THE BILINEAR MULTIPLIER PROBLEM FOR THE DISC

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Abstract. We present the main ideas of the proof of the following result: The characteristic function of the unit disc in $\mathbf{R}^2$ is the symbol of a bounded bilinear multiplier operator from $L^{p_1}(\mathbf{R}) \times L^{p_2}(\mathbf{R})$ into $L^p(\mathbf{R})$ when $2 \le p_1, p_2 < \infty$ and $1 < p = \frac{p_1 p_2}{p_1+p_2} \le 2$.

1. Introduction

The subject of multilinear operators was extensively studied by Coifman and Meyer [2], [3], [4], [5] in the seventies but remained stagnant during the late eighties and early nineties; it was not until the work of Lacey and Thiele [11], [12] on the bilinear Hilbert transform that renewed interest in the subject was spurred. One of the main problems in the area is to study bilinear multipliers, in particular to determine which functions are bounded bounded bilinear multipliers.

An L^∞ function $m(\xi_1, \xi_2)$ on $\mathbf{R}^2$ is said to be a bounded bilinear (L^{p_1}, L^{p_2}, L^p) multiplier (or symbol) if the expression

$$T_m(f_1, f_2)(x) = \int_{\mathbf{R}} \int_{\mathbf{R}} \widehat{f_1}(\xi)\widehat{f_2}(\eta) e^{2\pi i(\xi_1+\xi_2)x} m(\xi, \xi_2)\, d\xi_1\, d\xi_2\,,$$

gives rise to a bounded bilinear operator, i.e. there exists a constant $C_{p_1,p_2,p}$ such that for all Schwartz functions f_1 and f_2 on the line we have the estimate

$$\|T_m(f_1, f_2)\|_p \le C_{p_1,p_2,p} \|f\|_{p_1} \|f\|_{p_2}\,.$$

The smallest such constant $C_{p_1,p_2,p}$ is called the norm of the bilinear multiplier m. The bilinear Hilbert transforms are multiplier operators whose bilinear symbols are the functions $m_\alpha(\xi, \eta) = -i\pi\, \mathrm{sgn}\,(\xi - \alpha\eta)$ in $\mathbf{R}^2$, where α is a real parameter. These functions have jump discontinuities along the lines $\xi = \alpha\eta$ which present significant difficulties in the analysis of the corresponding operators. This analysis requires a careful decomposition based on sensitive time and frequency considerations and delicate combinatorial arguments, see [11], [12].

It is natural to ask whether characteristic functions of other geometric figures are bounded bilinear symbols. In this expository article we present some ideas of the proof of the fact that the characteristic function of the unit disc in $\mathbf{R}^2$ is such a bounded bilinear multiplier. To facilitate the understanding of the proof we include several figures. The full details of the proof of this result can be found in the article [9] by the same authors.

Recall the classical theorem of C. Fefferman [7] which says that the characteristic function of the unit disc is not a (linear) multiplier on $L^p(\mathbf{R}^2)$ unless $p = 2$. So, it may come as a surprise that the disc is a bounded bilinear (L^{p_1}, L^{p_2}, L^p) multiplier for some open set of indices (p_1, p_2, p) which satisfy $1/p_1 + 1/p_2 = 1/p$. Consideration of this function as a

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bilinear multiplier is motivated by the problem of uniform bounds for the family of bilinear Hilbert transforms studied by Thiele [17], Grafakos and Li [8], and Li [13]. The fact that the characteristic function of the unit disc is the symbol of a bounded pseudodifferential operator has some remarkable consequences that we state below. We set

$$T_D(f_1,f_2)(x)=\int_{\mathbf{R}}\int_{\mathbf{R}}\widehat{f_1}(\xi)\widehat{f_2}(\eta)e^{2\pi i(\xi+\eta)x}1_{\xi^2+\eta^2<1}\,d\xi\,d\eta\,,\qquad x\in\mathbf{R},$$

for Schwartz functions f_1, f_2 on the line. (We use the notation 1_A for the characteristic function of the set A and $\|h\|_q$ for the L^q norm of a function h over the real line.) Our main result is the following:

Theorem 1. [9] *Let $2\le p_1,p_2<\infty$ and $1<p=\frac{p_1p_2}{p_1+p_2}\le 2$. Then there is a constant $C=C(p_1,p_2)$ such that for all f_1, f_2 Schwartz functions on $\mathbf{R}$ we have*

$$\|T_D(f_1,f_2)\|_p\le C\|f_1\|_{p_1}\|f_2\|_{p_2}.$$

Theorem 1 above provides a strengthening of Theorem 1 in [8] which claims that the operators H_α are bounded from $L^{p_1}\times L^{p_2}\to L^p$ uniformly in $\alpha\in[-\infty,+\infty]$ whenever $2<p_1,p_2<\infty$ and $1<p=\frac{p_1p_2}{p_1+p_2}<2$. This assertion follows from a classical idea due to Y. Meyer: Observing that translations and dilations of bilinear symbols preserve their multiplier norms, we obtain that any disc has the same multiplier norm as the unit disc. Also Fatou's lemma gives that a pointwise limit of a sequence of bounded bilinear symbols is bounded. Given any half-plane one can find a sequence of increasing discs converging pointwise to it as in Figure 1. Thus the norm of the disc as a bilinear multiplier controls that of the indicator function of any such half-plane. Clearly this control is uniform in the slope of the half-planes, thus uniform bounds for the bilinear Hilbert transforms follow.

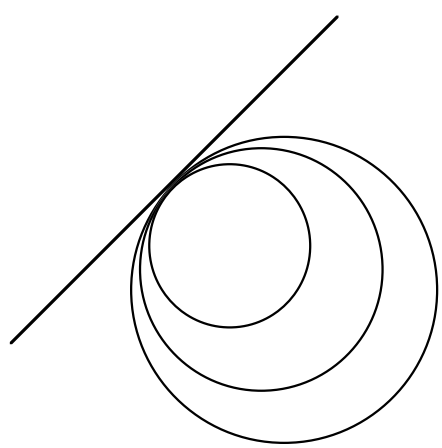

FIGURE 1. A sequence of discs approaching a line.

Building on this idea, we can obtain the following stronger result.

Corollary 1. *Let $2\le p_1,p_2<\infty$ and $1<p=\frac{p_1p_2}{p_1+p_2}\le 2$. Then there is a constant $C=C(p_1,p_2)$ such that for all sequences of Schwartz functions f_j,g_j on $\mathbf{R}$ we have*

$$\sup_{\{\alpha_j\}\subset\mathbf{R}}\Big\|\Big(\sum_j|H_{\alpha_j}(f_j,g_j)|^2\Big)^{1/2}\Big\|_p\le C\Big\|\Big(\sum_j|f_j|^2\Big)^{1/2}\Big\|_{p_1}\Big\|\Big(\sum_j|g_j|^2\Big)^{1/2}\Big\|_{p_2},$$

where H_{α_j} is the bilinear Hilbert transform, i.e. the bilinear operator with symbol the function $m_\alpha(\xi,\eta)=-i\pi\,sgn\,(\xi-\alpha\eta)$.

The corollary can be easily obtained from Theorem 1 using the two-dimensional generalization of Khintchine's inequality and an adaptation of the idea in the linear setting (c.f. [7].) We skip this easy argument but we refer the reader to [9] for details.

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2. An orthogonality lemma and two easy facts

We state and prove an orthogonality result which allows us to obtain that a sum of bounded bilinear operators is bounded under certain conditions on the supports of the symbols. This lemma will enable us to control a variety of errors that appear in our decomposition.

Suppose that a sequence of bilinear symbols σ_m is supported in sets S_m that have disjoint projections on the lines $\xi = 0$, $\eta = 0$, and $\xi = \eta$ as in Figure 2. If each σ_m is symbol of a bounded bilinear operator L_m and if all the operators L_m are uniformly bounded, we show below that the sum of the L_m's is also a bounded bilinear operator.

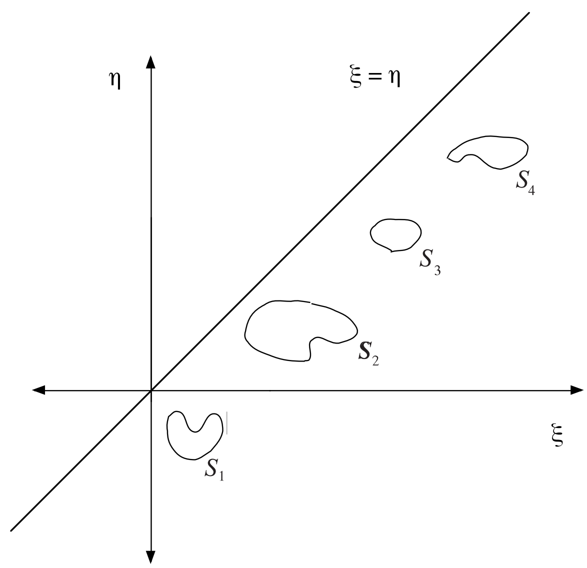

Figure 2. The sets S_m have disjoint projections on the lines $\xi = 0$, $\eta = 0$, and $\xi = \eta$.

Lemma 1. *Let* $2 \le p_1, p_2 < \infty$, $1 < p \le 2$, *and* $1/p_1 + 1/p_2 = 1/p$. *Suppose that* $\{L_m\}_{m\in\mathbf{Z}}$ *is a family of uniformly bounded bilinear operators from* $L^p(\mathbf{R}) \times L^q(\mathbf{R})$ *into* $L^r(\mathbf{R})$. *Furthermore, suppose that for all functions* f, g, h *on the line we have*

$$\langle L_m(f,g), h\rangle = \langle L_m(\Delta_m^1 f, \Delta_m^2 g), \Delta_m^3 h\rangle \,,$$

where $\widehat{\Delta_m^1 f} = \widehat{f}\chi_{A_m}$, $\widehat{\Delta_m^2 g} = \widehat{f}\chi_{B_m}$, $\widehat{\Delta_m^3 f} = \widehat{h}\chi_{C_m}$, *and* $\{A_m\}_m$, $\{B_m\}_m$, $\{C_m\}_m$ *are sets of intervals such that the* A_m*'s being pairwise disjoint, the* B_m*'s being pairwise disjoint,* *and the C_m's being pairwise disjoint. Then there is a constant $C = C(p_1, p_2, p)$ such that for all functions f, g we have*

$$\Big\| \sum_m L_m(f,g)\Big\|_p \le C\|f\|_{p_1}\|g\|_{p_2}\,.$$

Proof. Denoting $p' = p/(p-1)$ we have

$$\begin{aligned}
\big|\langle \sum_m L_m(f,g), h\rangle\big| =& \big|\sum_m \langle L_m(\Delta_m^1 f, \Delta_m^2 g), \Delta_m^3 h\rangle\big| \\
\le& \int_{\mathbf{R}} \big(\sum_m |L_m(\Delta_m^1 f, \Delta_m^2 g)|^2\big)^{1/2}\big(\sum_m |\Delta_m^3|^2\big)^{1/2} dx \\
\le& \Big\|\big(\sum_m |L_m(\Delta_m^1 f, \Delta_m^2 g)|^2\big)^{1/2}\Big\|_p \Big\|\big(\sum_m |\Delta_m^3 h|^2\big)^{1/2}\Big\|_{p'} \\
\le& \Big\|\big(\sum_m |L_m(\Delta_m^1 f, \Delta_m^2 g)|^2\big)^{1/2}\Big\|_p \big\|h\big\|_{p'},
\end{aligned}$$

where the last inequality follows from Rubio de Francia's Littlewood-Paley inequality for arbitrary disjoint intervals ($p' \ge 2$), see [14]. It suffices to estimate the square function above. We have

$$\begin{aligned}
&\Big\|\big(\sum_m |L_m(\Delta_m^1 f, \Delta_m^2 g)|^2\big)^{1/2}\Big\|_p^p \\
\le& \int_{\mathbf{R}} \sum_m |L_m(\Delta_m^1 f, \Delta_m^2 g)|^p\, dx \qquad \text{(since } p/2 \le 1) \\
=& \sum_m \|L_m(\Delta_m^1 f, \Delta_m^2 g)\|_p^p \\
\le& C\sum_m \|\Delta_m^1 f\|_{p_1}^p \|\Delta_m^2 g\|_{p_2}^p \qquad \text{(by unif. boundedness of } L_m) \\
\le& C\big(\sum_m \|\Delta_m^1 f\|_{p_1}^{p_1}\big)^{p/p_1}\big(\sum_m \|\Delta_m^2 g\|_{p_2}^{p_2}\big)^{p/p_2} \qquad \text{(Hölder)} \\
\le& C\big\|\big(\sum_m |\Delta_m^1 f|^2\big)^{1/2}\big\|_{p_1}^p \big\|\big(\sum_m |\Delta_m^2 g|^2\big)^{1/2}\big\|_{p_2}^p \qquad \text{(since } p_1, p_2 \ge 2) \\
\le& C\|f\|_{p_1}^p \|g\|_{p_2}^p,
\end{aligned}$$

where the last inequality also follows from Rubio de Francia's Littlewood-Paley inequality for arbitrary disjoint intervals ($p_1, p_2 \ge 2$), see [14]. □

We observe that Lemma 1 holds even when the intervals A_m are not necessarily disjoint, provided the intervals $A_{m+100}, A_{m+200}, A_{m+300}, \ldots$ are disjoint for all $m \in \mathbf{Z}$. We are going to use this lemma under such conditions on the intervals A_m, B_m, and C_m.

We denote by $f^\vee$ the inverse Fourier transform of a function f defined by $f^\vee(\xi) = \widehat{f}(-\xi)$. We will also need the following trivial lemma whose proof can be easily obtained by Minkowski's integral inequality and Hölder's inequality.

Lemma 2. *Suppose T is a bilinear operator with symbol $\sigma(\xi,\eta)$, $\xi, \eta \in \mathbf{R}$, i.e.*

$$T(f,g)(x) = \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f}(\xi)\widehat{g}(\eta)\sigma(\xi,\eta)e^{2\pi i(\xi+\eta)x}\, d\xi\, d\eta\,.$$

Assume that the inverse Fourier transform $\sigma^\vee$ (in $\mathbf{R}^2$) satisfies

$$\int_{\mathbf{R}}\int_{\mathbf{R}} |\sigma^\vee(x,y)|\,dx\,dy = C_0 < \infty\,.$$

Then T maps $L^p(\mathbf{R}) \times L^q(\mathbf{R}) \to L^r(\mathbf{R})$ when $1 \le p,q,r \le \infty$ and $1/p+1/q = 1/r$ with constant at most C_0.

Finally, we have the following lemma, whose proof is standard and also omitted. We use the notation $\partial_{radial} = \frac{\partial}{\partial r}$ and $\partial_{angular} = \frac{\partial}{\partial \theta}$, where (r,θ) are polar coordinates in $\mathbf{R}^2$.

Lemma 3. *Let k,l be real numbers greater than or equal to 1.*
(a) Let $\widehat{\phi}(\xi,\eta)$ be a smooth function supported in a rectangle of dimensions $2^{-k}\times 2^{-l}$ with sides parallel to the axes in $\mathbf{R}^2$. Assume that $|\partial_\xi^\alpha \widehat{\phi}| \le C_\alpha 2^{k\alpha}$ and $|\partial_\eta^\beta \widehat{\phi}| \le C_\beta 2^{l\beta}$ for all $\alpha,\beta \ge 0$. Then we have the estimate

$$|\phi(x,y)| \le \frac{C_N\,2^{-k-l}}{(1+(2^{-k}|x|)^2+(2^{-l}|y|)^2)^N}$$

for all $N \ge 0$. Thus ϕ has L^1 norm bounded by some constant independent of k and l.
(b) Let $\widehat{\phi}$ be a smooth function on $\mathbf{R}^2$ supported inside the intersection of an annulus of width 2^{-k} and a sector of angle 2^{-l}. Suppose that $|\partial^\alpha_{radial}\widehat{\phi}| \le C_\alpha 2^{k\alpha}$ and $|\partial^\beta_{angular}\widehat{\phi}| \le C_\beta 2^{l\beta}$ for all $\alpha,\beta\ge 0$. Then we have the estimate

$$|\phi(x,y)| \le \frac{C_N\,2^{-k-l}}{(1+(2^{-k}|(x,y)\cdot \mathbf{e}_r|)^2+(2^{-l}|(x,y)\cdot \mathbf{e}_a|)^2)^N}$$

for all $N\ge 0$, where $\mathbf{e}_r$ is a unit vector in the radial direction of the support of $\widehat{\phi}$ and $\mathbf{e}_a$ is a unit vector perpendicular to $\mathbf{e}_r$, while $\cdot$ is the usual inner product in $\mathbf{R}^2$. Therefore ϕ has L^1 norm bounded by some constant independent of k and l.

3. The decomposition of the disc

We start with a nonnegative smooth function ζ on $[0,1]$ which is identically equal to 1 on $[0,\frac{1}{2}-\frac{1}{2^{10}}]$, is supported in $[0,\frac{1}{2}+\frac{1}{2^{10}}]$, and satisfies $\zeta(t)+\zeta(1-t)=1$ for all $0\le t\le 1$. Define

$$\zeta_k(t) = \zeta(2^{k-1}(1-t)) - \zeta(2^k(1-t))$$

for k in $\mathbf{Z}^+$. Then each function ζ_k is supported in the interval

$$1-2^{-k}(1+2^{-9}) \le t \le 1-2^{-(k+1)}(1-2^{-9})$$

and we have the identity

$$\zeta(t)+\sum_{k=1}^{\infty}\zeta_k(t) = 1_{[0,1]}(t)\,.$$

Let ψ_0, ψ_1, ψ_2, … be radial Schwartz functions on $\mathbf{R}^2$ whose Fourier transforms are

$$\widehat{\psi_k}(\xi,\eta) = \zeta_k(|(\xi,\eta)|)\,.$$

It follows that for $k\ge 1$, each $\widehat{\psi_k}$ is supported in the annulus

$$1-2^{-k}(1+2^{-9}) \le |(\xi,\eta)| \le 1-2^{-(k+1)}(1-2^{-9})$$

and that

$$1_D = \widehat{\psi_0} + \sum_{k=1}^{\infty} \widehat{\psi_k}, \tag{3.1}$$

where $D = D(0,1)$ is the unit disc. This way we have a decomposition of the characteristic function of the unit disc as an infinite sum of smooth functions supported in annuli whose width becomes smaller as they get closer to the boundary of the disc.

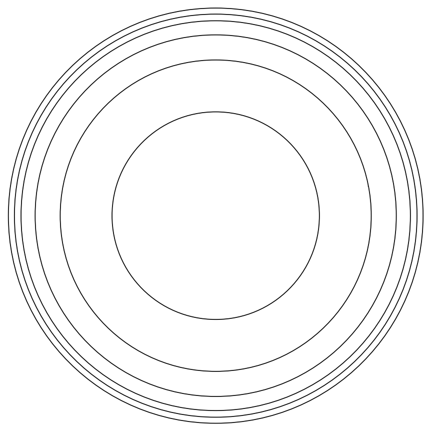

FIGURE 3. The disc is smoothly decomposed as a union of annuli whose width becomes smaller as they get closer to the boundary of the disc.

We now introduce a smooth function χ on the real line supported in $[-\frac{\pi}{8} - \frac{1}{2^{10}}, \frac{\pi}{8} + \frac{1}{2^{10}}]$ and equal to 1 on the interval $[-\frac{\pi}{8} + \frac{1}{2^{10}}, \frac{\pi}{8} - \frac{1}{2^{10}}]$, such that

$$\sum_{j \in \mathbf{Z}} \chi(x + \tfrac{\pi}{4} j) = 1 \tag{3.2}$$

for all $x \in \mathbf{R}$. For each $\ell \in \{1,2,3,4,5,6,7,8\}$ we introduce a function ϕ_ℓ on $\mathbf{R}^2$ whose Fourier transform is defined by

$$\widehat{\phi_\ell}(\xi, \eta) = \chi\Big(\text{Argument}\Big(\frac{\xi + i\eta}{\sqrt{\xi^2 + \eta^2}} - e^{i\frac{\pi}{4}(\ell - 1)}\Big)\Big) \tag{3.3}$$

and we also define functions

$$\widehat{b_k^\ell} = \widehat{\psi_k}\widehat{\phi_\ell}.$$

Observe that each $\widehat{\phi_\ell}$ is a homogeneous of degree zero function and that each $\widehat{\psi_k}$ is a radial function whose α^{th} derivative (in the radial direction) blows up like $C_\alpha 2^{k\alpha}$. Using (3.3), it follows that for all $\alpha, \beta \geq 0$

$$\big|\partial^\alpha_{radial}\partial^\beta_{angular}(\widehat{b_k^\ell})\big| \leq C_{\alpha,\beta} 2^{k\alpha}\,. \tag{3.4}$$

For all $k \geq 1$ and $\ell \in \{1,2,3,4,5,6,7,8\}$ we now introduce bilinear operators

$$T_{D(\ell)}(f,g)(x) = \sum_{k=1}^{\infty} \int_{\mathbf{R}} \int_{\mathbf{R}} \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i(\xi+\eta)x} \widehat{b_k^\ell}(\xi,\eta)\, d\xi\, d\eta\,.$$

Because of (3.1) and (3.2) we have obtained the following decomposition

$$T_D = T_0 + \sum_{\ell=1}^{8} T_{D(\ell)}\,,$$

where T_0 is the bilinear operator whose symbol is $\widehat{\psi_0}$. See Figure 4. Using Lemma 2, it follows that T_0 is a bounded bilinear operator and we therefore need to concentrate on the $T_{D(\ell)}$'s.

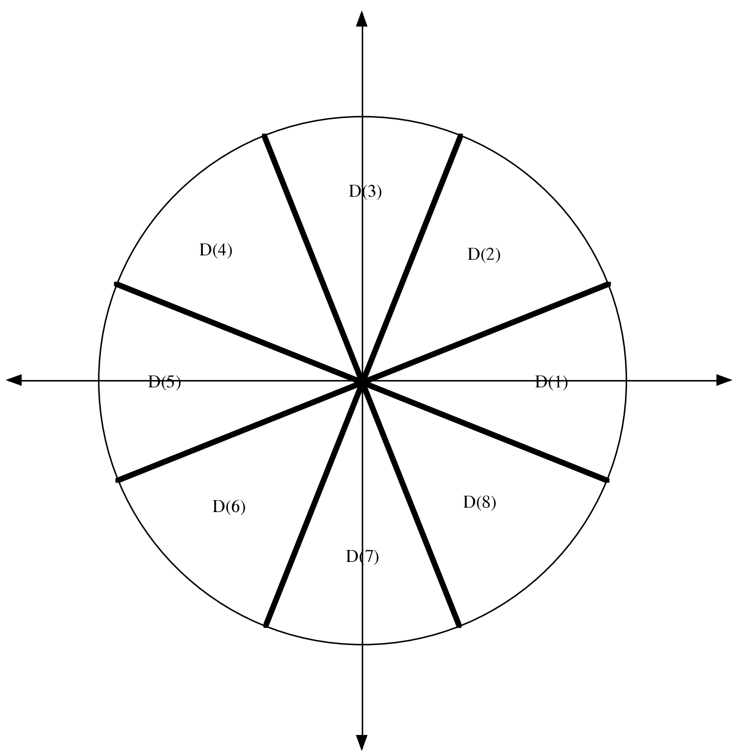

FIGURE 4. The eight pieces of the decomposition of the disc.

It is easy to see that if $\sigma(\xi,\eta)$ is a bounded bilinear symbol, then so is $\sigma(-\xi,-\eta)$. Therefore, it suffices to obtain estimates for the bilinear operators $T_{D(1)}$, $T_{D(2)}$, $T_{D(3)}$, and $T_{D(4)}$, since these imply the same estimates for $T_{D(5)}$, $T_{D(6)}$, $T_{D(7)}$, and $T_{D(8)}$ respectively. Moreover, the symbol of $T_{D(3)}$ can be obtained from that of $T_{D(1)}$ by interchanging ξ and η. Since the set of (p_1,p_2,p) for which we plan to obtain boundedness for $T_{D(1)}$ from $L^{p_1}\times L^{p_2}$ into L^p is symmetric in p_1 and p_2, the estimates for $T_{D(3)}$ can be obtained from those for $T_{D(1)}$ by symmetry. It therefore suffices to obtain estimates for $T_{D(1)}$, $T_{D(2)}$, and $T_{D(4)}$.

We now describe the decomposition of the operator $T_{D(1)}$ whose symbol is essentially supported in a neighborhood of the sector $D(1)$.

For $-\frac{\pi}{7}\le a<b\le\frac{\pi}{7}$ let us denote by $\Sigma\big(a,b\big)$ the sector

$$a\le\theta\le b.$$

For every $k \geq 1$ and $\mu \in \{1, 2, \dots, k+1\}$, we introduce functions ρ_k^μ on $\mathbf{R}^2$ such that $\widehat{\rho_k^\mu}$ are homogeneous of degree zero, smooth (away from the origin), satisfying for all $\beta \geq 0$

$$|\partial^\beta_{angular} \widehat{\rho_k^\mu}| \leq C_\beta 2^{\beta \frac{\mu}{2}}\,, \qquad 1 \leq \mu \leq k+1\,, \tag{3.5}$$

$$\widehat{\rho_k^{k+1}} + \widehat{\rho_k^k} + \widehat{\rho_k^{k-1}} + \cdots + \widehat{\rho_k^1} = 1 \quad \text{on} \quad \Sigma\big(-\tfrac{\pi}{8}(1+2^{-8}), \tfrac{\pi}{8}(1+2^{-8})\big), \tag{3.6}$$

and such that for any $2 \leq \mu \leq k$ the functions $\widehat{\rho_k^\mu}$ are supported in

$$\Sigma\big(-2^{-\frac{\mu-1}{2}}\tfrac{\pi}{8}(1+2^{-9}), 2^{-\frac{\mu-1}{2}}\tfrac{\pi}{8}(1+2^{-9})\big) \setminus \Sigma\big(-2^{-\frac{\mu}{2}}\tfrac{\pi}{8}(1-2^{-9}), 2^{-\frac{\mu}{2}}\tfrac{\pi}{8}(1-2^{-9})\big)$$

and are equal to 1 on

$$\Sigma\big(-2^{-\frac{\mu-1}{2}}\tfrac{\pi}{8}(1-2^{-9}), 2^{-\frac{\mu-1}{2}}\tfrac{\pi}{8}(1-2^{-9})\big) \setminus \Sigma\big(-2^{-\frac{\mu}{2}}\tfrac{\pi}{8}(1+2^{-9}), 2^{-\frac{\mu}{2}}\tfrac{\pi}{8}(1+2^{-9})\big)\,,$$

while for $\mu = 1$ the function $\widehat{\rho_k^1}$ is supported in

$$\Sigma\big(-\tfrac{\pi}{8}(1+2^{-9}), \tfrac{\pi}{8}(1+2^{-9})\big) \setminus \Sigma\big(-2^{-\frac{1}{2}}\tfrac{\pi}{8}(1-2^{-9}), 2^{-\frac{1}{2}}\tfrac{\pi}{8}(1-2^{-9})\big)$$

and is equal to 1 on

$$\Sigma\big(-\tfrac{\pi}{8}(1+2^{-8}), \tfrac{\pi}{8}(1+2^{-8})\big) \setminus \Sigma\big(-2^{-\frac{1}{2}}\tfrac{\pi}{8}(1+2^{-9}), 2^{-\frac{1}{2}}\tfrac{\pi}{8}(1+2^{-9})\big)\,,$$

and for $\mu = k+1$ the function $\widehat{\rho_k^{k+1}}$ is supported in

$$\Sigma\big(-2^{-\frac{k}{2}}\tfrac{\pi}{8}(1+2^{-9}), 2^{-\frac{k}{2}}\tfrac{\pi}{8}(1+2^{-9})\big)$$

and is equal to 1 on

$$\Sigma\big(-2^{-\frac{k}{2}}\tfrac{\pi}{8}(1-2^{-9}), 2^{-\frac{k}{2}}\tfrac{\pi}{8}(1-2^{-9})\big)\,.$$

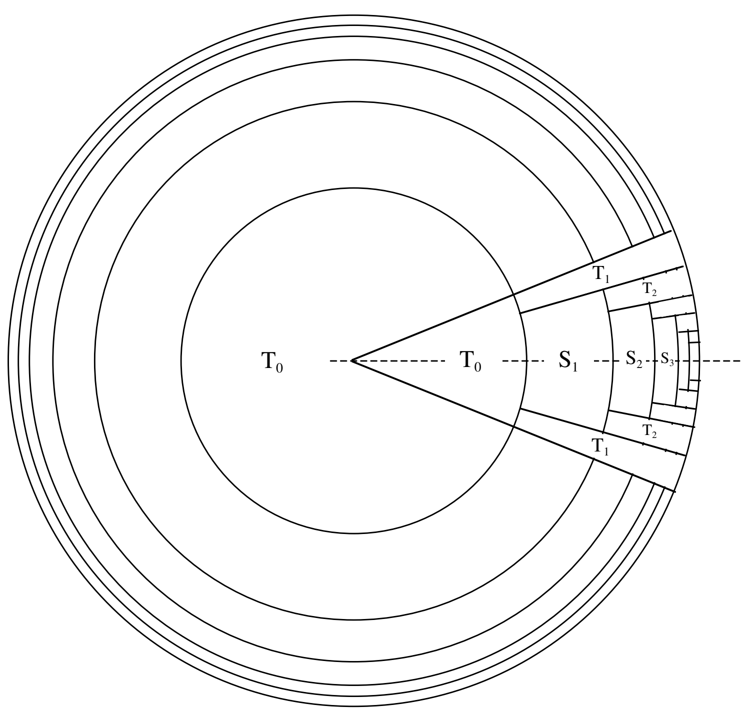

FIGURE 5. The decomposition of $D(1)$

In view of (3.6) we have the identity

$$\widehat{b_k^1} = \widehat{b_k^1}\big(\widehat{\rho_k^{k+1}} + \widehat{\rho_k^k} + \widehat{\rho_k^{k-1}} + \cdots + \widehat{\rho_k^1}\big) \tag{3.7}$$

since the set $\Sigma\big(-\frac{\pi}{8}(1+2^{-8}), \frac{\pi}{8}(1+2^{-8})\big)$ (on which the sum inside the parenthesis in (3.7) is equal to 1) contains the support of $\widehat{b_k^1}$ for all $k \geq 1$.

It follows from estimates (3.4) (with $\beta = 0$) and (3.5) that

$$\begin{aligned} |\partial^\alpha_{radial}(\widehat{b_k^1}\widehat{\rho_k^\mu})| &\leq C_\alpha 2^{\alpha k} \\ |\partial^\beta_{angular}(\widehat{b_k^1}\widehat{\rho_k^\mu})| &\leq C_\beta 2^{\beta\frac{\mu}{2}} \end{aligned} \tag{3.8}$$

for all $1 \leq \mu \leq k+1$ and $\alpha, \beta \geq 0$. Moreover the function $\widehat{b_k^1}\widehat{\rho_k^\mu}$ is supported inside an annulus of width approximately 2^{-k} and inside a sector of "length" approximately $2^{-\frac{\mu}{2}}$.

For each $k, \mu \geq 1$, we introduce bilinear operators S_k, T_μ as follows

$$S_k(f,g) = \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f}(\xi)\widehat{g}(\eta)\widehat{b_k^1}(\xi,\eta)\widehat{\rho_k^{k+1}}(\xi,\eta)\, e^{2\pi i(\xi+\eta)x} d\xi\, d\eta\,,$$

$$T_\mu(f,g) = \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f}(\xi)\widehat{g}(\eta)\sum_{k=\mu}^{\infty}\widehat{b_k^1}(\xi,\eta)\widehat{\rho_k^{\mu}}(\xi,\eta)\, e^{2\pi i(\xi+\eta)x} d\xi\, d\eta\,,$$

We have now achieved the following decomposition of $T_{D(1)}$:

$$T_{D(1)} = \sum_{k=1}^{\infty} S_k + \sum_{\mu=1}^{\infty} T_\mu\,,$$

and it will suffice to show that both sums above are bounded on the required product of L^p spaces. This decomposition is shown in Figure 5.

In the rest of the paper we present the main ideas of the proof of the boundedness of $T_{D(1)}$. We refer the reader to [9] for the treatment of the pieces $T_{D(2)}$ and $T_{D(4)}$.

4. The boundedness of $\sum_{k=1}^{\infty} S_k$

Let us denote the operator $\sum_{k=1}^{\infty} S_k(f_1,f_2)$ by $S(f_1,f_2)$. In this section we will prove the boundedness of S.

For each $k \geq 1$ we pick a Schwartz function $\Phi_{1,k}$ on the line whose Fourier transform $\widehat{\Phi_{1,k}}$ is supported in the interval $[-\frac{101}{100}\cdot 2^{-k}, -\frac{99}{100}\cdot 2^{-k-1}]$ and satisfies $|\frac{d^\alpha}{d\xi^\alpha}\widehat{\Phi_{1,k}}(\xi)| \leq C_\alpha 2^{k\alpha}$ for all $\alpha \geq 0$. Moreover we select these functions so that

$$\sum_{k=1}^{\infty}\widehat{\Phi_{1,k}}(\xi) = 1 \tag{4.1}$$

for all $-\frac{99}{200} < \xi < 0$. For each $k \geq 1$ we pick another Schwartz function $\Phi_{2,k}$ on the line whose Fourier transform $\widehat{\Phi_{2,k}}$ is equal to 1 on the interval $[-\frac{4}{5}\cdot 2^{-\frac{k}{2}}, \frac{4}{5}\cdot 2^{-\frac{k}{2}}]$, is supported in the interval $[-2^{-\frac{k}{2}}, 2^{-\frac{k}{2}}]$, and satisfies $|\frac{d^\alpha}{d\eta^\alpha}\widehat{\Phi_{2,k}}(\eta)| \leq C_\alpha 2^{\frac{k}{2}\alpha}$ for all $\alpha \geq 0$. We introduce a bilinear operator $\widetilde{S}'$ by setting

$$\widetilde{S}'(f_1,f_2)(x) = \sum_{k=1}^{\infty}\int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f_1}(\xi)\widehat{f_2}(\eta)\widehat{\Phi_{1,k}}(\xi-1)\widehat{\Phi_{2,k}}(\eta)e^{2\pi i(\xi+\eta)x}d\xi d\eta$$

and we prove the following result regarding it.

Lemma 4. *For all* $2 \le p_1, p_2 < \infty$ *and* $1 < p \le 2$ *satisfying* $1/p_1 + 1/p_2 = 1/p$*, there exists a constant* $C = C(p_1, p_2, p)$ *such that*

$$\|S(f_1, f_2) - \widetilde{S}'(f_1, f_2)\|_p \le C\|f_1\|_{p_1}\|f_2\|_{p_2} .$$

Proof. Let $L = S - \widetilde{S}'$. Using condition (4.1) we obtain that the symbol of the bilinear operator L consists of a smooth function with compact support plus a sum of smooth functions σ_k (k large) each supported in $A_k \times B_k$ union $A_k \times (-B_k)$, where

$$A_k = [1 - 5 \cdot 2^{-k}, 1 - \tfrac{1}{100} \cdot 2^{-k}]$$
$$B_k = [\tfrac{1}{8} \cdot 2^{-\frac{k}{2}}, 2 \cdot 2^{-\frac{k}{2}}] .$$

Because of conditions (3.8), the support properties of $\widehat{\rho_k^{k+1}}$, and the properties of $\Phi_{1,k}$ and $\Phi_{2,k}$, Lemma 3 implies that the σ_k's have uniformly integrable inverse Fourier transforms. Lemma 2 implies that the bilinear operators with symbols σ_k are uniformly bounded from $L^p \times L^q \to L^r$ for all $1 < p, q, r < \infty$ satisfying $1/p + 1/q = 1/r$. Now observe that the intervals $A_k, A_{k+10}, A_{k+20}, \dots$ are pairwise disjoint and the same property holds for the intervals B_k, $A_k + B_k$, and $A_k - B_k$. Using Lemma 1 we obtain that L is bounded from $L^{p_1} \times L^{p_2} \to L^p$ where the exponents p_1, p_2, p are as in the announcement of Lemma 4. $\square$

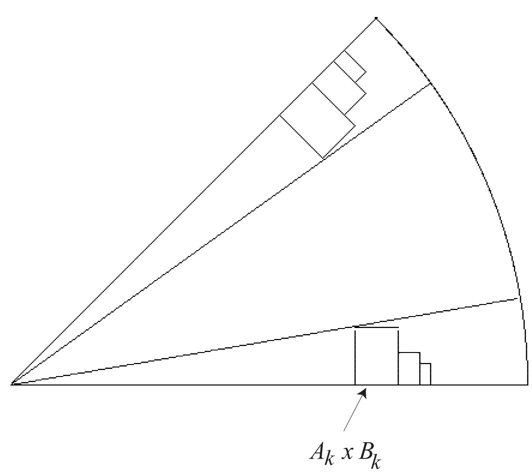

FIGURE 6. A closer look at the union of intervals $A_k \times B_k$.

We now turn our attention to the boundedness of $\widetilde{S}'$. Observe that

$$\widetilde{S}'(f_1, f_2)(x) = e^{2\pi i x}\widetilde{S}(f_1 e^{-2\pi i(\cdot)}, f_2)(x),$$

where

$$\widetilde{S}(f_1,f_2)(x) = \sum_{k=1}^{\infty}\int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f_1}(\xi)\widehat{f_2}(\eta)\widehat{\Phi_{1,k}}(\xi)\widehat{\Phi_{2,k}}(\eta)e^{2\pi i(\xi+\eta)x}d\xi d\eta$$
$$= \sum_{k=1}^{\infty}(f_1 * \Phi_{1,k})(x)(f_2 * \Phi_{2,k})(x).$$

Therefore the boundedness of $\widetilde{S}'$ is equivalent to that of $\widetilde{S}$. We now have the following.

Lemma 5. *For each $1 < p,q,r < \infty$ satisfying $1/p + 1/q = 1/r$, there exists a constant $C = C(p,q,r)$ such that*

$$\|\widetilde{S}(f_1,f_2)\|_{L^r} \le C\|f_1\|_{L^p}\|f_2\|_{L^q}\,.$$

Proof. For each $k \ge 2$, we pick a third Schwartz function $\Phi_{3,k}$ whose Fourier transform is supported in the interval $[-2\cdot 2^{-\frac{k}{2}}, 2\cdot 2^{-\frac{k}{2}}]$, which is identically equal to 1 on the interval $[-\frac{8}{5}\cdot 2^{-\frac{k}{2}}, \frac{8}{5}\cdot 2^{-\frac{k}{2}}]$, and which satisfies $|\frac{d^\alpha}{d\xi^\alpha}\widehat{\Phi_{3,k}}(\xi)| \le C_\alpha 2^{\frac{k}{2}\alpha}$ for all $\alpha \ge 0$. For $k=1$, pick $\Phi_{3,1}$ so that its Fourier transform is equal to 1 on the set $[-\frac{13}{10}, \frac{8}{5\sqrt{2}}]$ and supported in $[-\frac{13}{10} - \frac{1}{100}, \frac{8}{5\sqrt{2}} + \frac{1}{100}]$. It is easy to see that for all $k \ge 1$, the algebraic sum of the supports of $\widehat{\Phi_{1,k}}$ and $\widehat{\Phi_{2,k}}$ is contained in the interval $[-\frac{8}{5}\cdot 2^{-\frac{k}{2}}, \frac{8}{5}\cdot 2^{-\frac{k}{2}}]$ on which $\widehat{\Phi_{3,k}}$ is equal to one. It follows that

$$\widetilde{S}(f_1,f_2)(x) = \sum_{k=1}^{\infty}\int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f_1}(\xi)\widehat{f_2}(\eta)\widehat{\Phi_{1,k}}(\xi)\widehat{\Phi_{2,k}}(\eta)\widehat{\Phi_{3,k}}(\xi+\eta)e^{2\pi i(\xi+\eta)x}d\xi d\eta\,,$$

and pairing with another function f_3 we write the inner product $\langle \widetilde{S}(f_1,f_2), f_3\rangle$ as

$$\int_{\mathbf{R}} \widetilde{S}(f_1,f_2)(x)\overline{f_3(x)}\,dx = \sum_{k=1}^{\infty}\int_{\mathbf{R}}(f_1 * \Phi_{1,k})(x)(f_2 * \Phi_{2,k})(x)\overline{(f_3 * \Phi_{3,k})(x)}\,dx\,. \tag{4.2}$$

We now use a telescoping argument, inspired by [18], to write

$$\int_{\mathbf{R}} \widetilde{S}(f_1,f_2)(x)\overline{f_3(x)}\,dx = \sum_{k=1}^{\infty}\int_{\mathbf{R}}(f_1 * \Phi_{1,k})(x)(f_2 * \Phi_{2,k})(x)\overline{(f_3 * \Phi_{3,k})(x)}\,dx$$
$$= \sum_{k=1}^{\infty}\int_{\mathbf{R}}(f_1 * \Phi_{1,k})(x)\sum_{m=0}^{k+5}\Big\{(f_2 * \Phi_{2,k+m})(x)\overline{(f_3 * \Phi_{3,k+m})(x)}$$
$$- (f_2 * \Phi_{2,k+m+1})(x)\overline{(f_3 * \Phi_{3,k+m+1})(x)}\Big\}\,dx$$
$$+ \sum_{k=1}^{\infty}\int_{\mathbf{R}}(f_1 * \Phi_{1,k})(x)(f_2 * \Phi_{2,2k+6})(x)\overline{(f_3 * \Phi_{3,2k+6})(x)}\,dx\,.$$

We begin by claiming that the last sum above is identically equal to zero. Indeed, we have that the support of $\widehat{\Phi_{1,k}}$ is contained in $[-\frac{101}{100}\cdot 2^{-k}, -\frac{99}{100}\cdot 2^{-k-1}]$, the support of $\widehat{\Phi_{2,2k+6}}$ is contained in $[-2^{-k-3}, 2^{-k-3}]$, while the support of $\widehat{\Phi_{3,2k+6}}$ is contained in $[-2^{-k-2}, 2^{-k-2}]$. It follows that

$$\big(\text{supp }\widehat{\Phi_{1,k}} + \text{supp }\widehat{\Phi_{2,2k+6}}\big) \cap \text{supp }\widehat{\Phi_{3,2k+6}} = \emptyset\,,$$

which establishes our claim. It follows that

$$\begin{aligned}&\int_{\mathbf{R}} \widetilde{S}(f_1,f_2)(x)\overline{f_3(x)}\,dx\\ =&\sum_{k=1}^{\infty}\int_{\mathbf{R}}(f_1*\Phi_{1,k})(x)\sum_{m=0}^{k+5}\Big\{(f_2*\Phi_{2,k+m})(x)\overline{(f_3*\Phi_{3,k+m})(x)}\\ &\qquad\qquad - (f_2*\Phi_{2,k+m+1})(x)\overline{(f_3*\Phi_{3,k+m+1})(x)}\Big\}\,dx\,,\end{aligned}$$

which by a change of variables, we write as

$$\begin{aligned}&\int_{\mathbf{R}} \widetilde{S}(f_1,f_2)(x)\overline{f_3(x)}\,dx\\ =&\sum_{k=7}^{\infty}\int_{\mathbf{R}}\Big\{\sum_{m=0}^{\frac{k+5}{2}}(f_1*\Phi_{1,k-m})(x)\Big\}\Big\{(f_2*\Phi_{2,k})(x)\overline{(f_3*\Phi_{3,k})(x)}\\ &\qquad\qquad - (f_2*\Phi_{2,k+1})(x)\overline{(f_3*\Phi_{3,k+1})(x)}\Big\}\,dx\\ +&\sum_{k=1}^{6}\sum_{m=0}^{k-1}\int_{\mathbf{R}}(f_1*\Phi_{1,k-m})(x)\Big\{(f_2*\Phi_{2,k})(x)\overline{(f_3*\Phi_{3,k})(x)}\\ &\qquad\qquad - (f_2*\Phi_{2,k+1})(x)\overline{(f_3*\Phi_{3,k+1})(x)}\Big\}\,dx\,.\end{aligned}$$

Now the last double sum above is indeed a finite sum which is easily controlled by a constant multiple of $\|Mf_1\|_p\|Mf_2\|_q\|Mf_3\|_{r'}$ and the required estimate easily follows for it. (M here denotes the Hardy-Littlewood maximal operator.) We therefore concentrate our attention to the sum over $k\ge 7$ above. We set

$$\begin{aligned}I=&\sum_{k=7}^{\infty}\int_{\mathbf{R}}\Big\{\sum_{m=0}^{\frac{k+5}{2}}(f_1*\Phi_{1,k-m})(x)\Big\}\Big\{(f_2*\Phi_{2,k})(x)\overline{(f_3*\Phi_{3,k})(x)}\\ &\qquad\qquad - (f_2*\Phi_{2,k+1})(x)\overline{(f_3*\Phi_{3,k+1})(x)}\Big\}\,dx\,,\end{aligned}$$

and we write $I=I_1+I_2+I_3$, where

$$\begin{aligned}I_1=&\sum_{k=7}^{\infty}\int_{\mathbf{R}}\Big\{\sum_{m=0}^{\frac{k+5}{2}}(f_1*\Phi_{1,k-m})(x)\Big\}(f_2*\Phi_{2,k})(x)\,\overline{(f_3*(\Phi_{3,k}-\Phi_{3,k+1})(x)}\,dx\,,\\ I_2=&\sum_{k=7}^{\infty}\int_{\mathbf{R}}\Big\{\sum_{m=0}^{\frac{k+5}{2}}(f_1*\Phi_{1,k-m})(x)\Big\}\\ &\qquad (f_2*(\Phi_{2,k}-\Phi_{2,k+1}))(x)\,\overline{(f_3*(\Phi_{3,k+1}-\Phi_{3,k+6}))(x)}\,dx\,,\\ I_3=&\sum_{k=7}^{\infty}\int_{\mathbf{R}}\Big\{\sum_{m=0}^{\frac{k+5}{2}}(f_1*\Phi_{1,k-m})(x)\Big\}(f_2*(\Phi_{2,k}-\Phi_{2,k+1}))(x)\,\overline{(f_3*\Phi_{3,k+6})(x)}\,dx\,.\end{aligned}$$

We write I_1 as $I_{11} + I_{12}$, where

$$I_{11} = \sum_{k=7}^{\infty} \int_{\mathbf{R}} \Big\{ \sum_{m=0}^{\frac{k-4}{2}} (f_1 * \Phi_{1,k-m})(x) \Big\} (f_2 * \Phi_{2,k})(x) \, \overline{(f_3 * (\Phi_{3,k} - \Phi_{3,k+1})(x)} \, dx \, ,$$

$$I_{12} = \sum_{k=7}^{\infty} \int_{\mathbf{R}} \Big\{ \sum_{\frac{k-4}{2} < m \leq \frac{k+5}{2}} (f_1 * \Phi_{1,k-m})(x) \Big\} (f_2 * \Phi_{2,k})(x) \, \overline{(f_3 * (\Phi_{3,k} - \Phi_{3,k+1})(x)} \, dx \, .$$

We begin by observing that I_{11} is identically equal to zero. Indeed, let us calculate the supports of the functions the appear in I_{11}. We have

$$\operatorname{supp} \Big(\sum_{m=0}^{\frac{k-4}{2}} \widehat{\Phi_{1,k-m}} \Big) \subset \bigcup_{m=0}^{\frac{k-4}{2}} [-\tfrac{101}{100} \cdot 2^{-k+m}, -\tfrac{99}{100} \cdot 2^{-k+m-1}] \subset [-\tfrac{101}{400} \cdot 2^{-\frac{k}{2}}, -\tfrac{99}{200} \cdot 2^{-k}]$$

$$\operatorname{supp} \widehat{\Phi_{2,k}} \subset [-2^{-\frac{k}{2}}, 2^{-\frac{k}{2}}]$$

$$\operatorname{supp} \big(\widehat{\Phi_{3,k}} - \widehat{\Phi_{3,k+1}} \big) \subset [-2 \cdot 2^{-\frac{k}{2}}, -\tfrac{9}{5\sqrt{2}} \cdot 2^{-\frac{k}{2}}] \bigcup [\tfrac{9}{5\sqrt{2}} \cdot 2^{-\frac{k}{2}}, 2 \cdot 2^{-\frac{k}{2}}],$$

where in the last inclusion we used the fact that $\widehat{\Phi_{3,k}}$ is equal to one on a substantially large subset of its support. It is easy to see that

$$\operatorname{supp} \Big(\sum_{m=0}^{\frac{k-4}{2}} \widehat{\Phi_{1,k-m}} \Big) + \operatorname{supp} \widehat{\Phi_{2,k}} \subset [-\tfrac{501}{400} \cdot 2^{-\frac{k}{2}}, 2^{-\frac{k}{2}}],$$

from which it follows that

$$\Big(\operatorname{supp} \Big(\sum_{m=0}^{\frac{k-4}{2}} \widehat{\Phi_{1,k-m}} \Big) + \operatorname{supp} \widehat{\Phi_{2,k}} \Big) \bigcap \operatorname{supp} \big(\widehat{\Phi_{3,k}} - \widehat{\Phi_{3,k+1}} \big) = \emptyset,$$

since $\widehat{\Phi_{3,k}} - \widehat{\Phi_{3,k+1}}$ is supported in $[-2 \cdot 2^{-\frac{k}{2}}, -\frac{8}{5\sqrt{2}} \cdot 2^{-\frac{k}{2}}] \bigcup [\frac{8}{5\sqrt{2}} \cdot 2^{-\frac{k}{2}}, 2 \cdot 2^{-\frac{k}{2}}]$. We conclude that $I_{11} = 0$ and we don't need to worry about this term. We proceed with term I_{12} which is equal to a finite sum of expressions of the form

$$\sum_{k=7}^{\infty} \int_{\mathbf{R}} (f_1 * \Phi_{1,k-m(k)})(x) \, (f_2 * \Phi_{2,k})(x) \, \overline{(f_3 * (\Phi_{3,k} - \Phi_{3,k+1})(x)} \, dx \, , \tag{4.3}$$

where $m(k)$ is an integer in the interval $(\frac{k-4}{2}, \frac{k+5}{2}]$. We can now control I_{12} by

$$\int_{\mathbf{R}} \big(\sum_{k=7}^{\infty} |f_1 * \Phi_{1,k-m(k)}|^2 \big)^{\frac{1}{2}} \sup_k |f_2 * \Phi_{2,k}| \big(\sum_{k=7}^{\infty} |f_3 * (\Phi_{3,k} - \Phi_{3,k+1})|^2 \big)^{\frac{1}{2}} \, dx$$

and this is easily seen to be bounded by a constant multiple of

$$\|f_1\|_p \|f_2\|_q \|f_3\|_{r'}$$

in view of the Littlewood-Paley theorem and the fact that $\sup_k |f_2 * \Phi_{2,k}|$ is bounded by the Hardy-Littlewood maximal function of f_2.

We continue with term I_2. We first claim that the estimate below is valid:

$$\Big\| \sup_k \Big| \sum_{m=0}^{\frac{k+5}{2}} f_1 * \Phi_{1,k-m} \Big| \Big\|_p \leq C_p \|f_1\|_p \, . \tag{4.4}$$

To see this we bound the left side of (4.4) by

$$\Big\|\sup_k\Big|\sum_{m=\frac{k-5}{2}}^{k} f_1*\Phi_{1,m}\Big|\Big\|_p \le \Big\|\sup_k\Big|\sum_{\substack{m=\frac{k-5}{2}\\ m\text{ odd}}}^{k} f_1*\Phi_{1,m}\Big|\Big\|_p + \Big\|\sup_k\Big|\sum_{\substack{m=\frac{k-5}{2}\\ m\text{ even}}}^{k} f_1*\Phi_{1,m}\Big|\Big\|_p. \tag{4.5}$$

We let a_k (resp. c_k) be the infimum of the left points of the supports of the functions $\widehat{\Phi_{1,m}}$ with m odd (resp. even) in $[\frac{k-5}{2},k]$ and b_k (resp. d_k) is the supremum of the right points of the supports of the functions $\widehat{\Phi_{1,m}}$ with m odd (resp. even) in $[\frac{k-5}{2},k]$. Then the right hand side of (4.5) is equal to

$$\Big\|\sup_k\Big|\Big(\big(\sum_{m\text{ odd}} f_1*\Phi_{1,m}\big)^{\wedge}1_{[a_k,b_k]}\Big)^{\vee}\Big|\Big\|_p + \Big\|\sup_k\Big|\Big(\big(\sum_{m\text{ even}} f_1*\Phi_{1,m}\big)^{\wedge}1_{[c_k,d_k]}\Big)^{\vee}\Big|\Big\|_p$$

and this is bounded by

$$C_p\Big(\big\|\sum_{k\text{ odd}} f_1*\Phi_{1,k}\big\|_p + \big\|\sum_{k\text{ even}} f_1*\Phi_{1,k}\big\|_p\Big) \tag{4.6}$$

in view of the Carleson-Hunt theorem. The expression in (4.6) is easily controlled by $C_p'\|f_1\|_p$ via a simple orthogonality argument, and the proof of our claim in (4.4) is complete. It follows that I_2 is controlled by

$$\int_{\mathbf{R}}\sup_k\Big|\sum_{m=0}^{\frac{k+5}{2}} f_1*\Phi_{1,k-m}\Big|\Big(\sum_{k=7}^{\infty}|f_2*(\Phi_{2,k}-\Phi_{2,k+1})|^2\Big)^{\frac12}\Big(\sum_{k=7}^{\infty}|f_3*(\Phi_{3,k}-\Phi_{3,k+6})|^2\Big)^{\frac12}\,dx$$

and this is in turn bounded by a constant multiple of

$$\|f_1\|_p\|f_2\|_q\|f_3\|_{r'}$$

in view of the Littlewood-Paley theorem and the discussion above.

Before we turn our attention to I_3, we make a couple of observations regarding the supports of the Fourier transforms of the functions $\Phi_{1,k}$, $\Phi_{2,k}$, and $\Phi_{3,k}$. First we observe that

$$\begin{aligned}
&\text{supp}\,\Big(\sum_{m=0}^{\frac{k-4}{2}}\widehat{\Phi_{1,k-m}}\Big)\subset\bigcup_{m=0}^{\frac{k-4}{2}}[-\tfrac{101}{100}\cdot2^{-k+m},-\tfrac{99}{100}\cdot2^{-k+m-1}]\subset[-\tfrac{101}{400}\cdot2^{-\frac{k}{2}},-\tfrac{99}{200}\cdot2^{-k}]\,,\\
&\text{supp}\,\big(\widehat{\Phi_{2,k}}-\widehat{\Phi_{2,k+1}}\big)\subset[-2^{-\frac{k}{2}},-\tfrac{4}{5\sqrt2}\cdot2^{-\frac{k}{2}}]\cup[\tfrac{4}{5\sqrt2}\cdot2^{-\frac{k}{2}},2^{-\frac{k}{2}}]\,,\\
&\text{supp}\,\widehat{\Phi_{3,k+6}}\subset[-\tfrac14\cdot2^{-\frac{k}{2}},\tfrac14\cdot2^{-\frac{k}{2}}]\,.
\end{aligned}$$

Therefore, the algebraic sum

$$\text{supp}\,\Big(\sum_{m=0}^{\frac{k-4}{2}}\widehat{\Phi_{1,k-m}}\Big)+\text{supp}\,\big(\widehat{\Phi_{2,k}}-\widehat{\Phi_{2,k+1}}\big)$$

is contained in the union of the intervals

$$[-\tfrac{501}{400}\cdot2^{-\frac{k}{2}},-\tfrac{4}{5\sqrt2}\cdot2^{-\frac{k}{2}}-\tfrac{99}{200}\cdot2^{-k}]\cup[(\tfrac{4}{5\sqrt2}-\tfrac{101}{400})\cdot2^{-\frac{k}{2}},\tfrac{101}{200}\cdot2^{-\frac{k}{2}}]$$

from which it easily follows that

$$\Big(\mathrm{supp}\,\Big(\sum_{m=0}^{\frac{k-4}{2}}\widehat{\Phi_{1,k-m}}\Big)+\mathrm{supp}\,\big(\widehat{\Phi_{2,k}}-\widehat{\Phi_{2,k+1}}\big)\Big)\cap\mathrm{supp}\,\widehat{\Phi_{3,k+6}}=\emptyset\,.$$

Therefore I_3 reduces to the sum

$$\sum_{k=7}^{\infty}\int_{\mathbf{R}}\Big\{\sum_{\frac{k-4}{2}<m\leq\frac{k+5}{2}}(f_1*\Phi_{1,k-m})(x)\Big\}(f_2*(\Phi_{2,k}-\Phi_{2,k+1}))(x)\,\overline{(f_3*\Phi_{3,k+6})(x)}\,dx\,,$$

in which m ranges only through a finite set (depending on k). For every such $m=m(k)$, we can estimate I_3 by

$$\int_{\mathbf{R}}\big(\sum_{k=7}^{\infty}|f_1*\Phi_{1,k-m(k)}|^2\big)^{\frac{1}{2}}\big(\sum_{k=7}^{\infty}|f_2*(\Phi_{2,k}-\Phi_{2,k+1})|^2\big)^{\frac{1}{2}}\sup_k\big|f_3*\Phi_{1,k+6}\big|\,dx$$

and this is clearly bounded by a constant multiple of $\|f_1\|_p\|f_2\|_q\|f_3\|_{r'}$ via the Littlewood-Paley theorem and the $L^{r'}$ boundedness of the Hardy-Littlewood maximal operator. This estimate completes the proof of the boundedness of $\widetilde{S}$ and thus of Lemma 5. □

The proof of the boundedness of $S=\sum_{k=1}^{\infty}S_k$ is now complete.

5. The boundedness of $\sum_{\mu=1}^{\infty}T_\mu$

Throughout this section we will fix $2\leq p_1,p_2<\infty$, $1<p\leq 2$ with $1/p_1+1/p_2=1/p$. The main difficulty is to show that the operators T_μ are uniformly bounded from $L^{p_1}\times L^{p_2}\to L^p$. Once this is established we can use Lemma 1 to control the sum $\sum_{\mu=1}^{\infty}T_\mu$. To do so, it will suffice to check that the hypotheses of Lemma 1 apply for the operators T_μ. Indeed, one can easily see that the support of the symbol of T_μ is contained in the set $A_\mu\times B_\mu$ union the set $A_\mu\times(-B_\mu)$, where

$$\begin{aligned}A_\mu=&\big[\big(1-(1+2^{-9})2^{-\mu}\big)\cos\big(\tfrac{\pi}{8}(1+2^{-9})2^{\frac{1}{2}-\frac{\mu}{2}}\big),\cos\big(\tfrac{\pi}{8}(1-2^{-9})2^{-\frac{\mu}{2}}\big)\big]\\ B_\mu=&\big[\big(1-(1+2^{-9})2^{-\mu}\big)\sin\big(\tfrac{\pi}{8}(1-2^{-9})2^{-\frac{\mu}{2}}\big),\sin\big(\tfrac{\pi}{8}(1+2^{-9})2^{\frac{1}{2}-\frac{\mu}{2}}\big)\big]\,.\end{aligned}\tag{5.1}$$

It is now elementary to check that the sets $A_\mu, A_{\mu+10}, A_{\mu+20},\dots$ are disjoint, and similarly for the sets B_μ, $A_\mu+B_\mu$, and $A_\mu-B_\mu$. (To apply Lemma 1 one may consider the "upper" and the "lower" part of T_μ separately.)

We now turn our attention to the crucial issue of the uniform boundedness of the T_μ's. We fix a large μ and we break up the support of T_μ as the disjoint union of the "curved rectangles" $D_{\mu,k}$ for $k\geq\mu$, defined as follows

$$\begin{aligned}D_{\mu,\mu}=&\big\{(\xi,\eta):\ 1-2^{-\mu}(1+2^{-9})\leq|(\xi,\eta)|<1-2^{-(\mu+1)}\big\}\\ &\bigcap\big\{(\xi,\eta):\ \tfrac{\pi}{8}2^{-\frac{\mu}{2}}(1-2^{-9})\leq|\mathrm{Argument}(\xi,\eta)|<\tfrac{\pi}{8}2^{-\frac{\mu-1}{2}}(1+2^{-9})\big\}\,,\end{aligned}$$

while for $k\geq\mu+1$

$$\begin{aligned}D_{\mu,k}=&\big\{(\xi,\eta):\ 1-2^{-k}\leq|(\xi,\eta)|<1-2^{-(k+1)}\big\}\\ &\bigcap\big\{(\xi,\eta):\ \tfrac{\pi}{8}2^{-\frac{\mu}{2}}(1-2^{-9})\leq|\mathrm{Argument}(\xi,\eta)|<\tfrac{\pi}{8}2^{-\frac{\mu-1}{2}}(1+2^{-9})\big\}\,.\end{aligned}$$

In the sequel, rectangles will be products of intervals of the form $[a,b)\times[c,d)$. The quantity $(b-a)\times(d-c)$ will be referred to as the size of a rectangle. We tile up the plane as the union of rectangles of size $2^{-\mu-5}\times 2^{-\frac{\mu}{2}-5}$ and we let $\mathcal{E}_\mu$ be the set of all such rectangles. Let $\mathcal{E}_\mu^{select}$ be the subset of $\mathcal{E}_\mu$ consisting of all rectangles that intersect $D_{\mu,\mu}$. We denote by $\mathcal{E}_{\mu+1}$ the set of all rectangles obtained by subdividing each rectangle in $\mathcal{E}_\mu\setminus\mathcal{E}_\mu^{select}$ into four rectangles, each of size $2^{-\mu-6}\times 2^{-\frac{\mu}{2}-6}$, by halving its sides. Let $\mathcal{E}_{\mu+1}^{select}$ be the subset of $\mathcal{E}_{\mu+1}$ consisting of all rectangles that intersect $D_{\mu,\mu+1}$. Next, we denote by $\mathcal{E}_{\mu+2}$ the set of all rectangles obtained by subdividing each rectangle in $\mathcal{E}_{\mu+1}\setminus\mathcal{E}_{\mu+1}^{select}$ into four rectangles, each of size $2^{-\mu-7}\times 2^{-\frac{\mu}{2}-7}$, by halving its sides. Continue this way by induction. Then we have "essentially covered" each $D_{\mu,k}$ by disjoint rectangles of size $2^{-k-5}\times 2^{\frac{\mu}{2}-k-5}$ and the set of all such rectangles is denoted by $\mathcal{E}_{\mu+k}^{select}$. Since each $D_{\mu,k}$ has area about $2^{-\frac{\mu}{2}-k}$, we have used approximately $2^{k-\mu}$ rectangles of size $2^{-k-5}\times 2^{\frac{\mu}{2}-k-5}$ to "cover" $D_{\mu,k}$. In other words, the cardinality of each set $\mathcal{E}_{\mu+k}^{select}$ is of the order of $2^{k-\mu}$.

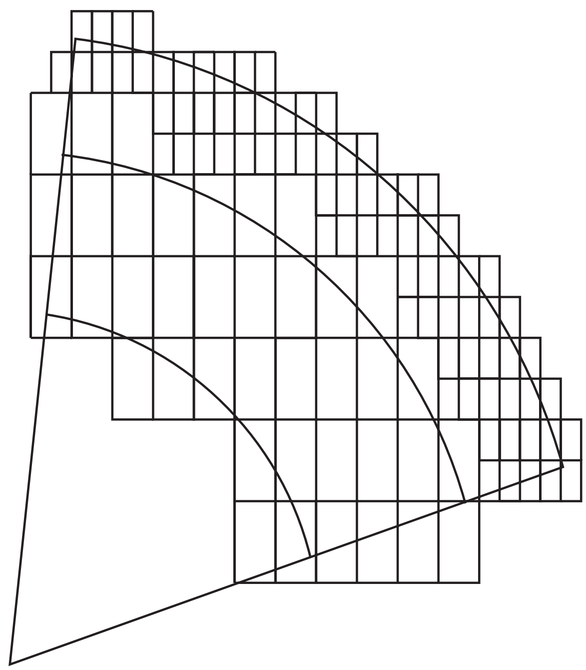

FIGURE 7. The selected rectangles of two successive generations.

Elements of $\mathcal{E}_{\mu+k}^{select}$ will be denoted by $R_{k,l,m}$; explicitly

$$R_{k,l,m}=[\tfrac{1}{32}2^{-k}l,\tfrac{1}{32}2^{-k}(l+1)]\times[\tfrac{1}{32}2^{-k+\frac{\mu}{2}}m,\tfrac{1}{32}2^{-k+\frac{\mu}{2}}(m+1)]\,.$$

So the first index k indicates that the rectangle $R_{k,l,m}$ has size $2^{-k-5}\times 2^{\frac{\mu}{2}-k-5}$. The second index l indicates the horizontal location of the rectangle $R_{k,l,m}$, while the third index m indicates its vertical location. Furthermore, if $R_{k,l,m}$ is selected, then for any integer $k\geq\mu+1$, l ranges in the interval

$$32\cdot(2^k-1)\cos\left(\tfrac{\pi}{8}(1+2^{-9})2^{-\frac{\mu}{2}+\frac{1}{2}}\right)-1\leq l\leq 32\cdot(2^k-\tfrac{1}{2})\cos\left(\tfrac{\pi}{8}(1-2^{-9})2^{-\frac{\mu}{2}}\right), \tag{5.2}$$

with the left inequality above only slightly changed to

$$32\cdot(2^k-(1+2^{-9}))\cos\left(\tfrac{\pi}{8}(1+2^{-9})2^{-\frac{\mu}{2}+\frac{1}{2}}\right)-1\leq l$$

when $k=\mu$. Moreover, for fixed k and l, the range of m is specified by the inequalities

$$2^{5+k-\frac{\mu}{2}}\sqrt{(1-2^{-k+1})^2-(\tfrac{2^{-k}(l+1)}{32})^2}-1<|m|<2^{5+k-\frac{\mu}{2}}\sqrt{(1-2^{-k-1})^2-(\tfrac{2^{-k}l}{32})^2}\,. \tag{5.3}$$

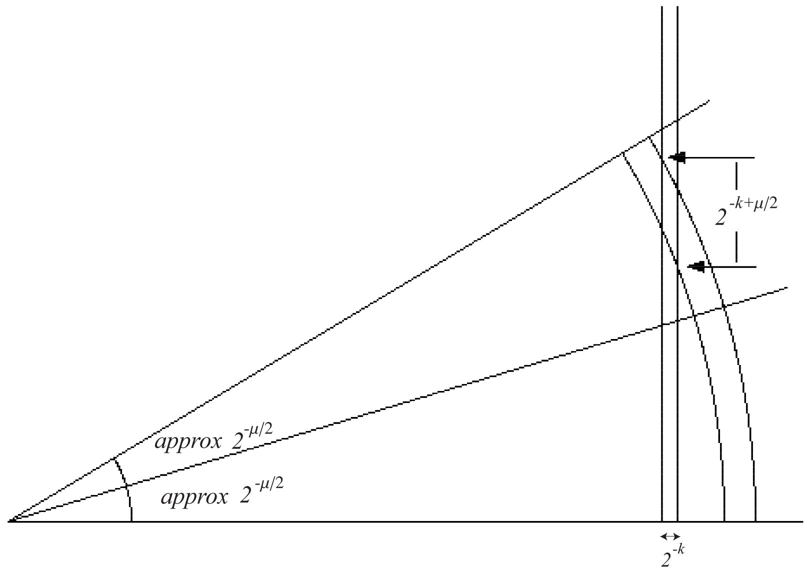

FIGURE 8. This picture indicates that there exist at most finitely many selected rectangles in $\mathcal{E}^{select}_{\mu+k}$ that intersect a vertical strip of width 2^{-k}.

It is a very important geometric fact that given a fixed k and l, there exist at most 64 integers m such that the rectangles $R_{k,l,m}$ in $\mathcal{E}^{select}_{\mu+k}$ intersect $D_{\mu,k}$. The verification of this fact is a simple geometric exercise shown in Figure 8 and is left to the reader. Therefore in the sequel, we will think of $m=m(k,l,\mu)$ as a function of k, l, and μ whose range is a set of integers with at most 64 elements.

Let $\varepsilon>0$ be a very small number. Pick Schwartz functions $\Phi_{1,k,l}$ and $\Phi_{2,k,m}$ (unrelated to the ones in the previous section) such that

$$\text{supp }\widehat{\Phi_{1,k,l}}\subset(1+\varepsilon)[\tfrac{1}{32}2^{-k}l,\tfrac{1}{32}2^{-k}(l+1)]\,,$$
$$\text{supp }\widehat{\Phi_{2,k,m}}\subset(1+\varepsilon)[\tfrac{1}{32}2^{-k+\frac{\mu}{2}}m,\tfrac{1}{32}2^{-k+\frac{\mu}{2}}(m+1)]\,,$$

$$\widehat{\Phi_{1,k,l}}=1\quad\text{on}\quad(1-\varepsilon)[\tfrac{1}{32}2^{-k}l,\tfrac{1}{32}2^{-k}(l+1)]\,,$$
$$\widehat{\Phi_{2,k,m}}=1\quad\text{on}\quad(1-\varepsilon)[\tfrac{1}{32}2^{-k+\frac{\mu}{2}}m,\tfrac{1}{32}2^{-k+\frac{\mu}{2}}(m+1)]\,,$$

such that for all $\beta\geq 0$

$$|\frac{d^\beta}{d\xi^\beta}\widehat{\Phi_{1,k,l}}(\xi)|\leq C_\beta 2^{k\beta},\qquad |\frac{d^\beta}{d\eta^\beta}\widehat{\Phi_{2,k,m}}(\eta)|\leq C_\beta 2^{(k-\frac{\mu}{2})\beta}\,, \tag{5.4}$$

and such that the function

$$\sum_{k=\mu}^{\infty} \sum_{\substack{l,m \text{ such that} \\ R_{k,l,m}\in\mathcal{E}^{select}_{\mu+k}}} \widehat{\Phi_{1,k,l}}(\xi)\widehat{\Phi_{2,k,m}}(\eta)$$

is equal to 1 on the union of all $R_{k,l,m}$ in $\mathcal{E}^{select}_{\mu+k}$ that do not intersect the boundary of the support of T_μ. For an explicit construction of these functions, we refer the reader to the appendix in [8]. The basic idea of this decomposition is that the functions

$$(\xi,\eta) \to \widehat{\Phi_{1,k,l}}(\xi)\widehat{\Phi_{2,k,m}}(\eta)$$

form a smooth partition of unity adapted to the rectangles $R_{k,l,m}$.

Recall that the symbol of the bilinear operator T_μ is

$$\sigma_\mu(\xi,\eta) = \sum_{k=\mu}^{\infty} \widehat{b^1_k}(\xi,\eta)\widehat{\rho^\mu_k}(\xi,\eta)\,.$$

We now write σ_μ as

$$\sigma_\mu(\xi,\eta) = \sum_{k=\mu}^{\infty} \sum_{\substack{l,m \text{ such that} \\ R_{k,l,m}\in\mathcal{E}^{select}_{\mu+k}}} \widehat{\Phi_{1,k,l}}(\xi)\widehat{\Phi_{2,k,m}}(\eta) + E^{(1)}_\mu(\xi,\eta)\,, \tag{5.5}$$

where $E^{(1)}_\mu$ is an error.

We start by studying the error $E^{(1)}_\mu$. Let (r,θ) denote polar coordinates in the (ξ,η) plane. The function $E^{(1)}_\mu$ consists of six pieces: The piece $E^{(1)}_{\mu,1}$ supported near the line $\theta = \frac{\pi}{8}2^{-\frac{\mu}{2}}$, the piece $E^{(1)}_{\mu,2}$ supported near the line $\theta = \frac{\pi}{8}2^{-\frac{\mu-1}{2}}$, the piece $E^{(1)}_{\mu,3}$ supported near the circle $r = 1-2^{-\mu}$ between these two lines, the piece $E^{(1)}_{\mu,4}$ supported near the line $\theta = -\frac{\pi}{8}2^{-\frac{\mu}{2}}$, the piece $E^{(1)}_{\mu,5}$ supported near the line $\theta = -\frac{\pi}{8}2^{-\frac{\mu-1}{2}}$, and the piece $E^{(1)}_{\mu,6}$ supported near the circle $r = 1-2^{-\mu}$ between these last two lines. The error $E^{(1)}_{\mu,3}+E^{(1)}_{\mu,6}$ is the easiest to control. Since $\mathcal{E}^{select}_\mu$ consists only of finitely many rectangles (independent of μ), $E^{(1)}_{\mu,3}+E^{(1)}_{\mu,6}$ is equal to a finite sum of smooth functions ϕ_μ which are supported in a small dilate of $D_{\mu,\mu}$ and which satisfy the estimates

$$|\frac{d^\beta}{d\xi^\beta}\phi_\mu| \le C_\beta 2^{\mu\beta}, \qquad |\frac{d^\beta}{d\eta^\beta}\phi_\mu| \le C_\beta 2^{\frac{\mu}{2}\beta}\,,$$

because of (3.8) and (5.4). It follows from Lemma 3 that the inverse Fourier transforms of the ϕ_μ's are in L^1 uniformly in μ. Using Lemma 2 we obtain the uniform (in μ) boundedness of the operators whose symbols are $E^{(1)}_{\mu,3}+E^{(1)}_{\mu,6}$.

We write $E^{(1)}_{\mu,1}+E^{(1)}_{\mu,2}+E^{(1)}_{\mu,4}+E^{(1)}_{\mu,5} = \sum_{k=\mu}^{\infty} E_{\mu,k}$, where each $E_{\mu,k}$ consists of the (smooth) piece of this function inside the annulus $1-2^{-k} \le r \le 1-2^{-k-1}$. An easy calculation shows that the support of $E_{\mu,k}$ is contained in the union of the sets $A_{\mu,k}\times B_{\mu,k}$, $A'_{\mu,k}\times B'_{\mu,k}$,

$A_{\mu,k} \times (-B_{\mu,k})$, and $A'_{\mu,k} \times (-B'_{\mu,k})$, where

$$\begin{aligned} A_{\mu,k} &= \big[(1-2^{-k+1})\cos\big(\tfrac{\pi}{8}(1+2^{-9})2^{-\frac{\mu}{2}}\big), (1-2^{-k-2})\cos\big(\tfrac{\pi}{8}(1-2^{-9})2^{-\frac{\mu}{2}}\big)\big] \\ B_{\mu,k} &= \big[(1-2^{-k+1})\sin\big(\tfrac{\pi}{8}(1-2^{-9})2^{-\frac{\mu}{2}}\big), (1-2^{-k-2})\sin\big(\tfrac{\pi}{8}(1+2^{-9})2^{-\frac{\mu}{2}}\big)\big] \\ A'_{\mu,k} &= \big[(1-2^{-k+1})\cos\big(\tfrac{\pi}{8}(1+2^{-9})2^{\frac{1}{2}-\frac{\mu}{2}}\big), (1-2^{-k-2})\cos\big(\tfrac{\pi}{8}(1-2^{-9})2^{\frac{1}{2}-\frac{\mu}{2}}\big)\big] \\ B'_{\mu,k} &= \big[(1-2^{-k+1})\sin\big(\tfrac{\pi}{8}(1-2^{-9})2^{\frac{1}{2}-\frac{\mu}{2}}\big), (1-2^{-k-2})\sin\big(\tfrac{\pi}{8}(1+2^{-9})2^{\frac{1}{2}-\frac{\mu}{2}}\big)\big]\,. \end{aligned}$$

We now observe that the sets $A_{\mu,k}, A_{\mu,k+10}, A_{\mu,k+20}, \dots$ are pairwise disjoint, and the same disjointness property also holds for the families $\{B_{\mu,k}\}_k$, $\{A_{\mu,k}+B_{\mu,k}\}_k$, $\{A_{\mu,k}-B_{\mu,k}\}_k$, $\{A'_{\mu,k}\}_k$, $\{B'_{\mu,k}\}_k$, $\{A'_{\mu,k}+B'_{\mu,k}\}_k$, and $\{A'_{\mu,k}-B'_{\mu,k}\}_k$. Using (3.8) and (5.4) we obtain that the inverse Fourier transforms of the functions $E_{\mu,k}$ are uniformly integrable in μ and k. Then Lemmata 1 and 2 imply that the operators with symbols $E^{(1)}_{\mu}$ are bounded from $L^p \times L^q$ into L^r uniformly in μ. (We apply Lemma 1 for each of the four pieces above separately.)

For every selected rectangle $R_{k,l,m}$ we now choose a third Schwartz function $\Phi_{3,k,l,m}$ such that

$$|\frac{d^\beta}{d\xi^\beta}\widehat{\Phi_{3,k,l,m}}(\xi)| \le C_\alpha 2^{(k-\frac{\mu}{2})\beta}$$

for all $\beta \ge 0$, $\widehat{\Phi_{3,k,l,m}}$ is equal to 1 on the interval

$$[\tfrac{1}{32}2^{-k}l + 2^{-k+\frac{\mu}{2}}m, \tfrac{1}{32}2^{-k}(l+1) + 2^{-k+\frac{\mu}{2}}(m+1)]$$

and also on the interval

$$[\tfrac{1}{32}2^{-k}l - 2^{-k+\frac{\mu}{2}}m, \tfrac{1}{32}2^{-k}(l+1) - 2^{-k+\frac{\mu}{2}}(m+1)]$$

and is supported in an $(1+\varepsilon)$-neighborhood of the union of the two intervals above. We now denote by $\widetilde{T}_\mu$ the bilinear operator whose symbol is $\sigma_\mu - E^{(1)}_\mu$. Then

$$\widetilde{T}_\mu(f,g)(x) = \sum_{k=\mu}^{\infty} \sum_{\substack{l,m \text{ such that} \\ R_{k,l,m}\in\mathcal{E}^{select}_{\mu+k}}} \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f}(\xi)\widehat{g}(\eta)\widehat{\Phi_{1,k,l}}(\xi)\widehat{\Phi_{2,k,m}}(\eta)e^{2\pi i(\xi+\eta)x}d\xi d\eta\,.$$

and because of the properties of the function $\Phi_{3,k,l,m}$, $\widetilde{T}_\mu(f,g)(x)$ is also equal to

$$\sum_{k=\mu}^{\infty} \sum_{\substack{l,m \text{ such that} \\ R_{k,l,m}\in\mathcal{E}^{select}_{\mu+k}}} \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f}(\xi)\widehat{g}(\eta)\widehat{\Phi_{1,k,l}}(\xi)\widehat{\Phi_{2,k,m}}(\eta)\widehat{\Phi_{3,k,l,m}}(\xi+\eta)e^{2\pi i(\xi+\eta)x}d\xi d\eta\,. \tag{5.6}$$

We now let $J_{k,\mu}$ be the set of all integers l satisfying (5.2) and for each k, l, μ we set

$$\lambda = \lambda(k,l,\mu) = \left[32\cdot 2^{k-\frac{\mu}{2}}\sqrt{(1-2^{-k-1})^2 - (\tfrac{2^{-k}l}{32})^2}\,\right] + 1\,,$$

where the square brackets above denote the integer part. We can therefore write $\widetilde{T}_\mu(f,g)(x)$ as a finite sum (over $1 \le s \le 64$) of operators of the form

$$I_\mu(f,g)(x) = \sum_{k=\mu}^{\infty}\sum_{l\in J_{k,\mu}} \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f}(\xi)\widehat{g}(\eta)\widehat{\Phi_{1,k,l}}(\xi)\widehat{\Phi_{2,k,\lambda-s}}(\eta)\widehat{\Phi_{3,k,l,\lambda-s}}(\xi+\eta)e^{2\pi i(\xi+\eta)x}d\xi\, d\eta\,.$$

The uniform boundedness of the I_μ's for the claimed range of exponents will be a consequence of the results in [8] and [13] once we have established Lemma 6 stated below. This lemma will allow us to use Lemma 4 in [8] and Lemma 3 in [13] to obtain uniform boundedness for each I_μ from $L^{p_1} \times L^{p_2} \to L^p$, where p_1, p_2, p are as in Theorem 1. The results in [13] are only needed to cover the endpoint cases in which $p_1 = 2$, $p_2 = 2$, or $p = 2$.

For uniformity we replaced $\Phi_{2,k,\lambda(k,l,\mu)-s}$ by $\Phi_{2,k,l,\lambda(k,l,\mu)-s}$ in the lemma below.

Lemma 6. *Let be $\Phi_{1,k,l}$, $\Phi_{2,k,l,\lambda-s}$, and $\Phi_{3,k,l,\lambda-s}$ be as above and let $|k-k'| \geq 100$.*
(1) If supp $\widehat{\Phi_{1,k,l}} \subsetneq$ supp $\widehat{\Phi_{1,k',l'}}$, then for $j \in \{2,3\}$ we have

$$\text{supremum}\big(\textit{supp } \widehat{\Phi_{j,k,l,\lambda-s}}\big) < \text{infimum}\big(\textit{supp } \widehat{\Phi_{j,k',l',\lambda-s'}}\big)$$

and

$$\frac{1}{16} \cdot 2^{-k'+\frac{\mu}{2}} < \textit{dist}\big(\textit{supp } \widehat{\Phi_{j,k,l,\lambda-s}}, \textit{supp } \widehat{\Phi_{j,k',l',\lambda-s'}}\big) < 5 \cdot 2^{-k'+\frac{\mu}{2}} . \tag{5.7}$$

(2) If supp $\widehat{\Phi_{j,k,l,\lambda-s}} \subsetneq$ supp $\widehat{\Phi_{j,k',l',\lambda-s'}}$ for $j \in \{2,3\}$, then

$$\text{supremum}\big(\textit{supp } \widehat{\Phi_{1,k',l'}}\big) < \text{infimum}\big(\textit{supp } \widehat{\Phi_{1,k,l}}\big)$$

and

$$\frac{5}{64} \cdot 2^{-k'} < \textit{dist}\big(\textit{supp } \widehat{\Phi_{1,k,l}}, \textit{supp } \widehat{\Phi_{1,k',l'}}\big) < 4 \cdot 2^{-k'} . \tag{5.8}$$

The proof of Lemma 6 can be obtained by a sequence of algebraic manipulations and is omitted in this exposition.

6. Concluding remarks

The range of indices we studied in Theorem 1 dealt with the spaces L^{p_1}, L^{p_2}, and $L^{p'}$ in which all functions are locally in L^2 since $p_1, p_2, p' \geq 2$. In the linear case, if $p, p' \geq 2$, we must have $p = 2$, which is the only exponent for which the disc multiplier is bounded. So, it is conceivable that there is an analogy with the linear case if the Theorem 1 is false for other exponents p_1, p_2, p. But it is still unknown to us whether Theorem 1 remains valid for other indices. This issue will be addressed in future work.

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