Radial Maximal Function Characterizations for Hardy Spaces on RD-spaces

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Abstract An RD-space $\mathcal{X}$ is a space of homogeneous type in the sense of Coifman and Weiss with the additional property that a reverse doubling property holds. The authors prove that for a space of homogeneous type $\mathcal{X}$ having "dimension" n, there exists a $p_0 \in (n/(n+1), 1)$ such that for certain classes of distributions, the $L^p(\mathcal{X})$ quasi-norms of their radial maximal functions and grand maximal functions are equivalent when $p \in (p_0, \infty]$. This result yields a radial maximal function characterization for Hardy spaces on $\mathcal{X}$.

1 Introduction

The theory of Hardy spaces on Euclidean spaces plays an important role in harmonic analysis and partial differential equations and has been systematically studied and developed; see, for example, [22, 7, 21, 8]. It is well known that spaces of homogeneous type, in the sense of Coifman and Weiss [3], are a natural setting of the Calderón-Zygmund theory of singular integrals; see also [4].

A space of homogeneous type is a set $\mathcal{X}$ equipped with a metric d and a regular Borel measure μ having the doubling property. Coifman and Weiss [4] introduced the atomic Hardy space $H^p_{\rm at}(\mathcal{X})$ for $p \in (0, 1]$ and further established a molecular characterization for $H^1_{\rm at}(\mathcal{X})$. Moreover, under the assumption that the measure of any ball in $\mathcal{X}$ is equivalent to its radius (i. e., $\mathcal{X}$ is an Ahlfors 1-regular metric measure space), when $p \in (1/2, 1]$, Macías and Segovia [15] used distributions acting on certain spaces of Lipschitz functions to obtain a grand maximal function characterization for $H^p_{\rm at}(\mathcal{X})$; Han [10] further established a Lusin-area characterization for $H^p_{\rm at}(\mathcal{X})$, and Duong and Yan [5] characterized these atomic Hardy spaces in terms of Lusin-area functions associated with certain Poisson semigroups. Also in this setting, a deep result of Uchiyama [23] states that if $p \in (p_0, 1]$ for some p_0 near 1, for functions in $L^1(\mathcal{X})$, the $L^p(\mathcal{X})$ quasi-norms of the grand maximal functions as in [15] are equivalent to the $L^p(\mathcal{X})$ quasi-norms of the radial maximal functions defined via some kernels in [4].

An important special class of spaces of homogeneous type is called RD-spaces, which is introduced in [11] (see also [12, 16]) and modeled on Euclidean spaces with A_∞-weights

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(Muckenhoupt's class), Ahlfors n-regular metric measure spaces (see, for example, [14]), Lie groups of polynomial growth (see, for example, [1, 24, 25]) and Carnot-Carathéodory spaces with doubling measure (see, for example, [19, 17, 18, 6, 20]). A Littlewood-Paley theory of Hardy spaces on RD-spaces was established in [12], and these Hardy spaces are shown to coincide with some of Triebel-Lizorkin spaces in [11]. The grand, nontangential and dyadic maximal function characterizations of these Hardy spaces have recently been established in [9].

The main purpose of this paper is twofold: first to generalize the results of Uchiyama [23] to the setting of RD-spaces and second to replace the space $L^1(\mathcal{X})$ used by Uchiyama in [23] by certain spaces of distributions developed in [12, 11]. In other words, we build on the work of Uchiyama [23] to establish a radial maximal function characterization for the Hardy spaces in [12].

To state our main results, we need to recall some definitions and notation. We begin with the classical notions of spaces of homogeneous type ([3], [4]) and RD-spaces ([11]).

Definition 1.1 *Let $(\mathcal{X}, d)$ be a metric space with a regular Borel measure μ such that all balls defined by d have finite and positive measures. For any $x \in \mathcal{X}$ and $r > 0$, set $B(x, r) \equiv \{y \in \mathcal{X} : d(x, y) < r\}$.*

(i) The triple $(\mathcal{X}, d, \mu)$ is called a space of homogeneous type if there exists a constant $C_0 \geq 1$ such that for all $x \in \mathcal{X}$ and $r > 0$,

$$\mu(B(x, 2r)) \leq C_0 \mu(B(x, r)) \text{ (doubling property)}. \tag{1.1}$$

(ii) Let $0 < \kappa \leq n$. The triple $(\mathcal{X}, d, \mu)$ is called a (κ, n)-space if there exist constants $0 < C_1 \leq 1$ and $C_2 \geq 1$ such that for all $x \in \mathcal{X}$, $0 < r < \operatorname{diam}(\mathcal{X})/2$ and $1 \leq \lambda < \operatorname{diam}(\mathcal{X})/(2r)$,

$$C_1 \lambda^\kappa \mu(B(x, r)) \leq \mu(B(x, \lambda r)) \leq C_2 \lambda^n \mu(B(x, r)), \tag{1.2}$$

where $\operatorname{diam}(\mathcal{X}) \equiv \sup_{x,\, y \in \mathcal{X}} d(x, y)$.

A space of homogeneous type is called an RD-space, if it is a (κ, n)-space for some $0 < \kappa \leq n$, i. e., some "reverse" doubling condition holds.

Remark 1.2 *(i) A regular Borel measure μ has the property that open sets are measurable and every set is contained in a Borel set with the same measure; see [14].*

(ii) The number n in some sense measures the "dimension" of $\mathcal{X}$. Obviously any (κ, n) space is a space of homogeneous type with $C_0 = C_2 2^n$. Conversely, any space of homogeneous type satisfies the second inequality of (1.2) with $C_2 = C_0$ and $n = \log_2 C_0$.

(iii) If μ is doubling, then μ satisfies (1.2) if and only if there exist constants $a_0 > 1$ and $\widetilde{C}_0 > 1$ such that for all $x \in \mathcal{X}$ and $0 < r < \operatorname{diam}(\mathcal{X})/a_0$,

$$\mu(B(x, a_0 r)) \geq \widetilde{C}_0 \mu(B(x, r)) \quad \text{(reverse doubling property)}$$

(If $a_0 = 2$, this is the classical reverse doubling condition), and equivalently, for all $x \in \mathcal{X}$ and $0 < r < \operatorname{diam}(\mathcal{X})/a_0$,

$$B(x, a_0 r) \setminus B(x, r) \neq \emptyset;$$

see [11]. From this, it follows that if $\mathcal{X}$ is an RD-space, then $\mu(\{x\}) = 0$ for all $x \in \mathcal{X}$.

Throughout the whole paper, we always assume that $\mathcal{X}$ is an RD-space and $\mu(\mathcal{X})=\infty$. For any $x,\ y\in\mathcal{X}$ and $\delta>0$, set $V_\delta(x)\equiv\mu(B(x,\delta))$ and $V(x,y)\equiv\mu(B(x,d(x,y)))$. It follows from (1.1) that $V(x,y)\sim V(y,x)$. The following notion of approximations of the identity on RD-spaces is a variant of that in [11, Definitions 2.1, 2.2]; see also [12].

Definition 1.3 *Let $\epsilon_1\in(0,1]$, $\epsilon_2>0$ and $\epsilon_3>0$. A sequence $\{S_k\}_{k\in\mathbb{Z}}$ of bounded linear integral operators on $L^2(\mathcal{X})$ is said to be a special approximation of the identity of order $(\epsilon_1,\epsilon_2,\epsilon_3)$ (for short, $(\epsilon_1,\epsilon_2,\epsilon_3)$-* SAOTI *), if there exists a constant $C_3>\sqrt{2}$ such that for all $k\in\mathbb{Z}$ and all x, x', y and $y'\in\mathcal{X}$, $S_k(x,y)$, the integral kernel of S_k is a function from $\mathcal{X}\times\mathcal{X}$ into $[0,\infty)$ satisfying*

(i) $S_k(x,y)\le C_3\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)}\frac{2^{-k\epsilon_2}}{(2^{-k}+d(x,y))^{\epsilon_2}}$;

(ii) $|S_k(x,y)-S_k(x',y)|\le C_3\frac{d(x,x')^{\epsilon_1}}{(2^{-k}+d(x,y))^{\epsilon_1}}\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)}\frac{2^{-k\epsilon_2}}{(2^{-k}+d(x,y))^{\epsilon_2}}$ *for* $d(x,x')\le(2^{-k}+d(x,y))/2$;

(iii) Property (ii) holds with x and y interchanged;

(iv) $|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]|\le C_3\frac{d(x,x')^{\epsilon_1}}{(2^{-k}+d(x,y))^{\epsilon_1}}\frac{d(y,y')^{\epsilon_1}}{(2^{-k}+d(x,y))^{\epsilon_1}}$ $\times\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)}\frac{2^{-k\epsilon_3}}{(2^{-k}+d(x,y))^{\epsilon_3}}$ *for* $d(x,x')\le(2^{-k}+d(x,y))/3$ *and* $d(y,y')\le(2^{-k}+d(x,y))/3$;

(v) $C_3V_{2^{-k}}(x)S_k(x,x)>1$ *for all $x\in\mathcal{X}$ and $k\in\mathbb{Z}$;*

(vi) $\int_{\mathcal{X}}S_k(x,y)\,d\mu(y)=1=\int_{\mathcal{X}}S_k(x,y)\,d\mu(x)$.

We remark that (i) and (v) of Definition 1.3 imply that $C_3>\sqrt{2}$. The existence of $(\epsilon_1,\epsilon_2,\epsilon_3)$- SAOTI 's was proved in [11, Theorem 2.1].

The following spaces of test functions play a key role in the theory of function spaces on RD-spaces; see [12, 11].

Definition 1.4 *Let $x_1\in\mathcal{X}$, $r\in(0,\infty)$, $\beta\in(0,1]$ and $\gamma\in(0,\infty)$. A function φ on $\mathcal{X}$ is said to be a test function of type (x_1,r,β,γ) if there exists a nonnegative constant C such that*

(i) $|\varphi(x)|\le C\frac{1}{V_r(x_1)+V(x_1,x)}\left(\frac{r}{r+d(x_1,x)}\right)^\gamma$ *for all $x\in\mathcal{X}$;*

(ii) $|\varphi(x)-\varphi(y)|\le C\left(\frac{d(x,y)}{r+d(x_1,x)}\right)^\beta\frac{1}{V_r(x_1)+V(x_1,x)}\left(\frac{r}{r+d(x_1,x)}\right)^\gamma$ *for all $x,\ y\in\mathcal{X}$ satisfying* $d(x,y)\le(r+d(x_1,x))/2$.

We denote by $\mathcal{G}(x_1,r,\beta,\gamma)$ the set of all test functions of type (x_1,r,β,γ). If $\varphi\in\mathcal{G}(x_1,r,\beta,\gamma)$, we define its norm by $\|\varphi\|_{\mathcal{G}(x_1,r,\beta,\gamma)}\equiv\inf\{C:\ (i)\ and\ (ii)\ hold\}$. The space $\mathcal{G}(x_1,r,\beta,\gamma)$ is called the space of test functions.

Throughout the whole paper, we fix $x_1\in\mathcal{X}$. Let $\mathcal{G}(\beta,\gamma)\equiv\mathcal{G}(x_1,1,\beta,\gamma)$. It is easy to see that for any $x_2\in\mathcal{X}$ and $r>0$, we have $\mathcal{G}(x_2,r,\beta,\gamma)=\mathcal{G}(\beta,\gamma)$ with equivalent norms (but with constants depending on x_1, x_2 and r). Moreover, $\mathcal{G}(\beta,\gamma)$ is a Banach space.

For any given $\epsilon\in(0,1]$, let $\mathcal{G}_0^\epsilon(\beta,\gamma)$ be the completion of the space $\mathcal{G}(\epsilon,\epsilon)$ in $\mathcal{G}(\beta,\gamma)$ when $\beta,\ \gamma\in(0,\epsilon]$. Obviously $\mathcal{G}_0^\epsilon(\epsilon,\epsilon)=\mathcal{G}(\epsilon,\epsilon)$. Moreover, $\varphi\in\mathcal{G}_0^\epsilon(\beta,\gamma)$ if and only if $\varphi\in\mathcal{G}(\beta,\gamma)$ and there exists $\{\phi_i\}_{i\in\mathbb{N}}\subset\mathcal{G}(\epsilon,\epsilon)$ such that $\|\varphi-\phi_i\|_{\mathcal{G}(\beta,\gamma)}\to0$ as $i\to\infty$. If $\varphi\in\mathcal{G}_0^\epsilon(\beta,\gamma)$, define $\|\varphi\|_{\mathcal{G}_0^\epsilon(\beta,\gamma)}\equiv\|\varphi\|_{\mathcal{G}(\beta,\gamma)}$. Obviously $\mathcal{G}_0^\epsilon(\beta,\gamma)$ is a Banach space and

$\|\varphi\|_{\mathcal{G}_0^\epsilon(\beta,\gamma)} = \lim_{i\to\infty} \|\phi_i\|_{\mathcal{G}(\beta,\gamma)}$ for the above chosen $\{\phi_i\}_{i\in\mathbb{N}}$. It is known that $\mathcal{G}_0^\epsilon(\beta,\gamma)$ is dense in $L^p(\mathcal{X})$ for $p\in[1,\infty)$; see [11, Corollary 2.1]. Let $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ be the set of all bounded linear functionals f from $\mathcal{G}_0^\epsilon(\beta,\gamma)$ to $\mathbb{C}$. Denote by $\langle f,\varphi\rangle$ the natural pairing of elements $f\in(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and $\varphi\in\mathcal{G}_0^\epsilon(\beta,\gamma)$.

Let $\epsilon\in(0,1)$, $\beta,\,\gamma\in(0,\epsilon)$ and $p\in(0,\infty]$. If $f\in(\mathcal{G}_0^\epsilon(\beta,\gamma))'$, then for all $x\in\mathcal{X}$, we define the grand maximal function of f to be

$$f^*(x)\equiv\sup\left\{|\langle f,\varphi\rangle|:\,\varphi\in\mathcal{G}(\epsilon,\epsilon),\,\|\varphi\|_{\mathcal{G}(x,r,\epsilon,\epsilon)}\le 1 \text{ for some } r>0\right\}.$$

Define the corresponding Hardy space by

$$H^{*,\,p}(\mathcal{X})\equiv\left\{f\in(\mathcal{G}_0^\epsilon(\beta,\gamma))':\,\|f^*\|_{L^p(\mathcal{X})}<\infty\right\}.$$

For any $f\in H^{*,\,p}(\mathcal{X})$, set $\|f\|_{H^{*,\,p}(\mathcal{X})}\equiv\|f^*\|_{L^p(\mathcal{X})}$. Let $\{Q_\alpha^k:\,k\in\mathbb{Z},\,\alpha\in I_k\}$ be the Christ dyadic cube collection of $\mathcal{X}$, where I_k is some index set; see [2]. For any $f\in(\mathcal{G}_0^\epsilon(\beta,\gamma))'$, we define the dyadic maximal function $\mathcal{M}_d(f)$ of f by setting, for all $x\in\mathcal{X}$,

$$\mathcal{M}_d(f)(x)\equiv\sup_{k\in\mathbb{Z},\,\alpha\in I_k}\left\{\frac{1}{\mu(Q_\alpha^k)}\int_{Q_\alpha^k}|S_k(f)(y)|\,d\mu(y)\right\}\chi_{Q_\alpha^k}(x),$$

and define $H_d^p(\mathcal{X})$ to be the corresponding Hardy space; see [9, Definition 2.9]. When $p\in(1,\infty]$, it was proved in [9, Corollary 3.12] that $H^{*,\,p}(\mathcal{X})=H_d^p(\mathcal{X})=L^p(\mathcal{X})$ with equivalent norms.

Definition 1.5 *Let $\epsilon_1\in(0,1]$, $\epsilon_2>0$, $\epsilon_3>0$, $\epsilon\in(0,\epsilon_1\wedge\epsilon_2)$ and $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1,\epsilon_2,\epsilon_3)$-* SAOTI*. Let $p\in(0,\infty]$ and $f\in(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ with $\beta,\,\gamma\in(0,\epsilon)$. Define the radial maximal function of f to be $\mathcal{M}_0(f)(x)\equiv\sup_{k\in\mathbb{Z}}|S_k(f)(x)|$ for all $x\in\mathcal{X}$. The corresponding Hardy spaces are defined by*

$$H_0^p(\mathcal{X})\equiv\left\{f\in(\mathcal{G}_0^\epsilon(\beta,\gamma))':\,\|\mathcal{M}_0(f)\|_{L^p(\mathcal{X})}<\infty\right\},$$

and moreover, we define $\|f\|_{H_0^p(\mathcal{X})}\equiv\|\mathcal{M}_0(f)\|_{L^p(\mathcal{X})}$.

The properties (i) and (ii) in Definition 1.3 imply that $\mathcal{M}_0(f)(x)\lesssim f^*(x)$ for all $x\in\mathcal{X}$. In what follows, for simplicity of presentation, for any $t>0$, we use the notation

$$S_t(x,y)\equiv\sum_{k\in\mathbb{Z}}S_k(x,y)\chi_{(2^{-k-1},\,2^{-k}]}(t). \tag{1.3}$$

By (1.3) and Definition 1.5, it is easy to see that for all $x\in\mathcal{X}$,

$$\mathcal{M}_0(f)(x)=\sup_{t>0}|S_t(f)(x)|. \tag{1.4}$$

Obviously, S_t satisfies (i) through (vi) in Definition 1.3 with 2^{-k} replaced by t. From (iv) and (v) in Definition 1.3, it follows easily that there exist constants $C_4\in(0,(C_3)^{-2/\epsilon_1})$ and $C_5>1$ such that for all $t>0$ and all $x,\,y\in\mathcal{X}$ satisfying $d(x,y)<C_4t$,

$$C_5V_t(x)S_t(x,y)>1. \tag{1.5}$$

This observation is used in applications below.

Denote by $\mathcal{M}$ the centered Hardy-Littlewood maximal operator. To be precise, for any $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and $x \in \mathcal{X}$, set

$$\mathcal{M}(f)(x) \equiv \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\, d\mu(y).$$

Then $\mathcal{M}$ is weak-type $(1,1)$ and bounded on $L^p(\mathcal{X})$ for $p \in (1,\infty]$ in [3, 4]. It is not so difficult to show that for all $x \in \mathcal{X}$, $\mathcal{M}_0(f)(x) \lesssim \mathcal{M}(f)(x)$ and $\mathcal{M}_d(f)(x) \lesssim \mathcal{M}(\mathcal{M}_0(f))(x)$ by their definitions and Lemma 2.1 (iv) below. Therefore, we have $H^p_0(\mathcal{X}) = L^p(\mathcal{X})$ with equivalent norms when $p \in (1,\infty]$.

The main result of this paper concerns the spaces $H^p_0(\mathcal{X})$ and $H^{*,p}(\mathcal{X})$, and is as follows.

Theorem 1.6 *Let $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$, $\epsilon_3 > 0$ and $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$. Let $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1,\epsilon_2,\epsilon_3)$-SAOTI and $\mathcal{M}_0$ be as in (1.4). Then there exist $\sigma \in (0,1/2)$ and $\eta \in (0,(1-\sigma)^{1/\epsilon} \wedge (1/2))$, both depending only on $\mathcal{X}$ and ϵ, such that for any given $p \in (n/(n+\log_\eta(1-\sigma)),\infty]$ and all $f \in (\mathcal{G}^\epsilon_0(\beta,\gamma))'$ with $\beta \in (0,\log_\eta(1-\sigma))$ and $\gamma \in (0,\epsilon)$,*

$$\|f^*\|_{L^p(\mathcal{X})} \le C\|\mathcal{M}_0(f)\|_{L^p(\mathcal{X})},$$

where C is a positive constant independent of f.

Theorem 1.6 will be a consequence of the following key proposition.

Proposition 1.7 *With the notation of Theorem 1.6, for any $\delta_0 \in (0,\log_\eta(1-\sigma))$, there exists a positive constant C, depending on $\mathcal{X}$, ϵ and δ_0, such that for all $f \in (\mathcal{G}^\epsilon_0(\beta,\gamma))'$ with $\beta \in (0,\log_\eta(1-\sigma))$ and $\gamma \in (0,\epsilon)$, and all $\varphi \in \mathcal{G}(x_0,r_0,\epsilon,\epsilon)$ satisfying $\|\varphi\|_{\mathcal{G}(x_0,r_0,\epsilon,\epsilon)} \le 1$ for some $x_0 \in \mathcal{X}$ and $r_0 > 0$,*

$$|\langle f,\varphi\rangle| \le C\left[\mathcal{M}([\mathcal{M}_0(f)]^{n/(n+\delta_0)})(x_0)\right]^{(n+\delta_0)/n}.$$

We remark that in Theorem 1.6 and Proposition 1.7, it is not necessary to assume that $\{S_k\}_{k\in\mathbb{Z}}$ has the property (vi) of Definition 1.3. Moreover, Theorem 1.6 follows easily from Proposition 1.7; see Section 3 below. The main ingredient in the proof of Proposition 1.7 is to expand φ as in Proposition 1.7 into a sum of certain S_t as in (1.3); see (3.11) below. When $\mathcal{X}$ is an Ahlfors 1-regular metric measure space, for any Lipschitz function with bounded support, Uchiyama in [23] established an expansion similar to (3.11), which holds pointwise. Unlike [23], we prove that for any φ as in Proposition 1.7, (3.11) also holds in $\mathcal{G}^\epsilon_0(\beta,\gamma)$ with β and γ as in Proposition 1.7. This allows us to relax the assumption $f \in L^1(\mathcal{X})$ to $f \in (\mathcal{G}^\epsilon_0(\beta,\gamma))'$.

From the fact $\mathcal{M}_0(f) \lesssim f^*$ and Theorem 1.6, it follows that for p in a certain range of $(0,1]$, $H^p_0(\mathcal{X})$ coincides with $H^{*,p}(\mathcal{X})$ as a subspace of certain distribution spaces $(\mathcal{G}^\epsilon_0(\beta,\gamma))'$. Recall that when $p \in (n/(n+1),1]$, [9, Remark 3.16] and [9, Corollary 4.19] tell us that the definition of $H^{*,p}(\mathcal{X})$ is independent of the choices of $(\mathcal{G}^\epsilon_0(\beta,\gamma))'$ with β, $\gamma \in (n(1/p-1),\epsilon)$. Therefore, we deduce the following conclusion.

Corollary 1.8 *Let ϵ_1, ϵ_2, ϵ_3, σ and η be as in Theorem 1.6 and $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$. Let $p_0 \equiv n/(n + \log_\eta(1-\sigma))$ and $p \in (p_0, 1]$. Then $H_0^p(\mathcal{X}) = H^{*,p}(\mathcal{X})$ with equivalent quasi-norms, where $H_0^p(\mathcal{X})$ and $H^{*,p}(\mathcal{X})$ are defined via $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ with some $\beta \in (n(1/p-1), n(1/p_0 - 1))$ and $\gamma \in (n(1/p-1), \epsilon)$. Consequently, the definition of $H_0^p(\mathcal{X})$ is independent of the choices of $(\epsilon_1, \epsilon_2, \epsilon_3)$-SAOTI and $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ with β and γ as above.*

Remark 1.9 *We point out that in Theorem 1.6 and Proposition 1.7, it is not necessary to assume that $\mathcal{X}$ satisfies the reverse doubling condition determined by the first inequality of (1.2). However, the assertion $H_0^p(\mathcal{X}) = L^p(\mathcal{X})$ when $p \in (1,\infty]$ and Corollary 1.8 do need this assumption, since, to obtain these conclusions, we need to use the Calderón reproducing formulae in [11], which depend on the reverse doubling condition.*

The organization of this paper is as follows. In Section 2, we give some technical lemmas which will be used in the proof of Proposition 1.7. Section 3 is the main part of this paper, which contains a proof of Proposition 1.7 and also of Theorem 1.6.

In this paper we use the following notation: $\mathbb{N} \equiv \{1, 2, \cdots\}$, $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$ and $\mathbb{R}_+ \equiv [0,\infty)$. For any $p \in [1,\infty]$, we denote by p' the conjugate index, namely, $1/p + 1/p' = 1$. We also denote by C positive constant independent of main parameters involved, which may vary at different occurrences. Constants with subscripts do not change through the whole paper. We use $f \lesssim g$ and $f \gtrsim g$ to denote $f \le Cg$ and $f \ge Cg$, respectively. If $f \lesssim g \lesssim f$, we then write $f \sim g$. For any a, $b \in \mathbb{R}$, set $a \wedge b \equiv \min\{a, b\}$ and $a \vee b \equiv \max\{a, b\}$. For any set E, we denote by $\sharp E$ the cardinality of E.

2 Some technical lemmas

In this section, we establish several technical lemmas which will be used in the proof of Proposition 1.7. The following lemma includes some basic properties on RD-spaces, which are used throughout the paper; see, for example, [12, 11, 9].

Lemma 2.1 *Let $\delta > 0$, $a > 0$, $r > 0$ and $\theta \in (0,1)$. Then,*

(i) For all x, $y \in \mathcal{X}$ and $r > 0$, $V_r(x) + V(x,y) \sim V_r(y) + V(y,x) \sim \mu(B(y, r + d(y,x))) \sim \mu(B(x, r + d(x,y)))$.

(ii) If x, x', $x_1 \in \mathcal{X}$ satisfy $d(x,x') \le \theta(r + d(x,x_1))$, then $r + d(x,x_1) \sim r + d(x',x_1)$ and $\mu(B(x, r + d(x,x_1))) \sim \mu(B(x', r + d(x',x_1)))$.

(iii) $\int_{\mathcal{X}} \frac{1}{V_r(x)+V(x,y)} (\frac{r}{r+d(x,y)})^a d(x,y)^\eta \, d\mu(x) < Cr^\eta$ uniformly in $x \in \mathcal{X}$ and $r > 0$ if $a > \eta \ge 0$.

(iv) For all $f \in L^1_{\rm loc}(\mathcal{X})$ and $x \in \mathcal{X}$, $\int_{d(x,y)>\delta} \frac{1}{V(x,y)} \frac{\delta^a}{d(x,y)^a} |f(y)| \, d\mu(y) \le C\mathcal{M}(f)(x)$ uniformly in $\delta > 0$, $f \in L^1_{\rm loc}(\mathcal{X})$ and $x \in \mathcal{X}$.

When $\delta = 0$, the following lemma provides a property of Carleson measures on RD-spaces; see [11, Proposition 5.14].

Lemma 2.2 *Let $p \in (1,\infty]$ and $\delta \ge 0$. Let ν be a non-negative measure on $\mathcal{X} \times \mathbb{R}_+$ such that for all $x \in \mathcal{X}$ and $r > 0$,*

$$\nu(B(x,r) \times (0,r)) \le [\mu(B(x,r))]^{1+\delta}. \tag{2.1}$$

Then there exists a positive constant C such that for all $f \in L^p(\mathcal{X})$,

$$\left\{\int_{\mathcal{X}\times\mathbb{R}_+} |F(r,y,f)|^{p(1+\delta)}\,d\nu(y,r)\right\}^{1/(p(1+\delta))} \le C\|f\|_{L^p(\mathcal{X})},$$

where and in what follows, $F(r,x,f)\equiv S_r(f)(x)$ for all $r>0$ and $x\in\mathcal{X}$.

Proof. Fix $\lambda>0$ and let $W_\lambda\equiv\{(x,r)\in\mathcal{X}\times\mathbb{R}_+ : |F(r,x,f)|>\lambda\}$. For any $\ell\in\mathbb{Z}$, set

$$W_{\ell,\lambda}\equiv\left\{x\in\mathcal{X} : \sup_{2^{\ell-1}<r\le 2^\ell}|F(r,x,f)|>\lambda\right\}.$$

For each $N\in\mathbb{N}$, let $E_N\equiv\{x\in\mathcal{X} : \sup_{r>2^N}|F(r,x,f)|>\lambda\}$. It is easy to deduce that

$$\lim_{N\to\infty}E_N=\emptyset. \tag{2.2}$$

To prove (2.2), notice that $\lim_{N\to\infty}E_N=\cap_{N\in\mathbb{N}}E_N$ since $E_{N+1}\subset E_N$ for any $N\in\mathbb{N}$. Suppose that (2.2) fails, that is, there exists an $x\in\cap_{N\in\mathbb{N}}E_N$. Thus for any $N\in\mathbb{N}$, there exists $r_{_N}>2^N$ satisfying that $|F(r_{_N},x,f)|>\lambda$. By this, (1.3), Hölder's inequality and Lemma 2.1 (iii), we obtain

$$\lambda<|F(r_{_N},x,f)|=\left|\int_{\mathcal{X}}S_{r_{_N}}(x,y)f(y)\,d\mu(y)\right|\lesssim\frac{1}{[V_{2^N}(x)]^{1/p}}\left\{\int_{\mathcal{X}}|f(y)|^p\,d\mu(y)\right\}^{1/p},$$

which implies that $V_{2^N}(x)\lesssim\lambda^{-p}\|f\|^p_{L^p(\mathcal{X})}<\infty$ for all $N\in\mathbb{N}$, and hence $\mu(\mathcal{X})<\infty$. This contradicts the assumption $\mu(\mathcal{X})=\infty$. Thus, (2.2) holds.

It is not so difficult to prove that for any given $N\in\mathbb{N}$, there exist $L_N<N$ with $L_N\in\mathbb{Z}$ or $L_N=-\infty$, a set of indices $I_{N,\ell}$ with $\ell\in\{L_N+1,\cdots,N\}$, and disjoint balls $\{B(y^N_{\ell,j},2^\ell)\}_{L_N<\ell\le N,\,j\in I_{N,\ell}}$ satisfying

(i) $y^N_{\ell,j}\in W_{\ell,\lambda}$;

(ii) $B(y^N_{\ell,j},2^\ell)\cap(\cup_{m=\ell+1}^N\cup_{i\in I_{N,m}}B(y^N_{m,i},2^m))=\emptyset$;

(iii) for any $x\in W_{\ell,\lambda}$, $B(x,2^\ell)\cap(\cup_{m=\ell}^N\cup_{i\in I_{N,m}}B(y^N_{m,i},2^m))\ne\emptyset$.

In fact, we start with $\ell=N$ and choose an arbitrary point in $W_{N,\lambda}$ as $y^N_{N,1}$. Then we find a point $y^N_{N,2}\in W_{N,\lambda}\setminus B(y^N_{N,1},2^N)$ such that $B(y^N_{N,2},2^N)\cap B(y^N_{N,1},2^N)=\emptyset$. Continuing in this way, by Zorn's lemma and the doubling property of the measure μ, we arrive at $I_{N,N}=\mathbb{N}$ or $I_{N,N}$ will be a finite set. We then consider $\ell=N-1$. In this way, one finds the desired balls.

From (i), (ii) and (iii), it follows that for each $N\in\mathbb{N}$,

$$W_\lambda\subset\left(\bigcup_{\ell=L_N+1}^N\bigcup_{j\in I_{N,\ell}}\left[B(y^N_{\ell,j},2^{\ell+1})\times(0,2^\ell)\right]\right)\bigcup\left(E_N\times(2^N,\infty)\right). \tag{2.3}$$

To see (2.3), notice that for any $(x,r)\in W_\lambda$, $|F(r,x,f)|>\lambda$. If $r>2^N$, then $(x,r)\in E_N\times(2^N,\infty)$; otherwise there exists $\ell\le N$ such that $2^{\ell-1}<r\le 2^\ell$, which implies that

$x \in W_{\ell,\lambda}$. By Property (iii) above, there exist integers $\ell \le m \le N$ and $j \in I_{N,m}$ such that $B(x,2^\ell) \cap B(y^N_{m,j}, 2^m) \neq \emptyset$. Noticing that $\ell \le m$, we have $x \in B(y^N_{m,j}, 2^{m+1})$ and thus $(x,r) \in B(y^N_{m,j}, 2^{m+1}) \times (0, 2^m)$, which yields (2.3) then.

For any $N \in \mathbb{N}$, by (2.3),

$$\nu(W_\lambda) \le \sum_{\ell=L_N+1}^{N} \sum_{j \in I_{N.\ell}} \nu(B(y^N_{\ell,j}, 2^{\ell+1}) \times (0, 2^\ell)) + \nu(E_N \times (2^N, \infty)).$$

Letting $N \to \infty$ in the formula above, then using (2.1) and (2.2), we obtain

$$\begin{aligned} \nu(W_\lambda) &\le \lim_{N\to\infty} \sum_{\ell=L_N+1}^{N} \sum_{j \in I_{N.\ell}} \left[\mu(B(y^N_{\ell,j}, 2^{\ell+1}))\right]^{1+\delta} \\ &\le \lim_{N\to\infty} \left\{ \sum_{\ell=L_N+1}^{N} \sum_{j \in I_{N.\ell}} \mu(B(y^N_{\ell,j}, 2^{\ell+1})) \right\}^{1+\delta}, \end{aligned} \tag{2.4}$$

where in the second step we use the fact $(\sum_{j\in\mathbb{N}} |a_j|)^\kappa \le \sum_{j\in\mathbb{N}} |a_j|^\kappa$ for any $\kappa \in (0,1]$. Choose $\widetilde{p} \in (1,p)$. For any N, $\ell \in (L_N, N]$ and any given $j \in I_{N,\ell}$, by Property (i), the size condition of $S_{-\ell}$, (1.2) and Hölder's inequality, we have

$$\begin{aligned} \lambda &< \sup_{2^{\ell-1}<r\le 2^\ell} |F(r, y^N_{\ell,j}, f)| \\ &\lesssim \frac{1}{V_{2^\ell}(y^N_{\ell,j})} \int_{B(y^N_{\ell,j}, 2^\ell)} |f(z)|\, d\mu(z) \\ &\quad + \sum_{k=1}^{\infty} \int_{2^{\ell+k-1} \le d(y^N_{\ell,j}, z) < 2^{\ell+k}} \frac{1}{V_{2^\ell}(y^N_{\ell,j}) + V(y^N_{\ell,j}, z)} \left(\frac{2^\ell}{2^\ell + d(y^N_{\ell,j}, z)} \right)^{\epsilon_2} |f(z)|\, d\mu(z) \\ &\lesssim \sum_{k=0}^{\infty} 2^{-k\epsilon_2} \left\{ \frac{1}{V_{2^{\ell+k}}(y^N_{\ell,j})} \int_{B(y^N_{\ell,j}, 2^{\ell+k})} |f(z)|^{p/\widetilde{p}}\, d\mu(z) \right\}^{\widetilde{p}/p} \\ &\lesssim \inf \left\{ \left[\mathcal{M}\left(|f|^{p/\widetilde{p}}\right)(z^N_{\ell,j}) \right]^{\widetilde{p}/p} : z^N_{\ell,j} \in B(y^N_{\ell,j}, 2^\ell) \right\}, \end{aligned}$$

which together with the pairwise disjointness of the balls $\{B(y^N_{\ell,j}, 2^\ell)\}_{L_N<\ell\le N,\, j\in I_{N,\ell}}$ and $L^{\widetilde{p}}(\mathcal{X})$-boundedness of $\mathcal{M}$ yields that for all $N \in \mathbb{N}$,

$$\sum_{\ell=L_N+1}^{N} \sum_{j \in I_{N.\ell}} \mu(B(y^N_{\ell,j}, 2^{\ell+1}))\lambda^p \lesssim \int_{\mathcal{X}} \left[\mathcal{M}\left(|f|^{p/\widetilde{p}}\right)(z) \right]^{\widetilde{p}} d\mu(z) \lesssim \|f\|^p_{L^p(\mathcal{X})}.$$

Combining this with (2.4) shows that $\lambda^{p(1+\delta)} \nu(W_\lambda) \lesssim \|f\|^{p(1+\delta)}_{L^p(\mathcal{X})}$. Then the desired conclusion follows from the Marcinkiewicz interpolation theorem, which completes the proof of Lemma 2.2.

Lemma 2.3 *Let $x_0 \in \mathcal{X}$, $r_0 > 0$ and g be a non-negative function on $\mathcal{X}$. Then for any $t \in (0,1]$, there exist $\{x_j\}_j \subset \mathcal{X}$ with $x_j \equiv x_j(g,t,x_0,r_0)$ and positive constants C_6 and C_7 depending only on $\mathcal{X}$ such that for all $x \in \mathcal{X}$,*

$$1 \le \sum_j \chi_{B(x_j,\,C_4tr_j)}(x) \le C_6 \tag{2.5}$$

and

$$g(x_j)^{1/2} \le C_7 F(tr_j, x_j, g^{1/2}\chi_{B(x_j,\,C_4tr_j)}), \tag{2.6}$$

where $r_j \equiv r_0 + d(x_j,x_0)$ and F is as in Lemma 2.2. In particular, there exists a constant $C_8 > 1$ such that for all j and all $x \in B(x_j, C_4tr_j)$,

$$\begin{aligned} &\frac{2^a C_8}{V_{r_j}(x_j)}\left(\frac{r_0}{r_j}\right)^a V_{tr_j}(x_j)S_{tr_j}(x_j,x)\chi_{B(x_j,\,C_4tr_j)}(x) \\ &\quad \ge \frac{1}{V_{r_0}(x_0)+V(x_0,x)}\left(\frac{r_0}{r_0+d(x_0,x)}\right)^a, \end{aligned} \tag{2.7}$$

where S_{tr_j} is as in (1.3). Moreover, C_6 through C_8 are independent of x_0, r_0 and g.

Proof. For any $x \in \mathcal{X}$, set $r_x \equiv r_0 + d(x_0,x)$. By Zorn's lemma, there exists a set of points $\{y_j\}_j \subset \mathcal{X}$ satisfying that $y_j \notin \cup_{i=1}^{j-1} B(y_i, C_4tr_{y_i}/4)$ and $\mathcal{X} = \cup_j B(y_j, C_4tr_{y_j}/4)$. From the selection of $\{y_j\}_j$, it follows that for any $i \neq j$,

$$d(y_i,y_j) \ge \frac{1}{4}C_4 t \min\{r_{y_i}, r_{y_j}\}, \tag{2.8}$$

and that for all $x \in \mathcal{X}$,

$$\sum_j \chi_{B(y_j,C_4tr_{y_j}/4)}(x) \ge 1. \tag{2.9}$$

By (1.2) and the disjointness of $\{B(y_i, C_4tr_0/8)\}_i$, we know that $\{y_j\}_j$ is at most countable. For every y_j, choose $x_j \equiv x_j(g,t,x_0,r_0)$ satisfying that

$$d(x_j,y_j) < C_4tr_{y_j}/4 \tag{2.10}$$

and

$$g(x_j)^{1/2} \le \frac{1}{\mu(B(y_j, C_4tr_{y_j}/4))}\int_{B(y_j,\,C_4tr_{y_j}/4)} g(z)^{1/2}\,d\mu(z). \tag{2.11}$$

Let $r_j \equiv r_{x_j}$. By (2.10) and the triangle inequality for d together with $C_4 < 1/2$ and $t \in (0,1]$, we have

$$r_{y_j}/2 \le r_j \le 2r_{y_j}, \tag{2.12}$$

which together with (2.9) implies the left-hand side inequality of (2.5).

For any $x \in \mathcal{X}$, set $J(x) \equiv \{j : d(x_j,x) < C_4tr_j\}$. Notice that for any $j \in J(x)$, by (2.12), $r_j/2 \le r_x \le 2r_j$. This together with (2.10) and (2.12) yields that $J(x) \subset \widetilde{J}(x)$, where $\widetilde{J}(x) \equiv \{j : d(y_j,x) < 3C_4tr_x\}$. It follows from (1.2) and the pairwise disjointness of $\{B(y_i, C_4tr_0/8)\}_i$ that $\sharp\widetilde{J}(x) < \infty$. Thus, we may assume that $r_{y_1} = \min\{r_{y_j} : j \in J(x)\}$.

Therefore, by (2.8), $\{B(y_j, C_4tr_{y_1}/8)\}_{j\in J(x)}$ are mutually disjoint. Furthermore, for any $j \in J(x)$, by (2.12) and $r_j/2 \le r_x \le 2r_j$, we have

$$B(y_j, C_4tr_x/32) \subset B(y_j, C_4tr_{y_1}/8) \subset B(x, 4C_4tr_x) \subset B(y_j, 7C_4tr_x).$$

From this and (1.2), it follows that $\sharp J(x)$ is bounded by a positive constant which depends only on $\mathcal{X}$. This implies the validity of (2.5).

The fact that $B(y_j, C_4tr_{y_j}/4) \subset B(x_j, C_4tr_j)$ together with (2.11) and (1.5) implies (2.6). Since for any $x \in B(x_j, C_4tr_j)$, we have $r_j/2 \le r_x \le 2r_j$. Using this fact, (1.5), (2.5) and Lemma 2.1 (i), we obtain (2.7), which completes the proof of Lemma 2.3.

Lemma 2.4 *Let $t \in (0,1]$, $a \in [0,\infty)$, $b \in (a,\infty)$, $M \in [0,\infty)$ and $\{x_j\}_j \subset \mathcal{X}$ satisfying*

$$\sum_j \chi_{B(x_j, C_4tr_j)}(x) \le C_6, \tag{2.13}$$

where $r_j \equiv r_0 + d(x_j, x_0)$ with $r_0 > 0$ and $x_0 \in \mathcal{X}$ and C_6 is as in (2.5). For any j and $x \in \mathcal{X}$, set

$$\begin{aligned} u_j(x) \equiv & \frac{1}{V_{r_j}(x_j)} \left(\frac{1}{r_j}\right)^a \frac{V_{C_4tr_j}(x_j)}{V_{tr_j}(x_j) + V_{tr_j}(x) + V(x_j,x)} \\ & \times \left(\frac{tr_j}{tr_j + d(x_j,x)}\right)^b \chi_{[M,\infty)}\left(\frac{d(x_j,x)}{tr_j}\right), \end{aligned}$$

where $\chi_{[M,\infty)}$ is the characteristic function of the interval $[M,\infty)$. Then there exists $C_9 > 1$ independent of x_0, r_0 and M such that for all $x \in \mathcal{X}$,

$$\sum_j u_j(x) \le C_9 \max\{t^b, (1+M)^{-b}\} \frac{1}{V_{r_0}(x_0) + V(x_0,x)} \left(\frac{1}{r_0 + d(x_0,x)}\right)^a.$$

Proof. For any $k \in \mathbb{Z}$, set $J(k) \equiv \{j : 2^{k-1} \le r_j < 2^k\}$ and $v_k \equiv \sum_{j\in J(k)} u_j$. For any fixed $x \in \mathcal{X}$, let

$$\mathrm{W}_1 \equiv \{k \in \mathbb{Z} : (r_0 + d(x_0,x))/2 \le 2^k < 4(r_0 + d(x_0,x))\},$$

$$\mathrm{W}_2 \equiv \{k \in \mathbb{Z} : 2^k < (r_0 + d(x_0,x))/2\},$$

and

$$\mathrm{W}_3 \equiv \{k \in \mathbb{Z} : 2^k \ge 4(r_0 + d(x_0,x))\}.$$

We then write,

$$\sum_j u_j(x) = \sum_{k\in \mathrm{W}_1} \sum_{j\in J(k)} u_j(x) + \sum_{k\in \mathrm{W}_2} \sum_{j\in J(k)} u_j(x) + \sum_{k\in \mathrm{W}_3} \sum_{j\in J(k)} u_j(x) \equiv \mathrm{Z}_1 + \mathrm{Z}_2 + \mathrm{Z}_3.$$

To estimate Z_1, notice that for any $k \in \mathrm{W}_1$ and $j \in J(k)$, we have

$$\mu(B(x_j, r_0 + d(x_j,x_0))) \sim \mu(B(x_0, r_j)) \sim \mu(B(x_0, 2^k)) \sim V_{r_0}(x_0) + V(x_0,x). \tag{2.14}$$

For any $j \in J(k)$ and $z \in B(x_j, C_4 t r_j)$, we have $d(z, x_j) < C_4 t r_j < C_4 t 2^k$, which together with Lemma 2.1 further implies that for all $z \in B(x_j, C_4 t r_j)$,

$$t2^k + d(x_j, x) \sim t2^k + d(z, x), \quad V_{t2^k}(x_j) + V(x_j, x) \sim V_{t2^k}(z) + V(z, x) \tag{2.15}$$

and

$$\chi_{[M,\infty)}\left(\frac{d(x_j, x)}{tr_j}\right) \le \chi_{[M,\infty)}\left(\frac{d(x_j, x)}{t2^{k-1}}\right) \le \chi_{[M-2C_4,\infty)}\left(\frac{d(z, x)}{t2^{k-1}}\right). \tag{2.16}$$

From (2.13) through (2.16), it follows that

$$\begin{aligned} v_k(x) &\lesssim \frac{1}{2^{a(k-1)}} \frac{1}{V_{r_0}(x_0) + V(x_0, x)} \\ &\quad \times \sum_{j \in J(k)} \frac{V_{C_4 t r_j}(x_j)}{V_{t2^k}(x_j) + V_{t2^k}(x) + V(x_j, x)} \left(\frac{t2^k}{t2^k + d(x_j, x)}\right)^b \chi_{[M,\infty)}\left(\frac{d(x_j, x)}{t2^{k-1}}\right) \\ &\lesssim \frac{1}{2^{ak}} \frac{1}{V_{r_0}(x_0) + V(x_0, x)} \\ &\quad \times \int_{\mathcal{X}} \frac{1}{V_{t2^k}(z) + V_{t2^k}(x) + V(z, x)} \left(\frac{t2^k}{t2^k + d(z, x)}\right)^b \chi_{[M-2C_4,\infty)}\left(\frac{d(z, x)}{t2^{k-1}}\right) d\mu(z). \end{aligned}$$

Denote by J the integral in the last formula. When $0 \le M \le 4C_4 + 1$, by Lemma 2.1,

$$J \le \int_{\mathcal{X}} \frac{1}{V_{t2^k}(z) + V_{t2^k}(x) + V(z, x)} \left(\frac{t2^k}{t2^k + d(z, x)}\right)^b d\mu(z) \lesssim 1 \lesssim (1 + M)^{-b}.$$

When $M > 1 + 4C_4$, we have $M - 2C_4 > (1 + M)/2$. For any $i \in \mathbb{N}$ and $x \in \mathcal{X}$, set

$$\mathrm{R}_i \equiv \{z \in \mathcal{X} : 2^{i+k-2}(M - 2C_4)t \le d(z, x) < 2^{i+k-1}(M - 2C_4)t\}.$$

We then obtain

$$\begin{aligned} J &= \sum_{i=1}^{\infty} \int_{\mathrm{R}_i} \frac{1}{V_{t2^k}(z) + V_{t2^k}(x) + V(z, x)} \left(\frac{t2^k}{t2^k + d(z, x)}\right)^b d\mu(z) \\ &\lesssim \sum_{i=1}^{\infty} [1 + 2^{i-1}(M - 2C_4)]^{-b} \lesssim (M - 2C_4)^{-b} \lesssim (1 + M)^{-b}. \end{aligned}$$

Therefore, for all $k \in \mathrm{W}_1$ and $x \in \mathcal{X}$, we have

$$v_k(x) \lesssim \frac{1}{2^{ak}} \frac{(1 + M)^{-b}}{V_{r_0}(x_0) + V(x_0, x)},$$

which together with the fact $\sharp \mathrm{W}_1 \le 5$ yields that

$$\mathrm{Z}_1 \le \sum_{k \in \mathrm{W}_1} \frac{1}{2^{ak}} \frac{(1 + M)^{-b}}{V_{r_0}(x_0) + V(x_0, x)} \lesssim \frac{1}{(r_0 + d(x_0, x))^a} \frac{(1 + M)^{-b}}{V_{r_0}(x_0) + V(x_0, x)}.$$

To estimate Z_2, notice that for any $k \in \mathrm{W}_2$ and $j \in J(k)$, we have

$$r_0 + d(x_0, x) \le r_0 + d(x_0, x_j) + d(x_j, x) < 2^k + d(x, x_j) \le (r_0 + d(x_0, x))/2 + d(x, x_j).$$

From this, it follows that for any $j \in J(k)$,

$$d(x, x_j) > (r_0 + d(x_0, x))/2 > 2^k \tag{2.17}$$

and thus

$$V(x_j, x) \gtrsim \mu(B(x, r_0 + d(x_0, x))) \sim V_{r_0}(x_0) + V(x_0, x). \tag{2.18}$$

For $k \in \mathbb{Z}$ and $j \in J(k)$, we have $V_{2^k}(x_j) \sim V_{2^k}(x_0)$ and $B(x_j, C_4 t r_j) \subset B(x_0, (1+C_4 t)2^k)$, which together with (1.2) and (2.13) yields that

$$\sum_{j\in J(k)} \frac{V_{C_4 t 2^k}(x_j)}{V_{2^k}(x_j)} \lesssim \int_{B(x_0,\,(1+C_4t)2^k)} \frac{1}{V_{2^k}(x_0)} \sum_{j\in J(k)} \chi_{B(x_j, C_4 t r_j)}(x)\, d\mu(x) \lesssim 1. \tag{2.19}$$

For any $k \in \mathbb{Z}$ and $j \in J(k)$, we have $V_{r_j}(x_j) \gtrsim V_{2^k}(x_j)$, which together with (2.17), (2.18), and (2.19) implies that

$$\begin{aligned}\mathrm{Z}_2 &\lesssim \sum_{k\in \mathrm{W}_2} \sum_{j\in J(k)} \frac{1}{2^{ak}} \frac{1}{V_{2^k}(x_j)} \frac{V_{C_4 t 2^k}(x_j)}{V_{r_0}(x_0) + V(x_0, x)} \left(\frac{t2^k}{t2^k + r_0 + d(x_0, x)}\right)^b \\ &\lesssim t^b \frac{1}{V_{r_0}(x_0) + V(x_0, x)} \left(\frac{1}{r_0 + d(x_0, x)}\right)^a.\end{aligned}$$

To estimate Z_3, notice that for any $k \in \mathrm{W}_3$ and $j \in J(k)$,

$$V_{r_j}(x_j) \sim \mu(B(x_j, r_0 + d(x_j, x_0))) \gtrsim \mu(B(x_0, 2^k)) \gtrsim V_{r_0}(x_0) + V(x_0, x). \tag{2.20}$$

Moreover, since

$$2^{k-1} \le r_0 + d(x_j, x_0) \le r_0 + d(x_j, x) + d(x, x_0) \le 2^{k-2} + d(x_j, x),$$

we have $d(x_j, x) \ge 2^{k-2} \ge r_j/4$. Consequently, for any $k \in \mathrm{W}_3$ and $j \in J(k)$,

$$V_{tr_j}(x_j) + V_{tr_j}(x) + V(x_j, x) \ge V(x_j, x) \gtrsim V_{r_j/2}(x_j). \tag{2.21}$$

An argument similar to (2.19) yields that

$$\sum_{j\in J(k)} \frac{V_{C_4 t r_j}(x_j)}{V_{r_j/2}(x_j)} \lesssim 1. \tag{2.22}$$

Applying (2.20) through (2.22) and the fact that $d(x_j, x) \ge r_j/4$, we obtain

$$\begin{aligned}\mathrm{Z}_3 &\lesssim \sum_{k\in \mathrm{W}_3} \sum_{j\in J(k)} \frac{1}{2^{ka}} \frac{1}{V_{r_0}(x_0) + V(x_0, x)} \frac{V_{C_4 t r_j}(x_j)}{V_{r_j/2}(x_j)} \left(\frac{t}{1+t}\right)^b \\ &\lesssim t^b \frac{1}{V_{r_0}(x_0) + V(x_0, x)} \left(\frac{1}{r_0 + d(x_0, x)}\right)^a,\end{aligned}$$

which completes the proof of Lemma 2.4.

3 Proofs of Theorem 1.6 and Proposition 1.7

Assuming Proposition 1.7 for the moment, we now prove Theorem 1.6.

Proof of Theorem 1.6 Choose $\delta_0 \in (0, \log_\eta(1-\sigma))$ satisfying $p > n/(n+\delta_0)$. By Proposition 1.7 and the $L^{p(n+\delta_0)/n}(\mathcal{X})$-boundedness of $\mathcal{M}$, we obtain

$$\|f^*\|_{L^p(\mathcal{X})} \lesssim \left\| \left[\mathcal{M}([\mathcal{M}_0(f)]^{n/(n+\delta_0)})\right]^{(n+\delta_0)/n} \right\|_{L^p(\mathcal{X})} \lesssim \|\mathcal{M}_0(f)\|_{L^p(\mathcal{X})},$$

which completes the proof of Theorem 1.6.

The rest of this section is devoted to the proof of Proposition 1.7. The key for the proof of Proposition 1.7 is to obtain a desired expansion (3.11) for any $\varphi \in \mathcal{G}(x_0, r_0, \epsilon, \epsilon)$ in terms of the given $(\epsilon_1, \epsilon_2, \epsilon_3)$- SAOTI and to show this expansion converges in $\mathcal{G}_0^\epsilon(\beta, \gamma)$. To this end, for a given $\varphi \in \mathcal{G}(x_0, r_0, \epsilon, \epsilon)$, we construct a sequence of functions $\{\varphi_s\}_{s\in\mathbb{Z}_+}$ and obtain some desired estimates for these functions.

Proof of Proposition 1.7 Let $\varphi \in \mathcal{G}(x_0, r_0, \epsilon, \epsilon)$ satisfy $\|\varphi\|_{\mathcal{G}(x_0,r_0,\epsilon,\epsilon)} \le 1$ for some $x_0 \in \mathcal{X}$ and $r_0 > 0$. Write φ as

$$\varphi = (\Re(\varphi) \vee 0) - (\Re(\varphi) \wedge 0) + i[(\Im(\varphi) \vee 0) - (\Im(\varphi) \wedge 0)],$$

where $\Re(\varphi)$ and $\Im(\varphi)$ represent the real part and the imaginary part of the function φ, respectively. Since $\|\varphi\|_{\mathcal{G}(x_0,r_0,\epsilon,\epsilon)} \le 1$, it follows easily that each term of the decomposition above has a norm in $\mathcal{G}(x_0, r_0, \epsilon, \epsilon)$ at most 1. Thus, we may assume that φ is non-negative.

Fix $A \equiv 2^{\epsilon+1}C_8$ and some $\sigma \in (0, 1/(1 + AC_3C_9))$. Choose positive numbers H and η such that

$$H \le \min\left\{ \frac{1}{2}\left(\frac{1-\sigma-AC_3C_9\sigma}{2}\right)^{1/\epsilon}, \frac{1}{2}\left(\frac{(1-\sigma-AC_3C_9\sigma)(1-\sigma)}{4AC_3C_9\sigma}\right)^{1/\epsilon_1} \right\}$$

and

$$\eta < \min\left\{ (1-\sigma)^{1/\epsilon}, \frac{H}{C_4}, \frac{1}{AC_3C_9}, \left(\frac{1-\sigma}{2}\right)^{1/\epsilon_1}, H\left(\frac{1}{AC_3C_9}\right)^{1/\epsilon_2} \right\}.$$

Choose $\delta_0 \in (0, \log_\eta(1-\sigma))$. For any $s \in \mathbb{N}$, applying Lemma 2.3 with $t = \eta^s$ and $g = \mathcal{M}_0(f)$, we obtain $\{x_{s,j} : s \in \mathbb{N}, j = 1, \cdots, j(s)\} \subset \mathcal{X}$, where $j(s)$ can be finite or ∞. For each $s \in \mathbb{N}$ and $j = 1, \cdots, j(s)$, let $r_{s,j} \equiv r_0 + d(x_{s,j}, x_0)$. Lemma 2.3 further implies that

(A) for any $x \in \mathcal{X}$ and $s \in \mathbb{N}$, $1 \le \sum_{j=1}^{j(s)} \chi_{B_{s,j}}(x) \le C_6$, where $B_{s,j} \equiv B(x_{s,j}, C_4\eta^s r_{s,j})$;

(B) $[\mathcal{M}_0(f)(x_{s,j})]^{1/2} \le C_7 F(\eta^s r_{s,j}, x_{s,j}, [\mathcal{M}_0(f)]^{1/2}\chi_{B_{s,j}})$.

Let us now inductively construct $\{\epsilon_{s,j} : s \in \mathbb{N}, j = 1, \cdots, j(s)\} \subset \{-1, 0, 1\}$ and functions $\{\varphi_s\}_{s\in\mathbb{Z}_+}$ satisfying that for all $x \in \mathcal{X}$,

(C) for all $s \in \mathbb{Z}_+$, $|\varphi_s(x)| \le (1-\sigma)^s \dfrac{1}{V_{r_0}(x_0) + V(x_0, x)} \left(\dfrac{r_0}{r_0 + d(x_0, x)}\right)^\epsilon$;

(D) $\varphi_0(x) \equiv \varphi(x)$, and for any $s \in \mathbb{N}$,

$$\varphi_s(x) \equiv \varphi(x) - A\sigma \sum_{i=1}^{s} \sum_{j=1}^{j(i)} (1-\sigma)^{i-1} \epsilon_{i,j} \frac{V_{C_4 \eta^s r_{i,j}}(x_{i,j})}{V_{r_{i,j}}(x_{i,j})} \left(\frac{r_0}{r_{i,j}}\right)^{\epsilon} S_{\eta^i r_{i,j}}(x_{i,j}, x). \quad (3.1)$$

In fact, obviously, φ_0 satisfies (C). Suppose that $\{\epsilon_{i,j} : i = 1, \cdots, s-1; j = 1, \cdots, j(i)\}$ and $\{\varphi_i\}_{i=0}^{s-1}$ satisfying (C) have been constructed. Then for each $j = 1, \cdots, j(s)$, let $\epsilon_{s,j} \equiv \mathrm{sign}(\varphi_{s-1}(x_{s,j}))$, and φ_s be as in (D). Now it remains to verify that φ_s satisfies (C). To this end, for all $x \in \mathcal{X}$, set

$$\omega_s(x) \equiv A\sigma \sum_{j=1}^{j(s)} (1-\sigma)^{s-1} \epsilon_{s,j} \frac{V_{C_4 \eta^s r_{s,j}}(x_{s,j})}{V_{r_{s,j}}(x_{s,j})} \left(\frac{r_0}{r_{s,j}}\right)^{\epsilon} S_{\eta^s r_{s,j}}(x_{s,j}, x). \quad (3.2)$$

By the size condition of S_t, $\epsilon_2 > \epsilon$ and Lemma 2.4 with $t = \eta^s$, $a = \epsilon$, $b = \epsilon_2$ and $M = 0$, we obtain

$$\begin{aligned} |\omega_s(x)| &\le AC_3\sigma \sum_{j=1}^{j(s)} (1-\sigma)^{s-1} \frac{V_{C_4 \eta^s r_{s,j}}(x_{s,j})}{V_{r_{s,j}}(x_{s,j})} \left(\frac{r_0}{r_{s,j}}\right)^{\epsilon} \\ &\quad \times \frac{1}{V_{\eta^s r_{s,j}}(x_{s,j}) + V_{\eta^s r_{s,j}}(x) + V(x_{s,j}, x)} \left(\frac{\eta^s r_{s,j}}{\eta^s r_{s,j} + d(x_{s,j}, x)}\right)^{\epsilon_2} \\ &\le AC_3C_9\sigma(1-\sigma)^{s-1} \frac{1}{V_{r_0}(x_0) + V(x_0, x)} \left(\frac{r_0}{r_0 + d(x_0, x)}\right)^{\epsilon}. \end{aligned} \quad (3.3)$$

If $d(x, y) \le \eta^{s-1}(r_0 + d(x_0, x))/2$, then for any $i = 1, \cdots, s-1$, we have that $d(x, y) \le (\eta^i r_{i,j} + d(x, x_{i,j}))/2$. This combined with the regularity of $S_{\eta^i r_{i,j}}$ and Lemma 2.4 with $t = \eta^i$, $a = \epsilon + \epsilon_1$, $b = \epsilon_1 + \epsilon_2$ and $M = 0$ yields that

$$\begin{aligned} &|\varphi_{s-1}(x) - \varphi_{s-1}(y)| \\ &\le |\varphi(x) - \varphi(y)| + AC_3\sigma \sum_{i=1}^{s-1} \sum_{j=1}^{j(i)} (1-\sigma)^{i-1} \\ &\quad \times \frac{V_{C_4 \eta^i r_{i,j}}(x_{i,j})}{V_{r_{i,j}}(x_{i,j})} \left(\frac{r_0}{r_{i,j}}\right)^{\epsilon} |S_{\eta^i r_{i,j}}(x_{i,j}, x) - S_{\eta^i r_{i,j}}(x_{i,j}, y)| \\ &\le |\varphi(x) - \varphi(y)| + AC_3\sigma \sum_{i=1}^{s-1} \sum_{j=1}^{j(i)} \frac{(1-\sigma)^{i-1}}{V_{r_{i,j}}(x_{i,j})} \left(\frac{r_0}{r_{i,j}}\right)^{\epsilon} \left(\frac{\eta^i r_{i,j}}{\eta^i r_{i,j} + d(x_{i,j}, x)}\right)^{\epsilon_2} \\ &\quad \times \frac{V_{C_4 \eta^i r_{i,j}}(x_{i,j})}{V_{\eta^i r_{i,j}}(x_{i,j}) + V_{\eta^i r_{i,j}}(x) + V(x_{i,j}, x)} \left(\frac{d(x, y)}{\eta^i r_{i,j} + d(x_{i,j}, x)}\right)^{\epsilon_1} \\ &\le |\varphi(x) - \varphi(y)| + \frac{AC_3C_9\sigma}{1 - \sigma - \eta^{\epsilon_1}} \left[\left(\frac{1-\sigma}{\eta^{\epsilon_1}}\right)^{s-1} - 1\right] \\ &\quad \times \frac{1}{V_{r_0}(x_0) + V(x_0, x)} \left(\frac{d(x, y)}{r_0 + d(x_0, x)}\right)^{\epsilon_1} \left(\frac{r_0}{r_0 + d(x, x_0)}\right)^{\epsilon}. \end{aligned} \quad (3.4)$$

When
$$d(x,y) < H\eta^{s-1}(r_0 + d(y,x_0)), \tag{3.5}$$
the assumption that $H < 1/4$ gives $d(x,y) < 2H\eta^{s-1}(r_0+d(x_0,x)) < \eta^{s-1}(r_0+d(x_0,x))/2$, which together with (3.4) and the regularity of φ yields that

$$\begin{aligned}|\varphi_{s-1}(x) - \varphi_{s-1}(y)| &\le \left[(2H)^\epsilon \left(\frac{\eta^\epsilon}{1-\sigma}\right)^{s-1} + \frac{(2H)^{\epsilon_1} AC_3C_9\sigma}{1-\sigma-\eta^{\epsilon_1}}\left(1 - \left(\frac{\eta^{\epsilon_1}}{1-\sigma}\right)^{s-1}\right)\right] \\ &\quad\times \frac{(1-\sigma)^{s-1}}{V_{r_0}(x_0) + V(x_0,x)}\left(\frac{r_0}{r_0 + d(x_0,x)}\right)^\epsilon \\ &\le \lambda_H \frac{(1-\sigma)^{s-1}}{V_{r_0}(x_0) + V(x_0,x)}\left(\frac{r_0}{r_0 + d(x_0,x)}\right)^\epsilon, \end{aligned}\tag{3.6}$$

where
$$\lambda_H \equiv (2H)^\epsilon + \frac{(2H)^{\epsilon_1} AC_3C_9\sigma}{1-\sigma-\eta^{\epsilon_1}}.$$

For any $\lambda \in \mathbb{R}$ and $s \in \mathbb{N}$, set
$$\Omega_{s,\lambda} \equiv \left\{x \in \mathcal{X} :\ \varphi_{s-1}(x) > \lambda(1-\sigma)^{s-1}\frac{1}{V_{r_0}(x_0) + V(x_0,x)}\left(\frac{r_0}{r_0 + d(x_0,x)}\right)^\epsilon\right\}.$$

If $\varphi_{s-1}(x_{s,j}) < 0$, then by (3.6) and the definition of $\Omega_{s,\lambda}$, we obtain
$$B(x_{s,j}, \eta^{s-1} r_{s,j}/3) \bigcap \Omega_{s,\lambda_H} = \emptyset. \tag{3.7}$$

In fact, for any $x \in B(x_{s,j}, \eta^{s-1} r_{s,j}/3)$, by (3.6),
$$\begin{aligned}\varphi_{s-1}(x) &\le \varphi_{s-1}(x_{s,j}) + \lambda_H \frac{(1-\sigma)^{s-1}}{V_{r_0}(x_0) + V(x_0,x)}\left(\frac{r_0}{r_0 + d(x_0,x)}\right)^\epsilon \\ &< \lambda_H \frac{(1-\sigma)^{s-1}}{V_{r_0}(x_0) + V(x_0,x)}\left(\frac{r_0}{r_0 + d(x_0,x)}\right)^\epsilon,\end{aligned}$$
which implies that $x \notin \Omega_{s,\lambda_H}$ and thus (3.7) holds.

For any $x \in \mathcal{X}$, by (3.2), we write
$$\begin{aligned}\omega_s(x) = &\sum_{\{j:\, \varphi_{s-1}(x_{s,j})>0\}} A\sigma(1-\sigma)^{s-1}\frac{V_{C_4\eta^s r_{s,j}}(x_{s,j})}{V_{r_{s,j}}(x_{s,j})}\left(\frac{r_0}{r_{s,j}}\right)^\epsilon S_{\eta^s r_{s,j}}(x_{s,j},x) \\ &- \sum_{\{j:\, \varphi_{s-1}(x_{s,j})<0\}} \cdots \equiv \mathrm{I} - \mathrm{II}.\end{aligned}$$

We now turn to obtain a desired upper bound for φ_s by considering two cases: $x \in \Omega_{s,\lambda_H}$ and $x \notin \Omega_{s,\lambda_H}$. For any $x \in \Omega_{s,\lambda_H}$, Property (A) implies that $x \in B(x_{s,j_0}, C_4\eta^s r_{s,j_0})$ for some j_0. By this and the assumption $\eta < H/C_4$, we have $d(x, x_{s_{j_0}}) < C_4\eta^s r_{s,j_0} \le H\eta^{s-1} r_{s,j_0}$, which together with (3.7) implies that $\varphi_{s-1}(x_{s,j_0}) > 0$. Thus, by (2.7),
$$\mathrm{I} \ge A\sigma(1-\sigma)^{s-1}\frac{V_{C_4\eta^s r_{s,j_0}}(x_{s,j_0})}{V_{r_{s,j_0}}(x_{s,j_0})}\left(\frac{r_0}{r_{s,j_0}}\right)^\epsilon S_{\eta^s r_{s,j_0}}(x_{s,j_0},x)$$

$$\geq \frac{A\sigma(1-\sigma)^{s-1}}{C_8 2^{\epsilon}} \frac{1}{V_{r_0}(x_0)+V(x_0,x)} \left(\frac{r_0}{r_0+d(x_0,x)}\right)^{\epsilon}.$$

On the other hand, by (3.7) and Lemma 2.4,

$$\mathrm{II} \leq \frac{AC_3C_9\sigma(1-\sigma)^{s-1}}{V_{r_0}(x_0)+V(x_0,x)} \left(\frac{r_0}{r_0+d(x_0,x)}\right)^{\epsilon} \max\{\eta^{s\epsilon_2}, (1+H\eta^{-1})^{-\epsilon_2}\}.$$

From the assumption on η, it follows easily that

$$\frac{A}{C_8 2^{\epsilon}} - AC_3C_9 \max\{\eta^{s\epsilon_2}, (1+H\eta^{-1})^{-\epsilon_2}\} \geq 1,$$

which together with the estimates for I and II yields that when $x \in \Omega_{s,\lambda_H}$,

$$\omega_s(x) \geq \frac{\sigma(1-\sigma)^{s-1}}{V_{r_0}(x_0)+V(x_0,x)} \left(\frac{r_0}{r_0+d(x_0,x)}\right)^{\epsilon}. \tag{3.8}$$

Thus, by (2.1) and the fact that (C) holds for $s-1$, we obtain that when $x \in \Omega_{s,\lambda_H}$,

$$\varphi_s(x) = \varphi_{s-1}(x) - \omega_s(x) \leq \frac{(1-\sigma)^s}{V_{r_0}(x_0)+V(x_0,x)} \left(\frac{r_0}{r_0+d(x_0,x)}\right)^{\epsilon}.$$

Let now $x \notin \Omega_{s,\lambda_H}$. Notice that (C) holds for $s-1$. From this and (3.3), we deduce that

$$\begin{aligned}
\varphi_s(x) &= \varphi_{s-1}(x) - \omega_s(x) \\
&\leq \lambda_H \frac{(1-\sigma)^{s-1}}{V_{r_0}(x_0)+V(x_0,x)} \left(\frac{r_0}{r_0+d(x_0,x)}\right)^{\epsilon} \\
&\quad + AC_3C_9\sigma(1-\sigma)^{s-1} \frac{1}{V_{r_0}(x_0)+V(x_0,x)} \left(\frac{r_0}{r_0+d(x_0,x)}\right)^{\epsilon} \\
&\leq \frac{(1-\sigma)^s}{V_{r_0}(x_0)+V(x_0,x)} \left(\frac{r_0}{r_0+d(x_0,x)}\right)^{\epsilon},
\end{aligned}$$

where in the last inequality, we used the fact $\lambda_H + AC_3C_9\sigma \leq 1-\sigma$. Thus, we obtain a desired upper bound of φ_s.

Let us now show that φ_s has the desired lower bound also by considering two cases: $x \notin \Omega_{s,-\lambda_H}$ and $x \in \Omega_{s,-\lambda_H}$. For every $x_{s,j}$ satisfying $\varphi_{s-1}(x_{s,j}) > 0$, by (3.6) and an argument similar to the proof of (3.7), we have $B(x_{s,j}, H\eta^{s-1}r_{s,j}) \cap (\mathcal{X} \setminus \Omega_{s,-\lambda_H}) = \emptyset$. From this, Lemmas 2.3 and 2.4 together with an argument similar to the estimates for I and II, it follows that when $x \notin \Omega_{s,-\lambda_H}$,

$$\omega_s(x) \leq -\frac{\sigma(1-\sigma)^{s-1}}{V_{r_0}(x_0)+V(x_0,x)} \left(\frac{r_0}{r_0+d(x_0,x)}\right)^{\epsilon}. \tag{3.9}$$

Therefore, (3.9) and the lower bound of φ_{s-1} yields that when $x \notin \Omega_{s,-\lambda_H}$,

$$\varphi_s(x) = \varphi_{s-1}(x) - \omega_s(x) \geq -\frac{(1-\sigma)^s}{V_{r_0}(x_0)+V(x_0,x)} \left(\frac{r_0}{r_0+d(x_0,x)}\right)^{\epsilon}; \tag{3.10}$$

and the validity of (C) for $s-1$ together with (3.3) and the assumption $\lambda_H + AC_3C_9\sigma \le 1-\sigma$ implies that when $x \in \Omega_{s,-\lambda_H}$,

$$\begin{aligned}\varphi_s(x) &\ge \varphi_{s-1}(x) - \omega_s(x)\\ &\ge -\lambda_H \frac{(1-\sigma)^{s-1}}{V_{r_0}(x_0)+V(x_0,x)}\left(\frac{r_0}{r_0+d(x_0,x)}\right)^{\epsilon}\\ &\quad -AC_3C_9\sigma(1-\sigma)^{s-1}\frac{1}{V_{r_0}(x_0)+V(x_0,x)}\left(\frac{r_0}{r_0+d(x_0,x)}\right)^{\epsilon}\\ &\ge -\frac{(1-\sigma)^{s}}{V_{r_0}(x_0)+V(x_0,x)}\left(\frac{r_0}{r_0+d(x_0,x)}\right)^{\epsilon},\end{aligned}$$

which together with (3.10) further yields the desired lower bound of φ_s. This finishes the proofs of (C) and (D).

It follows from (C) and (D) that for all $x \in \mathcal{X}$,

$$\varphi(x) = A\sigma \sum_{i=1}^{\infty}\sum_{j=1}^{j(i)} (1-\sigma)^{i-1}\epsilon_{i,j}\frac{V_{C_4\eta^i r_{i,j}}(x_{i,j})}{V_{r_{i,j}}(x_{i,j})}\left(\frac{r_0}{r_{i,j}}\right)^{\epsilon} S_{\eta^i r_{i,j}}(x_{i,j},x). \tag{3.11}$$

Set

$$\Phi_L(x) \equiv A\sigma \sum_{i=1}^{L}\sum_{\{j:\,\eta^i r_{i,j}\le L^i\}} (1-\sigma)^{i-1}\epsilon_{i,j}\frac{V_{C_4\eta^i r_{i,j}}(x_{i,j})}{V_{r_{i,j}}(x_{i,j})}\left(\frac{r_0}{r_{i,j}}\right)^{\epsilon} S_{\eta^i r_{i,j}}(x_{i,j},x).$$

By (A) and (1.2), it is easy to see that the series in Φ_L has only finitely many terms. To verify that (3.11) holds in $\mathcal{G}_0^{\epsilon}(\beta,\gamma)$, it suffices to show that Φ_L converges to φ in $\mathcal{G}_0^{\epsilon}(\beta,\gamma)$ as $L\to\infty$. Notice that for any i and j, $S_{\eta^i r_{i,j}}(x_{i,j},\cdot) \in \mathcal{G}(x_{i,j},\eta^i r_{i,j},\epsilon,\epsilon)$. Thus $\Phi_L \in \mathcal{G}(\epsilon,\epsilon)$ since it has only finite terms. Now it remains to show

$$\lim_{L\to\infty} \|\varphi - \Phi_L\|_{\mathcal{G}(x_0,r_0,\beta,\gamma)} = 0.$$

To this end, we write

$$\begin{aligned}|\varphi(x)-\Phi_L(x)| &\le \left|A\sigma\sum_{i=1}^{\infty}\sum_{\{j:\,\eta^i r_{i,j}>L^i\}}(1-\sigma)^{i-1}\epsilon_{i,j}\frac{V_{C_4\eta^i r_{i,j}}(x_{i,j})}{V_{r_{i,j}}(x_{i,j})}\left(\frac{r_0}{r_{i,j}}\right)^{\epsilon}S_{\eta^i r_{i,j}}(x_{i,j},x)\right|\\ &\quad+\left|A\sigma\sum_{i=L}^{\infty}\sum_{\{j:\,\eta^i r_{i,j}\le L^i\}}(1-\sigma)^{i-1}\epsilon_{i,j}\frac{V_{C_4\eta^i r_{i,j}}(x_{i,j})}{V_{r_{i,j}}(x_{i,j})}\left(\frac{r_0}{r_{i,j}}\right)^{\epsilon}S_{\eta^i r_{i,j}}(x_{i,j},x)\right|\\ &\equiv \Phi_L^1(x)+\Phi_L^2(x).\end{aligned}$$

Let us first prove that Φ_L^1 converges to zero in $\mathcal{G}(x_0,r_0,\beta,\gamma)$ as $L\to\infty$. Notice that $\epsilon>\gamma$ and $r_{i,j}=r_0+d(x_{i,j},x_0)$. By the size condition of S_t and Lemma 2.4, we obtain that for all $x\in\mathcal{X}$,

$$|\Phi_L^1(x)| \lesssim \sum_{i=1}^{\infty}\sum_{\{j:\,\eta^i r_{i,j}>L^i\}}(1-\sigma)^{i-1}\left(\frac{r_0\eta^i}{L^i}\right)^{\epsilon-\gamma}\frac{V_{C_4\eta^i r_{i,j}}(x_{i,j})}{V_{r_{i,j}}(x_{i,j})}\left(\frac{r_0}{r_{i,j}}\right)^{\gamma}S_{\eta^i r_{i,j}}(x_{i,j},x)$$

$$\lesssim \frac{(1-\sigma)\eta^{\epsilon-\gamma}}{L^{\epsilon-\gamma}-(1-\sigma)\eta^{\epsilon-\gamma}} \frac{1}{V_{r_0}(x_0)+V(x_0,x)} \left(\frac{r_0}{r_0+d(x_0,x)}\right)^{\gamma}. \tag{3.12}$$

For any $x,\, y \in \mathcal{X}$ satisfying $d(x,y) \le (r_0+d(x_0,x))/2$, we write

$$\begin{aligned} &|\Phi_L^1(x)-\Phi_L^1(y)| \\ &\quad \le A\sigma \sum_{i=1}^{\infty} \sum_{j\in \mathrm{W}_i^1} (1-\sigma)^{i-1} \frac{V_{C_4\eta^i r_{i,j}}(x_{i,j})}{V_{r_{i,j}}(x_{i,j})} \left(\frac{r_0}{r_{i,j}}\right)^{\epsilon} |S_{\eta^i r_{i,j}}(x_{i,j},x)-S_{\eta^i r_{i,j}}(x_{i,j},y)| \\ &\qquad + \sum_{i=1}^{\infty} \sum_{j\in \mathrm{W}_i^2} \cdots + \sum_{i=1}^{\infty} \sum_{j\in \mathrm{W}_i^3} \cdots \equiv \sum_{i=1}^{\infty} \left(Z_i^1+Z_i^2+Z_i^3\right), \end{aligned}$$

where

$$\mathrm{W}_i^1 \equiv \{j\in\mathbb{N}:\, \eta^i r_{i,j} > L^i,\, d(x,y) \le (\eta^i r_{i,j}+d(x,x_{i,j}))/2\},$$

$$\mathrm{W}_i^2 \equiv \{j\in\mathbb{N}:\, \eta^i r_{i,j} > L^i,\, d(x,y) > (\eta^i r_{i,j}+d(x,x_{i,j}))/2,\, d(y,x_{i,j}) \ge d(x,x_{i,j})\}$$

and

$$\mathrm{W}_i^3 \equiv \{j\in\mathbb{N}:\, \eta^i r_{i,j} > L^i,\, d(x,y) > (\eta^i r_{i,j}+d(x,x_{i,j}))/2,\, d(y,x_{i,j}) < d(x,x_{i,j})\}.$$

By the regularity of $S_{\eta^i r_{i,j}}$, Lemma 2.4 and $\eta^i(r_0+d(x_{i,j},x_0)) > L^i$, we obtain

$$\begin{aligned} \mathrm{Z}_i^1 &\lesssim \sum_{j\in\mathrm{W}_i^1} (1-\sigma)^{i-1} \frac{V_{C_4\eta^i r_{i,j}}(x_{i,j})}{V_{r_{i,j}}(x_{i,j})} \left(\frac{r_0\eta^i}{L^i}\right)^{\epsilon-\gamma} \left(\frac{r_0}{r_{i,j}}\right)^{\gamma} \\ &\quad \times \frac{1}{V_{\eta^i r_{i,j}}(x_{i,j})+V_{\eta^i r_{i,j}}(x)+V(x_{i,j},x)} \left(\frac{d(x,y)}{\eta^i r_{i,j}+d(x_{i,j},x)}\right)^{\epsilon_1} \left(\frac{\eta^i r_{i,j}}{\eta^i r_{i,j}+d(x_{i,j},x)}\right)^{\epsilon_2} \\ &\lesssim \left(\frac{(1-\sigma)\eta^{\epsilon-\gamma-\epsilon_1}}{L^{\epsilon-\gamma}}\right)^{i} \frac{1}{V_{r_0}(x_0)+V(x_0,x)} \left(\frac{d(x,y)}{r_0+d(x,x_0)}\right)^{\epsilon_1} \left(\frac{r_0}{r_0+d(x_0,x)}\right)^{\gamma}. \end{aligned}$$

By the size condition of $S_{\eta^i r_{i,j}}$, $d(x,y) > (\eta^i r_{i,j}+d(x,x_{i,j}))/2$, $d(y,x_{i,j}) \ge d(x,x_{i,j})$, $r_{i,j} \ge r_0$ and Lemma 2.4,

$$\begin{aligned} \mathrm{Z}_i^2 &\lesssim \sum_{j\in\mathrm{W}_i^2} (1-\sigma)^{i-1} \frac{V_{C_4\eta^i r_{i,j}}(x_{i,j})}{V_{r_{i,j}}(x_{i,j})} \left(\frac{r_0}{r_{i,j}}\right)^{\epsilon} \left[S_{\eta^i r_{i,j}}(x_{i,j},x)+S_{\eta^i r_{i,j}}(x_{i,j},y)\right] \\ &\lesssim \sum_{j\in\mathrm{W}_i^2} (1-\sigma)^{i-1} \left(\frac{r_0\eta^i}{L^i}\right)^{\epsilon-\gamma} \left(\frac{r_0}{r_{i,j}}\right)^{\gamma} \frac{V_{C_4\eta^i r_{i,j}}(x_{i,j})}{V_{r_{i,j}}(x_{i,j})} \left(\frac{d(x,y)}{\eta^i r_0}\right)^{\epsilon_1} \\ &\quad \times \frac{1}{V_{\eta^i r_{i,j}}(x_{i,j})+V_{\eta^i r_{i,j}}(x)+V(x_{i,j},x)} \left(\frac{\eta^i r_{i,j}}{\eta^i r_{i,j}+d(x_{i,j},x)}\right)^{\epsilon_2+\epsilon_1} \\ &\lesssim \left(\frac{(1-\sigma)\eta^{\epsilon-\gamma-\epsilon_1}}{L^{\epsilon-\gamma}}\right)^{i} \frac{1}{V_{r_0}(x_0)+V(x_0,x)} \left(\frac{d(x,y)}{r_0+d(x,x_0)}\right)^{\epsilon_1} \left(\frac{r_0}{r_0+d(x_0,x)}\right)^{\gamma}. \end{aligned}$$

From the assumption $d(x,y) \le (r_0 + d(x_0,x))/2$, it follows that

$$r_0 + d(x_0,x) \sim r_0 + d(y,x_0) \quad \text{and} \quad V_{r_0}(x_0) + V(x_0,x) \sim V_{r_0}(x_0) + V(x_0,y), \tag{3.13}$$

which together with the definition of W_i^3 and an argument similar to the estimation of Z_i^2 yields that

$$\mathrm{Z}_i^3 \lesssim \left(\frac{(1-\sigma)\eta^{\epsilon-\gamma-\epsilon_1}}{L^{\epsilon-\gamma}}\right)^i \frac{1}{V_{r_0}(x_0)+V(x_0,x)} \left(\frac{d(x,y)}{r_0+d(x_0,x)}\right)^{\epsilon_1} \left(\frac{r_0}{r_0+d(x_0,x)}\right)^{\gamma}.$$

By the estimates of $\mathrm{Z}_{\mathrm{i}}^1$, Z_i^2 and Z_i^3, we obtain that when L is large enough and satisfies $L^{\epsilon-\gamma} > (1-\sigma)\eta^{\epsilon-\gamma-\epsilon_1}$,

$$\begin{aligned} |\Phi_L^1(x) - \Phi_L^1(y)| \lesssim &\frac{(1-\sigma)\eta^{\epsilon-\gamma-\epsilon_1}}{L^{\epsilon-\gamma}-(1-\sigma)\eta^{\epsilon-\gamma-\epsilon_1}} \left(\frac{d(x,y)}{r_0+d(x_0,x)}\right)^{\epsilon_1} \\ &\times \frac{1}{V_{r_0}(x_0)+V(x_0,x)} \left(\frac{r_0}{r_0+d(x_0,x)}\right)^{\gamma}, \end{aligned}$$

which together with (3.12) implies that $\lim_{L\to\infty} \|\Phi_L^1\|_{\mathcal{G}(x_0,r_0,\beta,\gamma)} = 0$.

Now we consider $\|\Phi_L^2\|_{\mathcal{G}(x_0,r_0,\beta,\gamma)}$. By the size condition of $S_{\eta^i r_{i,j}}$ and Lemma 2.4, we obtain that for all $x \in \mathcal{X}$,

$$\begin{aligned} |\Phi_L^2(x)| &\lesssim \sum_{i=L}^{\infty} \sum_j (1-\sigma)^{i-1} \frac{1}{V_{r_{i,j}}(x_{i,j})} \left(\frac{r_0}{r_{i,j}}\right)^{\gamma} \\ &\quad \times \frac{V_{C_4\eta^i r_{i,j}}(x_{i,j})}{V_{\eta^i r_{i,j}}(x_{i,j}) + V_{\eta^i r_{i,j}}(x) + V(x_{i,j},x)} \left(\frac{\eta^i r_{i,j}}{\eta^i r_{i,j} + d(x_{i,j},x)}\right)^{\epsilon_2} \\ &\lesssim (1-\sigma)^L \frac{1}{V_{r_0}(x_0)+V(x_0,x)} \left(\frac{r_0}{r_0+d(x_0,x)}\right)^{\gamma}. \end{aligned} \tag{3.14}$$

For any x, $y \in \mathcal{X}$ satisfying $d(x,y) \le (r_0 + d(x_0,x))/2$, we write

$$\begin{aligned} &|\Phi_L^2(x) - \Phi_L^2(y)| \\ &\quad \le A\sigma \sum_{i=L}^{\infty} \sum_{j\in \mathrm{W}_i^4} (1-\sigma)^{i-1} \frac{V_{C_4\eta^i r_{i,j}}(x_{i,j})}{V_{r_{i,j}}(x_{i,j})} \left(\frac{r_0}{r_{i,j}}\right)^{\epsilon} |S_{\eta^i r_{i,j}}(x_{i,j},x) - S_{\eta^i r_{i,j}}(x_{i,j},y)| \\ &\qquad + \sum_{i=L}^{\infty} \sum_{j\in \mathrm{W}_i^5} \cdots + \sum_{i=L}^{\infty} \sum_{j\in \mathrm{W}_i^6} \cdots \equiv \sum_{i=L}^{\infty} \left(\mathrm{Z}_i^4 + \mathrm{Z}_i^5 + \mathrm{Z}_i^6\right), \end{aligned}$$

where

$$\mathrm{W}_i^4 \equiv \{j \in \mathbb{N} :\ d(x,y) \le (\eta^i r_{i,j} + d(x,x_{i,j}))/2\},$$

$$\mathrm{W}_i^5 \equiv \{j \in \mathbb{N} :\ d(x,y) > (\eta^i r_{i,j} + d(x,x_{i,j}))/2,\ d(y,x_{i,j}) \ge d(x,x_{i,j})\}$$

and

$$\mathrm{W}_i^6 \equiv \{j \in \mathbb{N} :\ d(x,y) > (\eta^i r_{i,j} + d(x,x_{i,j}))/2,\ d(y,x_{i,j}) < d(x,x_{i,j})\}.$$

For any $i \in \mathbb{N}$ and $k = 4, 5, 6$, by $d(x,y) \le (r_0 + d(x_0,x))/2$, (3.2), (3.3) and (3.13),

$$\mathrm{Z}_i^k \le |\omega_i(x)| + |\omega_i(y)| \lesssim (1-\sigma)^i \frac{1}{V_{r_0}(x_0) + V(x_0,x)} \left(\frac{r_0}{r_0 + d(x_0,x)}\right)^{\epsilon}. \tag{3.15}$$

The regularity of $S_{\eta^i r_{i,j}}$ and Lemma 2.4 show that

$$\begin{aligned}\mathrm{Z}_i^4 &\lesssim \sum_{j\in \mathrm{W}_i^4} (1-\sigma)^{i-1} \frac{V_{C_4\eta^i r_{i,j}}(x_{i,j})}{V_{r_{i,j}}(x_{i,j})} \left(\frac{r_0}{r_{i,j}}\right)^{\epsilon} \left(\frac{d(x,y)}{\eta^i r_{i,j} + d(x_{i,j},x)}\right)^{\epsilon_1} \\ &\quad \times \frac{1}{V_{\eta^i r_{i,j}}(x_{i,j}) + V_{\eta^i r_{i,j}}(x) + V(x_{i,j},x)} \left(\frac{\eta^i r_{i,j}}{\eta^i r_{i,j} + d(x_{i,j},x)}\right)^{\epsilon_2} \\ &\lesssim \left(\frac{1-\sigma}{\eta^{\epsilon_1}}\right)^i \frac{1}{V_{r_0}(x_0) + V(x_0,x)} \left(\frac{d(x,y)}{r_0 + d(x_0,x)}\right)^{\epsilon_1} \left(\frac{r_0}{r_0 + d(x_0,x)}\right)^{\epsilon}. \end{aligned} \tag{3.16}$$

For any $j \in \mathrm{W}_i^5$, we have $d(y,x_{i,j}) \ge d(x,x_{i,j})$. From this, the size condition of $S_{\eta^i r_{i,j}}$ and Lemma 2.4, we deduce that

$$\begin{aligned}\mathrm{Z}_i^5 &\lesssim \sum_{j\in \mathrm{W}_i^5} (1-\sigma)^{i-1} \frac{V_{C_4\eta^i r_{i,j}}(x_{i,j})}{V_{r_{i,j}}(x_{i,j})} \left(\frac{r_0}{r_{i,j}}\right)^{\epsilon} \left(\frac{d(x,y)}{\eta^i r_{i,j} + d(x_{i,j},x)}\right)^{\epsilon_1} \\ &\quad \times \frac{1}{V_{\eta^i r_{i,j}}(x_{i,j}) + V_{\eta^i r_{i,j}}(x) + V(x_{i,j},x)} \left(\frac{\eta^i r_{i,j}}{\eta^i r_{i,j} + d(x_{i,j},x)}\right)^{\epsilon_2} \\ &\lesssim \left(\frac{1-\sigma}{\eta^{\epsilon_1}}\right)^i \frac{1}{V_{r_0}(x_0) + V(x_0,x)} \left(\frac{d(x,y)}{r_0 + d(x_0,x)}\right)^{\epsilon_1} \left(\frac{r_0}{r_0 + d(x_0,x)}\right)^{\epsilon}, \end{aligned} \tag{3.17}$$

By an argument similar to the estimate of Z_i^3, we obtain

$$\mathrm{Z}_i^6 \lesssim \left(\frac{1-\sigma}{\eta^{\epsilon_1}}\right)^i \frac{1}{V_{r_0}(x_0) + V(x_0,x)} \left(\frac{d(x,y)}{r_0 + d(x_0,x)}\right)^{\epsilon_1} \left(\frac{r_0}{r_0 + d(x_0,x)}\right)^{\epsilon}. \tag{3.18}$$

The geometric means of (3.16) and (3.15), (3.17) and (3.15), (3.18) and (3.15), respectively, give that for all $i \in \mathbb{N}$ and $k = 4, 5, 6$,

$$\mathrm{Z}_i^k \lesssim \left(\frac{1-\sigma}{\eta^{\beta}}\right)^i \frac{1}{V_{r_0}(x_0) + V(x_0,x)} \left(\frac{d(x,y)}{r_0 + d(x_0,x)}\right)^{\beta} \left(\frac{r_0}{r_0 + d(x_0,x)}\right)^{\gamma}.$$

Using this and the assumption $\eta^\beta > 1 - \sigma$, we obtain, for all $x, y \in \mathcal{X}$ satisfying $d(x,y) \le (r_0 + d(x_0,x))/2$, the following estimate:

$$|\Phi_L^2(x) - \Phi_L^2(y)| \lesssim \left(\frac{1-\sigma}{\eta^{\beta}}\right)^L \frac{1}{V_{r_0}(x_0) + V(x_0,x)} \left(\frac{d(x,y)}{r_0 + d(x_0,x)}\right)^{\beta} \left(\frac{r_0}{r_0 + d(x_0,x)}\right)^{\gamma}.$$

This together with (3.14) yields that $\lim_{L\to\infty} \|\Phi_L^2\|_{\mathcal{G}(x_0,r_0,\beta,\gamma)} = 0$. Therefore, we obtain $\lim_{L\to\infty} \|\Phi_L\|_{\mathcal{G}(x_0,r_0,\beta,\gamma)} = 0$ and thus $\lim_{L\to\infty} \|\Phi_L\|_{\mathcal{G}(x_1,1,\beta,\gamma)} = 0$, which implies that (3.11) holds in $\mathcal{G}_0^\epsilon(\beta,\gamma)$.

Set

$$\nu \equiv \sum_{i=1}^{\infty}\sum_{j=1}^{j(i)}(1-\sigma)^{i-1}\frac{V_{C_4\eta^i r_{i,j}}(x_{i,j})}{V_{r_{i,j}}(x_{i,j})}\left(\frac{r_0}{r_{i,j}}\right)^{\epsilon}\delta_{(x_{i,j},\eta^i r_{i,j})},$$

where $\delta_{(x,r)}$ is the Dirac measure of point $(x,r) \in \mathcal{X} \times \mathbb{R}_+$. Notice that for any given i and j, by Property (B) above, we have

$$|\langle S_{\eta^i r_{i,j}}(x_{i,j},\cdot), f\rangle| \le \mathcal{M}_0(f)(x_{i,j}) \lesssim [F(\eta^i r_{i,j},\, x_{i,j},\, [\mathcal{M}_0(f)]^{1/2}\chi_{B_{i,j}})]^2.$$

Then, by this and (3.11),

$$\begin{aligned}|\langle f,\varphi\rangle| &\lesssim \sum_{i=1}^{\infty}\sum_{j=1}^{j(i)}(1-\sigma)^{i-1}\frac{V_{C_4\eta^i r_{i,j}}(x_{i,j})}{V_{r_{i,j}}(x_{i,j})}\left(\frac{r_0}{r_{i,j}}\right)^{\epsilon}\left[F\left(\eta^i r_{i,j},\, x_{i,j},\, [\mathcal{M}_0(f)]^{1/2}\chi_{B_{i,j}}\right)\right]^2\\ &= \int_{\mathcal{X}\times\mathbb{R}_+}\left[F\left(r,x,[\mathcal{M}_0(f)]^{1/2}\chi_{B(x,C_4 r)}\right)\right]^2\,d\nu(x,r).\end{aligned}$$

Denote by $\lfloor \log_2 r_0\rfloor$ the largest integer no more than $\log_2 r_0$. Since $r_{i,j} \ge r_0$, we then have

$$\begin{aligned}\nu &\lesssim (r_0)^{\epsilon}\sum_{t=\lfloor \log_2 r_0\rfloor+1}^{\infty}2^{-t\epsilon}V_{2^t}(x_0)^{-1}\sum_{\{i,j:\ 2^{t-1}\le r_{i,j}<2^t\}}(1-\sigma)^i V_{C_4\eta^i r_{i,j}}(x_{i,j})\delta_{(x_{i,j},\eta^i r_{i,j})}\\ &\equiv (r_0)^{\epsilon}\sum_{t=\lfloor \log_2 r_0\rfloor+1}^{\infty}2^{-t\epsilon}\nu_t. \qquad (3.19)\end{aligned}$$

If $2^{t-1} \le r_{i,j} < 2^t$, then for any $z \in B(x_{i,j}, C_4\eta^i r_{i,j})$, we have

$$d(z,x_0) \le d(z,x_{i,j}) + d(x_{i,j},x_0) \le 2r_{i,j} < 2^{t+1},$$

which implies that $B(x_{i,j}, C_4\eta^i r_{i,j}) \subset B(x_0, 2^{t+1})$. From this, it follows that for any $(x,r) \in \operatorname{supp}\nu_t$,

$$F(r,x,[\mathcal{M}_0(f)]^{1/2}\chi_{B(x,C_4 r)}) \le F(r,x,[\mathcal{M}_0(f)]^{1/2}\chi_{B(x_0,2^{t+1})}). \qquad (3.20)$$

By (3.19) and (3.20),

$$|\langle f,\varphi\rangle| \lesssim (r_0)^{\epsilon}\sum_{t=\lfloor \log_2 r_0\rfloor}^{\infty}2^{-t\epsilon}\int_{\mathcal{X}\times\mathbb{R}_+}\left[F\left(r,x,[\mathcal{M}_0(f)]^{1/2}\chi_{B(x,2^{t+1})}\right)\right]^2\,d\nu_t(x,r). \qquad (3.21)$$

We claim that for any fixed $\delta_0 \in (0, \log_\eta(1-\sigma))$, there exists a positive constant C_{δ_0} independent of t such that for all $x \in \mathcal{X}$ and $r > 0$,

$$\nu_t(B(x,r)\times(0,r)) \le C_{\delta_0}\left(\frac{V_r(x)}{V_{2^t}(x_0)}\right)^{(n+\delta_0)/n}. \qquad (3.22)$$

Assume this claim for the moment. By this claim, (3.21) and Lemma 2.2, we have

$$\begin{aligned}|\langle f,\varphi\rangle| &\lesssim (r_0)^\epsilon \sum_{t=\lfloor \log_2 r_0\rfloor+1}^{\infty} 2^{-t\epsilon}\left\{\frac{1}{V_{2^t}(x_0)}\int_{B(x_0,\, C_4 2^{t+1})}[\mathcal{M}_0(f)(x)]^{n/(n+\delta_0)}\,d\mu(x)\right\}^{(n+\delta_0)/n}\\ &\lesssim \left[\mathcal{M}\left((\mathcal{M}_0(f))^{n/(n+\delta_0)}\right)(x_0)\right]^{(n+\delta_0)/n},\end{aligned}$$

where in the last step, we use $(r_0)^\epsilon \sum_{t=\lfloor \log_2 r_0\rfloor+1}^{\infty} 2^{-t\epsilon} \lesssim 1$. Thus the desired conclusion of Proposition 1.7 holds.

To finish the proof of Proposition 1.7, we still need to verify the validity of (3.22). For any $x \in \mathcal{X}$ and $r > 0$, set

$$\mathrm{W} \equiv \{(i,j) :\, 2^{t-1} \le r_{i,j} < 2^t,\, \eta^i r_{i,j} < r,\, d(x, x_{i,j}) < r\}.$$

For any $(i,j) \in W$, since $C_4 < 1$, we have $B(x_{i,j}, C_4\eta^i r_{i,j}) \subset B(x, 2r)$, which further implies that

$$\sum_{\{j:\,(i,j)\in \mathrm{W}\}} V_{C_4\eta^i r_{i,j}}(x_{i,j}) \lesssim V_r(x). \tag{3.23}$$

If $(i,j) \in \mathrm{W}$, then $2^t < 2r_{i,j} < 2r\eta^{-i}$. From this, we deduce that for any $z \in B(x_0, 2^t)$,

$$d(z,x) \le d(z,x_0) + d(x_0,x_{i,j}) + d(x_{i,j},x) \le 2^t + 2^t + r \le (4\eta^{-i}+1)r,$$

and thus $B(x_0, 2^t) \subset B(x, (4\eta^{-i}+1)r)$. Moreover, by (1.2),

$$V_{2^t}(x_0) \lesssim (4\eta^{-i}+1)^n V_r(x) \lesssim \eta^{-in} V_r(x). \tag{3.24}$$

Combining (3.19), (3.23), (3.24) and the assumption $\delta_0 < \log_\eta(1-\sigma)$ yields that

$$\begin{aligned}&\nu_t(B(x,r)\times(0,r))\\ &\quad= [V_{2^t}(x_0)]^{-1}\int_{B(x,r)\times(0,r)} \sum_{\{i,\,j:\,2^{t-1}\le r_{i,j}<2^t\}} (1-\sigma)^i V_{C_4\eta^i r_{i,j}}(x_{i,j})\,d\delta_{(x_{i,j},\eta^i r_{i,j})}\\ &\quad\lesssim [V_{2^t}(x_0)]^{-1}\sum_{(i,\,j)\in\mathrm{W}} (1-\sigma)^i V_{C_4\eta^i r_{i,j}}(x_{i,j})\left(\frac{\eta^{-in}V_r(x)}{V_{2^t}(x_0)}\right)^{\delta_0/n}\\ &\quad\lesssim \left(\frac{V_r(x)}{V_{2^t}(x_0)}\right)^{(n+\delta_0)/n}\sum_{i\in\mathbb{N}}\left(\frac{1-\sigma}{\eta^{\delta_0}}\right)^i \lesssim \left(\frac{V_r(x)}{V_{2^t}(x_0)}\right)^{(n+\delta_0)/n},\end{aligned}$$

which implies the claim, and hence completes the proof of Proposition 1.7.

Remark 3.1. *By the proof of Theorem 1.6, we need the assumption* $p > n/(n+\delta_0)$*, while in the proof of Proposition 1.7, we also need* $\delta_0 < \log_\eta(1-\sigma)$*. Thus, Theorem 1.6 holds for any* $p \in (n/(n+\log_\eta(1-\sigma)), \infty]$*. In fact, Theorem 1.6 and Proposition 1.7 hold for all* $\sigma > 0$ *and* $\eta > 0$*, if* σ*,* η*,* $A > 0$ *in* (3.1) *and* $H > 0$ *in* (3.5) *satisfy the following* *five inequalities:* $H \le 1/3$; $C_4\eta < H$; $\eta^{\epsilon_1} < 1-\sigma < \eta^{\beta}$; $\frac{A}{C_8 2^{\epsilon}} - AC_3C_9 \max\{\eta^{\epsilon_2}, (1 + H\eta^{-1})^{-\epsilon_2}\} \ge 1$; *and*

$$\left(\frac{H}{1-H}\right)^{\epsilon} + \left(\frac{H}{1-H}\right)^{\epsilon_1} \frac{AC_3C_9\sigma}{1-\sigma-\eta^{\epsilon_1}} \le 1-\sigma - AC_3C_9\sigma.$$

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