{\epsilon,N,M}(f,g)(x) = \chi_{|x|\le M}\int_{\epsilon\le|y|\le N} f(x+y)g(x-y)|y|^{\alpha-n}dy\,.$$

For $1<p<\infty$ it is easy to see that

$$\begin{aligned}\left\|I_\alpha^{\epsilon,N,M}(\chi_{A_1},\chi_{A_2})\right\|_{L^p} &\le C_{\epsilon,N,M}\min(1,|A_1|)^{\frac{1}{p}}\min(1,|A_2|)^{\frac{1}{p}}\min(1,|A_1|,|A_2|)^{\frac{1}{p'}}\\ &\le C_{\epsilon,N,M}|A_1|^{\frac{1}{p_1}}|A_2|^{\frac{1}{p_2}},\end{aligned}$$

and from this (1.1) follows for $I_\alpha^{\epsilon,N,M}$ via Hölder's inequality. Here $p'=\frac{p}{p-1}$ and $0<\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p}<1$. Analogous estimates hold for the two adjoints of $I_\alpha^{\epsilon,N,M}$. For instance

$$\left(I_\alpha^{\epsilon,N,M}\right)^{*1}(h,g)(x) = \int_{\epsilon\le|y|\le N} h(x-y)g(x-2y)|y|^{\alpha-n}\chi_{|x-y|\le M}\,dy\,,$$

which is bounded by

$$\chi_{|x|\le M+N}\int_{\epsilon\le|y|\le N} h(x-y)g(x-2y)\,|y|^{\alpha-n}dy,$$

when $g,h\ge 0$, and thus a similar estimate holds for it. Then Theorem 1.1 and Remark 2 yield boundedness for $I_\alpha^{\epsilon,N,M}$ in H with bounds as in (1.2) i.e., independent of ϵ, N, M. Letting $\epsilon\downarrow 0$ and $N,M\uparrow\infty$ we obtain the same conclusion for I_α, via the Lebesgue monotone theorem.

References

[1] J. Bergh and J. Löfström, *Interpolation Spaces, An Introduction*, Grundlehren der Mathematischen Wissenschaften, No. 223, Springer-Verlag, Berlin–New York, 1976.

[2] L. Grafakos, *Modern Fourier Analysis, Third edition*, Graduate Texts in Math., no 250, Springer, New York, 2014.

[3] L. Grafakos and N. Kalton, *Some remarks on multilinear maps and interpolation*, Math. Ann. **319** (2001), 151–180.

[4] L. Grafakos, L. Liu, S. Lu, and F. Zhao, *The multilinear Marcinkiewicz interpolation theorem revisited: The behavior of the constant* J. Funct. Anal. **262** (2012), 2289–2313.

[5] L. Grafakos and T. Tao, *Multilinear interpolation between adjoint operators*, J. Funct. Anal. **199** (2003), 379–385.

[6] S. Janson, *On interpolation of multilinear operators*, Function Spaces and Applications (Lund, 1986), pp. 290–302, Lect. Notes in Math. 1302, Springer, Berlin–New York, 1988.

[7] C. Kenig and E. M. Stein, *Multilinear estimates and fractional integration*, Math. Res. Lett. **6** (1999), 1–15.

[8] R. Sharpley, *Interpolation of n pairs and counterexamples employing indices*, J. Approx. Theory **13** (1975), 117–127.

[9] R. Sharpley, *Multilinear weak type interpolation of mn-tuples with applications*, Studia Math. **60** (1977), 179–194.

[10] R. Strichartz, *A multilinear version of the Marcinkiewicz interpolation theorem*, Proc. Amer. Math. Soc. **21** (1969), 441–444.

[11] M. Zafran, *A multilinear interpolation theorem*, Studia Math. **62** (1978), 107–124.

[12] A. Zygmund, *Trigonometric Series*, Vol. II, 2nd ed., Cambridge University Press, New York, 1959.

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