THE LATTICE BUMP MULTIPLIER PROBLEM

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ABSTRACT. We study the lattice bump multiplier problem. Precisely, given a smooth bump supported in a ball centered at the origin, we consider the multiplier formed by adding the translations of this bump centered at N distinct lattice points. We investigate the dependence on N of the L^p norm of the linear and bilinear operators associated with this multiplier. We obtain sharp dependence on N in the linear case and in the bilinear case when $p > 1$.

1. INTRODUCTION

The theory of bilinear multipliers was recently enriched by a surge of interesting activity; see for instance [16], [7], [13], [11], [3], [4], [12]. The optimal smoothness required of the symbol to have boundedness on a given L^p is closely related to questions about the boundedness of a multiplier given by finite sum of translations of a given bump. In this paper we promote this point of view and we study multiplier operators associated with Fourier multipliers of this type. We focus on the bilinear case, although we briefly discuss the L^p behavior of linear multipliers of this sort.

We fix a smooth bump ϕ supported in the ball $|\xi| \leq \frac{1}{10}$ in $\mathbb{R}^n$. Consider the linear operator defined for $k \in \mathbb{Z}^n$

$$S_{k,\phi}(f)(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)\phi(\xi - k)e^{2\pi i x \cdot \xi} \, d\xi,$$

where $\widehat{f}(x) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x \cdot \xi} dx$ is the Fourier transform of a Schwartz function f.

Suppose we are given a finite subset E of $\mathbb{Z}^n$. Associated with E and ϕ we define a linear operator acting on Schwartz functions

$$L_{E,a,\phi} := \sum_{k \in E} a_k S_{k,\phi}(f),$$

where $a = \{a_k\}_{k \in \mathbb{Z}}$ is a sequence of complex numbers satisfying $|a_k| \leq 1$ for all k. We pose the following problem about $L_{E,a,\phi}$.

Problem 1: Given $p \in [1, \infty]$ what is the smallest value $\alpha(p)$ such that for all subsets E of $\mathbb{Z}^n$ with $|E| = N$ we have

$$\|L_{E,a,\phi}\|_{L^p \to L^p} \leq C_{p,n,\phi} \, N^{\alpha(p)} \, ?$$

The trivial L^2 and L^∞ estimates yield by interpolation that

$$(1) \qquad \alpha(p) \leq 2 \left| \frac{1}{p} - \frac{1}{2} \right|.$$

But estimate (1) is not sharp as it can be improved by a factor of $1/2$. Results similar to the following are known in different formulations in the literature, e.g. [10, inequality (7)], but for the sake of completeness we include a proof for it in the next section.

Proposition 1.1. *For any $p \in [1, \infty]$ and $\alpha(p) = |\frac{1}{p} - \frac{1}{2}|$, we have*

$$(2) \qquad \|L_{E,a,\phi}\|_{L^p \to L^p} \leq C N^{\alpha(p)}.$$

Conversely, we have

$$(3) \qquad \sup_{a: \|a\|_{\ell^\infty} \leq 1} \sup_{E: |E| = N} \|L_{E,a,\phi}\|_{L^p \to L^p} \geq C N^{\alpha(p)}.$$

Next, we consider the analogous, but more difficult, bilinear problem. We fix a smooth bump Φ supported in the ball $|\xi| \leq \frac{1}{20}$ in $\mathbb{R}^{2n}$. For a subset E of $\mathbb{Z}^{2n}$ we consider the following bilinear operator

$$B_{E,\Phi}(f, g)(x) := \sum_{(k,l) \in E} S_{(k,l),\Phi}(f \otimes g)(x, x).$$

Problem 2: Given p_1, p_2 with $1 \leq p_1, p_2 \leq \infty$, what is the smallest value $\alpha(p_1, p_2)$ such that for all subsets E of $\mathbb{R}^n$ with $|E| = N$ we have

$$\|B_{E,\Phi}\|_{L^{p_1} \times L^{p_2} \to L^p} \leq C_{p_1,p_2,n,\Phi} \, N^{\alpha(p_1,p_2)} \, ?$$

We focus on the situation where p, p_1, p_2 satisfy $1/p = 1/p_1 + 1/p_2$.

It is very natural to study multipliers appearing as sums of bumps supported in lattices, as many decompositions in analysis lead to such objects, e.g., wavelets, frames, φ transform, etc. Our present study is motivated by the solution of boundedness of rough bilinear singular integrals in the largest possible open set of exponents, obtained in [6]. This solution relies on standard techniques of harmonic analysis but also uses a new treatment of columns of coefficients. A discretization and adaptation of this idea builds the foundation of the proof of our following main result that addresses Problem 2.

Theorem 1.2. *Let $1 \leq p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$.*
(i) If $p \leq 1$, then there is a constant $C = C_{p_1, p_2}$ such that

$$\|B_{E,\Phi}\|_{L^{p_1} \times L^{p_2} \to L^p} \leq C N^{\frac{1}{\min(p_1, p_2)} + \frac{1}{2\max(p_1, p_2)} - \frac{1}{2}}.$$

(ii) If $p > 1$, then there is a constant $C = C_{p_1, p_2}$ such that

$$\|B_{E,\Phi}\|_{L^{p_1} \times L^{p_2} \to L^p} \leq C N^{\alpha(p_1, p_2)},$$

where

$$(4) \quad \alpha(p_1, p_2) = \frac{1}{2}\left[\max\left(\frac{1}{p} - \frac{1}{2}, 0\right) - \min\left(\frac{1}{p_1} - \frac{1}{2}, 0\right) - \min\left(\frac{1}{p_2} - \frac{1}{2}, 0\right)\right].$$

Moreover the power of N cannot be reduced in estimate (ii).

Remark 1.1. *Let $B_{E,a,\Phi}(f,g)(x) := \sum_{(k,l) \in E} a_{k,l} S_{(k,l),\Phi}(f \otimes g)(x,x)$ with $|a_{k,l}| \leq 1$. One see easily that all claims in Theorem 1.2 are valid for $B_{E,a,\Phi}$ as well.*

In the local L^2 case, i.e., the case where $2 \leq p_1, p_2, p' \leq \infty$, Theorem 1.2 (ii) yields the constant $\alpha(p_1, p_2) = 1/4$. Also when $p \geq 2$ we have $\alpha(p_1, p_2) = 1/2p'$. Here $p' = p/(p-1)$.

We also have results concerning upper and lower bounds for the constant $\alpha(p_1, p_2)$ for indices outside the local L^2 case. These are discussed in Sections 5 and 6.

Throughout this paper, C will denote a constant independent of N and dependent only on auxiliary parameters which may vary in different occurrences.

2. The linear case: The proof of Proposition 1.1

We begin with the positive direction of Proposition 1.1, namely (2).

Proof of (2). We observe that the multiplier of $L_{E,a,\phi}$ is

$$\sigma(\xi) = \sum_{k \in E} a_k \phi(\xi - k),$$

whose Fourier transform $\sigma^\vee(x) = \phi^\vee(x) \sum_{k \in E} a_k e^{2\pi i x \cdot k}$ satisfies

$$\left\|\sigma^\vee\right\|_{L^1} \leq \left\|\sum_{k \in E} a_k e^{\pi i x \cdot k}\right\|_{L^1([0,1]^n)}$$

due to the rapid decay of $\phi^\vee$ and the periodicity of $\sum_{k \in E} a_k e^{\pi i x \cdot k}$. This in turn is bounded by

$$(5) \quad \left\|\sum_{k \in E} a_k e^{\pi i x \cdot k}\right\|_{L^2([0,1]^n)} = \left(\sum_k |a_k|^2\right)^{1/2} \leq N^{1/2},$$

having used that $|a_k| \leq 1$. This implies $\|L_{E,a,\phi}\|_{L^1 \to L^1} \leq CN^{1/2}$. Interpolating between the trivial L^2 estimate $\|L_{E,a,\phi}\|_{L^2 \to L^2} \leq C$ and (5) we obtain (2). $\qquad \square$

Remark 2.1. *We can prove a continuous version of Proposition 1.1. Define $\phi_E(\xi) = \phi * \chi_E$, where the Lebesgue measure of E is N. Then it is easy to verify that*

$$\|\phi_E^\vee\|_{L^1} \leq \|\phi\|_{L^2}\|\chi_E\|_{L^2} \leq C|E|^{1/2},$$

which implies that the associated operator given by convolution with ϕ_E is bounded on $L^p(\mathbf{R}^n)$ with bound $CN^{\alpha(p)}$.

We now turn to the proof of (3). We fix a smooth bump ϕ supported in the ball of radius $1/10$. We will need the following lemma.

Lemma 2.1. *Let $E = E_N = \{-N, -N+1, \ldots, N-1, N\}$. For any fixed $p \in [1,2]$, there exists a constant $C_p > 0$ and a sequence $a = \{a_k\}_{k \in E}$ such that for all positive integers N we have*

$$(6) \qquad \|L_{E,a,\phi}\|_{L^p \to L^p} \geq C_p N^{\alpha(p)}.$$

Proof. For simplicity we first consider the one-dimensional case.

We first take $p \in (1, 2]$. The following counterexample is inspired by an example in [8]. Let

$$m(\xi) = \sum_{|k| \leq N} a_k(t)\phi(\xi - k), \quad m_N(\xi) = \sum_{|k| \leq N} a_k(t)\phi(N\xi - k),$$

where a_j are the Rademacher functions. Take a smooth function φ supported in the support of ϕ such that $\varphi\phi \neq 0$ and define f, f_N via

$$\widehat{f}(\xi) = N^{-\frac{1}{p'}} \sum_{|k| \leq N} \varphi(\xi - k), \qquad \widehat{f_N}(\xi) = \sum_{|k| \leq N} \varphi(N\xi - k).$$

Then $\|f\|_{L^p} = \|f_N\|_{L^p} \leq C$ as the Dirichlet kernel $D_N(x) = \sum_{|k| \leq N} e^{2\pi ixk}$ has L^p norm comparable to $N^{1/p'}$. Let T_m be the linear operator associated with m in the form $T_m(f) = (\widehat{f}m)^\vee$, then $\|T_m(f)\|_{L^p} =$

$\|T_{m_N}(f_N)\|_{L^p}$, and by applying Khintchine's inequality we have

$$\int_0^1 \|T_{m_N}(f_N)\|_{L^p(\mathbb{R})}^p dt = \int_0^1 \int_{\mathbb{R}} \left| \sum_{|k| \leq N} a_k(t) N^{-1}(\phi\varphi)^{\vee}(N^{-1}x) e^{2\pi i x \frac{k}{N}} \right|^p dx dt$$

$$\sim \int_{\mathbb{R}} \left(\sum_{|k| \leq N} \left| N^{-1}(\phi\varphi)^{\vee}(N^{-1}x) e^{2\pi i x \frac{k}{N}} \right|^2 \right)^{\frac{p}{2}} dx$$

$$\sim N^{-p} \int_{\mathbb{R}} N^{\frac{p}{2}} |(\phi\varphi)^{\vee}(N^{-1}x)|^p dx$$

(7) $$\sim N^{(\frac{1}{p} - \frac{1}{2})p}.$$

Denote $\sup_a \|L_{E,a,\phi}\|_{L^p \to L^p}$ by $C_p(N)$, then

$$\int_0^1 \|T_m(f)\|_{L^p}^p dt \leq C_p(N)^p.$$

In summary $C_p(N) \geq C_p' N^{\alpha(p)}$. In particular, we can find a sequence a such that $\|L_{E,a,\phi}\|_{L^p \to L^p} \geq C_p N^{\alpha(p)}$ with $C_p < C_p'$ for $p \in (1, 2]$.

Notice that $\|f\|_1 \leq C \log N$ by the L^1-norm of D_N, therefore we can only show that $C_1(N) \geq C N^{1/2}(\log N)^{-1}$ by the same argument. On the other hand, interpolation can help us to remove the logarithmic term below.

We next consider the case when $p = 1$. Suppose that (6) fails for $p = 1$. This is equivalent to saying that for any $C > 0$ there exists a corresponding N_C such that

(8) $$\sup_a \|L_{E,a,\phi}\|_{L^1 \to L^1} \leq C N_C^{\alpha(1)} = C N_C^{\frac{1}{2}}.$$

Interpolating between $\sup_a \|L_{E,a,\phi}\|_{L^2 \to L^2} \leq C'$ and (8) we obtain that, when $p \in (1, 2)$, for any $C_p > 0$ there exists a number N (by choosing C in (8) small enough) such that

$$\sup_a \|L_{E,a,\phi}\|_{L^p \to L^p} \leq \frac{C_p}{2} N^{\alpha(p)};$$

this contradicts (6) for $p \in (1, 2)$. In other words (6) holds when $p = 1$.

We now consider the higher dimensional case. The idea is simply to consider products of the one-dimensional example. We briefly describe this example.

Taking

$$m(\xi) = \sum_{|k| \leq N} a_k(t) \phi(\xi_1 - k) \prod_{j=2}^n \phi(\xi_j - 1),$$

and defining

$$\widehat{f}(\xi) = N^{-1/p'} \sum_{|k| \le N} \varphi(\xi_1 - k) \prod_{j=2}^{n} \varphi(\xi_j - 1),$$

we have

$$T_m(f)(x) = N^{-1/p'} \Big(\sum_{|k| \le N} a_k(t)(\phi\varphi)^{\vee}(x_1)e^{2\pi i x_1 k} \Big) \prod_{j=2}^{n} (\phi\varphi)^{\vee}(x_j)e^{2\pi i x_j}.$$

Since the variables are separated, this is essentially the one-dimensional case, and running the same argument as before yields the same necessary condition.

Combining these results we obtain the proof of Proposition 1.1.   □

3. The bilinear problem

In this section we begin the study of the bilinear problem lattice bump in $2n$ dimensions. We apply the Fourier series method of Coifman and Meyer [1, 2] to express the smooth bump Φ as a sum of products of bumps in each half of the variables. As the function Φ is supported in the ball $B(0, 1/20)$, which is contained in $[-1/2, 1/2]^{2n}$ we can express it in Fourier series as

$$\Phi(\xi, \eta) = \sum_{r,s \in \mathbf{Z}^n} c_{r,s} e^{2\pi i r \cdot \xi} e^{2\pi i s \cdot \eta} \phi(\xi)\phi(\eta),$$

where $\phi(\xi)$ is smooth, is equal to 1 on $|\xi| \le 1/20$, and vanishes outside $|\xi| \le 1/10$. Moreover,

$$c_{r,s} = \int_{B(0,1/20)} \Phi(x, y)e^{-2\pi i(x \cdot r + y \cdot s)} dx dy$$

and an easy integration by parts shows that

$$|c_{r,s}| \le C_M(1 + |r| + |s|)^{-M}$$

for every $M > 0$, where C_M depends on the L^∞ norms of sufficiently many derivatives of Φ. Letting $\phi_r(\xi) = e^{2\pi i r \cdot \xi}\phi(\xi)$, we have that

$$S_{(k,l),\Phi}(f \otimes g)(x, x) = \sum_{r,s \in \mathbf{Z}^n} c_{r,s} S_{k,\phi_r}(f)(x) S_{l,\phi_s}(g)(x)$$

and in view of the rapid decay of $c_{r,s}$, it will suffice to study an analogous problem for $S_{k,\phi_r}(f)(x) S_{l,\phi_s}(g)(x)$ in place of $S_{(k,l)}(f \otimes g)(x, x)$ and obtain estimates for the norm that are independent of r and s.

We make the remark that the same approach can handle the two adjoints of B_E. Let us look at the first adjoint of $S_{(k,l)}(f \otimes g)$. This is associated with the multiplier

$$\Phi(-\xi - \eta - k, \eta - l) = \Phi(-(\xi + k + l) - (\eta - l), \eta - l)$$
$$= \Phi^{*1}(\xi - (-l - k), \eta - l),$$

where $\Phi^{*1}(\xi, \eta) = \Phi(-\xi - \eta, \eta)$. Now notice that as (k, l) varies over E, then $(-k - l, l)$ varies over

$$E^{1*} = \{(-k - l, l) : (k, l) \in E\}$$

and $|E^{1*}| = |E|$, while the bump Φ^{*1} is smooth, has L^∞ norm 1 and is supported in $\{(\xi, \eta) : |\xi + \eta|^2 + |\eta|^2 \le \frac{1}{400}\}$ which is contained in $\{(\xi, \eta) : |(\xi, \eta)| \le \frac{\sqrt{3}}{20}\}$, which is only slightly larger than $B(0, 1/20)$. Thus any theorem about $B_{E,\Phi}$ can also be applied to the first adjoint $B_{E,\Phi}^{*1} = B_{E^{*1}, \Phi^{*1}}$ of $B_{E,\Phi}$, which has the same characteristics as $B_{E,\Phi}$. This symmetry is one main advantage of $B_{E,\Phi}(f, g)(x)$ compared with $S_{k,\phi_r}(f)(x) S_{l,\phi_s}(g)(x)$.

4. THE CASE $p_1 = p_2 = 2$

In this section we prove the sufficiency part of Theorem 1.2. By duality and interpolation it will suffice to consider only the case $p_1 = p_2 = 2$ and $p = 1$. The trivial estimate is

$$\|B_{E,\Phi}\|_{L^2 \times L^2 \to L^1} \le C N,$$

but it turns out that the optimal value of the constant $\alpha(2, 2) = 1/4$. The consideration here is related to the proof of [9, Theorem 1.3], which enhances the combinatorial argument in [6]

Proof. We denote by E' the set of all $k \in \mathbb{Z}^n$ with the property that there exists an $l \in \mathbb{Z}^n$ such that the point $(k, l) \in E$. That is E' is the set of all first coordinates of elements of E. We think of the set E as a union of columns Col_k indexed by $k \in E'$ and we write

$$E = \bigcup_{k \in E'} Col_k.$$

By the argument in Section 3 it suffices to consider the case when $B_{E,\Phi}(f, g)$ is a sum of products of operators of the form

$$T_{\sigma_N}(f, g) := \sum_{k \in E'} S_k(f) \sum_{l : (k,l) \in Col_k} S_l(g),$$

where $\sigma_N := \sum_{(k,l) \in E} \phi_r(\xi - k) \phi_s(\eta - l)$, and we have dropped the dependence on ϕ_r and ϕ_s for notational convenience.

We split the columns in large and small. Precisely, we write

$$E = E_1 \cup E_2,$$

where E_1 contains all columns of size $\geq K$ and E_2 contains all columns of size $< K$, for some K to be chosen later. Analogously we split

$$E' = E_1' \cup E_2',$$

where E_1' and E_2' is the set of all first coordinates of columns in E_1 and E_2, respectively. Correspondingly we define:

$$T_{\sigma_N}^1(f,g) = \sum_{k \in E_1'} S_k(f) \sum_{l:\,(k,l) \in Col_k} S_l(g)$$

and

$$T_{\sigma_N}^2(f,g) = \sum_{k \in E_2'} S_k(f) \sum_{l:\,(k,l) \in Col_k} S_l(g)$$

$$= \sum_{l:\,\exists k\,(k,l) \in E_2} S_l(g) \sum_{k:\,(k,l) \in E_2} S_k(f)$$

so that

$$T_{\sigma_N}(f,g) = T_{\sigma_N}^1(f,g) + T_{\sigma_N}^2(f,g).$$

We start with $T_{\sigma_N}^1$. We have

$$\left\| T_{\sigma_N}^1(f,g) \right\|_{L^1} \leq \sum_{k \in E_1'} \left\| S_k(f) \sum_{l:\,(k,l) \in Col_k} S_l(g) \right\|_{L^1}$$

$$\leq \sum_{k \in E_1'} \left\| S_k(f) \right\|_{L^2} \left\| \sum_{l:\,(k,l) \in Col_k} S_l(g) \right\|_{L^2}$$

$$\leq \left(\sum_{k \in E_1'} \left\| S_k(f) \right\|_{L^2}^2 \right)^{\frac{1}{2}} \left(\sum_{k \in E_1'} \left\| \sum_{l:\,(k,l) \in Col_k} S_l(g) \right\|_{L^2}^2 \right)^{\frac{1}{2}}$$

$$\leq \|\phi\|_{L^\infty} \|f\|_{L^2} (\#E_1')^{\frac{1}{2}} \|\phi\|_{L^\infty} \|g\|_{L^2},$$

exploiting the orthogonality of S_k's on L^2.

Notice that as there are N points in E and each column in E_1' has least K elements, this means that there are at most N/K columns in E_1'. We conclude that

(9) $$\left\| T_{\sigma_N}^1(f,g) \right\|_{L^1} \leq (N/K)^{\frac{1}{2}} \|\phi\|_{L^\infty}^2 \|f\|_{L^2} \|g\|_{L^2}.$$

We continue with $T_{\sigma_N}^2$. We have

$$\left\| T_{\sigma_N}^2(f,g) \right\|_{L^1}$$

$$= \left\| \sum_{l:\,\exists k\,(k,l) \in E_2} S_l(g) \sum_{k:\,(k,l) \in E_2} S_k(f) \right\|_{L^1}$$

$$\leq \sum_{l:\, \exists k\, (k,l)\in E_2} \left\| S_l(g) \sum_{k:\, (k,l)\in E_2} S_k(f) \right\|_{L^1}$$

$$\leq \sum_{l:\, \exists k\, (k,l)\in E_2} \left\| S_l(g) \right\|_{L^2} \left\| \sum_{k:\, (k,l)\in E_2} S_k(f) \right\|_{L^2}$$

$$\leq \left[\sum_{l:\, \exists k\, (k,l)\in E_2} \left\| S_l(g) \right\|_{L^2}^2 \right]^{\frac{1}{2}} \left[\sum_{l:\, \exists k\, (k,l)\in E_2} \left\| \sum_{k:\, (k,l)\in E_2} S_k(f) \right\|_{L^2}^2 \right]^{\frac{1}{2}}$$

$$\leq \|\phi\|_{L^\infty} \|g\|_{L^2} \left[\sum_{l:\, \exists k\, (k,l)\in E_2} \sum_{k:\, (k,l)\in E_2} \left\| S_k(f) \right\|_{L^2}^2 \right]^{\frac{1}{2}}$$

$$= \|\phi\|_{L^\infty} \|g\|_{L^2} \left[\sum_{k\in E_2'} \sum_{l:\, (k,l)\in Col_k} \left\| S_k(f) \right\|_{L^2}^2 \right]^{\frac{1}{2}}$$

$$\leq \|\phi\|_{L^\infty} \|g\|_{L^2} K^{\frac{1}{2}} \left[\sum_{k\in E_2'} \left\| S_k(f) \right\|_{L^2}^2 \right]^{\frac{1}{2}}$$

$$\leq \|\phi\|_{L^\infty} \|g\|_{L^2} K^{\frac{1}{2}} \|\phi\|_{L^\infty} \|f\|_{L^2}.$$

This yields

$$(10) \qquad \left\| T_{\sigma_N}^2(f,g) \right\|_{L^1} \leq K^{\frac{1}{2}} \|\phi\|_{L^\infty}^2 \|f\|_{L^2} \|g\|_{L^2}.$$

In view of (9) and (10), the optimal choice of $K = N^{1/2}$. This proves

$$(11) \qquad \left\| T_{\sigma_N}(f,g) \right\|_{L^1} \leq N^{\frac{1}{4}} \|\phi\|_{L^\infty}^2 \|f\|_{L^2} \|g\|_{L^2}.$$

We have now proved the sufficiency direction in Theorem 1.2. $\qquad\square$

5. Bilinear case: Sufficiency

Recalling (4) we notice that $\alpha(1,1) = 3/4$ and $\alpha(1,2) = \alpha(2,1) = 1/2$. We begin with the following result which is nontrivial when $p < 1$.

Proposition 5.1. *If $E \subset \mathbb{Z}^{2n}$ has cardinality N, then*

$$\|B_{E,\Phi}\|_{L^{p_1} \times L^{p_2} \to L^p} \leq CN$$

for $1 \leq p_1, p_2 \leq \infty$ with $1/p = 1/p_1 + 1/p_2$.

Proof. Recall that

$$B_{E,\Phi}(f,g)(x) = \iint \widehat{f}(\xi)\widehat{g}(\eta) \sum_{(k,l)\in E} \Phi((\xi,\eta)-(k,l)) e^{2\pi i x \cdot (\xi+\eta)} d\xi d\eta$$

$$= \iint K(y,z) f(x-y) g(x-z)\, dy dz,$$

where $K(y,z) = \Phi^\vee(y,z) \sum_{(k,l) \in E} e^{2\pi i (y,z) \cdot (k,l)}$.

Setting $\psi(y) = (1 + |y|)^{-2n}$, as $|\Phi^\vee(y,z)| \leq C\psi(y)\psi(z)$ we have $|K(y,z)| \leq CN\psi(y)\psi(z)$, which implies that

$$|B_{E,\Phi}(f,g)(x)| \leq CN(|f| * \psi)(x)\,(|g| * \psi)(x).$$

As a result we obtain

$$\|B_{E,\Phi}(f,g)\|_{L^p} \leq CN\||f| * \psi\|_{L^{p_1}}\||g| * \psi\|_{L^{p_2}} \leq CN\|f\|_{L^{p_1}}\|g\|_{L^{p_2}},$$

hence the conclusion follows. $\qquad\square$

Remark 5.1. *This result is sharp for* $(p_1, p_2) = (1, 1)$ *by Proposition 6.1, discussed in the next section.*

Corollary 5.2. *Fix* $\frac{1}{2} < p < 1$. *There is a constant* C *such that*

$$\|B_{E,\Phi}\|_{L^{2p} \times L^{2p} \to L^p} \leq CN^{\frac{3}{4p} - \frac{1}{2}}.$$

Proof. Interpolating using [5, Theorem 7.2.9] between the estimate at the point $(1,1,1/2)$ [Proposition 5.1] and at the point $(2,2,1)$ obtained in Section 4, we deduce the conclusion. $\qquad\square$

For some endpoints, we can also reduce one half of the exponent of the estimate in Proposition 5.1 as in the linear case.

Lemma 5.3. *There exists a constant* $C > 0$ *such that*

$$\|B_{E,\Phi}\|_{L^\infty \times L^\infty \to L^\infty} \leq C\,N^{\frac{1}{2}}.$$

Proof. Obviously $\|B_{E,\Phi}(f,g)\|_\infty$ is bounded by $\|K\|_{L^1(\mathbb{R}^{2n})}\|f\|_\infty\|g\|_\infty$, where $K(y,z) = \Phi^\vee(y,z) \sum_{(k,l) \in E} e^{2\pi i(y,z) \cdot (k,l)}$. We argue as in the proof of (2) in the linear case to show that $\|K\|_{L^1(\mathbb{R}^{2n})}$ is bounded by $N^{1/2}$, which concludes the proof. $\qquad\square$

For the most general case $1 < p_1, p_2 < \infty$, we obtain the following nontrivial estimate from Lemma 5.3 via duality and multilinear interpolation.

Proposition 5.4. *(i) If* $p < 1$, *then there is a constant* $C = C_{p_1,p_2}$ *such that*

$$\|B_{E,\Phi}\|_{L^{p_1} \times L^{p_2} \to L^p} \leq CN^{\frac{1}{\min(p_1,p_2)} + \frac{1}{2\max(p_1,p_2)} - \frac{1}{2}}.$$

(ii) Fix $i \in \{1,2\}$. *If* $1 < p_i < 2$, *and* $1 < p < 2$, *then there is a constant* $C = C_{p_1,p_2}$ *such that*

$$\|B_{E,\Phi}\|_{L^{p_1} \times L^{p_2} \to L^p} \leq CN^{\frac{1}{2p_i}}.$$

(iii) If $p > 2$, then there is a constant $C = C_p$ such that

$$\|B_{E,\Phi}\|_{L^{p_1} \times L^{p_2} \to L^p} \leq CN^{\frac{1}{2} - \frac{1}{2p}}.$$

Proof of Proposition 5.4. It follows from Lemma 5.3 and duality that we have the rate of growth $N^{1/2}$ when (p_1, p_2, p) is one of (∞, ∞, ∞), $(1, \infty, 1)$, and $(\infty, 1, 1)$.

Estimate (i). It suffices to consider the case $1 < p_1 < p_2$, and $p < 1$, when the desired estimate is

$$\|B_{E,\Phi}\|_{L^{p_1} \times L^{p_2} \to L^p} \leq CN^{\frac{1}{p_1} + \frac{1}{2p_2} - \frac{1}{2}}.$$

It follows from interpolation between $(1, 1, \frac{1}{2})$, $(2, 2, 1)$, and $(1, \infty, 1)$.

Estimate (ii) follows by interpolating between $(2, 2, 1)$, $(2, \infty, 2)$, and $(1, \infty, 1)$ when $i = 1$. The case $i = 2$ follows by symmetry.

Estimate (iii) follows from interpolation between $(\infty, 2, 2)$, $(2, \infty, 2)$, and (∞, ∞, ∞) . □

6. BILINEAR CASE: NECESSITY

Our main result in this section, stated below, includes the necessity direction in Theorem 1.2.

Proposition 6.1. *Fix a smooth bump Φ supported in the ball $|\xi| \leq \frac{1}{20}$ in $\mathbb{R}^{2n}$. Then for all p_1, p_2 with $1 \leq p_1, p_2 < \infty$ satisfying $1/p_1 + 1/p_2 = 1/p$ we have*

$$\sup_E \|B_{E,\Phi}\|_{L^{p_1} \times L^{p_2} \to L^p} \geq CN^{\max(\alpha(p_1, p_2), \frac{1}{p} - 1)}.$$

In particular, $\|B_{E,\Phi}\|_{L^1 \times L^1 \to L^{1/2}} \geq CN$, and estimate (ii) in Theorem 1.2 is sharp when $p \geq 1$.

We recall that $B_{E,\Phi}(f, g)(x) := \sum_{(k,l) \in E} S_{(k,l),\Phi}(f \otimes g)(x, x)$. Via an argument similar to that used in the proof of Lemma 2.1, it suffices to consider the case $n = 1$, which we discuss below.

Let

$$\alpha'(p_1, p_2) = \frac{1}{2} \left[\frac{1}{p} - \frac{1}{2} - \min\left(\frac{1}{p_1} - \frac{1}{2}, 0 \right) - \min\left(\frac{1}{p_2} - \frac{1}{2}, 0 \right) \right].$$

Note that $\alpha'(p_1, p_2) = \alpha(p_1, p_2)$ when $p \leq 2$. We need two lemmas to prove Proposition 6.1.

Lemma 6.2. *For all $1 \leq p_1, p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$ we have*

$$\text{(12)} \qquad \sup_E \|B_{E,\Phi}\|_{L^{p_1} \times L^{p_2} \to L^p} \geq CN^{\alpha'(p_1, p_2)}.$$

Proof. It suffices to prove the conclusion for $\sum_{(k,l)\in E} a_{k,l} S_k(f) S_l(g)$ with $a_{k,l} \in \{1, -1\}$. Actually if we verify that

$$\left\| \sum_{(k,l)\in E} a_{k,l} S_k(f) S_l(g) \right\|_{L^{p_1} \times L^{p_2} \to L^p} \geq 3CN^{\alpha'(p_1,p_2)}$$

with $a_{k,l} \in \{-1, 1\}$, then we must have (12); otherwise we obtain that

$$\left\| \sum_{(k,l)\in E} a_{k,l} S_k(f) S_l(g) \right\|_{L^{p_1} \times L^{p_2} \to L^p} \leq 2CN^{\alpha'(p_1,p_2)}$$

since we can write

$$\sum_{(k,l)\in E} a_{k,l} S_k(f) S_l(g) = B_{E_1,\Phi}(f,g) - B_{E_2,\Phi}(f,g)$$

for appropriate sets E_1 and E_2.

Inspired by the examples in [7] for $n = 1$, we define[1]

$$m(\xi, \eta) = \sum_{k=1}^{\sqrt{N}} \sum_{l=1}^{\sqrt{N}} a_k(t_1) a_l(t_2) a_{k+l}(t_3) c_{k+l} \phi(\xi - k) \phi(\eta - l),$$

where $a_k(t)$ are Rademacher functions, and $c_l = 1$ when $9\sqrt{N}/10 \leq l \leq 11\sqrt{N}/10$ and 0 elsewhere. We also define

$$\widehat{f_N}(\xi) = N^{-\frac{1}{2p_1'}} \sum_{k=1}^{\sqrt{N}} a_k(t_1) \widehat{\varphi}(\xi - k), \quad \widehat{g_N}(\eta) = N^{-\frac{1}{2p_2'}} \sum_{l=1}^{\sqrt{N}} a_l(t_2) \widehat{\varphi}(\eta - l).$$

By a calculation analogous to that in (7) we obtain

$$\left(\int_0^1 \|T_m(f_N, g_N)\|_{L^p}^p dt_3 \right)^{\frac{1}{p}} \sim N^{\frac{1}{2}(\frac{1}{p} - \frac{1}{2})}.$$

On the other hand

$$\left(\int_0^1 \|f_N\|_{L^{p_1}}^{p_1} dt_1 \right)^{\frac{1}{p_1}} \sim N^{\frac{1}{2}(\frac{1}{p_1} - \frac{1}{2})}.$$

Let $C_0(N) = \sup_E \|B_{E,\Phi}\|_{L^{p_1} \times L^{p_2} \to L^p}$, where the supremum is taken over all E with $|E| = N$, then

$$N^{\frac{1}{2}(\frac{1}{p} - \frac{1}{2})} \sim \left\| \|T_m(f_N, g_N)\|_{L^p} \right\|_{L^p(dt_3)} \leq C_0(N) \|f_N\|_{L^{p_1}} \|g_N\|_{L^{p_2}}.$$

Taking $L^{p_1}(dt_1)$ and $L^{p_2}(dt_2)$ norms on both sides, we obtain that

$$C_0(N) \geq C \frac{N^{\frac{1}{2}(\frac{1}{p} - \frac{1}{2})}}{N^{\frac{1}{2}(\frac{1}{p_1} - \frac{1}{2}) + \frac{1}{2}(\frac{1}{p_2} - \frac{1}{2})}} = CN^{\frac{1}{4}},$$

---

[1]$\sum_{k=1}^{\sqrt{N}} a_k$ means $\sum_{k=1}^{[\sqrt{N}]} a_k$, where $[\sqrt{N}]$ is the integer part of $\sqrt{N}$.

using the estimates for f_N and g_N. This estimate works for all choices of indices p_1, p_2, p with $1/p_1 + 1/p_2 = 1/p$ but it is sharp only in the local L^2 case, i.e. in the case where $2 \leq p_1, p_2, p' \leq \infty$.

Now if all the coefficients $a_k(t)$ are equal to 1 in the definition of f_N, then $\|f_N\|_{p_1} \leq C$ for $p_1 \in (1,2]$, which is smaller than $N^{\frac{1}{2}(\frac{1}{p_1}-\frac{1}{2})}$ if $p_1 < 2$. So in the case $p_1 \leq 2 \leq p_2$, we modify the multiplier m by

$$m(\xi,\eta) = \sum_{k=1}^{\sqrt{N}} \sum_{l=1}^{\sqrt{N}} a_l(t_2)a_{k+l}(t_3)c_{k+l}\phi(\xi-k)\phi(\eta-l)$$

correspondingly, which gives then

$$C_0(N) \geq CN^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2}-\frac{1}{p_2}+\frac{1}{2})} = CN^{\frac{1}{2p_1}}.$$

By symmetry we have $C_0(N) \geq CN^{\frac{1}{2p_2}}$ when $p_2 \leq 2 \leq p_1$. In analogous way, when $1 \leq p_1, p_2 \leq 2$ we set $a_k(t_1) = a_l(t_2) = 1$ to obtain the lower bound $C_0(N) \geq CN^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})}$. Combining these estimates in one form, we obtain the lower bound $C N^{\alpha'(p_1,p_2)}$. $\qquad\square$

Lemma 6.3. *There exists a set $E \subset \mathbb{Z}^2$ with cardinality N such that*

$$\tag{13} \|B_{E,\Phi}\|_{L^{p_1} \times L^{p_2} \to L^p} \geq CN^{\frac{1}{p}-1}.$$

In particular $\|B_{E,\Phi}\|_{L^1 \times L^1 \to L^{1/2}} \geq CN$.

This estimate is stronger than (12) when $\frac{1}{2} \leq p < \frac{2}{3}$.

Proof. We consider the multiplier

$$m(\xi,\eta) := \sum_{j=-N}^{N} \phi(\xi-j)\phi(\eta+j),$$

whose inverse Fourier transform is

$$K(y,z) = \phi^{\vee}(y)\phi^{\vee}(z) \sum_{j=-N}^{N} e^{2\pi ij(y-z)}.$$

We remark that $\sum_{j=-N}^{N} e^{2\pi ijs}$ is real by symmetry. Moreover we have $|(2N+1) - \sum_{j=-N}^{N} e^{2\pi ijs}| \leq N$ for $s \leq 1/(50N)$. We now take

$$f(y) = g(y) = 100N\chi_{[0,(100N)^{-1}]}(y),$$

which satisfy $\|f\|_{L^1} = \|g\|_{L^1} = 1$. Then

$$T_m(f,g)(x) = \iint K(x-y,x-z)f(y)g(z)dydz$$

satisfies that

(14)
$$|T_m(f,g)(x)| \geq CN \qquad \text{for } |x| \leq (100)^{-1}$$

if we choose ϕ appropriately so that $|\phi^\vee(x)| \geq 1$ for $|x| \leq \frac{1}{50}$. This yields that $\|T_m(f,g)\|_{L^{1/2}} \geq CN$. In summary $\|T_m\|_{L^1 \times L^1 \to L^{1/2}} \geq CN$. From this example we also obtain that

$$\|T_m\|_{L^{p_1} \times L^{p_2} \to L^p} \geq C \frac{N}{N^{1/p_1'} N^{1/p_2'}} \geq CN^{\frac{1}{p}-1}.$$

This concludes the proof of the lemma. $\qquad\square$

We provide the following intuitive understanding of the proof of (13). As m is supported in a tube with dimensions $N \times 1$ along the antidiagonal, the kernel $K = m^\vee$ is essentially equal to the constant N in a tube of dimensions $1 \times N^{-1}$ along the diagonal, in view of the uncertainty principle. If $f \otimes g$ is supported in a square of length N^{-1} of height N^2, then $K * (f \otimes g)(x,x)$ is essentially $K * (f \otimes g)(0,0) \sim N^3 N^{-2} \sim N$ for $|x| \leq C$. This gives the claimed lower bound of $B_{E,\Phi}$.

Remark 6.1. *Suppose that $1 \leq p_1, p_2 \leq 2$. We have $\alpha'(p_1,p_2) = \frac{1}{2p} - \frac{1}{4}$. Note that $\alpha'(p_1,p_2) \geq \frac{1}{p} - 1$ if and only if $p \geq \frac{2}{3}$. In other words, the example in Lemma 6.3 provides a larger lower bound for $\|B_{E,\Phi}\|_{L^{p_1} \times L^{p_2} \to L^p}$ when $p < \frac{2}{3}$.*

We now provide the proof of Proposition 6.1.

Proof of Proposition 6.1. It follows from (12) and the discussion in Section 3 that
$$\sup_E \|B_{E,\Phi}^{*1}\|_{L^{p_1} \times L^{p_2} \to L^p} \geq CN^{\alpha'(p_1,p_2)}.$$
More precisely if $1 < p_1 \leq 2$ and $1 \leq p \leq 2$, we have
$$\sup_E \|B_{E,\Phi}^{*1}\|_{L^{p_1} \times L^{p_2} \to L^p} \geq CN^{1/(2p_1)},$$
which implies by duality that
$$\sup_E \|B_{E,\Phi}\|_{L^{p'} \times L^{p_2} \to L^{p_1'}} \geq CN^{1/(2p_1)}.$$
We can rephrase this estimate as
$$\sup_E \|B_{E,\Phi}\|_{L^{p_1} \times L^{p_2} \to L^p} \geq CN^{1/(2p')},$$
which matches $N^{\alpha(p_1,p_2)}$ as the upper bound, and is greater than $\alpha'(p_1,p_2) = N^{1/4}$ when $p \geq 2$, which happens exactly when $\alpha(p_1,p_2) \geq \alpha'(p_1,p_2)$.

In summary, we obtain that
$$\sup_E \|B_{E,\Phi}\|_{L^{p_1} \times L^{p_2} \to L^p} \geq CN^{\alpha(p_1,p_2)},$$
which combined with (13) finishes the proof. $\qquad\square$

We finish this section by giving the formal proof of Theorem 1.2; this was essentially done in last three sections.

Proof of Theorem 1.2. We refer where we discussed the sufficient part first. The local L^2 case is proved in Section 4. The case when $p > 1$ but the local L^2 case is given by Proposition 5.4 (ii) and (iii). The case when $p \leq 1$ is Proposition 5.4 (i). The necessity is provided by Proposition 6.1. $\qquad\square$

Remark 6.2. *One notices that $\frac{3}{4p} - \frac{1}{2} > \frac{1}{p} - 1$ when $p > \frac{1}{2}$, and $\frac{3}{4p} - \frac{1}{2} > \frac{1}{2p} - \frac{1}{4}$ for $p < 1$, therefore, as of this writing, there is a gap between the positive result from $L^p \times L^p \to L^{p/2}$ for $p < 1$ in Corollary 5.2 and our two counterexamples.*

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