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Maximal function characterizations of Hardy spaces on RD-spaces and their applications

Loukas GRAFAKOS[1], LIU LiGuang[2] & YANG DaChun[2†]

[1] Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
[2] School of Mathematical Sciences, Beijing Normal University; Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China
(email: loukas@math.missouri.edu, liuliguang@mail.bnu.edu.cn, dcyang@bnu.edu.cn)

Abstract Let $\mathcal{X}$ be an RD-space, i.e., a space of homogeneous type in the sense of Coifman and Weiss, which has the reverse doubling property. Assume that $\mathcal{X}$ has a "dimension" n. For $\alpha \in (0,\infty)$ denote by $H^p_\alpha(\mathcal{X})$, $H^p_{\rm d}(\mathcal{X})$, and $H^{*,p}(\mathcal{X})$ the corresponding Hardy spaces on $\mathcal{X}$ defined by the nontangential maximal function, the dyadic maximal function and the grand maximal function, respectively. Using a new inhomogeneous Calderón reproducing formula, it is shown that all these Hardy spaces coincide with $L^p(\mathcal{X})$ when $p \in (1,\infty]$ and with each other when $p \in (n/(n+1),1]$. An atomic characterization for $H^{*,p}(\mathcal{X})$ with $p \in (n/(n+1),1]$ is also established; moreover, in the range $p \in (n/(n+1),1]$, it is proved that the space $H^{*,p}(\mathcal{X})$, the Hardy space $H^p(\mathcal{X})$ defined via the Littlewood-Paley function, and the atomic Hardy space of Coifman and Weiss coincide. Furthermore, it is proved that a sublinear operator T uniquely extends to a bounded sublinear operator from $H^p(\mathcal{X})$ to some quasi-Banach space $\mathcal{B}$ if and only if T maps all (p,q)-atoms when $q \in (p,\infty)\cap[1,\infty)$ or continuous (p,∞)-atoms into uniformly bounded elements of $\mathcal{B}$.

Keywords: space of homogeneous type, Calderón reproducing formula, space of test function, maximal function, Hardy space, atom, Littlewood-Paley function, sublinear operator, quasi-Banach space

MSC(2000): 42B25, 42B30, 47B38, 47A30

1 Introduction

The theory of Hardy spaces on the Euclidean space $\mathbb{R}^n$ certainly plays an important role in harmonic analysis and has been systematically studied and developed; see, for example, [1–3]. It is well known that spaces of homogeneous type in the sense of Coifman and Weiss[4] present a natural setting for the Calderón-Zygmund theory of singular integrals; see, for instance, [5]. Recall that a space of homogeneous type $\mathcal{X}$ is a set equipped with a metric d and a Borel regular measure μ that satisfies the doubling property. Coifman and Weiss[5] introduced the atomic Hardy spaces $H^p_{\rm at}(\mathcal{X})$ for $p \in (0,1]$. Under the assumption that the measure of any ball in $\mathcal{X}$ is equivalent to its radius (i.e., $\mathcal{X}$ is an Ahlfors 1-regular metric measure space), Coifman and Weiss[5] established a molecular characterization for $H^1_{\rm at}(\mathcal{X})$. Also in this setting,

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† Corresponding author
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if $p \in (1/2, 1]$, Macías and Segovia[6] obtained the grand maximal function characterization for $H^p_{\rm at}(\mathcal{X})$ via distributions acting on certain spaces of Lipschitz functions; Han[7] established a Lusin-area characterization for $H^p_{\rm at}(\mathcal{X})$; Duong and Yan[8] characterized these atomic Hardy spaces in terms of Lusin area functions associated with certain Poisson semigroups; Li[9] also characterized $H^p_{\rm at}(\mathcal{X})$ by grand maximal functions defined via test functions introduced in [10].

Structures of spaces of homogeneous type encompass several important examples in harmonic analysis, such as Euclidean spaces with A_∞-weights (of the Muckenhoupt class), Ahlfors n-regular metric measure spaces (see, for example, [11, 12]), Lie groups of polynomial growth (see, for instance, [13–15]), and Carnot-Carathéodory spaces with doubling measures (see [16–22]). All these examples fall under the scope of the study of RD-spaces introduced in [23] (see also [24]). An RD-space $\mathcal{X}$ is a space of homogeneous type which has a "dimension" n and satisfies the following reverse doubling property: there exists a constant $a_0 > 1$ such that for all $x \in \mathcal{X}$ and $0 < r < \operatorname{diam}(\mathcal{X})/a_0$, $B(x, a_0 r) \setminus B(x, r) \neq \emptyset$, where in this article $\operatorname{diam}(\mathcal{X}) := \sup_{x,\, y \in \mathcal{X}} d(x, y)$. A Littlewood-Paley theory of Hardy spaces on RD-spaces was established in [24], and these Hardy spaces are known to coincide with some of Triebel-Lizorkin spaces in [23].

Let $\mathcal{X}$ be an RD-space with "dimension" n. In this paper we achieve two goals. First, we introduce various Hardy spaces on $\mathcal{X}$ via the nontangential maximal function, the dyadic maximal function, and the grand maximal function, and show that these Hardy spaces coincide with $L^p(\mathcal{X})$ when $p \in (1, \infty]$ and with each other when $p \in (n/(n+1), 1]$. When $p \in (n/(n+1), 1]$, we further identify these Hardy spaces with the Hardy space $H^p(\mathcal{X})$ defined via the Littlewood-Paley function in [23, 24] and with the atomic Hardy space of Coifman and Weiss in [5]. Secondly, we prove that a sublinear operator T uniquely extends to a bounded sublinear operator from $H^p(\mathcal{X})$ to some quasi-Banach space $\mathcal{B}$ if and only if T maps all (p, q)-atoms when $q \in (p, \infty) \cap [1, \infty)$ or continuous (p, ∞)-atoms into uniformly bounded elements of $\mathcal{B}$. This last result answers a question posed by Meda, Sjögren and Vallarino in [25].

To be precise, in Section 2, we introduce various Hardy spaces $H^p_\alpha(\mathcal{X})$ for $\alpha \in (0, \infty)$, $H^p_{\rm d}(\mathcal{X})$, and $H^{*,\,p}(\mathcal{X})$, defined in terms of the nontangential maximal function, the dyadic maximal function, and the grand maximal function, respectively. We also introduce a slight variant of test functions in [23, 24], which is crucial in applications; see Remark 2.9 (iii) below.

One of the contributions of this paper is to establish a new inhomogeneous discrete Calderón reproducing formula; see Theorem 3.3 of Section 3. This formula plays a key role in the whole paper and may be useful in the study of other problems. Moreover, in Section 3, we prove that all Hardy spaces mentioned above coincide with $L^p(\mathcal{X})$ when $p \in (1, \infty]$, and all these Hardy spaces are equivalent to each other when $p \in (n/(n+1), 1]$.

Section 4 is devoted to obtaining an atomic characterization for the Hardy space $H^{*,\,p}(\mathcal{X})$ when $p \in (n/(n+1), 1]$ by using certain ideas from [6].

Some applications are given in Section 5. First, when $p \in (n/(n+1), 1]$, we prove that $H^{*,\,p}(\mathcal{X})$ and $H^p(\mathcal{X})$ coincide by means of their atomic characterizations, where $H^p(\mathcal{X})$ is the Hardy space defined via the Littlewood-Paley function in [23, 24]; see Theorem 5.4. From a key observation (Lemma 5.3) which establishes the connection between the space of test functions and the Lipschitz space introduced by Coifman and Weiss (cf. [5, (2.2)]), it follows directly that $H^p(\mathcal{X})$ and the atomic Hardy space of Coifman and Weiss[5] coincide; see Remark 5.5 (ii). This also answers a question in [24, Remark 2.30].

Our second application concerns the boundedness of sublinear operators from $H^p(\mathcal{X})$ to some quasi-Banach space $\mathcal{B}$. It is well-known that atomic characterizations play an important role in obtaining boundedness of sublinear operators in Hardy spaces, namely, boundedness of these operators can be deduced from their behavior on atoms in principle. However, Meyer in [26, p. 513] (see also [27, 28]) gave an example of $f \in H^1(\mathbb{R}^n)$ whose norm cannot be achieved by its finite atomic decompositions via $(1, \infty)$-atoms. Based on this fact, Bownik[28, Theorem2] constructed a surprising example of a linear functional defined on a dense subspace of $H^1(\mathbb{R}^n)$, which maps all $(1, \infty)$-atoms into bounded scalars, but cannot extend to a bounded linear functional on the whole $H^1(\mathbb{R}^n)$. It follows that proving that a linear operator T maps all (p, ∞)-atoms into uniformly bounded elements of $\mathcal{B}$ does not guarantee the boundedness of T from the entire $H^p(\mathbb{R}^n)$ (with $p \in (0, 1]$) to some quasi-Banach space $\mathcal{B}$; this phenomenon was essentially observed by Meyer in [29, p. 19]. Motivated by this, Yabuta[30] gave some sufficient conditions for the boundedness of T from $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$ to $L^q(\mathbb{R}^n)$ with $q \geqslant 1$ or $H^q(\mathbb{R}^n)$ with $q \in [p, 1]$. It was proved in [31] that a linear operator T extends to a bounded linear operator from Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$ to some quasi-Banach space $\mathcal{B}$ if and only if T maps all $(p, 2, s)$-atoms for some $s \geqslant \lfloor n(1/p - 1) \rfloor$ into uniformly bounded elements of $\mathcal{B}$, where $\lfloor n(1/p-1) \rfloor$ denotes the maximal integer no more than $n(1/p-1)$. For $q \in (1, \infty]$, denote by $H^{1,q}_{\rm fin}(\mathbb{R}^n)$ the vector space of all finite linear combinations of $(1, q)$-atoms endowed with the following norm:

$$\|f\|_{H^{1,q}_{\rm fin}(\mathbb{R}^n)} := \inf\left\{ \sum_{j=1}^{N} |\lambda_j| : f = \sum_{j=1}^{N} \lambda_j a_j,\ N \in \mathbb{N},\ \{\lambda_j\}_{j=1}^N \subset \mathbb{C}, \text{ and } \{a_j\}_{j=1}^N \text{ are } (1, q)\text{-atoms} \right\}.$$

Recently, by means of the maximal characterization of $H^1(\mathbb{R}^n)$, Meda, Sjögren and Vallarino[25] proved that $\|\cdot\|_{H^1(\mathbb{R}^n)}$ and $\|\cdot\|_{H^{1,q}_{\rm fin}(\mathbb{R}^n)}$ are equivalent quasi-norms on $H^{1,q}_{\rm fin}(\mathbb{R}^n)$ with $q \in (1, \infty)$ or on $H^{1,\infty}_{\rm fin}(\mathbb{R}^n) \cap \mathcal{C}(\mathbb{R}^n)$, where $\mathcal{C}(\mathbb{R}^n)$ denotes the set of continuous functions. (It was also claimed in [25] that this equivalence of norms remains true for $H^p(\mathbb{R}^n)$ and (p, q)-atoms with $p \in (0, 1)$ and $q \in [1, \infty]$.) From this these authors deduced that a linear operator defined on $H^{1,q}_{\rm fin}(\mathbb{R}^n)$ which maps $(1, q)$-atoms or continuous $(1, \infty)$-atoms into uniformly bounded elements of some Banach space $\mathcal{B}$ uniquely extends to a bounded linear operator from $H^1(\mathbb{R}^n)$ to $\mathcal{B}$. In [32], the results in [25] were generalized to weighted Hardy spaces on $\mathbb{R}^n$ with a general expansive matrix dilation. In [25], Meda, Sjögren and Vallarino also pointed out that it is not evident whether their results for $H^p(\mathbb{R}^n)$ can be extended to Hardy spaces on spaces of homogeneous type. In Section 5, using some ideas from [25], we give an affirmative answer to this question; see Theorems 5.6 and 5.9 below. We should mention that the result of [30] was generalized to Ahlfors 1-regular metric measure spaces in [33], and the result of [31] was extended to RD-spaces in [34]. Also, by using the dual theory, Meda, Sjögren and Vallarino[25] showed that T extends uniquely to a bounded linear operator from $H^1(\mathcal{X})$ to $L^1(\mathcal{X})$ if and only if T maps all $(1, q)$-atoms with $q \in (1, \infty)$ to uniformly bounded elements of $L^1(\mathcal{X})$, where $\mathcal{X}$ is a space of homogeneous type. However, this result is valid only for linear operators and the

Hardy space $H^1(\mathcal{X})$.

Finally, we mention that a radial maximal function characterization of these Hardy spaces was also recently given in [35].

In this paper we use the following notation: $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_+ := [0, \infty)$. For any $p \in [1, \infty]$, we denote by p' the conjugate index, namely, $1/p + 1/p' = 1$. We also denote by C a positive constant independent of the main parameters involved, which may vary at different occurrences. Constants with subscripts do not change through the whole paper. We use $f \lesssim g$ and $f \gtrsim g$ to denote $f \leqslant Cg$ and $f \geqslant Cg$, respectively. If $f \lesssim g \lesssim f$, we then write $f \sim g$. For any $a, b \in \mathbb{R}$, set $(a \wedge b) := \min\{a, b\}$ and $(a \vee b) := \max\{a, b\}$.

2 Preliminaries

We first recall the notions of spaces of homogeneous type in the sense of [4, 5] and RD-spaces in [23].

Definition 2.1. *Let $(\mathcal{X}, d)$ be metric space with Borel regular measure μ such that all balls defined by d have finite and positive measures. For any $x \in \mathcal{X}$ and $r > 0$, set $B(x, r) := \{y \in \mathcal{X} : d(x, y) < r\}$.*

(i) *The triple $(\mathcal{X}, d, \mu)$ is called a space of homogeneous type if there exists a constant $C_1 \geqslant 1$ such that for all $x \in \mathcal{X}$ and $r > 0$,*

$$\mu(B(x, 2r)) \leqslant C_1 \mu(B(x, r)) \quad (\textit{doubling property}). \tag{2.1}$$

(ii) *Let $0 < \kappa \leqslant n$. The triple $(\mathcal{X}, d, \mu)$ is called a (κ, n)-space if there exist constants $0 < C_2 \leqslant 1$ and $C_3 \geqslant 1$ such that for all $x \in \mathcal{X}$, $0 < r < \operatorname{diam}(\mathcal{X})/2$ and $1 \leqslant \lambda < \operatorname{diam}(\mathcal{X})/(2r)$,*

$$C_2 \lambda^\kappa \mu(B(x, r)) \leqslant \mu(B(x, \lambda r)) \leqslant C_3 \lambda^n \mu(B(x, r)), \tag{2.2}$$

where $\operatorname{diam}(\mathcal{X}) = \sup_{x,\, y \in \mathcal{X}} d(x, y)$.

A space of homogeneous type is called an RD*-space, if it is a (κ, n)-space for some $0 < \kappa \leqslant n$, i. e., if some "reverse" doubling condition holds.*

Remark 2.2. (i) In some sense, n measures the "dimension" of $\mathcal{X}$. Obviously a (κ, n) space is a space of homogeneous type with $C_1 := C_3 2^n$. Conversely, a space of homogeneous type satisfies the second inequality of (2.2) with $C_3 := C_1$ and $n := \log_2 C_1$.

(ii) If μ is doubling, then μ satisfies (2.2) if and only if there exist constants $a_0 > 1$ and $\widetilde{C}_0 > 1$ such that for all $x \in \mathcal{X}$ and $0 < r < \operatorname{diam}(\mathcal{X})/a_0$,

$$\mu(B(x, a_0 r)) \geqslant \widetilde{C}_0 \mu(B(x, r)) \quad \text{(reverse doubling property)}$$

(If $a_0 = 2$, this is the classical reverse doubling condition), and equivalently, for all $x \in \mathcal{X}$ and $0 < r < \operatorname{diam}(\mathcal{X})/a_0$, $B(x, a_0 r) \setminus B(x, r) \neq \emptyset$; see [23, 36].

(iii) From (ii) of this remark, it follows that if $\mathcal{X}$ is an RD-space, then $\mu(\{x\}) = 0$ for all $x \in \mathcal{X}$.

(iv) Let d be a quasi-metric, which means that there exists $A_0 \geqslant 1$ such that for all x, y, $z \in \mathcal{X}$, $d(x, y) \leqslant A_0(d(x, z) + d(z, y))$. Recall that Macías and Segovia[37, Theorem 2] proved that there exists an equivalent quasi-metric $\widetilde{d}$ such that all balls corresponding to $\widetilde{d}$ are open in the topology induced by $\widetilde{d}$, and there exist constants $A_0' > 0$ and $\theta \in (0,1)$ such that for all x, y, $z \in \mathcal{X}$,

$$|\widetilde{d}(x,z) - \widetilde{d}(y,z)| \leqslant A_0' \widetilde{d}(x,y)^{\theta} [\widetilde{d}(x,z) + \widetilde{d}(y,z)]^{1-\theta}.$$

It is known that the approximation of the identity as in Definition 2.3 below also exists for $\widetilde{d}$; see [23]. Obviously, all results in this paper are invariant on equivalent quasi-metrics. From these facts, we deduce that all conclusions of this paper are still valid for quasi-metrics.

In this paper, we always assume that $\mathcal{X}$ is an RD-space and $\mu(\mathcal{X}) = \infty$. For any x, $y \in \mathcal{X}$ and $\delta > 0$, set $V_\delta(x) := \mu(B(x,\delta))$ and $V(x,y) := \mu(B(x,d(x,y)))$. It follows from (2.1) that $V(x,y) \sim V(y,x)$. The following notion of approximations of the identity on RD-spaces were first introduced in [23]; see also [24].

Definition 2.3. *Let $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$ and $\epsilon_3 > 0$. A sequence $\{S_k\}_{k\in\mathbb{Z}}$ of bounded linear integral operators on $L^2(\mathcal{X})$ is said to be an approximation of the identity of order $(\epsilon_1,\epsilon_2,\epsilon_3)$ (in short, $(\epsilon_1,\epsilon_2,\epsilon_3)$-* AOTI*), if there exists a positive constant C_4 such that for all $k \in \mathbb{Z}$ and all x, x', y and $y' \in \mathcal{X}$, $S_k(x,y)$, the integral kernel of S_k is a function from $\mathcal{X}\times\mathcal{X}$ into $\mathbb{C}$ satisfying*

(i) $|S_k(x,y)| \leqslant C_4 \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)} \frac{2^{-k\epsilon_2}}{(2^{-k}+d(x,y))^{\epsilon_2}}$;

(ii) $|S_k(x,y) - S_k(x',y)| \leqslant C_4 \frac{d(x,x')^{\epsilon_1}}{(2^{-k}+d(x,y))^{\epsilon_1}} \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)} \frac{2^{-k\epsilon_2}}{(2^{-k}+d(x,y))^{\epsilon_2}}$ *for* $d(x,x') \leqslant (2^{-k}+d(x,y))/2$;

(iii) *Property* (ii) *holds with x and y interchanged*;

(iv) $|[S_k(x,y) - S_k(x,y')] - [S_k(x',y) - S_k(x',y')]| \leqslant C_4 \frac{d(x,x')^{\epsilon_1}}{(2^{-k}+d(x,y))^{\epsilon_1}} \frac{d(y,y')^{\epsilon_1}}{(2^{-k}+d(x,y))^{\epsilon_1}}$ $\times \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)} \frac{2^{-k\epsilon_3}}{(2^{-k}+d(x,y))^{\epsilon_3}}$ *for* $d(x,x') \leqslant (2^{-k}+d(x,y))/3$ *and* $d(y,y') \leqslant (2^{-k}+d(x,y))/3$;

(v) $\int_{\mathcal{X}} S_k(x,y)\,d\mu(y) = \int_{\mathcal{X}} S_k(x,y)\,d\mu(x) = 1$.

Remark 2.4. It was proved in [23, Theorem 2.1] that for any ϵ_1, ϵ_2, ϵ_3 as in Definition 2.3, there exists an $(\epsilon_1,\epsilon_2,\epsilon_3)$- AOTI $\{S_k\}_{k\in\mathbb{Z}}$ with bounded support on $\mathcal{X}$, which means that there exists a positive constant C such that for all $k\in\mathbb{Z}$ and $d(x,y) \geqslant C2^{-k}$, $S_k(x,y) = 0$.

The following space of test functions plays a key role in this paper, which is an equivalent variant of the space of test functions in [23, Definition 2.3]; see also [24].

Definition 2.5. *Let $x_1 \in \mathcal{X}$, $r \in (0,\infty)$, $\beta \in (0,1]$ and $\gamma \in (0,\infty)$. A function φ on $\mathcal{X}$ is said to be a test function of type (x_1,r,β,γ) if*

(i) $|\varphi(x)| \leqslant C \frac{1}{\mu(B(x,r+d(x,x_1)))} \big(\frac{r}{r+d(x_1,x)}\big)^{\gamma}$ *for all* $x \in \mathcal{X}$;

(ii) $|\varphi(x) - \varphi(y)| \leqslant C \big(\frac{d(x,y)}{r+d(x_1,x)}\big)^{\beta} \frac{1}{\mu(B(x,r+d(x,x_1)))} \big(\frac{r}{r+d(x_1,x)}\big)^{\gamma}$ *for all* x, $y \in \mathcal{X}$ *satisfying* $d(x,y) \leqslant (r+d(x_1,x))/2$.

We denote by $\mathcal{G}(x_1,r,\beta,\gamma)$ the set of all test functions of type (x_1,r,β,γ). If $\varphi \in \mathcal{G}(x_1,r,\beta,\gamma)$, we define its norm by $\|\varphi\|_{\mathcal{G}(x_1,r,\beta,\gamma)} := \inf\{C :$ (i) and (ii) hold$\}$. The space $\mathcal{G}(x_1,r,\beta,\gamma)$ is called the space of test functions.

Remark 2.6. (i) If $\mu(B(x,r+d(x,\,x_1)))$ is replaced by $V_r(x_1)+V(x_1,x)$ in Definition 2.5, then $\mathcal{G}(x_1,r,\beta,\gamma)$ was introduced in [23, Definition 2.3]; see also [24].

(ii) Notice that $\mu(B(x,r+d(x,x_1))) \sim V_r(x_1)+V(x_1,x)$ by (2.2). This implies that both spaces of test functions in Definition 2.5 and [23, Definition 2.3] are equivalent and with equivalent norms. Moreover, see Remark 2.9 (iii) for the advantage of Definition 2.5.

Throughout the paper, we fix $x_1 \in \mathcal{X}$. Let $\mathcal{G}(\beta,\gamma) := \mathcal{G}(x_1,1,\beta,\gamma)$. It is easy to see that for any $x_2 \in \mathcal{X}$ and $r > 0$, we have $\mathcal{G}(x_2,r,\beta,\gamma) = \mathcal{G}(\beta,\gamma)$ with equivalent norms (but with constants depending on x_1, x_2 and r). Furthermore, $\mathcal{G}(\beta,\gamma)$ is a Banach space.

For any given $\epsilon \in (0,1]$, let $\mathcal{G}_0^\epsilon(\beta,\gamma)$ be the completion of the space $\mathcal{G}(\epsilon,\epsilon)$ in $\mathcal{G}(\beta,\gamma)$ when β, $\gamma \in (0,\epsilon]$. Obviously $\mathcal{G}_0^\epsilon(\epsilon,\epsilon) = \mathcal{G}(\epsilon,\epsilon)$. Moreover, $\varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma)$ if and only if $\varphi \in \mathcal{G}(\beta,\gamma)$ and there exists $\{\phi_i\}_{i\in\mathbb{N}} \subset \mathcal{G}(\epsilon,\epsilon)$ such that $\|\varphi - \phi_i\|_{\mathcal{G}(\beta,\gamma)} \to 0$ as $i \to \infty$. If $\varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma)$, define $\|\varphi\|_{\mathcal{G}_0^\epsilon(\beta,\gamma)} := \|\varphi\|_{\mathcal{G}(\beta,\gamma)}$. Obviously $\mathcal{G}_0^\epsilon(\beta,\gamma)$ is a Banach space and $\|\varphi\|_{\mathcal{G}_0^\epsilon(\beta,\gamma)} = \lim_{i\to\infty} \|\phi_i\|_{\mathcal{G}(\beta,\gamma)}$ for the above chosen $\{\phi_i\}_{i\in\mathbb{N}}$.

We define

$$\mathring{\mathcal{G}}(x_1,r,\beta,\gamma) := \left\{\varphi \in \mathcal{G}(x_1,r,\beta,\gamma) : \int_{\mathcal{X}} \varphi(x)\,d\mu(x) = 0\right\},$$

which is called the space of test functions with mean zero. The space $\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$ is defined to be the completion of $\mathring{\mathcal{G}}(\epsilon,\epsilon)$ in $\mathring{\mathcal{G}}(\beta,\gamma)$ when β, $\gamma \in (0,\epsilon]$. Moreover, if $\varphi \in \mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$, we then define $\|\varphi\|_{\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)} := \|\varphi\|_{\mathcal{G}(\beta,\gamma)}$.

The notation $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ denotes the dual space of $\mathcal{G}_0^\epsilon(\beta,\gamma)$, that is, the set of all linear functionals f from $\mathcal{G}_0^\epsilon(\beta,\gamma)$ to $\mathbb{C}$ with the property that there exists a positive constant C such that for all $\varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma)$, $|\langle f,\varphi\rangle| \leqslant C\|\varphi\|_{\mathcal{G}(\beta,\gamma)}$. We denote by $\langle f,\varphi\rangle$ the natural pairing of elements $f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and $\varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma)$. Similarly, $(\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$ denotes the set of all bounded linear functionals from $\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$ to $\mathbb{C}$.

The following cube constructions, which provide an analogue of the grid of Euclidean dyadic cubes on spaces of homogeneous type, were given by Christ[38].

Lemma 2.7. *Let $\mathcal{X}$ be a space of homogeneous type. Then there exists a collection $\{Q_\alpha^k \subset \mathcal{X} : k \in \mathbb{Z}, \alpha \in I_k\}$ of open subsets, where I_k is some index set, and constants $\delta \in (0,1)$ and C_5, $C_6 > 0$ such that*

(i) *$\mu(\mathcal{X} \setminus \cup_\alpha Q_\alpha^k) = 0$ for each fixed k and $Q_\alpha^k \cap Q_\beta^k = \emptyset$ if $\alpha \neq \beta$;*

(ii) *for any α, β, k, ℓ with $\ell \geqslant k$, either $Q_\beta^\ell \subset Q_\alpha^k$ or $Q_\beta^\ell \cap Q_\alpha^k = \emptyset$;*

(iii) *for each (k,α) and each $\ell < k$, there is a unique β such that $Q_\alpha^k \subset Q_\beta^\ell$;*

(iv) *$\operatorname{diam}(Q_\alpha^k) \leqslant C_5\delta^k$;*

(v) *each Q_α^k contains some ball $B(z_\alpha^k, C_6\delta^k)$, where $z_\alpha^k \in \mathcal{X}$.*

In fact, we can think of Q_α^k as being a dyadic cube with diameter rough δ^k centered at z_α^k. In what follows, for simplicity, we may assume that $\delta = 1/2$; see [23, p. 25] or [10, pp. 96–98] for how to remove this restriction. For any dyadic cube Q and any function g, set

$$m_Q(g) := \frac{1}{\mu(Q)} \int_Q g(x)\,d\mu(x).$$

Let ϵ_1, ϵ_2 and $\{S_k\}_{k\in\mathbb{Z}}$ be as in Definition 2.3 and $\epsilon \in (0,\epsilon_1 \wedge \epsilon_2)$. Observe that for any fixed $x \in \mathcal{X}$, $S_k(x,\cdot) \in \mathcal{G}_0^\epsilon(\beta,\gamma)$ with β, $\gamma \in (0,\epsilon)$. In what follows, for any $k \in \mathbb{Z}$, $f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and $x \in \mathcal{X}$, we define $S_k(f)(x) := \langle f, S_k(x,\cdot)\rangle$.

Definition 2.8. *Let $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$, $\epsilon_3 > 0$, $\epsilon \in (0,\epsilon_1 \wedge \epsilon_2)$ and $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1,\epsilon_2,\epsilon_3)$-* AOTI*. Let $p \in (0,\infty]$, $\alpha \in (0,\infty)$ and $f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))'$ with some β, $\gamma \in (0,\epsilon)$. For any $x \in \mathcal{X}$,* *the grand maximal function of f is defined by*

$$f^*(x) := \sup\{|\langle f, \varphi\rangle| : \varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma),\ \|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)} \leqslant 1 \text{ for some } r > 0\}; \tag{2.3}$$

the nontangential maximal function of f is defined by

$$\mathcal{M}_\alpha(f)(x) := \sup_{k\in\mathbb{Z}} \sup_{d(x,\,y)\leqslant \alpha 2^{-k}} |S_k(f)(y)|; \tag{2.4}$$

and the dyadic maximal function of f is defined by

$$\mathcal{M}_d(f)(x) := \sup_{k\in\mathbb{Z},\,\alpha\in I_k} m_{Q_\alpha^k}(|S_k(f)|)\chi_{Q_\alpha^k}(x), \tag{2.5}$$

where $\{Q_\alpha^k\}_{k\in\mathbb{Z},\,\alpha\in I_k}$ is as in Lemma 2.7.

The corresponding Hardy spaces are defined, respectively, by

$$H^{*,\,p}(\mathcal{X}) := \{f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))' : \|f^*\|_{L^p(\mathcal{X})} < \infty\},$$
$$H_\alpha^p(\mathcal{X}) := \{f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))' : \|\mathcal{M}_\alpha(f)\|_{L^p(\mathcal{X})} < \infty\}$$

and $H_{\rm d}^p(\mathcal{X}) := \{f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))' : \|\mathcal{M}_d(f)\|_{L^p(\mathcal{X})} < \infty\}$. Moreover, we define $\|f\|_{H^{*,\,p}(\mathcal{X})} := \|f^*\|_{L^p(\mathcal{X})}$, $\|f\|_{H_\alpha^p(\mathcal{X})} := \|\mathcal{M}_\alpha(f)\|_{L^p(\mathcal{X})}$ and $\|f\|_{H_{\rm d}^p(\mathcal{X})} := \|\mathcal{M}_d(f)\|_{L^p(\mathcal{X})}$.

Remark 2.9. (i) Lemma 2.7 implies that $\mathcal{M}_d(f)(x)$ can be equivalently defined by

$$\mathcal{M}_d(f)(x) := \sup\{m_{Q_\alpha^k}(|S_k(f)|) : k \in \mathbb{Z},\, \alpha \in I_k,\, Q_\alpha^k \ni x\}, \tag{2.6}$$

where $\{Q_\alpha^k\}_{k\in\mathbb{Z},\alpha\in I_k}$ is as in Lemma 2.7.

(ii) Let $\{S_k\}_{k\in\mathbb{Z}}$ and α be as in Definition 2.8. Observing that there exists a positive constant C_α such that $\sup_{k\in\mathbb{Z}} \sup_{d(x,\,y)\leqslant\alpha 2^{-k}} \|S_k(y,\cdot)\|_{\mathcal{G}(x,2^{-k},\epsilon_1,\epsilon_2)} =: C_\alpha < \infty$, we then obtain that for all $x \in \mathcal{X}$, $\mathcal{M}_\alpha(f)(x) \leqslant C_\alpha f^*(x)$.

(iii) For any $\lambda \in (0,\infty)$, the set $\Omega := \{x \in \mathcal{X} : f^*(x) > \lambda\}$ is open, which is a key fact used in Section 4. In fact, if $x_0 \in \Omega$, then there exists $\varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma)$ such that $\|\varphi\|_{\mathcal{G}(x_0,r,\beta,\gamma)} \leqslant 1$ for some $r > 0$ and $|\langle f,\varphi\rangle| > \lambda$. Let η be a sufficiently small positive quantity such that $|\langle f,\varphi\rangle| > \lambda(1+\eta)$. For any $x \in B(x_0, r - r/(1+\eta)^{1/\gamma})$, choose $s \in (0,\infty)$ satisfying $r/(1+\eta)^{1/\gamma} < s < r - d(x,x_0)$. It is easy to verify that $\|\varphi\|_{\mathcal{G}(x,s,\beta,\gamma)} < 1+\eta$. Thus $f^*(x) > \lambda$ for all $x \in B(x_0, r - r/(1+\eta)^{1/\gamma})$, which implies that Ω is open.

(iv) It is proved in Corollary 3.11 below that if $p \in (1,\infty]$, then $H^{*,\,p}(\mathcal{X})$, $H_\alpha^p(\mathcal{X})$ and $H_{\rm d}^p(\mathcal{X})$ are all equivalent to $L^p(\mathcal{X})$.

(v) Let $\epsilon \in (0,1)$ and $p \in (n/(n+\epsilon), 1]$. It is proved in Remark 3.15 below that if $n(1/p-1) < \beta,\ \gamma < \epsilon$, then the definitions of Hardy spaces $H^{*,\,p}(\mathcal{X})$, $H_\alpha^p(\mathcal{X})$ and $H_{\rm d}^p(\mathcal{X})$ are independent of the choices of AOTI's and $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$. Moreover, they coincide with each other.

3 Maximal function characterizations

The following basic properties concerned with RD-spaces are useful throughout the paper, which are proved in [23, Lemma 2.1, Propositions 2.2 and 3.1]. In what follows, for any $f \in L^1_{\rm loc}(\mathcal{X})$, the centered Hardy-Littlewood maximal function $\mathcal{M}(f)$ is defined by that for all $x \in \mathcal{X}$,

$$\mathcal{M}(f)(x) := \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,\,r)} |f(y)|\, d\mu(y).$$

Lemma 3.1. *Let $\delta > 0$, $a > 0$, $r > 0$ and $\theta \in (0,1)$. Then,*

(i) *For all x, $y \in \mathcal{X}$ and $r > 0$, $\mu(B(x, r + d(x,y))) \sim \mu(B(y, r + d(x,y)))$.*

(ii) *If x, x', $x_1 \in \mathcal{X}$ satisfy $d(x,x') \leqslant \theta(r + d(x,x_1))$, then $r + d(x,x_1) \sim r + d(x',x_1)$ and $\mu(B(x, r + d(x,x_1))) \sim \mu(B(x', r + d(x',x_1)))$.*

(iii) *$\int_{\mathcal{X}} \frac{1}{\mu(B(x,r+d(x,y)))}(\frac{r}{r+d(x,y)})^a d(x,y)^\eta \, d\mu(x) \leqslant Cr^\eta$ uniformly in $x \in \mathcal{X}$ and $r > 0$ if $a > \eta \geqslant 0$.*

(iv) *For all $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and $x \in \mathcal{X}$, $\int_{d(x,y)>\delta} \frac{1}{V(x,y)} \frac{\delta^a}{d(x,y)^a} |f(y)| \, d\mu(y) \leqslant C\mathcal{M}(f)(x)$ uniformly in $\delta > 0$, $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and $x \in \mathcal{X}$.*

(v) *Let $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1, \epsilon_2, \epsilon_3)$-AOTI with $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$ and $\epsilon_3 > 0$. Then S_k is bounded on $L^p(\mathcal{X})$ for $p \in [1,\infty]$ uniformly in $k \in \mathbb{Z}$. For any $f \in L^p(\mathcal{X})$ with $p \in (1,\infty)$, $\|S_k(f)\|_{L^p(\mathcal{X})} \to 0$ as $k \to -\infty$. For any $f \in L^p(\mathcal{X})$ with $p \in [1,\infty)$, $\|S_k(f) - f\|_{L^p(\mathcal{X})} \to 0$ as $k \to \infty$.*

We introduce the following inhomogeneous approximation of the identity on $(\mathcal{X}, d, \mu)$, which is a slight variant of the one in [23].

Definition 3.2. *Let $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$, $\epsilon_3 > 0$ and $\ell_0 \in \mathbb{Z}$. A sequence $\{S_k\}_{k=\ell_0}^\infty$ of linear operators is said to be an inhomogeneous approximation of the identity of order $(\epsilon_1, \epsilon_2, \epsilon_3)$ (for short, $(\epsilon_1, \epsilon_2, \epsilon_3)$-$\ell_0$-AOTI), if S_k satisfies* (i) *through* (v) *of Definition* 2.3 *for any $k \in \{\ell_0, \ell_0 + 1, \ldots\}$.*

In the following, for $k \in \mathbb{Z}$ and $\tau \in I_k$, we denote by $Q_\tau^{k,\nu}$, $\nu = 1, 2, \ldots, N(k,\tau)$, the set of all cubes $Q_{\tau'}^{k+j_0} \subset Q_\tau^k$, where Q_τ^k is the dyadic cube as in the Lemma 2.7 and j_0 is a positive integer satisfying

$$2^{-j_0} C_5 < 1/3. \tag{3.1}$$

Denote by $z_\tau^{k,\nu}$ the "center" of $Q_\tau^{k,\nu}$ and $y_\tau^{k,\nu}$ any point of $Q_\tau^{k,\nu}$.

For any $(\epsilon_1, \epsilon_2, \epsilon_3)$-$\ell_0$-AOTI, following the procedure of the proof for [23, Theorem 4.6], we obtain a new inhomogeneous discrete Calderón reproducing formula, which starts from any $\ell_0 \in \mathbb{Z}$. The details of the proof are omitted by similarity. This inhomogeneous discrete Calderón reproducing formula plays a key role in this paper, which has independent interest.

Theorem 3.3. *Let $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$, $\epsilon_3 > 0$, $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$ and $\ell_0 \in \mathbb{Z}$. Let $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1, \epsilon_2, \epsilon_3)$-$\ell_0$-AOTI. Set $D_{\ell_0} := S_{\ell_0}$ and $D_k := S_k - S_{k-1}$ for $k \geqslant \ell_0 + 1$. Then for any fixed j_0 satisfying* (3.1) *large enough, there exists a family of functions $\{\widetilde{D}_k(x,y)\}_{k=\ell_0}^\infty$ such that for any fixed $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$ with $k \geqslant \ell_0 + 1$, $\tau \in I_k$ and $\nu \in \{1, 2, \ldots, N(k,\tau)\}$, and all $f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))'$ with β, $\gamma \in (0,\epsilon)$,*

$$\begin{aligned}
f(x) &= \sum_{\tau\in I_{\ell_0}} \sum_{\nu=1}^{N(\ell_0,\tau)} \int_{Q_\tau^{\ell_0,\nu}} \widetilde{D}_{\ell_0}(x,y) \, d\mu(y) D_{\tau,1}^{\ell_0,\nu}(f) \\
&\quad + \sum_{k=\ell_0+1}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \widetilde{D}_k(x, y_\tau^{k,\nu}) D_k(f)(y_\tau^{k,\nu}) \\
&= \sum_{\tau\in I_{\ell_0}} \sum_{\nu=1}^{N(\ell_0,\tau)} \int_{Q_\tau^{\ell_0,\nu}} \widetilde{D}_{\ell_0}(x,y) \, d\mu(y) D_{\tau,1}^{\ell_0,\nu}(f)
\end{aligned}$$

$$+\sum_{k=\ell_0+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})\widetilde{D}_k(x,y_\tau^{k,\nu})D_{\tau,1}^{k,\nu}(f),$$

where the series converges in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$, *and for* $k\geqslant\ell_0$, $\tau\in I_k$ *and* $\nu=1,2,\ldots,N(k,\tau)$, $D_{\tau,1}^{k,\nu}(z)$ *is the corresponding integral operator with the kernel* $D_{\tau,1}^{k,\nu}(z):=m_{Q_\tau^{k,\nu}}(S_k(\cdot,z))$. *Moreover,* $\widetilde{D}_k$ *for* $k\geqslant\ell_0$ *satisfies* (i) *and* (ii) *of Definition* 2.3 *with* ϵ_1 *and* ϵ_2 *replaced by any* $\epsilon'\in[\epsilon,\epsilon_1\wedge\epsilon_2)$, *and*

$$\int_{\mathcal{X}}\widetilde{D}_k(x,y)\,d\mu(x)=\int_{\mathcal{X}}\widetilde{D}_k(x,y)\,d\mu(y)=1$$

when $k=\ell_0$; $=0$ *when* $k>\ell_0$.

Remark 3.4. Let $\ell_0\in\mathbb{Z}$. From the proof of Theorem 3.3, we obtain that the constant C, that appears in (i) and (ii) of Definition 2.3 for $\{\widetilde{D}_k\}_{k=\ell_0}^\infty$, depends on j_0 and ϵ', but not on ℓ_0. This observation plays a key role in applications. Moreover, a continuous version of Theorem 3.3 also holds by following the proof of [23, Theorem 3.4].

Now we show that all Hardy spaces defined in Definition 2.8 are equivalent to $L^p(\mathcal{X})$ when $p\in(1,\infty]$. We begin with some technical lemmas. Applying Theorem 2.3 of [23] yields the following result; we omit the details.

Lemma 3.5. *Let* $\epsilon_1\in(0,1]$, $\epsilon_2>0$, $\epsilon_3>0$, $\epsilon\in(0,\epsilon_1\wedge\epsilon_2)$, β, $\gamma\in(0,\epsilon]$ *and* $\{S_k\}_{k\in\mathbb{Z}}$ *be an* $(\epsilon_1,\epsilon_2,\epsilon_3)$-AOTI. *Then there exists a positive constant* C *such that for all* $k\geqslant 0$ *and* $\varphi\in\mathcal{G}(\beta,\gamma)$, $\|S_k(\varphi)\|_{\mathcal{G}(\beta,\gamma)}\leqslant C\|\varphi\|_{\mathcal{G}(\beta,\gamma)}$.

Lemma 3.6. *Let all the notation be as in Lemma* 3.5. *Then for all* $\varphi\in\mathcal{G}(\beta,\gamma)$ *and* $\beta'\in(0,\beta)$, $\lim_{k\to\infty}S_k(\varphi)=\varphi$ *in* $\mathcal{G}(\beta',\gamma)$.

Proof. Notice that Lemma 3.5 implies that $S_k(\varphi)\in\mathcal{G}(\beta,\gamma)\subset\mathcal{G}(\beta',\gamma)$ for any $k\geqslant 0$. It remains to show that $\lim_{k\to\infty}\|S_k(\varphi)-\varphi\|_{\mathcal{G}(\beta',\gamma)}=0$. Set $W_1:=\{z\in\mathcal{X}:\ d(z,x)\leqslant(1+d(x,x_1))/2\}$ and $W_2:=\{z\in\mathcal{X}:\ d(z,x)>(1+d(x,x_1))/2\}$. Using (v) of Definition 2.3, we write

$$\begin{aligned}|S_k(\varphi)(x)-\varphi(x)|&\leqslant\int_{W_1}|S_k(x,z)||\varphi(z)-\varphi(x)|\,d\mu(z)\\&\quad+|\varphi(x)|\int_{W_2}|S_k(x,z)|\,d\mu(z)+\int_{W_2}|S_k(x,z)||\varphi(z)|\,d\mu(z)\\&=:\mathrm{Z}_1+\mathrm{Z}_2+\mathrm{Z}_3.\end{aligned}$$

The size condition of S_k together with the regularity of φ and the assumption $\beta<\epsilon_2$ together with Lemma 3.1 (iii) yield that

$$\mathrm{Z}_1\lesssim 2^{-k\beta}\frac{1}{\mu(B(x,1+d(x,x_1)))}\left(\frac{1}{1+d(x,x_1)}\right)^\gamma.$$

By the definition of W_2, the size conditions of S_k and φ, and a procedure similar to the estimation of Z_1, we obtain

$$\mathrm{Z}_2\lesssim 2^{-k\gamma}\frac{1}{\mu(B(x,1+d(x,x_1)))}\left(\frac{1}{1+d(x,x_1)}\right)^\gamma.$$

Notice that $\int_{\mathcal{X}} |\varphi(x)|\, d\mu(x) \lesssim 1$. This together with the size condition of S_k and the definition of W_2 yields that

$$\mathrm{Z}_3 \lesssim 2^{-k\gamma} \frac{1}{\mu(B(x, 1+d(x,x_1)))} \left(\frac{1}{1+d(x,x_1)}\right)^{\gamma}.$$

Therefore, for all $x \in \mathcal{X}$,

$$|S_k(\varphi)(x) - \varphi(x)| \lesssim 2^{-k(\beta\wedge\gamma)} \frac{1}{\mu(B(x, 1+d(x,x_1)))} \left(\frac{1}{1+d(x,x_1)}\right)^{\gamma}. \tag{3.2}$$

For x, $x' \in \mathcal{X}$ satisfying $d(x,x') \leqslant (1+d(x,x_1))/2$, by Lemma 3.5 and the regularity of φ,

$$\begin{aligned} &|[S_k(\varphi)(x) - \varphi(x)] - [S_k(\varphi)(x') - \varphi(x')]| \\ &\quad \lesssim \left(\frac{d(x,x')}{1+d(x,x_1)}\right)^{\beta} \frac{1}{\mu(B(x, 1+d(x,x_1)))} \left(\frac{1}{1+d(x,x_1)}\right)^{\gamma}; \end{aligned}$$

on the other hand, by (3.2) and the assumption $d(x,x') \leqslant (1+d(x,x_1))/2$,

$$\begin{aligned} &|[S_k(\varphi)(x) - \varphi(x)] - [S_k(\varphi)(x') - \varphi(x')]| \\ &\quad \lesssim 2^{-k(\beta\wedge\gamma)} \frac{1}{\mu(B(x, 1+d(x,x_1)))} \left(\frac{1}{1+d(x,x_1)}\right)^{\gamma}. \end{aligned}$$

Then the geometric mean between the last two formulae above implies that

$$\begin{aligned} &|[S_k(\varphi)(x) - \varphi(x)] - [S_k(\varphi)(x') - \varphi(x')]| \\ &\quad \lesssim 2^{-k(\beta\wedge\gamma)(1-\sigma)} \left(\frac{d(x,x')}{1+d(x,x_1)}\right)^{\sigma\beta} \frac{1}{\mu(B(x, 1+d(x,x_1)))} \left(\frac{1}{1+d(x,x_1)}\right)^{\gamma}, \end{aligned} \tag{3.3}$$

where $\sigma \in (0,1)$. The estimates (3.2) and (3.3) imply the desired conclusion, which completes the proof of Lemma 3.6.

Proposition 3.7. *Let $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$, $\epsilon_3 > 0$, $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$, $\beta \in (0,\epsilon)$, $\gamma \in (0,\epsilon]$ and $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1,\epsilon_2,\epsilon_3)$-* AOTI*. Then $\lim_{k\to\infty} S_k(\varphi) = \varphi$ in $\mathcal{G}_0^{\epsilon}(\beta,\gamma)$ for all $\varphi \in \mathcal{G}_0^{\epsilon}(\beta,\gamma)$; and $\lim_{k\to\infty} S_k(f) = f$ in $(\mathcal{G}_0^{\epsilon}(\beta,\gamma))'$ for all $f \in (\mathcal{G}_0^{\epsilon}(\beta,\gamma))'$.*

Proof. We may assume that $k \in \mathbb{N}$. If $\varphi \in \mathcal{G}_0^{\epsilon}(\beta,\gamma)$, then $\varphi \in \mathcal{G}(\beta,\gamma)$ and there exists $\{\phi_j\}_{j\in\mathbb{N}} \subset \mathcal{G}(\epsilon,\epsilon)$ such that $\|\varphi - \phi_j\|_{\mathcal{G}(\beta,\gamma)} \to 0$ as $j \to \infty$. By Lemma 3.5, we know $S_k(\phi_j) \in \mathcal{G}(\epsilon,\epsilon)$ and

$$\lim_{j\to 0} \|S_k(\varphi) - S_k(\phi_j)\|_{\mathcal{G}(\beta,\gamma)} \lesssim \lim_{j\to 0} \|\varphi - \phi_j\|_{\mathcal{G}(\beta,\gamma)} = 0,$$

which implies that $S_k(\varphi) \in \mathcal{G}_0^{\epsilon}(\beta,\gamma)$. For any $j,\ k \in \mathbb{N}$, by Lemma 3.5 again, we have

$$\|S_k(\varphi) - \varphi\|_{\mathcal{G}(\beta,\gamma)} \lesssim \|S_k(\phi_j) - \phi_j\|_{\mathcal{G}(\beta,\gamma)} + \|\phi_j - \varphi\|_{\mathcal{G}(\beta,\gamma)},$$

which together with Lemma 3.6 (here we require $\beta < \epsilon$) implies that

$$\lim_{k\to\infty} \|S_k(\varphi) - \varphi\|_{\mathcal{G}(\beta,\gamma)} = 0.$$

A standard duality argument yields the second conclusion of Proposition 3.7.

Proposition 3.8. *Let $H^{*,1}(\mathcal{X})$ be as in Definition* 2.8*. Then $H^{*,1}(\mathcal{X}) \subset L^1(\mathcal{X})$.*

Proof. Let $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$, $\epsilon_3 > 0$ and $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1, \epsilon_2, \epsilon_3)$- AOTI. For any given $f \in H^{*,1}(\mathcal{X})$, by the definition, we have $\|f^*\|_{L^1(\mathcal{X})} < \infty$. Notice that (2.3) and Definition 2.3 imply that

$$\|\sup_{k\in\mathbb{Z}} |S_k(f)|\|_{L^1(\mathcal{X})} \lesssim \|f^*\|_{L^1(\mathcal{X})}.$$

Thus, $\{S_k(f)\}_{k\in\mathbb{Z}}$ is a bounded set in $L^1(\mathcal{X})$. Thus by [39, Theorem III.C.12], $\{S_k(f)\}_{k\in\mathbb{Z}}$ is relatively weakly compact in $L^1(\mathcal{X})$. The Eberlein-Šmulian theorem (cf. [39, II.C]) implies that there exists a subsequence $\{S_{k_j}(f)\}_{j\in\mathbb{N}}$ of $\{S_k(f)\}_{k\in\mathbb{Z}}$ which converges weakly in $L^1(\mathcal{X})$, and hence in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$. This together with Proposition 3.7 implies that $f \in L^1(\mathcal{X})$, which completes the proof of Proposition 3.8.

Proposition 3.9. *Let $p \in (0,\infty]$ and all the notation be as in Definition* 2.8. *Then for any given $\alpha \in (0,\infty)$, there exists a positive constant C such that for all $f \in L^p(\mathcal{X})$ and all $x \in \mathcal{X}$, $f^*(x) \leqslant C\mathcal{M}(f)(x)$ and $\mathcal{M}_d(f)(x) \leqslant C\mathcal{M}(\mathcal{M}_\alpha(f))(x)$. Furthermore, when $p \in (1,\infty]$, there exists a positive constant C such that for all $f \in L^p(\mathcal{X})$,*

$$\|\mathcal{M}_d(f)\|_{L^p(\mathcal{X})} \leqslant C\|\mathcal{M}_\alpha(f)\|_{L^p(\mathcal{X})} \leqslant C\|f^*\|_{L^p(\mathcal{X})} \leqslant C\|f\|_{L^p(\mathcal{X})}.$$

Proof. For any $x \in \mathcal{X}$ and $\varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma)$ satisfying $\|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)} \leqslant 1$ for some $r > 0$, by the size condition of φ and Lemma 3.1 (iv), we have

$$\begin{aligned}|\langle f,\varphi\rangle| &\lesssim \int_{d(z,x)\leqslant r} \frac{1}{\mu(B(x,r))}|f(z)|\,d\mu(z)\\ &\quad+\int_{d(z,x)>r}\frac{1}{\mu(B(z,d(z,x)))}\frac{r^\gamma}{d(x,z)^\gamma}|f(z)|\,d\mu(z) \lesssim \mathcal{M}(f)(x),\end{aligned} \tag{3.4}$$

which further implies that $f^*(x) \lesssim \mathcal{M}(f)(x)$ for all $x \in \mathcal{X}$.

For any $x \in \mathcal{X}$, if x is contained in some "dyadic" cube Q_α^k, then by Lemma 2.7,

$$B(z_\alpha^k, C_6 2^{-k}) \subset Q_\alpha^k \subset B(x, C_5 2^{-k}) \subset B(z_\alpha^k, 2C_5 2^{-k}).$$

Thus $\mu(Q_\alpha^k) \geqslant \mu(B(z_\alpha^k, C_6 2^{-k})) \gtrsim \mu(B(x, C_5 2^{-k}))$ by (2.2). From this and (2.6), it follows that for all $x \in \mathcal{X}$,

$$\mathcal{M}_d(f)(x) \lesssim \sup_{k\in\mathbb{Z}}\left\{\frac{1}{\mu(B(x,C_5 2^{-k}))}\int_{B(x,\,C_5 2^{-k})}\mathcal{M}_\alpha(f)(y)\,d\mu(y)\right\} \lesssim \mathcal{M}(\mathcal{M}_\alpha(f))(x).$$

This together with Remark 2.9 (ii) and the $L^p(\mathcal{X})$-boundedness of $\mathcal{M}$ with $p \in (1,\infty]$ (cf. [5]) then yields the desired norm inequalities, which completes the proof of Proposition 3.9.

Theorem 3.10. *Let $p \in (1,\infty]$. With the same notation as in Definition* 2.8, *we have $H_{\rm d}^p(\mathcal{X}) = L^p(\mathcal{X})$ in the following sense: there exists a positive constant C independent of f such that if $f \in L^p(\mathcal{X})$, then $f \in H_{\rm d}^p(\mathcal{X})$ and $\|f\|_{H_{\rm d}^p(\mathcal{X})} \leqslant C\|f\|_{L^p(\mathcal{X})}$; conversely, for any $f \in H_{\rm d}^p(\mathcal{X})$, there exists a function $\widetilde{f} \in L^p(\mathcal{X})$ such that for all $\varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma)$,*

$$\langle f,\varphi\rangle = \int_{\mathcal{X}} \widetilde{f}(x)\varphi(x)\,d\mu(x).$$

Moreover, there exists a positive constant C independent of f and x such that $\|\widetilde{f}\|_{L^p(\mathcal{X})} \leqslant C\|f\|_{H_{\rm d}^p(\mathcal{X})}$ and $|\widetilde{f}(x)| \leqslant Cf^*(x)$ almost everywhere.

Proof. The fact $L^p(\mathcal{X}) \subset (\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and Proposition 3.9 imply that $L^p(\mathcal{X}) \subset H^p_{\rm d}(\mathcal{X})$. Conversely, if $f \in H^p_{\rm d}(\mathcal{X})$, then we first show that for all $\varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma)$,

$$|\langle f, \varphi\rangle| \lesssim \|f\|_{H^p_{\rm d}(\mathcal{X})}\|\varphi\|_{L^{p'}(\mathcal{X})}, \tag{3.5}$$

where $1/p + 1/p' = 1$. To this end, fix j_0 as in (3.1) large enough. Then applying Theorem 3.3 with $\ell_0 = j \in \mathbb{Z}$ yields that

$$\begin{aligned} f(x) = &\sum_{\tau\in I_j}\sum_{\nu=1}^{N(j,\tau)}\int_{Q_\tau^{j,\nu}} \widetilde{D}_j(x,y)\,d\mu(y) D_{\tau,1}^{j,\nu}(f) \\ &+ \sum_{k=j+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu})\widetilde{D}_k(x,y_\tau^{k,\nu})D_{\tau,1}^{k,\nu}(f) \end{aligned} \tag{3.6}$$

in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$. Moreover, for any $Q_\tau^{k,\nu} \ni y_\tau^{k,\nu}$ with $k \geqslant j$, by the definition, we have that $Q_\tau^{k,\nu} \subset Q_{\tau'}^k$ for some $\tau' \in I_k$. From this and (2.2), we deduce that $\mu(Q_{\tau'}^k) \lesssim 2^{nj_0}\mu(Q_\tau^{k,\nu})$. Thus, for any $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$,

$$|D_{\tau,1}^{k,\nu}(f)| \lesssim 2^{nj_0}\frac{1}{\mu(Q_{\tau'}^k)}\int_{Q_{\tau'}^k} |S_k(f)(z)|\,d\mu(z) \lesssim \mathcal{M}_d(f)(y_\tau^{k,\nu}). \tag{3.7}$$

For any $\varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma)$, Proposition 3.7 implies that $S_j(\varphi) \in \mathcal{G}_0^\epsilon(\beta,\gamma)$ for any $j \in \mathbb{Z}$. By this, (3.6) and (3.7), we obtain that for all $\varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma)$, $j \in \mathbb{Z}$ and $y_\tau^{j,\nu} \in Q_\tau^{j,\nu}$,

$$\begin{aligned} |\langle f, S_j\varphi\rangle| \lesssim &\sum_{\tau\in I_j}\sum_{\nu=1}^{N(j,\tau)}\left|\int_{Q_\tau^{j,\nu}} \widetilde{D}_j^*(S_j\varphi)(y)\,d\mu(y)\right|\mathcal{M}_d(f)(y_\tau^{j,\nu}) \\ &+ \sum_{k=j+1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu})|\widetilde{D}_k^*(S_j\varphi)(y_\tau^{k,\nu})|\mathcal{M}_d(f)(y_\tau^{k,\nu}), \end{aligned} \tag{3.8}$$

where and in what follows, $\widetilde{D}_k^*$ denotes the integral operator with the kernel $\widetilde{D}_k^*(x,y) = \widetilde{D}_k(y,x)$ for all x, $y \in \mathcal{X}$. For any k, $j \in \mathbb{Z}$, by similar arguments as in [23, Lemma 3.8] when $k = j$ and [23, Lemma 3.1] when $k > j$, respectively, we obtain that for any $\epsilon_1' \in (0, \epsilon_1 \wedge \epsilon_2)$,

$$|\widetilde{D}_k^* S_j(y,z)| \lesssim 2^{-(k-j)\epsilon_1'}\frac{1}{V_{2^{-j}}(y) + V_{2^{-j}}(z) + V(y,z)}\frac{2^{-j\epsilon_1'}}{(2^{-j} + d(x,y))^{\epsilon_1'}}. \tag{3.9}$$

An argument similar to the proof of (3.4) together with (3.9) yields that for any $k \geqslant j$ and $x \in \mathcal{X}$,

$$|D_k^*(S_j\varphi)(x)| \lesssim 2^{-(k-j)\epsilon_1'}\mathcal{M}(\varphi)(x). \tag{3.10}$$

Moreover, it follows directly from (3.9) and Lemma 3.1 (iii) that for all $k \geqslant j$,

$$\left|\int_{\mathcal{X}} D_k^*(S_j\varphi)(y)\,d\mu(y)\right| \lesssim 2^{-(k-j)\epsilon_1'}\|\varphi\|_{L^1(\mathcal{X})}. \tag{3.11}$$

When $p \in (1,\infty)$, by (3.8) together with the arbitrariness of $y_\tau^{k,\nu}$, (3.10), Hölder's inequality and the $L^p(\mathcal{X})$-boundedness of $\mathcal{M}$ with $p \in (1,\infty]$, we further have

$$\begin{aligned} |\langle f, S_j\varphi\rangle| &\lesssim \sum_{k=j}^{\infty}\int_{\mathcal{X}} |D_k^*(S_j\varphi)(y)|\mathcal{M}_d(f)(y)\,d\mu(y) \\ &\lesssim \sum_{k=j}^{\infty} 2^{-(k-j)\epsilon_1'}\|\mathcal{M}(\varphi)\|_{L^{p'}(\mathcal{X})}\|\mathcal{M}_d(f)\|_{L^p(\mathcal{X})} \lesssim \|\varphi\|_{L^{p'}(\mathcal{X})}\|f\|_{H^p_{\rm d}(\mathcal{X})}. \end{aligned} \tag{3.12}$$

When $p = \infty$, this can be deduced directly from (3.8) and (3.11). For any $\varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma)$, by Proposition 3.7, we have $\lim_{j\to\infty} S_j(\varphi) = \varphi$ in $\mathcal{G}_0^\epsilon(\beta,\gamma)$, which together with (3.12) implies (3.5).

From (3.5) and the density of $\mathcal{G}_0^\epsilon(\beta,\gamma)$ in $L^{p'}(\mathcal{X})$ (see [23, Corollary 2.1]), it follows that f uniquely extends to a bounded linear functional on $L^{p'}(\mathcal{X})$, where $1/p+1/p'=1$. By the well-known Riesz representation theorem, there exists a unique function $\widetilde{f} \in L^p(\mathcal{X})$ such that for all $\varphi \in L^{p'}(\mathcal{X})$, $\langle f,\varphi\rangle = \int_{\mathcal{X}} \widetilde{f}(x)\varphi(x)\,d\mu(x)$, and moreover, $\|\widetilde{f}\|_{L^p(\mathcal{X})} \lesssim \|f\|_{H^p_{\rm d}(\mathcal{X})}$. Lemma 3.1 (v) together with the Riesz lemma further implies that $\widetilde{f}(x) = \lim_{k_i\to\infty} S_{k_i}\widetilde{f}(x)$ almost everywhere for some $\{k_i\}_i \subset \mathbb{N}$. This combined with (2.3) yields that for almost every $x \in \mathcal{X}$,

$$|\widetilde{f}(x)| \leqslant \lim_{k_i\to\infty} |S_{k_i}\widetilde{f}(x)| \lesssim \widetilde{f}^*(x) = f^*(x),$$

which completes the proof of Theorem 3.10.

Combining Proposition 3.9 and Theorem 3.10 yields the following conclusion.

Corollary 3.11. *Let all the notation be as in Definition* 2.8 *and* $p \in (1,\infty]$. *Then* $H^p_{\rm d}(\mathcal{X}) = H^p_\alpha(\mathcal{X}) = H^{*,p}(\mathcal{X}) = L^p(\mathcal{X})$ *with equivalent norms.*

To prove the equivalence of Hardy spaces defined as above when $p \leqslant 1$, we need the following technical lemma (see [23, Lemma 5.2]).

Lemma 3.12. *Let* $\epsilon > 0$, k', $k \in \mathbb{Z}$, *and* $y^{k,\nu}_\tau$ *be any point in* $Q^{k,\nu}_\tau$ *for* $\tau \in I_k$ *and* $\nu = 1,2,\dots,N(k,\tau)$. *If* $r \in (n/(n+\epsilon),1]$, *then there exists a positive constant* C *depending on* r *such that for all* $a^{k,\nu}_\tau \in \mathbb{C}$ *and all* $x \in \mathcal{X}$,

$$\begin{aligned}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} &\mu(Q^{k,\nu}_\tau)\frac{1}{V_{2^{-(k'\wedge k)}}(x)+V(x,y^{k,\nu}_\tau)}\frac{2^{-(k'\wedge k)\epsilon}}{(2^{-(k'\wedge k)}+d(x,y^{k,\nu}_\tau))^\epsilon}|a^{k,\nu}_\tau|\\ &\leqslant C2^{[(k'\wedge k)-k]n(1-1/r)}\left\{\mathcal{M}\left(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}|a^{k,\nu}_\tau|^r\chi_{Q^{k,\nu}_\tau}\right)(x)\right\}^{1/r},\end{aligned}$$

where C *is also independent of* k, k', τ *and* ν.

Theorem 3.13. *Let all the notation be as in Definition* 2.8. *Let* $p \in (n/(n+\epsilon),1]$ *and* β, $\gamma \in (n(1/p-1),\epsilon)$. *Then* $H^p_{\rm d}(\mathcal{X}) = H^p_\alpha(\mathcal{X}) = H^{*,p}(\mathcal{X})$ *with equivalent quasi-norms.*

Proof. Given any $\alpha \in (0,\infty)$, by Remark 2.9 (ii), $H^{*,p}(\mathcal{X}) \subset H^p_\alpha(\mathcal{X})$ and for all $f \in H^{*,p}(\mathcal{X})$, $\|f\|_{H^p_\alpha(\mathcal{X})} \lesssim \|f\|_{H^{*,p}(\mathcal{X})}$.

To prove $H^p_\alpha(\mathcal{X}) \subset H^{*,p}(\mathcal{X})$, we need only to show that for any $\kappa \in (n/[n+(\beta\wedge\gamma)],1]$, there exists a positive constant C_κ such that for all $f \in H^p_\alpha(\mathcal{X})$ and $x \in \mathcal{X}$,

$$f^*(x) \leqslant C_\kappa\{\mathcal{M}([\mathcal{M}_\alpha(f)]^\kappa)(x)\}^{1/\kappa}. \tag{3.13}$$

Suppose now that (3.13) holds for some $\kappa \in (n/[n+(\beta\wedge\gamma)],p)$. Then by this and the $L^p(\mathcal{X})$-boundedness of $\mathcal{M}$ with $p \in (1,\infty]$, we obtain that for all $f \in H^p_\alpha(\mathcal{X})$,

$$\|f\|_{H^{*,p}(\mathcal{X})} = \|f^*\|_{L^p(\mathcal{X})} \lesssim \|\{\mathcal{M}([\mathcal{M}_\alpha(f)]^\kappa)\}^{1/\kappa}\|_{L^p(\mathcal{X})} \lesssim \|\mathcal{M}_\alpha(f)\|_{L^p(\mathcal{X})} = \|f\|_{H^p_\alpha(\mathcal{X})},$$

which implies that $H^p_\alpha(\mathcal{X}) \subset H^{*,p}(\mathcal{X})$.

To show (3.13), fix any $x \in \mathcal{X}$. Then for any $\varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma)$ satisfying $\|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)} \leqslant 1$ with some $r > 0$, we choose $\ell_0 \in \mathbb{Z}$ such that $2^{-\ell_0} \leqslant r < 2^{-\ell_0+1}$. Then it is easy to verify that there exists a positive constant C which is independent of x and ℓ_0 such that $\|\varphi\|_{\mathcal{G}(x,2^{-\ell_0},\beta,\gamma)} \leqslant C$. Choose $j_0 \in \mathbb{N}$ satisfying (3.1) large enough and $C_5 2^{-j_0} < \alpha$. Applying Theorem 3.3 with such a j_0, we obtain that for any $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$,

$$|\langle f,\varphi\rangle| \leqslant \left|\sum_{\tau\in I_{\ell_0}} \sum_{\nu=1}^{N(\ell_0,\tau)} \int_{Q_\tau^{\ell_0,\nu}} \widetilde{D}_{\ell_0}^*(\varphi)(y)\,d\mu(y) D_{\tau,1}^{\ell_0;\nu}(f)\right| + \left|\sum_{k=\ell_0+1}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \widetilde{D}_k^*(\varphi)(y_\tau^{k,\nu}) D_k(f)(y_\tau^{k,\nu})\right|. \tag{3.14}$$

Since $\operatorname{diam}(Q_\tau^{k,\nu}) \leqslant C_5 2^{-k-j_0} < \alpha 2^{-k}$ for all $k \geqslant \ell_0$, $\tau \in I_k$ and $\nu = 1,2,\ldots,N(\tau,k)$, then for any $y_\tau^{\ell_0,\nu} \in Q_\tau^{\ell_0,\nu}$,

$$|D_{\tau,1}^{\ell_0;\nu}(f)| = \left|\frac{1}{\mu(Q_\tau^{\ell_0,\nu})} \int_{Q_\tau^{\ell_0,\nu}} S_{\ell_0}(f)(w)\,d\mu(w)\right| \leqslant \mathcal{M}_\alpha(f)(y_\tau^{\ell_0,\nu}); \tag{3.15}$$

and when $k > \ell_0$, for any $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$,

$$|D_k(f)(y_\tau^{k,\nu})| \leqslant |S_k(f)(y_\tau^{k,\nu})| + |S_{k-1}(f)(y_\tau^{k,\nu})| \leqslant 2\mathcal{M}_\alpha(f)(y_\tau^{k,\nu}). \tag{3.16}$$

By an argument similar to [23, Lemma 3.1], we obtain that for any $k \geqslant \ell_0$, $y \in \mathcal{X}$ and $\epsilon_1' \in (0,\beta\wedge\gamma)$,

$$|\widetilde{D}_k^*(\varphi)(y)| \lesssim 2^{-(k-\ell_0)\epsilon_1'} \frac{1}{\mu(B(y,2^{-\ell_0}+d(y,x)))} \frac{2^{-\ell_0\gamma}}{(2^{-\ell_0}+d(y,x))^\gamma}. \tag{3.17}$$

Furthermore, the triangle inequality of d and (2.2) imply that for any $y_\tau^{\ell_0,\nu}$, $z \in Q_\tau^{\ell_0,\nu}$,

$$2^{-\ell_0} + d(y_\tau^{\ell_0,\nu},x) \sim 2^{-\ell_0} + d(z,x) \tag{3.18}$$

and

$$\mu(B(y_\tau^{\ell_0,\nu}, 2^{-\ell_0} + d(y_\tau^{\ell_0,\nu},x))) \sim \mu(B(z, 2^{-\ell_0}+d(z,x))). \tag{3.19}$$

From (3.14) together with the estimates (3.15) through (3.19), Lemma 3.12 and the arbitrariness of $y_\tau^{k,\nu}$ in $Q_\tau^{k,\nu}$, we deduce that for any $\kappa \in (n/(n+(\beta\wedge\gamma)),1]$,

$$|\langle f,\varphi\rangle| \lesssim \sum_{k=\ell_0}^{\infty} 2^{-(k-\ell_0)[\epsilon_1'+n(1-1/\kappa)]} \left\{\mathcal{M}\left(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} [\mathcal{M}_\alpha(f)]^\kappa \chi_{Q_\tau^{k,\nu}}\right)(x)\right\}^{1/\kappa}. \tag{3.20}$$

Notice that

$$\mathcal{M}\left(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} [\mathcal{M}_\alpha(f)]^\kappa \chi_{Q_\tau^{k,\nu}}\right)(x) = \mathcal{M}([\mathcal{M}_\alpha(f)]^\kappa)(x). \tag{3.21}$$

Therefore, for any $\kappa \in (n/(n+\beta\wedge\gamma),\epsilon)$, if we choose $\epsilon_1' \in (0,\beta\wedge\gamma)$ with $\epsilon_1' > n(1/\kappa-1)$, then combining (3.20) with (3.21) yields (3.13). Thus, for any given $\alpha \in (0,\infty)$, $H^{*,p}(\mathcal{X}) = H_\alpha^p(\mathcal{X})$ with equivalent quasi-norms.

Instead of (3.14), (3.15) and (3.16), respectively, by (3.6) and (3.7), repeating the proof for $H^p_\alpha(\mathcal{X}) \subset H^{*,p}(\mathcal{X})$ then yields that $H^p_{\rm d}(\mathcal{X}) \subset H^{*,p}(\mathcal{X})$. By Lemma 2.7 (iv) and (2.6), we have $\mathcal{M}_d(f)(x) \leqslant \mathcal{M}_{C_5}(f)(x)$, which implies that $H^p_{C_5}(\mathcal{X}) \subset H^p_{\rm d}(\mathcal{X})$. This together with $H^p_{C_5}(\mathcal{X}) = H^{*,p}(\mathcal{X})$ further implies that $H^{*,p}(\mathcal{X}) \subset H^p_{\rm d}(\mathcal{X})$. Thus, $H^{*,p}(\mathcal{X}) = H^p_{\rm d}(\mathcal{X})$ with equivalent quasi-norms, which completes the proof of Theorem 3.13.

Using the Calderón reproducing formula in Theorem 3.3, we can verify that definitions of Hardy spaces are independent of the choices of underlying spaces of distributions; see [23, Proposition 5.3] for some details.

Proposition 3.14. *With the notation of Definition* 2.8, *let* $\alpha \in (0,\infty)$ *and* $p \in (n/(n+\epsilon), 1]$. *If* $f \in (\mathcal{G}^\epsilon_0(\beta_1,\gamma_1))'$ *with*

$$n(1/p-1) < \beta_1, \quad \gamma_1 < \epsilon, \tag{3.22}$$

and $\|f\|_{H^p_\alpha(\mathcal{X})} < \infty$, *then* $f \in (\mathcal{G}^\epsilon_0(\beta_2,\gamma_2))'$ *for every* β_2 *and* γ_2 *satisfying* (3.22).

Remark 3.15. Let $\epsilon \in (0,1)$ and $p \in (n/(n+\epsilon), 1]$. Then,

(i) For any given β and γ satisfying (3.22), Theorem 3.13 implies that the definitions of $H^p_\alpha(\mathcal{X})$ and $H^p_{\rm d}(\mathcal{X})$ are independent of the choices of any $\alpha \in (0,\infty)$ and any $(\epsilon_1,\epsilon_2,\epsilon_3)$- AOTI $\{S_k\}_{k\in\mathbb{Z}}$ satisfying $(\epsilon_1 \wedge \epsilon_2) > \epsilon$, $\epsilon_1 \leqslant 1$ and $\epsilon_3 > 0$.

(ii) Theorem 3.13 and Proposition 3.14 imply that the definitions of $H^{*,p}(\mathcal{X})$, $H^p_\alpha(\mathcal{X})$ with $\alpha \in (0,\infty)$ and $H^p_{\rm d}(\mathcal{X})$ are also independent of the choices of $(\mathcal{G}^\epsilon_0(\beta,\gamma))'$ with β and γ satisfying (3.22).

(iii) In the sequel, given $p \in (n/(n+1), 1]$, when we mention the Hardy spaces $H^{*,p}(\mathcal{X})$, $H^p_\alpha(\mathcal{X})$ with $\alpha \in (0,\infty)$, and $H^p_{\rm d}(\mathcal{X})$, we will assume that $\epsilon \in (n(1/p-1), 1)$ is chosen so that $p \in (n/(n+\epsilon), 1]$ and these Hardy spaces are defined via some $(\mathcal{G}^\epsilon_0(\beta,\gamma))'$ with β and γ satisfying (3.22), and some $(\epsilon_1,\epsilon_2,\epsilon_3)$- AOTI $\{S_k\}_{k\in\mathbb{Z}}$ with $(\epsilon_1 \wedge \epsilon_2) > \epsilon$, $\epsilon_1 \leqslant 1$ and $\epsilon_3 > 0$. By Theorem 3.13, these Hardy spaces coincide with each other.

By an argument similar to the proof of [6, Lemma (2.8)] (see also [9, Theorem 2.1]), we can verify the following conclusion, we omit the details.

Proposition 3.16. *Let* $p \in (n/(n+1), 1]$. *Then the space* $H^{*,p}(\mathcal{X})$ *is a complete quasi-Banach space.*

4 The atomic decomposition

The procedure to obtain the atomic decomposition for the Hardy space $H^{*,p}(\mathcal{X})$ when $p \in (n/(n+1), 1]$ is quite similar to that presented in [6] (see also [9]). To shorten the presentation of this paper, we only give an outline by beginning with the following notions of atoms and atomic Hardy spaces.

Definition 4.1. *Let* $p \in (0,1]$ *and* $q \in [1,\infty] \cap (p,\infty]$. *A function* $a \in L^q(\mathcal{X})$ *is said to be a* (p,q)*-atom if*

(A1) $\operatorname{supp} a \subset B(x_0,r)$ *for some* $x_0 \in \mathcal{X}$ *and* $r > 0$;

(A2) $\|a\|_{L^q(\mathcal{X})} \leqslant [\mu(B(x_0,r))]^{1/q-1/p}$;

(A3) $\int_{\mathcal{X}} a(x)\,d\mu(x) = 0$.

Definition 4.2. *Let* $p \in (0,1]$ *and* $q \in [1,\infty] \cap (p,\infty]$. *Let* $\epsilon \in (0,1)$ *and* β, $\gamma \in (0,\epsilon)$. *The distribution* $f \in (\mathcal{G}^\epsilon_0(\beta,\gamma))'$ *is an element of* $H^{p,q}_{at}(\mathcal{X})$, *if there exist* $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ *and*

(p, q)-atoms $\{a_j\}_{j\in\mathbb{N}}$ such that $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and $\sum_{j\in\mathbb{N}} |\lambda_j|^p < \infty$. Moreover, we define

$$\|f\|_{H^{p,\,q}_{at}(\mathbb{R}^n)} := \inf\left\{\left(\sum_{j\in\mathbb{N}} |\lambda_j|^p\right)^{1/p}\right\},$$

where the infimum is taken over all the decompositions of f as above.

Remark 4.3. (i) Let $\epsilon \in (0, 1)$, $p \in (n/(n+\epsilon), 1]$, $q \in [1,\infty] \cap (p,\infty]$ and β, $\gamma \in (n(1/p - 1), \epsilon)$. Since each (p,∞)-atom is a (p, q)-atom, $H^{p,\,\infty}_{at}(\mathcal{X}) \subset H^{p,\,q}_{at}(\mathcal{X})$. We will further show in Theorems 4.5 and 4.16 below that $H^{p,\,q}_{at}(\mathcal{X}) \subset H^{*,\,p}(\mathcal{X})$ and $H^{*,\,p}(\mathcal{X}) \subset H^{p,\,\infty}_{at}(\mathcal{X})$. This implies that $H^{p,\,\infty}_{at}(\mathcal{X}) = H^{p,\,q}_{at}(\mathcal{X})$.

(ii) We simply write $H^p_{\rm at}(\mathcal{X})$ instead of $H^{p,\,\infty}_{at}(\mathcal{X})$ without confusion.

An argument similar to the proof of [6, Lemma (2.3)] yields the following lemma, we omit the details.

Lemma 4.4. *Let $\epsilon \in (0, 1]$, $p \in (n/(n+\epsilon), 1]$, $q \in [1,\infty] \cap (p,\infty]$, $\beta \in (n(1/p-1), \epsilon]$ and $\gamma \in (0, \epsilon]$. There exists a positive constant $C_{p,\,q}$ such that for all $h \in L^q(\mathcal{X})$ with support contained in $B(x_0, r_0)$ with $x_0 \in \mathcal{X}$ and $r_0 > 0$, and $\int_{\mathcal{X}} h(x)\, d\mu(x) = 0$,*

$$\int_{\mathcal{X}} [h^*(x)]^p\, d\mu(x) \leqslant C_{p,\,q}[\mu(B(x_0, r_0))]^{1-p/q}\|h\|^p_{L^q(\mathcal{X})}.$$

Lemma 4.4 together with a standard argument yields the following proposition.

Proposition 4.5. *Let all the notation be as in Lemma 4.4. Then $H^{p,\,q}_{at}(\mathcal{X}) \subset H^{*,\,p}(\mathcal{X})$; moreover, there exists a positive constant C such that for all $f \in H^{p,\,q}_{at}(\mathcal{X})$, $\|f\|_{H^{*,\,p}(\mathcal{X})} \leqslant C\|f\|_{H^{p,\,q}_{at}(\mathcal{X})}$.*

The rest of this section is devoted to showing that $H^{*,\,p}(\mathcal{X}) \subset H^p_{\rm at}(\mathcal{X})$ by following the procedure of [6]. The following lemma when Ω is a bounded set (i. e., Ω is contained in some ball of $\mathcal{X}$) was proved in [4, pp. 70–71] and in [5, Theorem 3.2], although the current version was also claimed in [4, p. 70] without a proof; see also [6, p. 277] for another variant. A detailed proof can be given by borrowing some ideas from [3, pp. 15–16], we omit the details.

Lemma 4.6. *Let Ω be an open proper subset of $\mathcal{X}$ and let $d(x) := \inf\{d(x, y) : y \notin \Omega\}$. For any given $C \geqslant 1$, let $r(x) := d(x)/(2C)$. Then there exist a positive number M, which depends only on C and C_3 but independent of Ω, and a sequence $\{x_k\}_k$ such that if we denote $r(x_k)$ by r_k, then*

(i) *$\{B(x_k, r_k/4)\}_k$ are pairwise disjoint*;

(ii) $\cup_k B(x_k, r_k) = \Omega$;

(iii) *for every given k, $B(x_k, Cr_k) \subset \Omega$;*

(iv) *for every given k, $x \in B(x_k, Cr_k)$ implies that $Cr_k < d(x) < 3Cr_k$;*

(v) *for every given k, there exists a $y_k \notin \Omega$ such that $d(x_k, y_k) < 3Cr_k$;*

(vi) *for every given k, the number of balls $B(x_i, Cr_i)$ whose intersections with the ball $B(x_k, Cr_k)$ are non-empty is at most M.*

Remark 4.7. If Ω is an open bounded subset of $\mathcal{X}$ and $\{r_k\}_k$ in Lemma 4.6 is an infinite sequence of positive numbers, then by the proofs in [5, pp. 623–624] and [4, p. 69], we further have that $\lim_{k\to\infty} r_k = 0$.

Using Lemma 4.6, similarly to the proof of [6, Lemma (2.16)], we obtain the following technical lemma.

Lemma 4.8. *Let Ω be an open subset of $\mathcal{X}$ with finite measure. Consider the sequence $\{x_k\}_k$ and $\{r_k\}_k$ given in Lemma 4.6 for $C = 15$. Then there exist non-negative functions $\{\phi_k\}_k$ satisfying:*

(i) *for any given k, $0 \leqslant \phi_k \leqslant 1$, $\operatorname{supp} \phi_k \subset B(x_k, 2r_k)$ and $\sum_k \phi_k = \chi_\Omega$;*

(ii) *for any given k and $x \in B(x_k, r_k)$, $\phi_k(x) \geqslant 1/M$;*

(iii) *there exists a positive constant $\widetilde{C}$ independent of Ω such that for all k and all $\epsilon \in (0, 1]$, $\|\phi_k\|_{\mathcal{G}(x_k, r_k, \epsilon, \epsilon)} \leqslant \widetilde{C} V_{r_k}(x_k)$.*

From Lemma 4.8 and an argument similar to the proof of [6, Lemma (3.1)], we deduce the following conclusion.

Lemma 4.9. *Let $\epsilon \in (0, 1]$, β, $\gamma \in (0, \epsilon]$, and $\{\phi_k\}_k$ be the partition of unity as in Lemma 4.8 associated with some open set Ω. Then for any given k, the linear operator*

$$\Phi_k(\varphi)(x) := \phi_k(x) \left[\int_{\mathcal{X}} \phi_k(z)\, d\mu(z) \right]^{-1} \int_{\mathcal{X}} [\varphi(x) - \varphi(z)] \phi_k(z)\, d\mu(z)$$

is bounded on $\mathcal{G}_0^\epsilon(\beta, \gamma)$ with an operator norm depending on k.

Lemma 4.10. *Let $\beta > 0$, $q \in (n/(n + \beta), \infty)$ and $M \in \mathbb{N}$. Then there exists a positive constant C depending only on q, β and M such that for any given sequences of points $\{x_k\}_k \subset \mathcal{X}$ and any positive numbers $\{r_k\}_k$ satisfying that any point in $\mathcal{X}$ belongs to no more than M balls of $\{B(x_k, r_k)\}_k$, then*

$$\int_{\mathcal{X}} \left[\sum_k \frac{\mu(B(x_k, r_k))}{\mu(B(x, r_k + d(x, x_k)))} \left(\frac{r_k}{r_k + d(x_k, x)} \right)^\beta \right]^q d\mu(x) \leqslant C \mu\bigg(\bigcup_k B(x_k, r_k) \bigg). \tag{4.1}$$

Proof. By (2.2) and the definition of $\mathcal{M}$, we have

$$\frac{r_k}{r_k + d(x_k, x)} \lesssim \left(\frac{\mu(B(x_k, r_k))}{\mu(B(x, r_k + d(x, x_k)))} \right)^{1/n} \lesssim [\mathcal{M}(\chi_{B(x_k, r_k)})(x)]^{1/n}.$$

By this, the assumption $q(n+\beta)/n > 1$, the Fefferman-Stein inequality on RD-spaces (see [40]) and the finite intersection property of balls $\{B(x_k, r_k)\}_k$, we obtain that the left-hand side of (4.1) is dominated by a multiple of

$$\begin{aligned} &\int_{\mathcal{X}} \left[\sum_k (\mathcal{M}(\chi_{B(x_k, r_k)})(x))^{(n+\beta)/n} \right]^q d\mu(x) \\ &\qquad \lesssim \int_{\mathcal{X}} \left[\sum_k (\chi_{B(x_k, r_k)}(x))^{(n+\beta)/n} \right]^q d\mu(x) \lesssim \mu\bigg(\bigcup_k B(x_k, r_k) \bigg), \end{aligned}$$

which completes the proof of Lemma 4.10.

Let $\epsilon \in (0, 1)$, $p \in (n/(n + \epsilon), 1]$, β, $\gamma \in (n(1/p - 1), \epsilon)$ and $H^{*,p}(\mathcal{X})$ be as in (2.3). For $f \in H^{*,p}(\mathcal{X})$ and $t \in (0, \infty)$, set $\Omega := \{x \in \mathcal{X} : f^*(x) > t\}$. By Remark 2.9 (iii), Ω is open. Obviously $\mu(\Omega) < \infty$. Denote by $\{\phi_k\}_k$ the partition of unity in Lemma 4.8 associated to Ω.

Let $\{\Phi_k\}_k$ be the corresponding linear operator in Lemma 4.9. For any $\varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma)$, define the distribution b_k by setting

$$\langle b_k, \varphi\rangle := \langle f, \Phi_k(\varphi)\rangle. \tag{4.2}$$

By following an approach in [6, Lemma (3.2)] or [9, Lemma 3.7], we obtain the following Calderón-Zygmund type decomposition. To limit the length of the paper, we omit the details. In what follows, for any set $E \subset \mathcal{X}$, we denote $\mathcal{X} \setminus E$ by $E^{\complement}$.

Proposition 4.11. *With the previous notation, there exists a positive constant C such that for all k and $x \in \mathcal{X}$,*

$$\begin{aligned} b_k^*(x) \leqslant{} & Ct\frac{V_{r_k}(x_k)}{\mu(B(x_k, r_k + d(x,x_k)))}\left(\frac{r_k}{r_k + d(x_k,x)}\right)^\beta \chi_{B(x_k,10r_k)^{\complement}}(x) \\ & + Cf^*(x)\chi_{B(x_k,10r_k)}(x) \end{aligned} \tag{4.3}$$

and

$$\int_{\mathcal{X}} [b_k^*(x)]^p \, d\mu(x) \leqslant C\int_{B(x_k,10r_k)} [f^*(x)]^p \, d\mu(x); \tag{4.4}$$

moreover, the series $\sum_k b_k$ converges in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ to a distribution b satisfying

$$b^*(x) \leqslant Ct\sum_k \frac{V_{r_k}(x_k)}{\mu(B(x_k, r_k + d(x,x_k)))}\left(\frac{r_k}{r_k + d(x_k,x)}\right)^\beta + Cf^*(x)\chi_\Omega(x) \tag{4.5}$$

and

$$\int_{\mathcal{X}} [b^*(x)]^p \, d\mu(x) \leqslant C\int_{\Omega} [f^*(x)]^p \, d\mu(x); \tag{4.6}$$

the distribution $g := f - b$ satisfies that $g \in (\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and

$$g^*(x) \leqslant Ct\sum_k \frac{V_{r_k}(x_k)}{\mu(B(x_k, r_k + d(x,x_k)))}\left(\frac{r_k}{r_k + d(x_k,x)}\right)^\beta + Cf^*(x)\chi_{\Omega^{\complement}}(x). \tag{4.7}$$

Applying Proposition 4.11, Lemma 4.10 and Corollary 3.11, and following an argument similar to [6, Theorem (3.34)], we obtain the following density result on $H^{*,p}(\mathcal{X})$.

Proposition 4.12. *Let $\epsilon \in (0,1)$, $p \in (n/(n+\epsilon), 1]$, β, $\gamma \in (n(1/p-1), \epsilon)$ and $H^{*,p}(\mathcal{X})$ be defined via the distribution space $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ as in Definition* 2.8. *If $q \in (1,\infty)$, then $L^q(\mathcal{X}) \cap H^{*,p}(\mathcal{X})$ is dense in $H^{*,p}(\mathcal{X})$.*

Combining Lemmas 4.6, 4.8, 4.9 and Proposition 4.12, similarly to the proof of [6, Lemma (3.36)], we obtain the following proposition.

Proposition 4.13. *Let $\epsilon \in (0,1)$, $p \in (n/(n+\epsilon), 1]$, β, $\gamma \in (n(1/p-1), \epsilon)$, $q \in (1,\infty)$ and $f \in L^q(\mathcal{X}) \cap H^{*,p}(\mathcal{X})$. Assume that there exists a positive constant $\widetilde{C}$ such that for all $x \in \mathcal{X}$, $f(x) \leqslant \widetilde{C}f^*(x)$. With the same notation as in Proposition* 4.11, *then there exists a positive constant C independent of f, k and t such that*

(i) *if $m_k := [\int_{\mathcal{X}} \phi_k(\xi)\, d\mu(\xi)]^{-1} \int_{\mathcal{X}} f(\xi)\phi_k(\xi)\, d\mu(\xi)$, then $|m_k| \leqslant Ct$ for all k;*

(ii) *if $b_k := (f - m_k)\phi_k$, then $\operatorname{supp} b_k \subset B(x_k, 2r_k)$ and the distribution on $\mathcal{G}_0^\epsilon(\beta,\gamma)$ induced by b_k coincides with b_k in Proposition* 4.11;

(iii) *the series $\sum_k b_k$ converges in $L^q(\mathcal{X})$. It induces a distribution on $\mathcal{G}_0^\epsilon(\beta,\gamma)$ which coincides with b in Proposition* 4.11 *and is still denoted by b. Moreover,* $\operatorname{supp} b \subset \Omega_t$;

(iv) *set* $g := f - b$, *then* $g = f\chi_{\Omega^\complement} + \sum_k m_k\phi_k$ *and for all* $x \in \mathcal{X}$, $|g(x)| \leqslant Ct$. *Moreover,* g *induces a distribution on* $\mathcal{G}_0^\epsilon(\beta,\gamma)$ *which coincides with* g *appearing in Proposition* 4.11.

Remark 4.14. (a) If f does not satisfy $|f(x)| \leqslant \widetilde{C}f^*(x)$ for all $x \in \mathcal{X}$, then by Theorem 3.10, there exists a function h such that f and h induce the same distribution and $|h(x)| \leqslant \widetilde{C}f^*(x) = \widetilde{C}h^*(x)$ for all $x \in \mathcal{X}$. Therefore, we can replace f by h since we only need to consider the behavior of f as a distribution not as a function.

(b) Denote by $\mathcal{C}(\mathcal{X})$ the space of continuous functions on $\mathcal{X}$. If $f \in \mathcal{C}(\mathcal{X})$, by Proposition 4.13 (ii), then $b_k \in \mathcal{C}(\mathcal{X})$ for all k. Applying Lemma 4.6 (vi) to $\sum_k b_k$, we obtain that for any given $x \in \mathcal{X}$, $\sum_k b_k(x)$ has only finite terms when it is restricted to some small neighborhood of x, which further implies that $b \in \mathcal{C}(\mathcal{X})$.

Lemma 4.15. *Let* $\epsilon \in (0,1)$, $q \in (n/(n+\epsilon),1)$, $p \in (q,1]$ *and* β, $\gamma \in (n(1/p-1),\epsilon)$. *If* $h \in L^2(\mathcal{X})$ *satisfying* $|h(x)| \leqslant 1$ *for all* $x \in \mathcal{X}$ *and* $h^* \in L^q(\mathcal{X})$, *then there exist* $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ *and* (p,∞)*-atoms* $\{a_k\}_{k\in\mathbb{N}}$ *such that* $h = \sum_{k\in\mathbb{N}} \lambda_k a_k$ *in* $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ *and almost everywhere. Moreover, there exists a positive constant* C *independent of* h *such that* $\sum_{k\in\mathbb{N}} |\lambda_k|^p \leqslant C\|h^*\|^q_{L^q(\mathcal{X})}$, $\|\sum_{k\in\mathbb{N}} |\lambda_k a_k|\|_{L^\infty(\mathcal{X})} \leqslant C$ *and* $\|\sum_{k\in\mathbb{N}} |\lambda_k a_k|\|^r_{L^r(\mathcal{X})} \leqslant C\|h^*\|^q_{L^q(\mathcal{X})}$ *if* $r \in [1,\infty)$.

Proof. For any $\theta \in (0,1)$, we define a sequence of functions $\{H_m\}_{m\in\mathbb{N}}$ as follows. Set $H_0 := h$. Proceeding by induction, assume that H_{m-1} is defined. Then set $\Omega_m := \{x \in \mathcal{X} : (H_{m-1})^*(x) > \theta^m\}$ and define H_m as the function g in Proposition 4.13 associated to $f = H_{m-1}$ and $t = \theta^m$. Notice that each Ω_m has a decomposition of $\Omega_m = \bigcup_i B(x_{m,i}, r_{m,i})$ satisfying (i) through (vi) of Lemma 4.6. Then define $\{\phi_{m,i}\}_i$ in the same way as in Lemma 4.8. If $m \geqslant 1$, then we have

$$H_m = H_{m-1} - \sum_i b_{m,i}, \tag{4.8}$$

where $b_{m,i}$ is as in Proposition 4.12 (ii). Moreover, there exists a positive constant C_7 such that for all $m \in \mathbb{N}$ and all $x \in \mathcal{X}$,

$$|H_m(x)| \leqslant C_7\theta^m \tag{4.9}$$

and

$$H_m^*(x) \leqslant h^*(x) + C_7 \sum_{i=1}^m \theta^i \sum_j \frac{\mu(B(x_{i,j}, r_{i,j}))}{\mu(B(x_{i,j}, r_{i,j} + d(x_{i,j}, x)))} \left(\frac{r_{i,j}}{r_{i,j} + d(x_{i,j}, x)}\right)^\beta. \tag{4.10}$$

Notice that (4.9) follows immediately from the definition of H_m and Proposition 4.13 (iv). To prove (4.10), we first have that for any $x \notin \Omega_m$, by (4.8) and (4.5),

$$H_m^*(x) \leqslant H_{m-1}^*(x) + C\theta^m \sum_{j\in\mathbb{N}} \frac{\mu(B(x_{m,j}, r_{m,j}))}{\mu(B(x_{m,j}, r_{m,j} + d(x_{m,j}, x)))} \left(\frac{r_{m,j}}{r_{m,j} + d(x_{m,j}, x)}\right)^\beta, \tag{4.11}$$

where C is the constant given in (4.5). Observe that for any $x \in \Omega_m$, Lemma 4.6 (ii) implies that $x \in B(x_{m,j}, r_{m,j})$ for some $j \in \mathbb{N}$. From this and (4.7), we deduce that if $x \in \Omega_m$, then

$$H_m^*(x) \lesssim t \lesssim \theta^m \sum_{j\in\mathbb{N}} \frac{\mu(B(x_{m,j}, r_{m,j}))}{\mu(B(x_{m,j}, r_{m,j} + d(x_{m,j}, x)))} \left(\frac{r_{m,j}}{r_{m,j} + d(x_{m,j}, x)}\right)^\beta. \tag{4.12}$$

Combining (4.11) and (4.12) and repeating this process m times yield (4.10).

For every $k\in\mathbb{N}$, by (4.8), $h=H_k+\sum_{m=1}^k\sum_j b_{m,j}$ almost everywhere. By Proposition 4.13 (iii), Remark 4.14 and (4.9), we obtain $h=\sum_{m=1}^\infty\sum_j b_{m,j}$ almost everywhere. The assumption $h^*\in L^2(\mathcal{X})$ together with (4.10) and Lemma 4.10 further implies that $H_m\in L^2(\mathcal{X})$. Thus by (4.9) and the size condition of φ, we obtain that for any $\varphi\in\mathcal{G}_0^\epsilon(\beta,\gamma)$,

$$\left|\langle h,\varphi\rangle-\sum_{m=1}^k\sum_j\langle b_{m,j},\varphi\rangle\right|=|\langle H_k,\varphi\rangle|=\left|\int_{\mathcal{X}}H_k(x)\varphi(x)\,d\mu(x)\right|\lesssim\theta^k\|\varphi\|_{\mathcal{G}(\beta,\gamma)}\to 0,$$

as $k\to\infty$, which further implies that

$$h=\lim_{k\to\infty}\sum_{m=1}^k\sum_j b_{m,j}=\sum_{m\in\mathbb{N}}\sum_j b_{m,j}\tag{4.13}$$

in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$. By the expression given in Proposition 4.13 (ii) for $b_{m,j}$, (4.9) and Proposition 4.13 (i), we obtain that for all $x\in\mathcal{X}$,

$$\begin{aligned}|b_{m,j}(x)|&\leqslant|H_{m-1}(x)|+\left|\left[\int_{\mathcal{X}}\phi_{m,j}(x)\,d\mu(x)\right]^{-1}\int_{\mathcal{X}}H_{m-1}(x)\phi_{m,j}(x)\,d\mu(x)\right|\\&\leqslant 2C_7\theta^{m-1}.\end{aligned}\tag{4.14}$$

Let $\lambda_{m,j}:=2C_7\theta^{m-1}\mu(B(x_{m,j},2r_{m,j}))^{1/p}$ and $e_{m,j}:=(\lambda_{m,j})^{-1}b_{m,j}$. It is easy to verify that each $e_{m,j}$ is a (p,∞)-atom. Then from (4.13), we deduce that

$$h=\sum_{m\in\mathbb{N}}\sum_j\lambda_{m,j}e_{m,j}\tag{4.15}$$

holds in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and almost everywhere. To estimate $\sum_{m\in\mathbb{N}}\sum_j|\lambda_{m,j}|^p$, by Lemma 4.6,

$$\sum_{m\in\mathbb{N}}\sum_j|\lambda_{m,j}|^p\lesssim\sum_{m\in\mathbb{N}}\theta^{mp}\mu(\Omega_m).$$

By the definition of Ω_m, (4.10) and Lemma 4.10, we obtain that there exists a positive constant C_8 such that for all $m\in\mathbb{N}$,

$$\begin{aligned}\theta^{mq}\mu(\Omega_m)&\leqslant\int_{\mathcal{X}}[H_{m-1}^*(x)]^q\,d\mu(x)\\&\leqslant\int_{\mathcal{X}}[h^*(x)]^q\,d\mu(x)+(C_7)^q\sum_{i=1}^{m-1}\theta^{iq}\\&\quad\times\int_{\mathcal{X}}\left[\sum_j\frac{\mu(B(x_{i,j},r_{i,j}))}{\mu(B(x_{i,j},r_{i,j}+d(x_{i,j},x)))}\left(\frac{r_{i,j}}{r_{i,j}+d(x_{i,j},x)}\right)^\beta\right]^q d\mu(x)\\&\leqslant C_8\left[\int_{\mathcal{X}}[h^*(x)]^q\,d\mu(x)+\sum_{i=1}^{m-1}\theta^{iq}\mu(\Omega_i)\right].\end{aligned}$$

Let $b_0:=\int_{\mathcal{X}}h^*(x)^q\,d\mu(x)$ and $b_m:=\theta^{mq}\mu(\Omega_m)$ for $m\in\mathbb{N}$. Then the formula above can be written as $b_m\leqslant C_8\sum_{i=0}^{m-1}b_i$. By induction, we obtain $b_i\leqslant b_0(C_8+2)^i$ for every $i\in\mathbb{Z}_+$. That is, $\theta^{iq}\mu(\Omega_i)\leqslant(C_8+2)^i\int_{\mathcal{X}}[h^*(x)]^q\,d\mu(x)$. Choose $\theta>0$ small enough such that $\theta^{p-q}(C_8+2)<1$. Then,

$$\sum_{m\in\mathbb{N}}\sum_j|\lambda_{m,j}|^p\lesssim\sum_{m\in\mathbb{N}}\theta^{(p-q)m}(C_8+2)^m\int_{\mathcal{X}}[h^*(x)]^q\,d\mu(x)\lesssim\int_{\mathcal{X}}[h^*(x)]^q\,d\mu(x).$$

By the fact $\operatorname{supp} e_{m,j} \subset B(x_{m,j}, r_{m,j})$, (4.14) and Lemma 4.6, we obtain that for all $x \in \mathcal{X}$,

$$\sum_{m,j} |\lambda_{m,j} e_{m,j}(x)| \lesssim \sum_{m\in\mathbb{N}} \theta^m \chi_{\Omega_m}(x) \lesssim 1,$$

which implies that $\sum_{m,j} |\lambda_{m,j} e_{m,j}| \in L^\infty(\mathcal{X})$. When $r \in [1,\infty)$, by Hölder's inequality,

$$\Big\| \sum_{m,j} |\lambda_{m,j} e_{m,j}| \Big\|_{L^r(\mathcal{X})} \lesssim \sum_{m\in\mathbb{N}} \theta^m [\mu(\Omega_m)]^{1/r} \lesssim \Big[\sum_{m\in\mathbb{N}} \theta^{mp} \mu(\Omega_m) \Big]^{1/r}.$$

Thus $\| \sum_{m,j} |\lambda_{m,j} e_{m,j}| \|_{L^r(\mathcal{X})}^r \lesssim \|h^*\|_{L^q(\mathcal{X})}^q$, which completes the proof of Lemma 4.15.

Theorem 4.16. *Let $\epsilon \in (0,1)$, $p \in (n/(n+\epsilon), 1]$ and β, $\gamma \in (n(1/p-1), \epsilon)$. If $f \in H^{*,p}(\mathcal{X})$, then there exist $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ and (p,∞)-atoms $\{a_k\}_{k\in\mathbb{N}}$ such that $f = \sum_{k\in\mathbb{N}} \lambda_k a_k$ in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$. Moreover, there exists a constant $C \geqslant 1$ such that*

$$\frac{1}{C} \|f\|_{H^{*,p}(\mathcal{X})} \leqslant \Big\{ \sum_{k\in\mathbb{N}} |\lambda_k|^p \Big\}^{1/p} \leqslant C \|f\|_{H^{*,p}(\mathcal{X})}.$$

Proof. We only give an outline of the proof since it is similar to that of [6, Theorem (4.13)]. First assume that $f^* \in L^p(\mathcal{X}) \cap L^2(\mathcal{X})$, then Theorem 3.10 tells us that f can be represented by a function satisfying $|f(x)| \leqslant C f^*(x)$. For any given integer k, set $\Omega_k := \{x \in \mathcal{X} : f^*(x) > 2^k\}$. By Proposition 4.13, we have $f = B_k + G_k$, where B_k and G_k are the functions b and g in Proposition 4.13 corresponding to $t = 2^k$. Since $B_k + G_k = B_{k+1} + G_{k+1}$, we then define

$$h_k := G_{k+1} - G_k = B_k - B_{k+1}. \tag{4.16}$$

Thus for any $m \in \mathbb{N}$, we have $f - \sum_{k=-m}^m h_k = B_{m+1} + G_{-m}$. Notice that $|G_{-m}(x)| \lesssim 2^{-m}$ for all $x \in \mathcal{X}$ and $\operatorname{supp} B_m \subset \Omega_m$ by Proposition 4.13. Then following the arguments in [6, p. 300], we deduce that for any $q \in (n/(n+\epsilon), 1]$, $\|h_k^*\|_{L^q(\mathcal{X})} \lesssim 2^k \mu(\Omega_k)$ by Lemma 4.10 and Proposition 4.11; and moreover $f = \sum_{k\in\mathbb{Z}} h_k$ holds in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ (here, we need to use Proposition 3.16) and almost everywhere.

By (4.16), we obtain that $\operatorname{supp} h_k \subset \Omega_k$ and there exists a positive constant $\widetilde{C}$ such that $|h_k(x)| \leqslant \widetilde{C} 2^k$ for all $x \in \mathcal{X}$ and $k \in \mathbb{Z}$. Applying Lemma 4.15 to $\widetilde{C}^{-1} 2^{-k} h_k$ and $q \in (n/(n+\epsilon), p)$ yields that $\widetilde{C}^{-1} 2^{-k} h_k = \sum_{i\in\mathbb{N}} \lambda_{k,i} a_{k,i}$ in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and almost everywhere, where $\{a_{k,j}\}_j$ are (p,∞)-atoms and $\{\lambda_{k,j}\}_j \subset \mathbb{C}$ satisfying $\sum_i |\lambda_{k,i}|^p \lesssim \|2^{-k} h_k^*\|_{L^q(\mathcal{X})}^q \lesssim \mu(\Omega_k)$. Let $\rho_{k,i} := \widetilde{C} 2^k \lambda_{k,i}$. Then $h_k = \sum_i \rho_{k,i} a_{k,i}$, and thus $f = \sum_{k\in\mathbb{Z}} \sum_i \rho_{k,i} a_{k,i}$ in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and almost everywhere. Moreover, $\sum_{k\in\mathbb{Z}} \sum_i |\rho_{k,i}|^p \lesssim \sum_{k\in\mathbb{Z}} 2^{kp} \mu(\Omega_k) \lesssim \|f^*\|_{L^p(\mathcal{X})}^p$. This together with Proposition 4.5 shows the theorem with the additional assumption $f^* \in L^2(\mathcal{X})$. The general case of the theorem then follows from the density of $L^2(\mathcal{X}) \cap H^{*,p}(\mathcal{X})$ in $H^{*,p}(\mathcal{X})$ and standard arguments as in [6, pp. 301–302], which completes the proof of Theorem 4.16.

For $\epsilon \in (0,1]$ and β, $\gamma \in (0,\epsilon)$, let $\mathring{\mathcal{G}}_b^\epsilon(\beta,\gamma)$ be the set of functions in $\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$ with bounded support. Let $\mathcal{C}_0(\mathcal{X})$ be the set of continuous functions on $\mathcal{X}$ which tends to zero at infinity. From Theorem 4.16 and the existence of an approximation of the identity with bounded support (see [23, Theorem 2.1]), we can deduce the following density result. We omit the details.

Proposition 4.17. *Let $\epsilon \in (0,1)$, $p \in (n/(n+\epsilon), 1]$ and β, $\gamma \in (n(1/p-1), \epsilon)$. Then $\mathring{\mathcal{G}}_b^\epsilon(\beta,\gamma)$ is a dense subset of $H^{*,p}(\mathcal{X})$; for any $p \in [1,\infty)$, $\mathring{\mathcal{G}}_b^\epsilon(\beta,\gamma)$ is a dense subset of $L^p(\mathcal{X})$; and if $p = \infty$, $\mathring{\mathcal{G}}_b^\epsilon(\beta,\gamma)$ is a dense subset of $\mathcal{C}_0(\mathcal{X})$.*

Proposition 4.5, Theorem 4.16, Remarks 3.15 and 2.4 imply the following conclusion.

Corollary 4.18. *Let* $\alpha \in (0,\infty)$, $p \in (n/(n+1),1]$ *and* $q \in (p,\infty] \cap [1,\infty]$. *Then* $H_{at}^{p,q}(\mathcal{X}) = H^{*,p}(\mathcal{X}) = H_\alpha^p(\mathcal{X}) = H_{\rm d}^p(\mathcal{X})$ *with equivalent quasi-norms.*

Remark 4.19. Let ϵ, p, β and γ be as in Theorem 4.16. For any $f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))'$, define that for all $x \in \mathcal{X}$, $\widetilde{f}^*(x) := \sup\{|\langle f,\varphi\rangle| : \varphi \in \mathcal{G}(\epsilon,\epsilon),\ \|\varphi\|_{\mathcal{G}(x,r,\epsilon,\epsilon)} \leqslant 1 \text{ for some } r>0\}$ and $\widetilde{H}^{*,p}(\mathcal{X}) := \{f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))' : \|\widetilde{f}^*\|_{L^p(\mathcal{X})} < \infty\}$ with the norm $\|f\|_{\widetilde{H}^{*,p}(\mathcal{X})} := \|\widetilde{f}^*\|_{L^p(\mathcal{X})}$. Obviously, for all $x \in \mathcal{X}$, by the definitions, we have $\widetilde{f}^*(x) \lesssim f^*(x)$, which implies that $H^{*,p}(\mathcal{X}) \subset \widetilde{H}^{*,p}(\mathcal{X})$. Repeating the procedure of the atomic decompositions, we see that Theorem 4.16 still holds with $H^{*,p}(\mathcal{X})$ replaced by $\widetilde{H}^{*,p}(\mathcal{X})$, which further implies that $\widetilde{H}^{*,p}(\mathcal{X}) \subset H_{\rm at}^p(\mathcal{X})$. Combining this with Corollary 4.18 gives that $H^{*,p}(\mathcal{X}) = \widetilde{H}^{*,p}(\mathcal{X})$. We also point out that when $p \in (1,\infty]$, by a slight modification for the proofs of Proposition 3.9 and Corollary 3.11, we also have that for all β, $\gamma \in (0,\epsilon)$, $\widetilde{H}^{*,p}(\mathcal{X}) = L^p(\mathcal{X})$. This observation was used in [35].

5 Some applications

As the first application, we show that $H^{*,p}(\mathcal{X})$ coincides with the Hardy space $H^p(\mathcal{X})$ defined in terms of the Littlewood-Paley function in [23, 24], via the atomic characterizations of these spaces.

Definition 5.1. *Let* $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$, $\epsilon_3 > 0$ *and* $\{S_k\}_{k\in\mathbb{Z}}$ *be an* $(\epsilon_1,\epsilon_2,\epsilon_3)$-AOTI. *Let* $\epsilon \in (0,\epsilon_1 \wedge \epsilon_2)$, $p \in (n/(n+\epsilon),1]$ *and* β, $\gamma \in (n(1/p-1),\epsilon)$. *For* $k \in \mathbb{Z}$, *set* $D_k := S_k - S_{k-1}$. *For any* $f \in (\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$, *the Lusin-area function* (*also called the Littlewood-Paley S-function*) $\dot{S}(f)$ *is defined by*

$$\dot{S}(f)(x) := \left\{ \sum_{k=-\infty}^{\infty} \int_{d(x,y)<2^{-k}} |D_k(f)(y)|^2 \frac{d\mu(y)}{V_{2^{-k}}(x)} \right\}^{1/2}.$$

Define $H^p(\mathcal{X}) := \{f \in (\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))' : \|\dot{S}(f)\|_{L^p(\mathcal{X})} < \infty\}$ *and* $\|f\|_{H^p(\mathcal{X})} := \|\dot{S}(f)\|_{L^p(\mathcal{X})}$.

Let $\epsilon \in (0,1)$, $p \in (n/(n+\epsilon),1]$ and β, $\gamma \in (n(1/p-1),\epsilon)$. Define $\mathring{H}_{\rm at}^p(\mathcal{X})$ by replacing the distribution space $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ with $(\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$ in the definition of $H_{at}^{p,2}(\mathcal{X})$; see Definition 4.2 or [23, 24]. It follows from [24, Theorem 2.21] that $\mathring{H}_{\rm at}^p(\mathcal{X})$ coincides with $H^p(\mathcal{X})$ when $p \in (n/(n+1),1]$. This together with Theorem 4.16 tells us that in order to prove $H^p(\mathcal{X}) = H^{*,p}(\mathcal{X})$, it suffices to show $\mathring{H}_{\rm at}^p(\mathcal{X}) = H_{\rm at}^p(\mathcal{X})$. To this end, we start with the following two technical lemmas. The proof of the first lemma is trivial and we omit the details.

Lemma 5.2. *Let* $\epsilon \in (0,1)$ *and* β, $\gamma \in (0,\epsilon)$. *If* $f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))'$ *and* $\langle f,\varphi\rangle = 0$ *for all* $\varphi \in \mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$, *then* f *is a constant.*

Let $\alpha \in (0,\infty)$. Coifman and Weiss (see [5, (2.2)]) introduced the following Lipschitz space ${\rm Lip}_\alpha(\mathcal{X})$, namely, $\varphi \in {\rm Lip}_\alpha(\mathcal{X})$ if and only if for all x, $y \in \mathcal{X}$ and $x \neq y$, $|\varphi(x)-\varphi(y)| \leqslant C[\mu(B)]^\alpha$, where B is any ball containing both x and y and C is a positive constant depending only on φ. Define $\widetilde{H}_{\rm at}^p(\mathcal{X})$ by replacing $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ with $({\rm Lip}_{1/p-1}(\mathcal{X}))'$ in Definition 4.2, where $({\rm Lip}_{1/p-1}(\mathcal{X}))'$ is the dual space of ${\rm Lip}_{1/p-1}(\mathcal{X})$; see [5].

The following key observation establishes the connection between spaces of test functions and the Lipschitz spaces above. This plays a key role in establishing the connection between the Hardy spaces introduced in this paper and [23, 24] with the atomic Hardy spaces $\widetilde{H}_{\rm at}^p(\mathcal{X})$ of Coifman and Weiss; see Remark 5.5 below.

Lemma 5.3. *Let $\beta \in (0,1]$ and $\gamma \in (0,\infty)$. Then there exists a positive constant C such that for all $\varphi \in \mathcal{G}(\beta,\gamma)$ and $x, y \in \mathcal{X}$, $|\varphi(x)-\varphi(y)| \leqslant C\|\varphi\|_{\mathcal{G}(\beta,\gamma)}[\mu(B(x,d(x,y)))]^{\beta/n}$.*

Proof. For any $x, y \in \mathcal{X}$ satisfying $d(x,y) \leqslant (1+d(x,x_1))/2$, by the fact $\mu(B(x_1,1+d(x,x_1))) \sim \mu(B(x,1+d(x,x_1)))$ and (2.2), we obtain

$$\frac{\mu(B(x_1,1+d(x,x_1)))}{\mu(B(x,d(x,y)))} \lesssim \frac{\mu(B(x,1+d(x,x_1)))}{\mu(B(x,d(x,y)))} \lesssim \left(\frac{1+d(x,x_1)}{d(x,y)}\right)^n.$$

This together with the regularity of φ yields that

$$|\varphi(x)-\varphi(y)| \lesssim \|\varphi\|_{\mathcal{G}(\beta,\gamma)}[\mu(B(x,d(x,y)))]^{\beta/n}.$$

For any $x, y \in \mathcal{X}$ satisfying $d(x,y) > (1+d(x,x_1))/2$, using the fact $\mu(B(x_1,1+d(x,x_1))) \sim \mu(B(x,1+d(x,x_1)))$ and (2.2) again, we also obtain

$$\frac{\mu(B(x_1,1+d(x,x_1)))}{\mu(B(x,d(x,y)))} \lesssim \left(\frac{1+d(x,x_1)}{d(x,y)}\right)^\kappa.$$

This together with the size condition of φ also yields that

$$|\varphi(x)-\varphi(y)| \lesssim \|\varphi\|_{\mathcal{G}(\beta,\gamma)} \lesssim \|\varphi\|_{\mathcal{G}(\beta,\gamma)}[\mu(B(x,d(x,y)))]^{\beta/n},$$

which completes the proof of Lemma 5.3.

Theorem 5.4. *Let $\epsilon \in (0,1)$, $p \in (n/(n+\epsilon),1]$ and β, γ satisfy* (3.22). *Then $\mathring{H}^p_{\rm at}(\mathcal{X}) = H^p_{\rm at}(\mathcal{X})$ with equivalent quasi-norms.*

Proof. Since $(\mathcal{G}^\epsilon_0(\beta,\gamma))' \subset (\mathring{\mathcal{G}}^\epsilon_0(\beta,\gamma))'$, we then obviously have $H^{p,2}_{at}(\mathcal{X}) \subset \mathring{H}^p_{\rm at}(\mathcal{X})$. We still need to show $\mathring{H}^p_{\rm at}(\mathcal{X}) \subset H^p_{\rm at}(\mathcal{X})$ by Corollary 4.18. If $f \in \mathring{H}^p_{\rm at}(\mathcal{X})$, then $f \in (\mathring{\mathcal{G}}^\epsilon_0(\beta,\gamma))'$ with some β and γ as in (3.22) and there exist a sequence of $(p,2)$-atoms $\{a_k\}_{k\in\mathbb{N}}$ and $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ with $\sum_{k\in\mathbb{N}}|\lambda_k|^p < \infty$ such that $f = \sum_{k\in\mathbb{N}}\lambda_k a_k$ in $(\mathring{\mathcal{G}}^\epsilon_0(\beta,\gamma))'$. For any $\varphi \in \mathcal{G}^\epsilon_0(\beta,\gamma)$ with β and γ as in (3.22), set

$$\langle \widetilde{f},\varphi\rangle := \sum_{k\in\mathbb{N}}\lambda_k\langle a_k,\varphi\rangle. \tag{5.1}$$

Let a be a $(p,2)$-atom supported on $B := B(x_0,r_0)$ and $\varphi \in \mathcal{G}^\epsilon_0(\beta,\gamma)$. Notice that $\varphi \in \mathcal{G}(n(1/p-1),\gamma)$ since $\beta > n(1/p-1)$. Then applying (A3), Lemma 5.3, Hölder's inequality and (A2), we obtain

$$|\langle a,\varphi\rangle| = \left|\int_{\mathcal{X}} a(x)[\varphi(x)-\varphi(x_0)]\,d\mu(x)\right| \lesssim \|\varphi\|_{\mathcal{G}^\epsilon_0(\beta,\gamma)}.$$

Combining this with (5.1) yields that

$$|\langle \widetilde{f},\varphi\rangle| \lesssim \|\varphi\|_{\mathcal{G}^\epsilon_0(\beta,\gamma)}\sum_{k\in\mathbb{N}}|\lambda_k| \lesssim \|\varphi\|_{\mathcal{G}^\epsilon_0(\beta,\gamma)}\left(\sum_{k\in\mathbb{N}}|\lambda_k|^p\right)^{1/p},$$

which further tells us that $\widetilde{f} \in (\mathcal{G}^\epsilon_0(\beta,\gamma))'$, $\widetilde{f} = \sum_{k\in\mathbb{N}}\lambda_k a_k$ in $(\mathcal{G}^\epsilon_0(\beta,\gamma))'$ and $\widetilde{f} = f$ on $\mathring{\mathcal{G}}^\epsilon_0(\beta,\gamma)$. Moreover, $\widetilde{f} \in H^p_{\rm at}(\mathcal{X})$.

Suppose that there exists another extension of f, say $\widetilde{g} \in H^p_{\rm at}(\mathcal{X})$. Then $\widetilde{g} \in (\mathcal{G}^\epsilon_0(\beta,\gamma))'$ and $\widetilde{g} = f$ on $\mathring{\mathcal{G}}^\epsilon_0(\beta,\gamma)$. By Lemma 5.2, $\widetilde{f} - \widetilde{g}$ is a constant. Note that $\widetilde{f} \in H^p_{\rm at}(\mathcal{X})$ and the non-zero constant function does not belong to $H^p_{\rm d}(\mathcal{X}) = H^p_{\rm at}(\mathcal{X})$ (see Corollary 4.18). Hence, $\widetilde{g} \notin H^p_{\rm at}(\mathcal{X})$. This contradiction implies that $\widetilde{f} \in H^p_{\rm at}(\mathcal{X})$ is the unique extension of $f \in H^p(\mathcal{X})$. Taking over all decompositions of f yields that $\|\widetilde{f}\|_{H^p_{\rm at}(\mathcal{X})} \leqslant \|f\|_{\mathring{H}^p_{\rm at}(\mathcal{X})}$, which completes the proof of Theorem 5.4.

Remark 5.5. (i) Notice that $\epsilon \in (0,1)$ is arbitrary. By Theorem 5.4 and Corollary 4.18, we obtain that for any $p \in (n/(n+1), 1]$, $H^{*,p}(\mathcal{X})$ coincides to $H^p(\mathcal{X})$ with equivalent quasi-norms. As a consequence, $H^{*,p}(\mathcal{X})$ also coincide with the Triebel-Lizorkin spaces $\mathring{F}^0_{p,2}(\mathcal{X})$ with equivalent quasi-norms; see [23].

(ii) Let $\epsilon \in (0,1)$, $p \in (n/(n+\epsilon), 1]$ and $\beta,\ \gamma \in (n(1/p-1), \epsilon)$. Lemma 5.3 implies that $\mathcal{G}^\epsilon_0(\beta,\gamma) \subset {\rm Lip}_{1/p-1}(\mathcal{X})$ and thus $\widetilde{H}^p_{\rm at}(\mathcal{X}) \subset H^p_{\rm at}(\mathcal{X})$ by their definitions. Conversely, given any $f \in H^p_{\rm at}(\mathcal{X})$, f has a decomposition as $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $(\mathcal{G}^\epsilon_0(\beta,\gamma))'$, where $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$, $\{a_j\}_{j\in\mathbb{N}}$ are (p,∞)-atoms and $\sum_{j\in\mathbb{N}} |\lambda_j|^p < \infty$. It is not difficult (see Theorem 5.4) to show that $\sum_{j\in\mathbb{N}} \lambda_j a_j$ converges to an element in $({\rm Lip}_{1/p-1}(\mathcal{X}))'$, say $\widetilde{f}$. Thus $\widetilde{f} = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $(\mathcal{G}^\epsilon_0(\beta,\gamma))'$ by Lemma 5.3. Therefore, $f = \widetilde{f} \in \widetilde{H}^p_{\rm at}(\mathcal{X})$, which implies that $H^p_{\rm at}(\mathcal{X}) \subset \widetilde{H}^p_{\rm at}(\mathcal{X})$. So $\widetilde{H}^p_{\rm at}(\mathcal{X}) = H^p_{\rm at}(\mathcal{X}) = \mathring{H}^p_{\rm at}(\mathcal{X})$ with equivalent quasi-norms, which answers a question in [24, Remark 2.30].

(iii) From now on, for $p \in (n/(n+1), 1]$, we use $H^p(\mathcal{X})$ to denote $H^{*,p}(\mathcal{X})$, $H^p_\alpha(\mathcal{X})$ with $\alpha \in (0,\infty)$, $H^p_{\rm d}(\mathcal{X})$, $\widetilde{H}^p_{\rm at}(\mathcal{X})$, $H^p_{\rm at}(\mathcal{X})$ and $\mathring{H}^p_{\rm at}(\mathcal{X})$ if there exists no confusion.

As another application, we extend the results of [25] to RD-spaces. Suppose that $p \in (n/(n+1), 1]$ and $q \in [1,\infty] \cap (p,\infty]$. Denote by $H^{p,q}_{\rm fin}(\mathcal{X})$ the vector space of all finite linear combinations of (p,q)-atoms. Notice that $H^{p,q}_{\rm fin}(\mathcal{X})$ consists of all $L^q(\mathcal{X})$ functions with bounded support and integral 0. Clearly, $H^{p,q}_{\rm fin}(\mathcal{X})$ is a dense subset of $H^p_{\rm at}(\mathcal{X})$. Define the quasi-norm on $H^{p,q}_{\rm fin}(\mathcal{X})$ by

$$\|f\|_{H^{p,q}_{\rm fin}(\mathcal{X})} := \inf\left\{\left(\sum_{j=1}^N |\lambda_j|^p\right)^{1/p} : f = \sum_{j=1}^N \lambda_j a_j,\ N \in \mathbb{N}, \{\lambda_j\}_{j=1}^N \subset \mathbb{C}, \text{and } \{a_j\}_{j=1}^N \text{ are } (p,q)\text{-atoms}\right\}.$$

Motivated by [25], we obtain the following theorem by means of atomic characterizations for $H^{*,p}(\mathcal{X})$ in Section 4.

Theorem 5.6. *Let $p \in (n/(n+1), 1]$. Then the following hold*:

(a) *if $q \in (p,\infty) \cap [1,\infty)$, then $\|\cdot\|_{H^{p,q}_{\rm fin}(\mathcal{X})}$ and $\|\cdot\|_{H^p(\mathcal{X})}$ are equivalent quasi-norms on $H^{p,q}_{\rm fin}(\mathcal{X})$.*

(b) *$\|\cdot\|_{H^{p,\infty}_{\rm fin}(\mathcal{X})}$ and $\|\cdot\|_{H^p(\mathcal{X})}$ are equivalent quasi-norms on $H^{p,\infty}_{\rm fin}(\mathcal{X}) \cap \mathcal{C}(\mathcal{X})$.*

Proof. For any $f \in H^{p,q}_{\rm fin}(\mathcal{X})$ with $p \in (n/(n+1), 1]$ and $q \in (p,\infty] \cap [1,\infty]$, we obviously have $\|f\|_{H^p_{\rm at}(\mathcal{X})} \leqslant \|f\|_{H^{p,q}_{\rm fin}(\mathcal{X})}$. By Remark 5.5 (iii), we further have $\|f\|_{H^p(\mathcal{X})} \lesssim \|f\|_{H^{p,q}_{\rm fin}(\mathcal{X})}$. Thus, to complete the proof of the theorem, it suffices to show that for all $f \in H^{p,q}_{\rm fin}(\mathcal{X})$ when $q < \infty$ or $H^{p,\infty}_{\rm fin} \cap \mathcal{C}(\mathcal{X})$ when $q = \infty$, $\|f\|_{H^{p,q}_{\rm fin}(\mathcal{X})} \lesssim \|f\|_{H^{*,p}(\mathcal{X})}$.

Let x_1 be as in Definition 2.5. We may assume that $f \in H^{p,q}_{\rm fin}(\mathcal{X})$ with $\|f\|_{H^{*,p}(\mathcal{X})} = 1$, and further assume that ${\rm supp}\, f \subset B(x_1, R)$ for some $R > 0$. Let all the notation be as in Section 4. For each $k \in \mathbb{Z}$, set $\Omega_k := \{x \in \mathcal{X} : f^*(x) > 2^k\}$. Let $H^k_0 := \widetilde{C}^{-1} 2^{-k} h_k$, where h_k and $\widetilde{C}$ are as in Theorem 4.16. Replace h, θ, H_{m-1} and Ω_m for $m \in \mathbb{N}$ in the proof of Lemma 4.15, respectively, by H_0^k, θ_k, H_{m-1}^k and Ω_m^k. Repeating the proof of Lemma 4.15 for H_0^k, θ_k, H_{m-1}^k and Ω_m^k, we obtain that $\Omega_m^k := \{x \in \mathcal{X} : (H_{m-1}^k)^*(x) > (\theta_k)^m\} = \bigcup_i B(x_{m,i}^k, r_{m,i}^k)$, where the balls $\{B(x_{m,i}^k, r_{m,i}^k)\}_i$ satisfy (i) through (vi) of Lemma 4.6, and that $H_m^k = H_{m-1}^k - \sum_i b_{m,i}^k$, and

$$b_{m,i}^k = \left\{H_{m-1}^k - \left[\int_{\mathcal{X}} \phi_{m,i}^k(x)\, d\mu(x)\right]^{-1} \int_{\mathcal{X}} H_{m-1}^k(x)\phi_{m,i}^k(x)\, d\mu(x)\right\}\phi_{m,i}^k \tag{5.2}$$

with $\{\phi_{m,i}^k\}_i$ as in Lemma 4.8 associated to Ω_m^k.

By the procedure of the atomic decomposition of f in Theorem 4.16 and Lemma 4.15, we obtain that

$$f = \sum_{k\in\mathbb{Z}} h_k = \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{N}} \sum_i \rho_{m,i}^k a_{m,i}^k \tag{5.3}$$

in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and almost everywhere. Moreover, it also follows from the proofs of Theorem 4.16 and Lemma 4.15 that

(i) every $a_{m,i}^k$ is a (p,∞)-atom and $\operatorname{supp} a_{m,i}^k \subset B(x_{m,i}^k, 2r_{m,i}^k) \subset \Omega_m^k$;
(ii) $b_{m,i}^k = C2^{-k}\rho_{m,i}^k a_{m,i}^k$, where C is a positive constant independent of k, m and i;
(iii) for any $k \in \mathbb{Z}$, $h_k = \sum_{m\in\mathbb{N}} \sum_i \rho_{m,i}^k a_{m,i}^k$ and $\operatorname{supp} h_k \subset \Omega_k$;
(iv) for any $L \in \mathbb{Z}_+$, $H_L^k = \sum_{m=L+1}^\infty \sum_i b_{m,i}^k$ in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and almost everywhere;
(v) there exists a positive constant C independent of f such that

$$\sum_{k,m,i} |\rho_{m,i}^k|^p \leqslant C\|f\|_{H^{*,p}(\mathcal{X})}^p = C;$$

(vi) given any $r \in [1,\infty]$, there exists a positive constant C independent of f and $k \in \mathbb{Z}$ such that $\|\sum_{m,i} |\rho_{m,i}^k a_{m,i}^k|\|_{L^r(\mathcal{X})} \leqslant C2^k[\mu(\Omega_k)]^{1/r}$.

We claim that there exists a positive constant $\widetilde{C}$ depending only on $\mathcal{X}$ such that for all $x \in \mathcal{X} \setminus B(x_1, 16R)$,

$$f^*(x) \leqslant \widetilde{C}[\mu(B(x_1,R))]^{-1/p}. \tag{5.4}$$

Assume this claim for a moment, which will be proved at the end of the proof of this theorem. Denote by k' the largest integer k satisfying $2^k \leqslant \widetilde{C}[\mu(B(x_1,R))]^{-1/p}$. Then for any $k > k'$, we have $\Omega_k \subset B(x_1, 16R)$. Define h and ℓ, respectively, by

$$h := \sum_{k\leqslant k'} \sum_{m\in\mathbb{N}} \sum_i \rho_{m,i}^k a_{m,i}^k \quad \text{and} \quad \ell := \sum_{k>k'} \sum_{m\in\mathbb{N}} \sum_i \rho_{m,i}^k a_{m,i}^k. \tag{5.5}$$

Observe that $\operatorname{supp} \ell \subset B(x_1, 16R)$. This together with $\operatorname{supp} f \subset B(x_1, R)$ and $f = h + \ell$ yields that $\operatorname{supp} h \subset B(x_1, 16R)$. By Property (vi) above,

$$|h| \leqslant \sum_{k\leqslant k'} \sum_{m\in\mathbb{N}} \sum_i |\rho_{m,i}^k a_{m,i}^k| \lesssim \sum_{k\leqslant k'} 2^k \lesssim [\mu(B(x_1,R))]^{-1/p}.$$

Combining this with the fact $\operatorname{supp} h \subset B(x_1, 16R)$ and the Lebesgue dominated convergence theorem, we further obtain that $\sum_{k\leqslant k'} \sum_{m\in\mathbb{N}} \sum_i \rho_{m,i}^k a_{m,i}^k$ converges in $L^1(\mathcal{X})$ and thus h has integral 0. Therefore, h is a multiple of some (p,∞)-atom.

Now we assume that $q \in [1,\infty) \cap (p,\infty)$ and conclude the proof of (a). For any $N := (N_1, N_2) \in \mathbb{N} \times \mathbb{N}$, set

$$\ell_N := \sum_{k=k'+1}^{N_1} \sum_{i+m\leqslant N_2} \rho_{m,i}^k a_{m,i}^k.$$

Obviously $\ell_N \in H^{p,q}_{\rm fin}(\mathcal{X})$. Notice that $f = h + \ell_N + (\ell - \ell_N)$. Thus, to finish the proof of (a), it suffices to show that when N_1 and N_2 are large enough, $\ell - \ell_N$ is an arbitrary small multiple of some (p,q)-atom. Notice that $\ell - \ell_N$ has bounded support since both ℓ and ℓ_N do. Moreover,

$$\int_{\mathcal{X}} [\ell(x) - \ell_N(x)]\, d\mu(x) = \int_{\mathcal{X}} [f(x) - h(x) - \ell_N(x)]\, d\mu(x) = 0.$$

So it suffices to show that for any $\eta > 0$, there exist $N_1,\ N_2 \in \mathbb{N}$ large enough such that

$$\|\ell - \ell_N\|_{L^q(\mathcal{X})} < \eta. \tag{5.6}$$

To see this, notice that by (5.5),

$$\|\ell - \ell_N\|_{L^q(\mathcal{X})} \leqslant \left\| \sum_{k>N_1} \left| \sum_{m,\,i} \rho^k_{m,i} a^k_{m,i} \right| \right\|_{L^q(\mathcal{X})} + \left\| \sum_{k=k'+1}^{N_1} \left| \sum_{i+m>N_2} \rho^k_{m,i} a^k_{m,i} \right| \right\|_{L^q(\mathcal{X})} =: \mathrm{I} + \mathrm{II}.$$

For any given $s \in \mathbb{Z}$ and almost every $z \in (\Omega_s \setminus \Omega_{s+1})$, by (iii) and (vi) above, we have

$$\sum_{k>k'} \left| \sum_{m,\,i} \rho^k_{m,i} a^k_{m,i}(z) \right| \leqslant C \sum_{k'<k\leqslant s} 2^k \leqslant C 2^s \leqslant C f^*(z),$$

where C is a positive constant independent of f and s. It follows that for all $z \in \mathcal{X}$,

$$\sum_{k>k'} \left| \sum_{m,\,i} \rho^k_{m,i} a^k_{m,i}(z) \right| \lesssim f^*(z). \tag{5.7}$$

Assume first that $q > 1$. Since $f \in L^q(\mathcal{X})$, we have $f^* \in L^q(\mathcal{X})$, which together with (5.7) and the Lebesgue dominated convergence theorem gives that I tends to 0 as $N_1 \to \infty$. Let now $q = 1$. Notice that $\operatorname{supp} \sum_{k>N_1} |h_k| \subset \Omega_{N_1}$ by (iii). This combined with (5.3) yields that

$$\sum_{k>N_1} |h_k(z)| \leqslant \left| f(z) - \sum_{k\leqslant N_1} |h_k| \right| \leqslant |f(z)| + C2^{N_1}.$$

For each $k \in \mathbb{Z}$, set $f^k_1 := f\chi_{\{|f|>2^k\}}$ and $f^k_2 := f - f^k_1$. By this, the fact $f^*(x) \lesssim \mathcal{M}(f)(x)$ for all $x \in \mathcal{X}$, the boundedness of $\mathcal{M}$ from $L^1(\mathcal{X})$ to weak-$L^1(\mathcal{X})$ (see [5]), and $f \in L^1(\mathcal{X})$, we obtain that when $k \to \infty$,

$$\begin{aligned} 2^k \mu(\Omega_k) &\lesssim 2^k \mu(\{x \in \mathcal{X} : \mathcal{M}(f)(x) > 2^k\}) \\ &\lesssim 2^k \mu(\{x \in \mathcal{X} : \mathcal{M}(f^k_1)(x) > 2^{k-1}\}) \lesssim \|f^k_1\|_{L^1(\mathcal{X})} \to 0. \end{aligned}$$

This implies that when $q = 1$, we also have $\mathrm{I} \lesssim \int_{\Omega_{N_1}} [f(x) + 2^k]\, d\mu(x) \to 0$ as $N_1 \to \infty$. Property (vi) implies that for any given $N_1 \in \mathbb{N}$, there exists $N_2 \in \mathbb{N}$ large enough such that $\mathrm{II} < \eta$ for any $\eta > 0$. Therefore, (5.6) holds.

Then, using Property (v) above, we obtain

$$\|f\|^p_{H^{p,q}_{\rm fin}(\mathcal{X})} \leqslant C^p + \sum_{k=k'+1}^{N_1} |\rho^k_{m,i}|^p + \eta^p \lesssim 1,$$

which completes the proof of (a).

Now we turn to the proof of (b). Since $f^*(x) \leqslant C_{\mathcal{X}}\|f\|_{L^\infty(\mathcal{X})}$, where $C_{\mathcal{X}}$ is a positive constant depending only on $\mathcal{X}$, we have $\Omega_k = \emptyset$ for all k satisfying $2^k \geqslant C_{\mathcal{X}}\|f\|_{L^\infty(\mathcal{X})}$. Denote by k'' the largest integer for which the last inequality does not hold. Combining this with the definition of Ω_k and (5.5), we have

$$\ell = \sum_{k'<k\leqslant k''} \sum_{m\in\mathbb{N}} \sum_i \rho^k_{m,i} a^k_{m,i}, \tag{5.8}$$

since for every $k' < k \leqslant k''$, $\operatorname{supp} h_k \subset \Omega_k \subset B(x_1, 16R)$. Moreover, by (4.16) and Remark 4.14 (b), we have $h_k \in \mathcal{C}(\mathcal{X})$ for all $k \in \mathbb{Z}$. From this and (5.2), it follows that every $b^k_{1,i} \in \mathcal{C}(\mathcal{X})$. Thus $H^k_1 = h_k - \sum_i b^k_{1,i} \in \mathcal{C}(\mathcal{X})$ by Lemma 4.6 (vi). This combined with an argument of induction on m further tells us that $H^k_m \in \mathcal{C}(\mathcal{X})$ and $b^k_{m,i} \in \mathcal{C}(\mathcal{X})$ for all k, m, i. Therefore, every $a^k_{m,i} \in \mathcal{C}(\mathcal{X})$ by Property (ii) above.

For any $L \in \mathbb{N}$ and $\delta > 0$, set $F^L_1 := \{(k,m,i) : k' < k \leqslant k'', m > L\}$,

$$F^{L,\delta}_2 := \{(k,m,i) : k' < k \leqslant k'', 1 \leqslant m \leqslant L, 2r_{m,i} \leqslant \delta\}$$

and $F^{L,\delta}_3 := \{(k,m,i) : k' < k \leqslant k'', 1 \leqslant m \leqslant L, 2r_{m,i} > \delta\}$. By (5.8), we write $\ell = \ell^L_1 + \ell^{L,\delta}_2 + \ell^{L,\delta}_3$ with $\ell^L_1 := \sum_{(k,m,i)\in F^L_1} \rho^k_{m,i} a^k_{m,i}$ and for $j = 2, 3$,

$$\ell^{L,\delta}_j := \sum_{(k,m,i)\in F^{L,\delta}_j} \rho^k_{m,i} a^k_{m,i}.$$

We claim that for any given k and m, if $\{r^k_{m,i}\}_i$ is an infinite sequence, then $\lim_{i\to\infty} r^k_{m,i} = 0$. To see this, by Remark 4.7, it suffices to show that every Ω^k_m is bounded in $\mathcal{X}$. To verify that Ω^k_m is bounded, recall that [5, Lemma 3.9] implies that if $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ with bounded support, then for each $\alpha > 0$, the set $\{x \in \mathcal{X} : \mathcal{M}(f)(x) > \alpha\}$ is contained in a ball depending on α. Therefore by this and the fact $(H^k_{m-1})^* \lesssim \mathcal{M}(H^k_{m-1})$ (see Proposition 3.9), we only need to show that every H^k_{m-1} has bounded support, which follows from an inductive argument. In fact, since $\operatorname{supp} f \subset B(x_1, R)$, by [5, Lemma 3.9] and the fact $f^* \lesssim \mathcal{M}(f)$, we know that each Ω_k is bounded in $\mathcal{X}$, which together with $\operatorname{supp} H^k_0 = \operatorname{supp} h_k \subset \Omega_k$ yields that H^k_0 has bounded support. Notice that $H^k_m = H^k_{m-1} - \sum_i b^k_{m,i}$ and $\operatorname{supp}(\sum_i b^k_{m,i}) \subset \Omega^k_m$ by Proposition 4.13 (iii). Therefore, if $\operatorname{supp} H^k_{m-1}$ is bounded, then arguing similarly as above implies that Ω^k_m is bounded, and therefore $\operatorname{supp} H^k_m$ is bounded in $\mathcal{X}$. Thus, the claim holds. For any $L \in \mathbb{N}$ and $\delta > 0$, by this claim, we obtain that $F^{L,\delta}_3$ is finite and $\ell^{L,\delta}_3$ is a linear combination of finite continuous (p,∞)-atoms.

From Properties (ii), (iv) and (4.9), we obtain that for all $x \in \mathcal{X}$,

$$|\ell^L_1(x)| \lesssim \sum_{k'<k\leqslant k''} 2^k |H^k_L(x)| \lesssim \sum_{k'<k\leqslant k''} 2^k (\theta_k)^L,$$

which implies that $\lim_{L\to\infty} \|\ell^L_1\|_{L^\infty(\mathcal{X})} = 0$.

Given any $L \in \mathbb{N}$, let us now show that for any $\eta > 0$, there exists $\delta > 0$ small enough such that $\|\ell^{L,\delta}_2\|_{L^\infty(\mathcal{X})} < \eta$. Notice that for any given k and m, $H^k_m \in \mathcal{C}(\mathcal{X})$ with a bounded support. Therefore, for any $\eta > 0$, there exists $\delta > 0$ such that for any $k' < k \leqslant k''$, $0 \leqslant m \leqslant L$ and any $x, y \in \mathcal{X}$ satisfying $d(x,y) < \delta$, $|H^k_m(x) - H^k_m(y)| < \eta/L$. For any $(k,m,i) \in F^{L,\delta}_2$, by (5.2), we have

$$\begin{aligned}|b^k_{m,i}(x)| &= \bigg|H^k_{m-1}(x) - H^k_{m-1}(x^k_{m,i}) \\ &\quad - \bigg[\int_{\mathcal{X}} \phi^k_{m,i}(z)\,d\mu(z)\bigg]^{-1} \int_{\mathcal{X}} (H^k_{m-1}(z) - H^k_{m-1}(x^k_{m,i}))\phi^k_{m,i}(z)\,d\mu(z)\bigg|\phi^k_{m,i}(x) \\ &< 2\eta/L,\end{aligned}$$

which further implies that for any given $L \in \mathbb{N}$, there exists a positive δ such that $\|\ell_2^{L,\,\delta}\|_{L^\infty(\mathcal{X})} \lesssim \sum_{k'<k\leqslant k''} 2^k \sum_{m\leqslant L} \eta/L \lesssim \eta$. Hence, ℓ can be decomposed into a continuous part and a part that is uniformly arbitrarily small, say, $\ell_3^{L,\,\delta}$ and $\ell_1^L + \ell_2^{L,\,\delta}$. Therefore, ℓ is continuous. Furthermore, $h = f - \ell$ is also continuous and thus h is a multiple of some continuous (p,∞)-atom.

To find a finite atomic decomposition of ℓ, we use once more the decomposition $\ell = (\ell_1^L + \ell_2^{L,\,\delta}) + \ell_3^{L,\,\delta}$. Obviously $\ell_3^{L,\,\delta}$ is a finite linear combination of continuous (p,∞)-atoms and $\|\ell_3^{L,\,\delta}\|_{H^{p,\,\infty}_{\rm fin}(\mathcal{X})} \leqslant \|f\|_{H^{*,\,p}(\mathcal{X})}$. Observe that $\ell_1^L + \ell_2^{L,\,\delta} = \ell - \ell_3^{L,\,\delta}$. Thus $\ell_1^L + \ell_2^{L,\,\delta}$ has bounded support and integral 0 since ℓ and $\ell_3^{L,\,\delta}$ do. This together with the known fact that $\|\ell_1^L + \ell_2^{L,\,\delta}\|_{L^\infty(\mathcal{X})}$ can be arbitrary small implies that $\ell_1^L + \ell_2^{L,\,\delta}$ is a small multiple of some continuous (p,∞)-atom. Thus $f = h + \ell_3^{L,\,\delta} + (\ell_1^L + \ell_2^{L,\,\delta})$ and $\|f\|_{H^{p,\infty}_{\rm fin}(\mathcal{X})} \lesssim 1$, which completes the proof of (b).

To finish the proof of the theorem, we still need to prove the claim (5.4). Let $\epsilon \in (0,1)$ and $\beta,\gamma \in (n(1/p-1),\epsilon)$. Recall that

$$f^*(x) := \sup\{|\langle f,\varphi\rangle| :\ \varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma),\ \|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)} \leqslant 1 \text{ for some } r>0\}.$$

If $\varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma)$ satisfying $\|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)} \leqslant 1$ for some $r \geqslant 4d(x,x_1)/3$, then by an argument similar to [9, Lemma 2.2], we obtain that there exists a positive constant C depending only on $\mathcal{X}$, β and γ such that for any $y \in B(x,d(x,x_1))$, $\|\varphi\|_{\mathcal{G}(y,r,\beta,\gamma)} \leqslant C$. This implies that $|\langle f,\varphi\rangle| \lesssim f^*(y)$ for any $y \in B(x,d(x,x_1))$. Taking p-power average on the ball $B(x,d(x,x_1))$ and using $B(x_1,R) \subset B(x,2d(x,x_1))$, (2.2) and $\|f^*\|_{L^p(\mathcal{X})} \lesssim 1$, we obtain

$$|\langle f,\varphi\rangle| \lesssim \bigg\{\frac{1}{\mu(B(x,d(x,x_1)))}\int_{B(x,d(x,x_1))} [f^*(y)]^p\,d\mu(x)\bigg\}^{1/p} \lesssim [\mu(B(x_1,R))]^{-1/p}. \tag{5.9}$$

Next assume that $\varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma)$ satisfying $\|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)} \leqslant 1$ for some $r \in (0,4d(x,x_1)/3)$. We choose $\xi \in C_c^\infty(\mathbb{R})$ satisfying $0 \leqslant \xi \leqslant 1$, $\xi(x)=1$ if $|x| \leqslant 1$ and $\xi(x)=0$ if $|x| \geqslant 2$. Set $\widetilde{\varphi}(z) := \varphi(z)\xi(\frac{16d(z,x_1)}{d(x,x_1)})$. Obviously $\widetilde{\varphi} \in \mathcal{G}_0^\epsilon(\beta,\gamma)$. Moreover, there exists a positive constant C independent of φ such that for all $y \in B(x_1,d(x,x_1))$,

$$\|\widetilde{\varphi}\|_{\mathcal{G}(y,r,\beta,\gamma)} \leqslant C. \tag{5.10}$$

Assume (5.10) for a moment. Then using (5.10) and $\operatorname{supp} f \subset B(x_1,R)$, we obtain that $|\langle f,\varphi\rangle| = |\langle f,\widetilde{\varphi}\rangle| \leqslant f^*(y)$ for all $y \in B(x_1,R)$, which together with an argument similar to (5.9) yields that $|\langle f,\varphi\rangle| \lesssim [\mu(B(x_1,R))]^{-1/p}$. This combined with (5.9) yields (5.4).

Thus to finish the proof of (5.4), we still need to verify (5.10). Notice that if $\widetilde{\varphi}(z) \neq 0$, then $d(z,x) \geqslant d(x,x_1) - d(z,x_1) \geqslant 7d(x,x_1)/8$ and thus for all $y \in B(x_1,d(x,x_1))$, $d(z,y) \leqslant 23d(z,x)/7$. By this and the size condition of φ, we have

$$|\widetilde{\varphi}(z)| \leqslant |\varphi(z)|\chi_{\{d(z,x_1)\leqslant d(x,x_1)/8\}}(z) \lesssim \frac{1}{\mu(B(z,r+d(z,y)))}\bigg(\frac{r}{r+d(z,y)}\bigg)^\gamma. \tag{5.11}$$

For any z, $z' \in \mathcal{X}$ satisfying $d(z,z') \leqslant (r+d(z,y))/2$, we estimate $|\widetilde{\varphi}(z) - \widetilde{\varphi}(z')|$ in the following two cases. First, assume that $d(z,z') \leqslant (r+d(z,x))/4$. Note that if $\widetilde{\varphi}(z) - \widetilde{\varphi}(z') \neq 0$, then $\max\{d(z,x_1), d(z',x_1)\} \leqslant d(x,x_1)/8$. This together with $d(z,z') \leqslant (r+d(z,x))/4$ and $r < 4d(x,x_1)/3$ implies that $d(z,x_1) \leqslant 17d(x,x_1)/18$ and thus $d(x,x_1)/18 \leqslant d(z,x) \leqslant 35d(x,x_1)/18$. We further have $d(z,y) \leqslant 37d(z,x)$. From these and (2.2), it follows that when $d(z,z') \leqslant (r+d(z,x))/4$,

$$|\widetilde{\varphi}(z) - \widetilde{\varphi}(z')| \leqslant \varphi(z)\left[\xi\left(\frac{16d(z,x_1)}{d(x,x_1)}\right) - \xi\left(\frac{16d(z',x_1)}{d(x,x_1)}\right)\right] + |\varphi(z) - \varphi(z')|$$
$$\lesssim \left(\frac{d(z,z')}{r+d(z,y)}\right)^{\beta} \frac{1}{\mu(B(z,r+d(z,y)))}\left(\frac{r}{r+d(z,y)}\right)^{\gamma}.$$

Secondly, suppose that $d(z,z') > (r+d(z,x))/4$. Since $|\widetilde{\varphi}(z) - \widetilde{\varphi}(z')| \leqslant |\widetilde{\varphi}(z)| + |\widetilde{\varphi}(z')|$, we need only to show that $|\widetilde{\varphi}(z)| + |\widetilde{\varphi}(z')|$ is bounded by the last formula above. To this end, by the support condition of $\widetilde{\varphi}$, we may assume that $d(z,x_1) \leqslant d(x,x_1)/8$ and $d(z',x_1) \leqslant d(x,x_1)/8$. These assumptions together with $d(z,z') \leqslant (r+d(z,x))/4$ and $r < 4d(x,x_1)/3$ yield that $d(z,y) \leqslant 5d(z,x)$ and $r + d(z,y) \leqslant 10(r+d(z',x))$. From this and (2.2), it follows that when $d(z,z') \leqslant (r+d(z,y))/2$ and $d(z,z') > (r+d(z,x))/4$,

$$|\widetilde{\varphi}(z) - \widetilde{\varphi}(z')| \leqslant \left(\frac{d(z,z')}{r+d(z,x)}\right)^{\beta}\left[\frac{1}{\mu(B(z,r+d(z,x)))}\left(\frac{r}{r+d(z,x)}\right)^{\gamma}\right.$$
$$\left. + \frac{1}{\mu(B(z',r+d(z',x)))}\left(\frac{r}{r+d(z',x)}\right)^{\gamma}\right]$$
$$\lesssim \left(\frac{d(z,z')}{r+d(z,y)}\right)^{\beta} \frac{1}{\mu(B(z,r+d(z,y)))}\left(\frac{r}{r+d(z,y)}\right)^{\gamma}.$$

Combining this with (5.11) yields (5.10). Therefore, (5.4) holds. This finishes the proof of Theorem 5.6.

Remark 5.7. Let $p \in (n/(n+1), 1]$, $f \in \mathring{\mathcal{G}}_b^{\epsilon}(\beta,\gamma)$ for some $\epsilon \in (0,1)$ and let β, $\gamma \in (n(1/p-1), \epsilon)$. From Proposition 4.17 and the proof of (b) of Theorem 5.6, it follows easily that f admits an atomic decomposition of the form $f = \sum_{j=1}^N \lambda_j a_j$, where $N \in \mathbb{N}$, $\{a_j\}_{j=1}^N \subset \mathring{\mathcal{G}}_b^{\epsilon}(\beta,\gamma)$ are (p,∞)-atoms, $\{\lambda_j\}_{j=1}^N \subset \mathbb{C}$ and $\sum_{j=1}^N |\lambda_j|^p \lesssim \|f\|^p_{H^{*,p}(\mathcal{X})}$. This combined with the density of $\mathring{\mathcal{G}}_b^{\epsilon}(\beta,\gamma)$ in $H^{*,p}(\mathcal{X})$ and Remark 5.5 implies that $H^{p,\infty}_{\mathrm{fin}}(\mathcal{X}) \cap \mathcal{C}(\mathcal{X})$ is dense in $H^{p,\infty}_{at}(\mathcal{X})$.

Before turning to the boundedness of operators, we first recall some notions; see [34].

Definition 5.8. (i) *A quasi-Banach space $\mathcal{B}$ is a vector space endowed with a quasi-norm $\|\cdot\|_{\mathcal{B}}$ which is non-negative, non-degenerate* (*i.e.*, $\|f\|_{\mathcal{B}} = 0$ *if and only if* $f = 0$), *homogeneous, and obeys the quasi-triangle inequality, i. e., there exists a constant $K \geqslant 1$ such that for all f, $g \in \mathcal{B}$, $\|f+g\|_{\mathcal{B}} \leqslant K(\|f\|_{\mathcal{B}} + \|g\|_{\mathcal{B}})$.*

(ii) *Let $r \in (0,1]$. A quasi-Banach space $\mathcal{B}_r$ with the quasi norm $\|\cdot\|_{\mathcal{B}_r}$ is said to be a r-quasi-Banach space if $\|f+g\|^r_{\mathcal{B}_r} \leqslant \|f\|^r_{\mathcal{B}_r} + \|g\|^r_{\mathcal{B}_r}$ for all f, $g \in \mathcal{B}$.*

(iii) *For any given r-quasi-Banach space $\mathcal{B}_r$ with* $r \in (0,1]$ *and linear space $\mathcal{Y}$, an operator T from $\mathcal{Y}$ to $\mathcal{B}_r$ is called to be $\mathcal{B}_r$-sublinear if for any f, $g \in \mathcal{Y}$ and λ, $\nu \in \mathbb{C}$,*

$$\|T(\lambda f + \nu g)\|_{\mathcal{B}_r} \leqslant (|\lambda|^r \|T(f)\|^r_{\mathcal{B}_r} + |\nu|^r \|T(g)\|^r_{\mathcal{B}_r})^{1/r} \tag{5.12}$$

and

$$\|T(f)-T(g)\|_{\mathcal{B}_r} \leqslant \|T(f-g)\|_{\mathcal{B}_r}. \tag{5.13}$$

From Theorem 5.6, it is easy to deduce the following criterion on boundedness of sublinear operators in Hardy spaces $H^p(\mathcal{X})$.

Theorem 5.9. *Let $p \in (n/(n+1), 1]$ and $r \in [p, 1]$. Suppose that $\mathcal{B}_r$ is an r-quasi-Banach space and one of the following holds:*

(i) *$q \in (p, \infty) \cap [1, \infty)$ and $T: H^{p,q}_{\rm fin}(\mathcal{X}) \to \mathcal{B}_r$ is a $\mathcal{B}_r$-sublinear operator such that*

$$A := \sup\{\|Ta\|_{\mathcal{B}_r} : a \text{ is a } (p,q)\text{-atom}\} < \infty;$$

(ii) *$T: H^{p,\infty}_{\rm fin}(\mathcal{X}) \cap \mathcal{C}(\mathcal{X}) \to \mathcal{B}_r$ is a $\mathcal{B}_r$-sublinear operator such that*

$$A := \sup\{\|Ta\|_{\mathcal{B}_r} : a \text{ is a continuous } (p,\infty)\text{-atom}\} < \infty.$$

Then T uniquely extends to a bounded $\mathcal{B}_r$-sublinear operator from $H^p(\mathcal{X})$ to $\mathcal{B}_r$.

Proof. Assume that (i) holds. Then for any $f \in H^{p,q}_{\rm fin}(\mathcal{X})$, we write $f = \sum_{j=1}^N \lambda_j a_j$, where $N \in \mathbb{N}$, $\{\lambda_j\}_{j=1}^N \subset \mathbb{C}$ and $\{a_j\}_{j=1}^N$ are (p,q)-atoms. Using (i), (5.12) and the fact for all $\nu \in (0,1]$, $\sum_{i=1}^\infty |a_i| \leqslant \{\sum_{i=1}^\infty |a_i|^\nu\}^{1/\nu}$, we then obtain

$$\|T(f)\|_{\mathcal{B}_r} \lesssim \left(\sum_{j=1}^N |\lambda_j|^r\right)^{1/r} \leqslant \left(\sum_{j=1}^N |\lambda_j|^p\right)^{1/p}.$$

Taking the infimum over all finite atomic decompositions of f and using Theorem 5.6 and Remark 5.5, we deduce that for all $f \in H^{p,q}_{\rm fin}(\mathcal{X})$,

$$\|Tf\|_{\mathcal{B}_r} \lesssim \|f\|_{H^p(\mathcal{X})}. \tag{5.14}$$

For any $f \in H^p(\mathcal{X})$, by the density of $H^{p,q}_{\rm fin}(\mathcal{X})$ in $H^p(\mathcal{X})$, there exists a sequence $\{f_N\}_{N=1}^\infty \subset H^{p,q}_{\rm fin}(\mathcal{X})$ such that $\|f - f_N\|_{H^p(\mathcal{X})} \to 0$ as $N \to \infty$. This together with (5.13) and (5.14) yields that $\{T(f_N)\}_N$ is a Cauchy sequence in $\mathcal{B}_r$. Define $\widetilde{T}(f) := \lim_{N\to\infty} T(f_N)$ in $\mathcal{B}_r$. By (5.13) and (5.14), it is easy to see that $\widetilde{T}(f)$ is well defined, unique and satisfies (5.14). Thus, $\widetilde{T}$ gives the desired extension of T.

If (ii) holds, using Remark 5.7 and arguing as in the previous case, we also obtain the desired conclusion. This finishes the proof of Theorem 5.9.

Remark 5.10. (i) Notice that any Banach space is a 1-quasi-Banach space, and the quasi-Banach spaces ℓ^q, $L^q(\mathcal{X})$ and $H^q(\mathcal{X})$ with $q < 1$ are typical q-quasi-Banach spaces.

(ii) According to the Aoki-Rolewicz theorem (see [41, 42]), any quasi-Banach space is, in essential, a q-quasi-Banach space, where $q = 1/\log_2(2K)$ and K is as in Definition 5.8 (i). Thus, Theorem 5.9 actually holds for general quasi-Banach spaces.

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