OPTIMAL WEIGHTED ESTIMATES FOR HÖRMANDER MULTIPLIERS

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Abstract. We characterize weights for which Hörmander multiplier operators are bounded on the associated weighted Lebesgue spaces. The weights we consider lie in intersections of Muckenhoupt and reverse Hölder classes. Our proofs are based on delicate interpolation between weighted estimates with weights whose powers lie in such intersections. The sharpness of our results is indicated by counterexamples and extrapolation. Our results rely on but also extend those of Beltran–Roos–Seeger [3] and Bernicot–Frey–Petermichl [5].

1. Introduction

For a Schwartz function f on $\mathbb{R}^n$ we denote by $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx$ its Fourier transform. The inverse Fourier transform of f is denoted by $f^\vee(\xi)$ and equals $\widehat{f}(-\xi)$. A Fourier multiplier is an operator of the form $f \mapsto (\sigma\widehat{f}\,)^\vee$, where σ is a bounded function on $\mathbb{R}^n$. A Fourier multiplier associated with a function σ is denoted by T_σ and is called an L^p Fourier multiplier if it admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself, for some $p \in (1,\infty)$.

A condition on bounded functions σ which imply that T_σ is an L^p Fourier multiplier for some $p \in (1,\infty)$ is the following one due to Mikhlin [28]:

$$|\partial^\alpha \sigma(\xi)| \le C_\alpha |\xi|^{-|\alpha|}, \qquad |\alpha| \le \Big[\frac{n}{2}\Big] + 1. \tag{1}$$

Inspired by the work of Hörmander [17], condition (1) has been relaxed in a few ways, for instance, if $rs > n$, the requirement

$$\sup_{R>0} \left(R^{r|\alpha|-n} \int_{R<|x|<2R} |\partial^\alpha \sigma(x)|^r \, dx \right)^{1/r} < +\infty \quad \text{for all } |\alpha| \le s, \tag{2}$$

implies that the associated operator T_σ is L^p bounded. A natural generalization of (2) is

$$\sup_{j\in\mathbb{Z}} \big\| (I-\Delta)^{\frac{s}{2}} (\widehat{\Psi}\, \sigma(2^j\cdot)) \big\|_{L^r} < \infty, \qquad \text{where } rs > n. \tag{3}$$

Here $(I-\Delta)^{\frac{s}{2}}$ is a fractional power of the Laplacian Δ and throughout this paper Ψ is Schwartz function whose Fourier transform is supported in annulus $\{\xi \in \mathbb{R}^n : \frac{6}{7} \le |\xi| < 2\}$, is equal to 1 on $\{\xi \in \mathbb{R}^n : 1 \le |\xi| < \frac{12}{7}\}$, which satisfies $\sum_{j\in\mathbb{Z}} \hat{\Psi}(2^{-j}\xi) = 1$ for all $\xi \neq 0$.

Calderón–Torchinsky [7] showed that condition (3) implies that T_σ is bounded on L^p in the range

$$\left|\frac{1}{p} - \frac{1}{2}\right| = \frac{1}{r} < \frac{s}{n}, \qquad 1 < p < \infty. \tag{4}$$

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The appearance of r in (4) is not necessary. In fact, if $1 < p < \infty$, $0 < s < n$, and (3) holds, then T_σ is bounded on L^p whenever

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{s}{n}\,; \tag{5}$$

on this see [14]. Endpoint estimates involving Besov spaces $B^s_{p,q}$ (see [33]) were obtained by Seeger [30], [31], who for instance, for the Hardy space H^1 showed that

$$\|T_\sigma f\|_{L^{1,r}} \le C \sup_{j\in\mathbb{Z}} \|\widehat{\Psi}\,\sigma(2^j\cdot)\|_{B^{n/2}_{2,1}} \|f\|_{H^1}. \qquad r \ge 2. \tag{6}$$

On the critical line, Grafakos–He–Honzík–Nguyen [14] proved that if $1 < p < 2$, $0 < s < n/2$, and $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$, then T_σ is bounded from L^p to the Lorentz space $L^{p,2}$ provided

$$\|\widehat{\Psi}\,\sigma(2^j\cdot)\|_{B^s_{n/s,1}} < \infty.$$

Grafakos and Slavíková [13] improved the space in (3) from L^r to the Lorentz space $L^{n/s,1}$, proving that under the Lorentz–Sobolev condition

$$\sup_{j\in\mathbb{Z}} \big\|(I-\Delta)^{s/2}(\widehat{\Psi}\,\sigma(2^j\cdot))\big\|_{L^{n/s,1}(\mathbb{R}^n)} < \infty, \tag{7}$$

the operator T_σ is bounded on L^p for all

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{s}{n}. \tag{8}$$

Moreover, a counterexample of Slavíková [32] indicates that the boundedness of T_σ from L^p to L^p fails on the critical line $|1/p - 1/2| = s/n$.

In contrast with the unweighted case, the understanding of the weighted theory for Fourier multipliers has been rather limited. Kurtz and Wheeden [23] proved that if $1 < p < \infty$, $1 < r < 2$, $rs > n$ and $ps > n$, and σ satisfies (2), then the operator T_σ is bounded on $L^p(w)$ for every weight $w \in A_{ps/n}$. Additionally, Kurtz and Wheeden identified the sharp range of power weights $|x|^\beta$ ensuring the boundedness of T_σ under (2). Later, Muckenhoupt–Wheeden–Young [27] obtained a complete characterization of Fourier multipliers on power-weighted spaces $L^2(|x|^{2a})$. Recently, Park and Tomita [29] investigated necessary conditions for the $L^p(w)$ boundedness of Fourier multipliers under (2) when $2s > n$ and $r = 2$. They proved that $A_{ps/n}$ is the optimal Muckenhoupt class in this setting. In the radial setting, Lee and Seeger [25] further showed that for quasi-radial multipliers, weighted L^2 boundedness with power weights is equivalent to a one-dimensional Sobolev condition on the multiplier.

Extrapolation results for operators ([2], [9], [10]) show that weighted boundedness of an operator T at a single exponent provides weighted boundedness for other admissible exponents. These results give that weighted boundedness for multiplier operators T_σ with $s < n/2$, under conditions (3) and (7) must necessarily involve weights contained in both A_p and RH classes.

The recent work of Beltran-Roos-Seeger [3] shows that T_σ satisfies a sparse bound whenever (3) holds. Combining their result with Proposition 6.4 of Bernicot-Frey-Petermichl [5], one obtains the weighted estimate

$$T_\sigma : L^p(w) \to L^p(w) \quad \text{for all } w \in A_{p(\frac{1}{r}+\frac{1}{2})} \cap RH_q,$$

where the A_p and RH_q classes are described in Definitions 1.1 and 1.2 and $q > 1$ satisfies

$$\frac{s}{n} - \frac{1}{r} > \frac{1}{2} - \frac{1}{p} + \frac{1}{pq}. \tag{9}$$

In this paper, we make advances in this direction providing necessary and sufficient conditions for weights w so that T_σ is bounded from $L^p(w)$ to $L^p(w)$ under condition (3).

We now provide some definitions. Throughout this article, all cubes will have sides parallel to the axes. Recall that a weight is a locally integrable, a.e. positive function.

Definition 1.1 (Muckenhoupt A_p weights)**.** Let $1 < p < \infty$. A weight w belongs to the (Muckenhoupt) class A_p if

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)\, dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-\frac{1}{p-1}}\, dx \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. For $p = 1$, we say $w \in A_1$ if

$$[w]_{A_1} := \sup_Q \frac{1}{|Q|} \int_Q w(x)\, dx \cdot (\operatorname*{ess\,inf}_Q w)^{-1} < \infty.$$

We note that $[w]_{A_p} \geq 1$ for all $p \geq 1$.

Definition 1.2 (Reverse Hölder RH_q weights)**.** Let $1 < q < \infty$. We say $w \in RH_q$ if

$$[w]_{RH_q} := \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)^q\, dx \right)^{1/q} \left(\frac{1}{|Q|} \int_Q w(x)\, dx \right)^{-1}, < \infty$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. For $q = \infty$, we say $w \in RH_\infty$ if

$$[w]_{RH_\infty} := \sup_Q \frac{\operatorname{ess\,sup}_Q w}{\frac{1}{|Q|} \int_Q w(x)\, dx} < \infty.$$

In Section 6 we prove an interpolation result (Theorem 6.1) that allows us to vary the smoothness of a multiplier operator $T_\sigma f = (\sigma \widehat{f}\,)^\vee$ on weighted spaces. As an application of Theorem 6.1 we obtain a characterization of weights in Muckenhoupt intersection RH classes for which estimates for Fourier multipliers satisfying Hörmander's condition (3) are valid.

Main results: The main contribution of this paper are Theorem 7.1 and Theorem 7.3 which state that if $1 < p < \infty$, $0 < \frac{s}{n} \leq \frac{1}{2}$, $|\frac{1}{2} - \frac{1}{p}| < \frac{s}{n}$, $rs > n$, then the weighted estimate

$$\|T_\sigma f\|_{L^p(w)} \lesssim \sup_{j \in \mathbb{Z}} \left\| (I - \Delta)^{\frac{s}{2}} [\hat{\Psi} \sigma(2^j \cdot)] \right\|_{L^r} \|f\|_{L^p(w)}$$

is valid for all σ satisfying (3) and all weights w in intersections of Muckenhoupt and RH classes if and only if

$$w \in \bigcup_{\alpha \in I} \left(A_{p(\frac{s}{n} + \alpha)} \cap RH_{(\frac{1}{p\alpha})'} \right) \tag{10}$$

where

$$I := \begin{cases} \left[\frac{1}{2} - \frac{s}{n}, \frac{1}{p}\right] & \text{when } 2 \leq p < \infty, \\ \left[\frac{1}{p} - \frac{s}{n}, \frac{1}{2}\right] & \text{when } 1 < p < 2. \end{cases}$$

We adopt the conventions that $(1)' = \infty$, $(\infty)' = 1$, and $\frac{1}{0} = \infty$ in the above formula. The optimality of our result is indicated by counterexamples discussed in Section 2. The range of pairs $(\frac{1}{p}, \alpha)$ permitted in the union in (10) is depicted in the following figure.

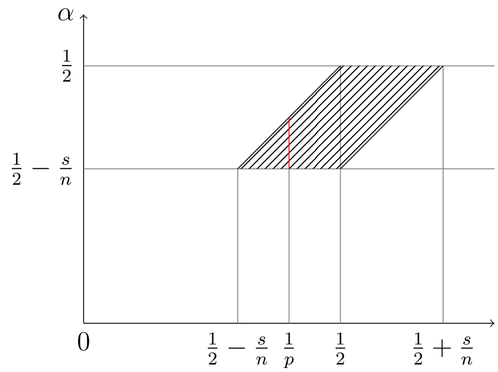

FIGURE 1. The vertical red line segment represents the corresponding range of α in the union in (10) for a fixed value of $\frac{1}{p}$.

Notation: Throughout this paper we use the symbol $A \lesssim B$ to denote that the quantity A is bounded by a constant multiple of B, where the constant only depends on inessential parameters. We use $A \sim B$ to indicate both $A \lesssim B$ and $B \lesssim A$. We also use the notation $C(a, b, \dots)$ to denote a constant depending on the indicated parameters. For an exponent $p \in (1, \infty)$ we denote by $p' = p/(p-1)$ its conjugate exponent. We also denote by χ_Q the characteristic function of a set Q. We use the standard L^p notation for Lebsegue spaces and $L^{p,q}$ for Lorentz spaces (for the latter, see [11, Definition 1.4.6]).

2. COUNTEREXAMPLE

In this section we discuss counterexamples that pose restrictions on the indices ℓ and q for weights w in $A_\ell \cap RH_q$ if the Fourier multiplier T_σ maps $L^p(w)$ to itself.

Lemma 2.1. *Let $s > 0$. Consider the multiplier*

$$\sigma(\xi) = e^{-2\pi i \xi_1}(1 + 4\pi^2|\xi|^2)^{-\frac{s}{2}} \tag{11}$$

defined on $\mathbb{R}^n$, where $\xi = (\xi_1, \dots, \xi_n)$. Then

$$\sup_{j \in \mathbb{Z}} \|(I - \Delta)^{\frac{s}{2}}[\widehat{\Psi}\sigma(2^j \cdot)]\|_{L^{\frac{n}{s},1}} < \infty. \tag{12}$$

Here $\widehat{\Psi}$ is a smooth function supported in an annulus centered at zero.

Proof. Case I: If s is an even integer, then we have

$$(I-\Delta)^{\frac{s}{2}}(\hat{\Psi}\sigma(2^j \cdot)) = \sum_{|\alpha| \le \frac{s}{2}} C_{\alpha,s}\partial^{2\alpha}(\hat{\Psi}\sigma(2^j\xi)) = \sum_{|\alpha| \le \frac{s}{2}} \sum_{\beta \le 2\alpha} C_{\alpha,s}\binom{2\alpha}{\beta}(\partial^{2\alpha-\beta}\hat{\Psi}(\xi))2^{j|\beta|}(\partial^\beta\sigma)(2^j\xi).$$

Moreover, one has

$$|\partial^\beta\sigma(\xi)| = \left|\sum_{\gamma \le \beta} \partial^{\beta-\gamma}(e^{-2\pi i \xi_1})\partial^\gamma(\frac{1}{(1 + 4\pi^2|\xi|^2)^{s/2}})\right| \le \sum_{\gamma \le \beta} \binom{\beta}{\gamma} C_\gamma(1 + |\xi|)^{-s-\gamma}.$$

In view of these facts we obtain

$$
\begin{aligned}
|(I-\Delta)^{\frac{s}{2}}(\hat{\Psi}\sigma(2^j\cdot))| &\leq \sum_{|\alpha|\leq s}\sum_{\beta\leq\alpha}\sum_{\gamma\leq\beta}\binom{\alpha}{\beta}\binom{\beta}{\gamma}C_{\gamma,\alpha,s}|\partial^{\alpha-\beta}\hat{\Psi}(\xi)|2^{j(|\beta|-s-\gamma)}\Big(\frac{1}{2^j}+|\xi|\Big)^{-s-\gamma} \\
&\leq \sum_{|\alpha|\leq s}\sum_{\beta\leq\alpha}\sum_{\gamma\leq\beta}\binom{\alpha}{\beta}\binom{\beta}{\gamma}C_{\gamma,\alpha,s}\,|\partial^{\alpha-\beta}\hat{\Psi}(\xi)|\,|\xi|^{-s-\gamma},
\end{aligned}
\tag{13}
$$

where we used that $2^{j(|\beta|-s-\gamma)}\leq 1$ since $|\beta|\leq|\alpha|\leq s$ and $j\geq 0$. We conclude that $|(I-\Delta)^{\frac{s}{2}}(\hat{\Psi}\sigma(2^j\cdot))|$ is bounded by the absolute value of a compactly supported smooth function, which certainly lies in $L^{\frac{n}{s},1}$ with norm independent on j. Thus we have

$$
\sup_{j\in\mathbb{Z}}\|(I-\Delta)^{\frac{s}{2}}(\hat{\Psi}\sigma(2^j\cdot))\|_{L^{\frac{n}{s},1}}<\infty.
$$

Case II. If s is not an even integer, we find an even integer N such that $N>s$. Define

$$
F(z)=\int_{\mathbb{R}^n}(I-\Delta)^{\frac{zN}{2}}[\hat{\Psi}\sigma(2^j\cdot)](\xi)\psi(\xi)\,d\xi,
$$

where $\psi\in L^{(\frac{n}{s})',\infty}$ with $\|\psi\|_{L^{(\frac{n}{s})',\infty}}=1$.

When $z=it$ we have

$$
\begin{aligned}
|F(it)| &\leq \left|\int_{\mathbb{R}^n}(I-\Delta)^{\frac{itN}{2}}[\hat{\Psi}\sigma(2^j\cdot)](\xi)\psi(\xi)d\xi\right| \\
&\leq \|\psi\|_{L^{(\frac{n}{s})',\infty}}\big\|(I-\Delta)^{\frac{itN}{2}}(\hat{\Psi}\sigma(2^j\cdot))\big\|_{L^{\frac{n}{s},1}} \\
&\leq C_s(1+|t|)^{[\frac{n}{2}]+2}\|\hat{\Psi}\sigma(2^j\cdot)\|_{L^{\frac{n}{s},1}} \\
&\leq \|\sigma\|_{L^\infty}C_s'(1+|t|)^{[\frac{n}{2}]+2}.
\end{aligned}
\tag{14}
$$

An application of Hölder's inequality for weak type spaces yields the second inequality above. For the third one we used [12, Corollary 5.3.3], which gives $\|(I-\Delta)^{it}\|_{L^p\to L^p}\leq C(n,p)(1+|t|)^{[\frac{n}{2}]+2}$ when $1<p<\infty$. And for the third we simply use that σ is a bounded function.

Similarly, when $z=1+it$, for the same choice of N we have

$$
\begin{aligned}
|F(1+it)| &= \left|\int_{\mathbb{R}^n}(I-\Delta)^{\frac{itN}{2}+\frac{N}{2}}[\hat{\Psi}\sigma(2^j\cdot)](\xi)\psi(\xi)d\xi\right| \\
&\leq \|\psi\|_{L^{(\frac{n}{s})',\infty}}\big\|(I-\Delta)^{\frac{itN}{2}+\frac{N}{2}}[\hat{\Psi}\sigma(2^j\cdot)]\big\|_{L^{\frac{n}{s},1}} \\
&\leq C_s(1+|t|)^{[\frac{n}{2}]+2}\big\|(I-\Delta)^{\frac{N}{2}}[\hat{\Psi}\sigma(2^j\cdot)]\big\|_{L^{\frac{n}{s},1}} \\
&\leq C_s'(1+|t|)^{[\frac{n}{2}]+2}.
\end{aligned}
\tag{15}
$$

We applied Case I (since N is an even integer) to deduce the last inequality above.

Assume momentarily that F is analytic on the unit strip $S=\{z:\ 0\leq\Re z\leq 1\}$ and continuous on $\bar{S}$. This allows us to apply [12, C.0.4] to obtain

$$
|F(\theta)|\leq C_{n,\theta}(C_{N,s})^{1-\theta}(C_{N,s}')^{\theta}\quad\forall\ \ \theta\in(0,1).
$$

Thus taking $\theta_0 \in (0,1)$ such that $s = \theta_0 N$, we deduce

$$\begin{aligned}\|(I-\Delta)^{\frac{s}{2}}(\hat{\Psi}\sigma(2^j\cdot))\|_{L^{\frac{n}{s},1}} &= \sup_{\|\psi\|_{L^{(\frac{n}{s})',\infty}}=1} \int_{\mathbb{R}^n} (I-\Delta)^{\frac{\theta_0 N}{2}}[\hat{\Psi}\sigma(2^j\cdot)](\xi)\psi(\xi)d\xi \\ &= \sup_{\|\psi\|_{L^{(\frac{n}{s})',\infty}}=1} |F(\theta_0)| \\ &\le C_{n,\theta_0}(C_{N,s})^{1-\theta_0}(C'_{N,s})^{\theta_0},\end{aligned}$$

which is the desired conclusion.

We now prove that F is analytic on S and continuous on $\bar{S}$. By Fourier inversion we write

$$\begin{aligned}F(z) &= \int_{\mathbb{R}^n} (I-\Delta)^{\frac{zN}{2}}[\hat{\Psi}\sigma(2^j\cdot)](\xi)\psi(\xi)d\xi \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (1+4\pi^2|\eta|^2)^{\frac{zN}{2}}(\widehat{\hat{\Psi}\sigma(2^j\cdot)})(\eta)e^{2\pi i\xi\cdot\eta}\,d\eta\,\psi(\xi)\,d\xi.\end{aligned} \tag{16}$$

The double integral converges absolutely and the continuity of F on $\bar{S}$ is straightforward. We now show that F is analytic on S. For any point z_0 in S, pick $\delta > 0$ such that

$$2\delta < \frac{N}{2} - \Re\Big(\frac{z_0 N}{2}\Big).$$

By [12, lemma 2.7.5] for $0 < \frac{N}{2}|z| < \delta$, we obtain

$$\left|\frac{(1+4\pi^2|\eta|^2)^{\frac{zN}{2}}-1}{z}\right| \le \frac{2N}{\delta}\max\Big\{(1+4\pi^2|\eta|^2)^{2\delta},(1+4\pi^2|\eta|^2)^{-2\delta}\Big\},$$

which yields

$$\left|\frac{(1+4\pi^2|\eta|^2)^{\frac{zN+z_0N}{2}}-(1+4\pi^2|\eta|^2)^{\frac{z_0N}{2}}}{z}\right| \le \frac{2N}{\delta}(1+4\pi^2|\eta|^2)^{\frac{N}{2}}.$$

Taking into account that $\hat{\Psi}$ and ψ are Schwartz functions and the preceding estimate we apply the Lebesgue dominated convergence theorem to deduce the analyticity of F on S. $\square$

Proposition 2.2. *Let $1 < p < \infty$, $0 < s < n$ and let σ be as in (11). Suppose that T_σ is bounded from $L^p(|x|^\beta)$ to itself for some $\beta \in \mathbb{R}$. Then we must have*

$$-sp \le \beta \le sp.$$

Proof. Let $e_1 = (1,0,\dots,0)$. Consider the function

$$f(x) = |x+e_1|^{-n/p}\left(\log\frac{1}{|x+e_1|}\right)^{-\delta}\chi_{\{|x+e_1|\le 1/5\}}(x)$$

for some δ satisfying $1/p < \delta < 1$. Then we have

$$\int_{\mathbb{R}^n}|f(x)|^p|x|^\beta\,dx \approx \int_{|x+e_1|\le 1/5}|x+e_1|^{-n}\left(\log\frac{1}{|x+e_1|}\right)^{-p\delta}dx < \infty.$$

Notice that

$$\sigma^\vee(x) = G_s(x-e_1),$$

where G_s is the Bessel potential. Then $T_\sigma(f)(x)$ is nonnegative (since G_s is positive) and equals

$$\begin{aligned} T_\sigma(f)(x) &= \int_{|y+e_1|\le 1/5} |y+e_1|^{-n/p}\left(\log\frac{1}{|y+e_1|}\right)^{-\delta} G_s(x-e_1-y)\,dy \\ &= \int_{|y|\le 1/5} |y|^{-n/p}\left(\log\frac{1}{|y|}\right)^{-\delta} G_s(x-y)\,dy \\ &\ge c\int_{\substack{|y|\le 1/5\\ |x-y|\le 2}} |y|^{-n/p}\left(\log\frac{1}{|y|}\right)^{-\delta} |x-y|^{-n+s}\,dy. \end{aligned}$$

Let us consider x in the region $|x|\le\frac{2}{5}$. Then if $|x|\ge 2|y|$, it follows that $|x-y|\approx|x|$. Moreover, if $|y|\le\frac{1}{5}$ then $|x-y|\le 2$. Then restricting the integral to $|x|\ge 2|y|$, we obtain

$$\begin{aligned} T_\sigma(f)(x) &\ge c\int_{|y|\le 1/5} |y|^{-n/p}\left(\log\frac{1}{|y|}\right)^{-\delta} |x-y|^{-n+s}\,dy \\ &\ge c\int_{\substack{|y|\le 1/5\\ |x|\ge 2|y|}} |y|^{-n/p}\left(\log\frac{1}{|y|}\right)^{-\delta} |x-y|^{-n+s}\,dy \\ &\approx c|x|^{-n+s}\int_{|y|\le |x|/2} |y|^{-n/p}\left(\log\frac{1}{|y|}\right)^{-\delta} dy. \end{aligned}$$

Since $|x|/2\le 1/5$, the last integral is equivalent to

$$|x|^{-n/p+s}\left(\log\frac{1}{|x|}\right)^{-\delta},$$

so we obtain

$$T_\sigma(f)(x)\ge c|x|^{-n/p+s}\left(\log\frac{1}{|x|}\right)^{-\delta}, \quad \text{when } |x|\le\frac{2}{5}.$$

Now recall that we are assuming T_σ maps $L^p(|x|^\beta)$ to itself. This implies that

$$\int_{|x|\le 2/5}\left[|x|^{-n/p+s}\left(\log\tfrac{1}{|x|}\right)^{-\delta}\right]^p |x|^\beta dx<\infty,$$

and this implies that $\beta\ge -ps$.

We now turn to the other assertion of the proposition. Consider the function

$$g(x)=|x|^{-(n+\beta)/p}\left(\log\frac{1}{|x|}\right)^{-\delta}\chi_{\{|x|\le 1/5\}}(x)$$

for some $1/p<\delta<1$. Since $\delta p>1$, we have $g\in L^p(|x|^\beta)$.

Arguing as in the preceding example, we write

$$\begin{aligned} T_\sigma(g)(x) &\ge c\int g(y)G_s(x-e_1-y)\,dy \\ &\ge c\int_{\substack{|y|\le 1/5\\ |x-e_1-y|\le 2}} |y|^{-(n+\beta)/p}\left(\log\frac{1}{|y|}\right)^{-\delta} |x-e_1-y|^{-n+s}dy. \end{aligned}$$

Let us consider x in the region $|x-e_1| \le \frac{2}{5}$. Then if $|x-e_1| \ge 2|y|$, it follows that $|x-e_1-y| \approx |x-e_1|$. Moreover, if $|y| \le \frac{1}{5}$ then $|x-e_1-y| \le 2$. Then restricting the integral to $|x-e_1| \ge 2|y|$, we obtain

$$T_\sigma(g)(x) \ge c|x-e_1|^{-n+s} \int_{|y|\le|x-e_1|/2} |y|^{-(n+\beta)/p} \left(\log \frac{1}{|y|}\right)^{-\delta} dy.$$

Hence,

$$T_\sigma(g)(x) \ge c|x-e_1|^{s-(n+\beta)/p} \left(\log \frac{1}{|x-e_1|}\right)^{-\delta}, \quad \text{when } |x-e_1| \le \frac{2}{5}.$$

Now, notice that the assumption

$$\int_{|x-e_1|\le 2/5} T_\sigma(g)(x)^p |x|^\beta dx < \infty$$

implies that $ps-\beta \ge 0$. $\square$

Using the information from Proposition 2.2 and the extrapolation theorem in [2, Theorem 4.9] we derive the following result.

Theorem 2.3. *Let* $0 < s \le n/2$.

When $p \ge 2$, *suppose that either of the following holds:*

(a) $\ell > \frac{ps}{n}+1$ *and* $1 \le r \le \infty$;

(b) $1 \le \ell \le \infty$ *and* $1 \le r < (\frac{2n}{p(n-2s)})'$.

When $p < 2$, *suppose that either of the following holds:*

(a) $\ell > p(\frac{s}{n}+\frac{1}{2})$ *and* $1 \le r \le \infty$;

(b) $1 \le \ell \le \infty$ *and* $1 \le r < \frac{n}{ps}$.

Then there is a weight $w \in A_\ell \cap RH_r$ *(for some* $\ell, r \in [1,\infty]$*) and a bounded function* σ *on* $\mathbb{R}^n$ *satisfying*

$$\sup_{j\in\mathbb{Z}} \|(I-\Delta)^{s/2}[\hat{\Psi}\,\sigma(2^j\cdot)]\|_{L^{\frac{n}{s},1}} < \infty \tag{17}$$

such that T_σ *is unbounded from* $L^p(w)$ *to itself.*

Proof. Case $p \ge 2$.

(a) If $\ell > \frac{ps}{n}+1$, then we choose β satisfying $ps < \beta < n(\ell-1)$, and we note that $|x|^\beta$ lies in $A_\ell \cap RH_\infty$. By Proposition 2.2, T_σ is unbounded from $L^p(w)$ to itself. The function σ here is as in (11).

(b) If $1 \le r < \left(\frac{2n}{p(n-2s)}\right)'$, suppose T_σ is bounded from $L^p(w)$ to itself for all $w \in A_\ell \cap RH_r$, then there exists $q_0 > \frac{2n}{n-2s}$ such that $\left(\frac{q_0}{p}\right)' = r < \left(\frac{2n}{p(n-2s)}\right)'$. By the extrapolation theorem in [2, Theorem 4.9], there exists q with

$$p < \frac{2n}{n-2s} < q < q_0$$

such that T_σ is bounded from $L^q(w)$ to itself for all $w \in A_{\frac{\ell q}{p}} \cap RH_{(\frac{q_0}{q})'}$. Note that $p < \frac{2n}{n-2s}$ since $|\frac{1}{2}-\frac{1}{p}| < \frac{s}{n}$ is a necessary condition to get the boundedness of T_σ satisfying (17) on unweighted spaces L^p; see [13, Theorem 1.1] and [32, Theorem 1]. In particular, taking $w = 1$, we obtain the unweighted case: T_σ is bounded from L^q to itself for

$$q > \frac{2n}{n-2s}, \quad i.e. \quad \frac{1}{2} - \frac{1}{q} > \frac{s}{n},$$

which is a contradiction.

Case $p < 2$.

(a) A similar argument to the case [$p \geq 2$ (b)] applies by combining the necessary condition

$$\frac{1}{p} - \frac{1}{2} < \frac{s}{n}$$

in the unweighted case with the extrapolation theorem in [2, Theorem 4.9].

(b) If $1 \leq r < \frac{n}{ps}$, then we use a similar argument to [$p \geq 2$ (a)] by choosing β such that

$$-\frac{n}{r} < \beta < -sp.$$

This concludes the proof of the theorem. □

We also remark that condition (17) in Theorem 2.3 can be replaced by the more restrictive condition

$$\sup_{j\in\mathbb{Z}} \|(I-\Delta)^{\frac{s}{2}}(\widehat{\Psi}\sigma(2^j\cdot))\|_{L^r} < \infty$$

for $r > n/s$. This is followed by a small adjustment in the proof of Lemma 2.1

3. Preliminaries

In this section we present some useful facts about the A_p and RH_s classes.

Lemma 3.1 ([11, Corollary 7.2.6])**.** *Let $1 < p < \infty$, then we have*

$$A_p = \bigcup_{q\in(1,p)} A_q.$$

Lemma 3.2 ([20, Lemma 3.1])**.** *We have $w \in RH_s$ if and only if $w^s \in A_\infty := \cup_{p>1} A_p$ with*

$$\frac{[w^s]_{A_\infty}^{\frac{1}{s}}}{[w]_{A_\infty}} \leq [w]_{RH_s} \leq [w^s]_{A_\infty}^{\frac{1}{s}}.$$

Throughout this paper we will make use of the notation

$$\mathcal{A}_p^q := \{\, w : [w^q]_{A_p} < \infty \,\} \ \text{ for } \ 1 < p < \infty \text{ and } 1 < q < \infty$$

introduced in [21, Section I].

Lemma 3.3 ([21, Section I])**.** *Let $0 < \sigma < \infty$, $1 < p < \infty$ and $v \in \mathcal{A}_p^\sigma$, then $v \in A_{1+\frac{p-1}{\sigma}}$ if and only if $v \in A_\infty$. In fact, we have*

$$[v]_{A_\infty} \leq [v]_{A_{1+\frac{p-1}{\sigma}}} \leq [v^\sigma]_{A_p}^{\frac{1}{\sigma}} \ \textit{if } \sigma \geq 1,$$

and

$$[v]_{A_\infty} \leq [v]_{A_{1+\frac{p-1}{\sigma}}} \leq [v^\sigma]_{A_p}^{\frac{1}{\sigma}} [v^\sigma]_{RH_{\frac{1}{\sigma}}}^{\frac{1}{\sigma}} \ \textit{if } 0 < \sigma < 1.$$

Lemma 3.4 ([8, Theorem 2.2], [21, Section I])**.** *For $1 \le s < \infty$ we have*

$$\mathcal{A}_p^q = A_{1+\frac{p-1}{q}} \cap RH_s$$

with the following norm estimates

$$\big([w]_{RH_q}[w]_{A_{1+\frac{p-1}{q}}}\big)^{\frac{q}{2}} \le [w^q]_{A_p} \le \big([w]_{RH_q}[w]_{A_{1+\frac{p-1}{q}}}\big)^{q}.$$

Proof. This follows by combining Lemmas 3.2 and 3.3. □

Lemma 3.5 ([12, Theorem 8.6.9])**.** *For any $w \in A_p, 1 < p < \infty$, and $K > 1$, and any r satisfying*

$$1 \le r \le 1 + \frac{K-1}{K}\frac{1}{2^{3/2}e^{1/e}6^n[w]_{A_p}}$$

we have $[w]_{RH_r} \le K$, i.e., for any cube Q in $\mathbf{R}^n$,

$$\left(\frac{1}{|Q|}\int_Q w^r\,dx\right)^{\frac{1}{r}} \le \frac{K}{|Q|}\int_Q w\,dx.$$

Lemma 3.6. *If $1 < q < \infty$, then we have*

$$\bigcup_{\bar{q}>q} RH_{\bar{q}} = RH_q.$$

Proof. Let $w \in RH_q$ be arbitrary. By Lemma 3.2, we know that

$$w^q \in A_\ell \quad \text{for some } \ell > 1.$$

It follows from Lemma 3.5 that there exists $r > 1$ such that

$$(w^q)^r = w^{qr} \in A_\ell.$$

Applying Lemma 3.2 again, we conclude that

$$w \in RH_{qr}.$$

Since $r > 1$, we have $qr > q$, and hence for every $w \in RH_q$ there exists $\bar{q} > q$ such that $w \in RH_{\bar{q}}$. On the other hand, it is well known that the Reverse Hölder classes are monotone, namely

$$RH_{\bar{q}} \subset RH_q \quad \text{for all } \bar{q} > q.$$

We conclude that

$$\bigcup_{\bar{q}>q} RH_{\bar{q}} = RH_q,$$

which completes the proof. □

Lemma 3.7. *Let $1 < p, p_i < \infty$, $0 < s^*, s_i^* < n$ and $1 < q, q_i < \infty$ for $i \in \{0,1\}$. Suppose $w_0 \in A_{\frac{p_0 s_0^*}{n}} \cap RH_{q_0}$ and $w_1 \in A_{\frac{p_1 s_1^*}{n}} \cap RH_{q_1}$ with $\frac{p_i s_i^*}{n} > 1$, and these parameters satisfy*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad s^* = (1-\theta)s_0^* + \theta s_1^*, \qquad \frac{1}{pq} = \frac{1-\theta}{p_0 q_0} + \frac{\theta}{p_1 q_1}$$

for some $\theta \in (0,1)$. Then we have

$$w := w_0^{\frac{1-\theta}{p_0}p}\, w_1^{\frac{\theta}{p_1}p} \in A_{\frac{ps^*}{n}} \cap RH_q$$

and moreover,

$$[w^q]_{A_\rho} \leq \big([w_0^{q_0}]_{A_{\rho_0}}\big)^{\frac{pq}{p_0q_0}(1-\theta)}\big([w_1^{q_1}]_{A_{\rho_1}}\big)^{\frac{pq}{p_1q_1}\theta}, \tag{18}$$

where

$$\rho_0 = \Big(\frac{p_0s_0^*}{n} - 1\Big)q_0 + 1, \quad \rho_1 = \Big(\frac{p_1s_1^*}{n} - 1\Big)q_1 + 1, \quad \rho = \Big(\frac{ps^*}{n} - 1\Big)q + 1. \tag{19}$$

Proof. The proof follows by a straightforward application of Hölder's inequality and the use of Lemma 3.4. □

Lemma 3.8 (The converse to Lemma 3.7)**.** *Let $1 < p, q < \infty$ and $0 < s^* < n$. Fix parameters $q_0, q_1, p_0, p_1 \in (1, \infty)$, $s_0^*, s_1^* \in (0, n)$, and $\theta \in (0, 1)$, with $\frac{p_is_i^*}{n} > 1$ for $i = 0, 1$, satisfying*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad s^* = (1-\theta)s_0^* + \theta s_1^*, \qquad \frac{1}{pq} = \frac{1-\theta}{p_0q_0} + \frac{\theta}{p_1q_1}.$$

Then for any $w \in A_{\frac{ps^*}{n}} \cap RH_q$ there exist weights

$$w_0 \in A_{\frac{p_0s_0^*}{n}} \cap RH_{q_0} \quad \text{and} \quad w_1 \in A_{\frac{p_1s_1^*}{n}} \cap RH_{q_1}$$

such that

$$w := w_0^{\frac{1-\theta}{p_0}p}\, w_1^{\frac{\theta}{p_1}p}$$

and

$$[w_0^{q_0}]_{A_{\rho_0}} \leq c_0\big([w^q]_{A_\rho}\big)^{1+\frac{\rho_0-1}{\rho-1}} \quad \text{and} \quad [w_1^{q_1}]_{A_{\rho_1}} \leq c_1\big([w^q]_{A_\rho}\big)^{1+\frac{\rho_1-1}{\rho-1}} \tag{20}$$

for some constants c_0, c_1 (depending on $p_0, p_1, q_0, q_1, s_0^*, s_1^*, n$), where ρ_0, ρ_1, ρ are as in (19).

Proof. By the classical factorization of A_p weights (see [22], [12, Proposition 8.5.1]) for a given $w \in A_{\frac{ps^*}{n}} \cap RH_q$, there exist $v_1, v_2 \in A_1$ such that

$$w^q = v_1\, v_2^{-q(\frac{ps^*}{n}-1)}$$

and

$$[v_1]_{A_1} \leq c_1'\,[w^q]_{A_{q(\frac{ps^*}{n}-1)+1}}, \qquad [v_2]_{A_1} \leq c_2'\,\big([w^q]_{A_{q(\frac{ps^*}{n}-1)+1}}\big)^{\frac{1}{q(\frac{ps^*}{n}-1)}},$$

for some constants c_1', c_2' that depend on p, s^*, n. Then we define

$$w_0 := \big(v_1\, v_2^{-q_0(\frac{p_0s_0^*}{n}-1)}\big)^{\frac{1}{q_0}}, \qquad w_1 := \big(v_1\, v_2^{-q_1(\frac{p_1s_1^*}{n}-1)}\big)^{\frac{1}{q_1}}$$

and we observe that

$$w_0 \in A_{\frac{p_0s_0^*}{n}} \cap RH_{q_0} \quad \text{and} \quad w_1 \in A_{\frac{p_1s_1^*}{n}} \cap RH_{q_1}.$$

Finally, we notice that

$$\begin{aligned} w^q &= v_1\, v_2^{-q(\frac{ps^*}{n}-1)} \\ &= \big(v_1\, v_2^{-q_0(\frac{p_0s_0^*}{n}-1)}\big)^{\frac{q}{q_0}\frac{1-\theta}{p_0}p}\big(v_1\, v_2^{-q_1(\frac{p_1s_1^*}{n}-1)}\big)^{\frac{q}{q_1}\frac{\theta}{p_1}p} \\ &= \big(w_0^{q_0}\big)^{\frac{1-\theta}{p_0q_0}pq}\big(w_1^{q_1}\big)^{\frac{\theta}{p_1q_1}pq}. \end{aligned}$$

The estimates (20) follow from the standard inequalities $[v_1v_2^{1-p}]_{A_p} \leq [v_1]_{A_1}[v_2]_{A_1}^{p-1}$. This concludes the proof. □

Remark 3.9. It follows from (20) that the converse of (18) holds with an extra square, i.e.,

$$\big([w_0^{q_0}]_{A_{\rho_0}}\big)^{\frac{pq}{p_0q_0}(1-\theta)}\big([w_1^{q_0}]_{A_{\rho_1}}\big)^{\frac{pq}{p_1q_1}\theta} \lesssim \big([w^q]_{A_\rho}\big)^2. \tag{21}$$

Lemma 3.10. *Let $0 < s < n$, $\frac{n}{s} < r_s < r_l < \infty$, and suppose that the multiplier σ satisfies*

$$\sup_{j\in\mathbb{Z}} \Big\|(I-\Delta)^{s/2}\Big[\hat{\Psi}\,\sigma(2^j\cdot)\Big]\Big\|_{L^{r_l}} < \infty.$$

Then we have

$$\sup_{j\in\mathbb{Z}} \Big\|(I-\Delta)^{s/2}\Big[\hat{\Psi}\,\sigma(2^j\cdot)\Big]\Big\|_{L^{r_s}} < \infty.$$

Proof. Let $s > 0$ and $1 < p_i, q_i < \infty$ with $i \in \{1,2\}$. For nice f, g we have

$$\|(I-\Delta)^{s/2}(fg)\|_{L^{p,1}} \lesssim \|(I-\Delta)^{s/2}f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}} + \|f\|_{L^{p_1,q_1}}\|(I-\Delta)^{s/2}g\|_{L^{p_2,q_2}}$$

where the indices are related as in

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad 1 = \frac{1}{q_1} + \frac{1}{q_2}$$

by [15, Page 21]. Let $p = r_s$, $p_1 = r_l$, $q_1 = r_l$, $q_2 = r_l'$, $f = \hat{\Psi}\sigma(2^j\cdot)$, $g = \hat{\theta}$ which is compactly supported and $\hat{\theta} \equiv 1$ on supp $(\hat{\Psi})$. Note that $\|\cdot\|_{L^r} = \|\cdot\|_{L^{r,r}}$ for $1 < r < \infty$. Choosing $p_2 > 1$ such that $\frac{1}{r_s} = \frac{1}{r_l} + \frac{1}{p_2}$ is satisfied, we obtain that

$$\begin{aligned}
&\|(I-\Delta)^{s/2}(\hat{\theta}\hat{\Psi}\sigma(2^j\cdot))\|_{L^{r_s}}\\
&\qquad \le \|(I-\Delta)^{s/2}(\hat{\theta}\hat{\Psi}\sigma(2^j\cdot))\|_{L^{r_s,1}}\\
&\qquad \lesssim \|(I-\Delta)^{s/2}(\hat{\Psi}\sigma(2^j\cdot))\|_{L^{r_l}}\|\hat{\theta}\|_{L^{p_2,r_l'}} + \|\hat{\Psi}\sigma(2^j\cdot)\|_{L^{r_l}}\|(I-\Delta)^{s/2}\hat{\theta}\|_{L^{p_2,r_l'}}\\
&\qquad \lesssim \|(I-\Delta)^{s/2}(\hat{\Psi}\sigma(2^j\cdot))\|_{L^{r_l}} < \infty.
\end{aligned}$$

This provides the required conclusion. ☐

Theorem 3.11 ([6], [12, Theorem 8.4.3])**.** *The Hardy-Littlewood maximal operator with respect to cubes M_c satisfies the estimate*

$$\big\|\mathrm{M}_c\|_{L^p(w)\to L^p(w)} \le 3^{(n+1)p'}(16^n pp')^{\frac{1}{p}}[w]_{A_p}^{\frac{1}{p-1}}$$

for $1 < p < \infty$. Consequently, the centered Hardy-Littlewood maximal operator with respect to balls M satisfies

$$\big\|M\|_{L^p(w)\to L^p(w)} \le (\sqrt{n}/2)^n 3^{(n+1)p'}(16^n pp')^{\frac{1}{p}}[w]_{A_p}^{\frac{1}{p-1}}$$

for $1 < p < \infty$.

Another tool that will be useful to us is the following sharp form of extrapolation.

Theorem 3.12 ([19])**.** *Let $\mathcal{F}$ be a set of pairs of functions*[1] *defined on $\mathbb{R}^n$. Let $1 < r < \infty$. Suppose that there exists an increasing function $\mathbf{N}_r$ such that for every $v \in A_r$ and every $(f,g) \in \mathcal{F}$ the following estimate is valid:*

$$\|g\|_{L^r(v)} \le \mathbf{N}_r([v]_{A_r})\,\|f\|_{L^r(v)}.$$

[1]The precise formulation in [19] is not stated in terms of pairs, but the proof easily carries through to this setting.

Then for any $1 < p < \infty$, every $(f,g) \in \mathcal{F}$, and every weight $w \in A_p$, one has

$$\|g\|_{L^p(w)} \le \mathbf{N}_p([w]_{A_p})\, \|f\|_{L^p(w)},$$

where for any $t > 0$,

$$\mathbf{N}_p(t) = \begin{cases} 2^{\frac{1}{r}}\, \mathbf{N}_r\Big(2\left(2^n 3^{(n+1)p}(16^n pp')^{\frac{1}{p'}}\right)^{\frac{p-r}{p-1}} t\Big) & \text{if} \quad p > r \\ 2^{\frac{1}{r'}}\, \mathbf{N}_r\Big(2^{r-1}\left(2^n 3^{(n+1)p'}(16^n pp')^{\frac{1}{p}}\right)^{\frac{(r-p)(r-1)}{p-1}} t^{\frac{r-1}{p-1}}\Big) & \text{if} \quad p < r\,. \end{cases} \tag{22}$$

Recall our fixed Schwartz function Ψ whose Fourier transform is supported in annulus $\{\xi \in \mathbb{R}^n : \frac{6}{7} \le |\xi| < 2\}$, equal to 1 on the annulus $\{\xi \in \mathbb{R}^n : 1 \le |\xi| < \frac{12}{7}\}$, and which satisfies $\sum_{j\in\mathbb{Z}} \hat{\Psi}(2^{-j}\xi) = 1$ for all $\xi \neq 0$. We now introduce the Littlewood-Paley operator Δ_j (associated with Ψ) by setting

$$\Delta_j f = f * \Psi_{2^{-j}},$$

where $j \in \mathbb{Z}$ and $\Psi_t(x) = t^{-n}\Psi(t^{-1}x)$. The Littlewood-Paley square function is defined by

$$Sf = \left(\sum_{j\in\mathbb{Z}} |\Delta_j f|^2\right)^{\frac{1}{2}}.$$

Theorem 3.13. *Let $1 < p < \infty$. Then if $w \in A_p$ and $f \in L^p(w)$, we have*

$$\|Sf\|_{L^p(w)} \le \left(c_n^{pp'}[w]_{A_p}\right)^{\max(1,\frac{1}{p-1})} \|f\|_{L^p(w)} \tag{23}$$

and

$$\|f\|_{L^p(w)} \le 3c_n^{p^2p'}[w]_{A_p}^{\max(1,\frac{1}{p-1})}\, \|Sf\|_{L^p(w)} \tag{24}$$

for some dimensional constant c_n.

Proof. We begin by noting that (23) holds when $p = 2$ using the Rademacher functions and the result of [18]. To deduce the case of a general $p \in (1,\infty)$ we use Theorem 3.12 which provides the norm estimate

$$\big\|S\big\|_{L^p(w)\to L^p(w)} \le \begin{cases} \sqrt{2}\,\mathbf{N}_2\big(2C(p')^{\frac{p-2}{p-1}}[w]_{A_p}\big) & \text{when } p > 2 \\ \\ \sqrt{2}\,\mathbf{N}_2\big(C(p)^{\frac{2-p}{p-1}}[w]_{A_p}^{\frac{1}{p-1}}\big) & \text{when } p < 2 \end{cases}$$

for our square function S. Here $C(p) = 2^n 3^{(n+1)p'}(16^n pp')^{\frac{1}{p}}$ from Theorem 3.11 and

$$\big\|S\big\|_{L^2(w)\to L^2(w)} \le \mathbf{N}_2([w]_{A_2})$$

which, by [18], is known to satisfy $\mathbf{N}_2([w]_{A_2}) \le c_n'[w]_{A_2}$, where c_n' depends only on the dimension. Therefore, we get $\left(c_n^{\max(p,p')}pp'[w]_{A_p}\right)^{\max(1,\frac{1}{p-1})}$ as the constant in (23), and we can choose an upper bound $(c_n^{pp'}[w]_{A_p})^{\max(1,\frac{1}{p-1})}$ showing in (23). Using these facts, we deduce the validity of (23). Now we can use the duality argument to get (24). We compute

$$\begin{aligned}\|f\|_{L^p(w)} &= \sup_{\|g\|_{L^{p'}(w^{1-p'})}\le 1} \left|\int fg\,dx\right| \\ &\le 3 \sup_{\|g\|_{L^{p'}(w^{1-p'})}\le 1} \left|\int_{\mathbb{R}^n} (Sf)(Sg)dx\right| \\ &\le 3 \sup_{\|g\|_{L^{p'}(w^{1-p'})}\le 1} \big\|Sf\big\|_{L^p(w)}\big\|Sg\big\|_{L^{p'}(w^{1-p'})} \\ &\le 3\big(c_n^{pp'}[w^{1-p'}]_{A'_p}\big)^{\max(1,p-1)}\big\|Sf\big\|_{L^p(w)} \qquad \text{by (23)} \\ &\le 3c_n^{p^2p'}[w]_{A_p}^{\max(1,\frac{1}{p-1})}\big\|Sf\big\|_{L^p(w)}\end{aligned} \tag{25}$$

in view of the fact that $[w^{1-p'}]_{A_{p'}} = [w]_{A_p}^{\frac{1}{p-1}}$. This proves (24). □

Theorem 3.14 ([12, Corollary 8.6.11])**.** *Let $1 < p < \infty$. Given $w \in A_p$ for any an index q in $[\frac{p-1}{1+(c_n[w]_{A_p}^{p'-1})^{-1}} + 1, \infty)$ we have $w \in A_q$. Moreover, one has $\frac{1}{q-1} \le \frac{2}{p-1}$ and*

$$[w]_{A_q} \le 2^{p-1}[w]_{A_p}.$$

Recall that M is the centered Hardy-Littlewood maximal operator with respect to balls. The next result we need is the so-called weighted Fefferman–Stein vector-valued estimate for the Hardy-Littlewood maximal operator. This was obtained in [1, Theorem 3.1] without explicit bounds on the constants. As we need the latter, we present a different proof based on sharp extrapolation.

Theorem 3.15. *Let $1 < p, r < \infty$. Then there exists a constant $C(p, r, n)$ such that for every $w \in A_p$ and every $f_k \in L^p(\mathbb{R}^n, w)$ we have*

$$\left\|\Big(\sum_k (Mf_k)^r\Big)^{1/r}\right\|_{L^p(w)} \le (c_n^{pp'rr'})^{\max(\frac{r-p}{p-1}+1,1)}[w]_{A_p}^{\max(1,\frac{1}{p-1},\frac{r-1}{p-1},\frac{1}{r-1})}\left\|\Big(\sum_k |f_k|^r\Big)^{1/r}\right\|_{L^p(w)}, \tag{26}$$

where c_n is a constant depending only on the dimension n.

Proof. The case $p = r$ comes from Theorem 3.11. For the remaining cases we use the estimates (22) contained in Theorem 3.12 to get the constant for the estimate (26):

$$\begin{cases} (\frac{\sqrt{n}}{2})^n 3^{(n+1)r'}(rr')^{\frac{1}{r}}2^{\max(1,r-1)}(2^n3^{(n+1)p'}16^npp')^{\frac{(r-p)(r-1)}{p-1}\max(1,\frac{1}{r-1})}[w]_{A_p}^{\max(\frac{1}{p-1},\frac{r-1}{p-1})} & \text{if } p<r \\ (\frac{\sqrt{n}}{2})^n 3^{(n+1)r'}(16^n rr')^{\frac{1}{r}}[w]_{A_r}^{\max(1,\frac{1}{r-1})} & \text{if } p=r \\ (\frac{\sqrt{n}}{2})^n 3^{(n+1)r'}(rr')^{\frac{1}{r}}2^{(n+1)\max(1,\frac{1}{r-1})}3^{(n+1)p\max(1,\frac{1}{r-1})}(16^npp')^{\max(1,\frac{1}{r-1})}[w]_{A_r}^{\max(1,\frac{1}{r-1})} & \text{if } p>r. \end{cases}$$

Choosing an upper bound for the above formulas, such as for example

$$(c_n^{pp'rr'})^{\max(\frac{r-p}{p-1}+1,1)}[w]_{A_p}^{\max(1,\frac{1}{p-1},\frac{r-1}{p-1},\frac{1}{r-1})}$$

with $c_n > (100n)^n$, yields the claimed assertion. □

In the next result, we set $M_q f = (M[|f|^q])^{1/q}$ for $1 < q < \infty$.

Lemma 3.16 ([13, Theorem 8.6.9])**.** *Assume that $0 < s < n$ and $q > n/s$. Then there is a constant $C_0(s,n)$ such that for all local L^q functions f on $\mathbb{R}^n$ we have*

$$\sup_{j\in\mathbb{Z}} \left\| \frac{f(x+2^{-j}\cdot)}{(1+|\cdot|)^s} \right\|_{L^{\frac{n}{s},\infty}} \le \frac{C_0(s,n)}{1-2^{n-sq}} M_q f(x), \qquad x \in \mathbb{R}^n.$$

The stated value of the constant simply follows by tracing the flow of the constants in the proof in [13, Theorem 8.6.9].

Lemma 3.17 ([13, Lemma 3.3])**.** *Let $1 < p, p_0, p_1 < \infty$ be related as in*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

for some $\theta \in (0,1)$. Given $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $\varepsilon > 0$, there exist smooth functions h_j^ε, $j = 1, \dots, N_\varepsilon$, supported in cubes on $\mathbb{R}^n$ with pairwise disjoint interiors, and nonzero complex constants c_j^ε such that

$$f_z^\varepsilon = \sum_{j=1}^{N_\varepsilon} |c_j^\varepsilon|^{\frac{p}{p_0}(1-z)+\frac{p}{p_1}z} h_j^\varepsilon, \tag{27}$$

$$\|f_\theta^\varepsilon - f\|_{L^2} < \varepsilon, \quad \|f_{k+it}^\varepsilon\|_{L^{p_k}} \le (\|f\|_{L^p} + \varepsilon)^{\frac{p}{p_k}}, \quad k = 0, 1.$$

Having completed this introductory material, we now proceed to stating some known results concerning estimates of multipliers on weighted spaces that will be useful to us. This is accomplished in the next section.

4. Known results on weighted estimates for multiplier operators

We use the definition of sparse collections given in [26] and [3]. Let $0 < \gamma < 1$ and $S_{\mathbb{R}^n}$ denote the set of simple functions on $\mathbb{R}^n$. We say that a linear operator T mapping $S_{\mathbb{R}^n}$ to weakly measurable valued functions on $\mathbb{R}^n$ satisfies a sparse (p_1, p_2) bound (in the sense of [26]) if there exists a constant $C > 0$ such that for all $f, g \in S_{\mathbb{R}^n}$ we have

$$\left| \int (Tf) g\, dx \right| \le C \sup_{\mathfrak{S} \text{ is } \gamma\text{-sparse}} \sum_{Q\in\mathfrak{S}} |Q| \left(\frac{1}{|Q|} \int_Q |f(x)|^{p_1}\, dx \right)^{1/p_1} \left(\frac{1}{|Q|} \int_Q |g(x)|^{p_2}\, dx \right)^{1/p_2}.$$

The smallest constant C such that the above estimate is valid is denoted by $\|T\|_{\mathrm{Sp}_\gamma(p_1,p_2)}$.

Proposition 4.1 ([3, Proposition 7.5])**.** *Let $1 \le p_0 \le 2$, $n > s > 0$, $1/r = 1/p_0 - 1/2$, and σ satisfies*

$$\sup_{j\in\mathbb{Z}} \|(I-\Delta)^{s/2}[\hat{\Psi}\sigma(2^j\cdot)]\|_{L^r} < \infty.$$

Suppose one of these followings holds:
(i) $1 \le p_0 \le q_0 \le 2$, and $s > n(1/p_0 - 1/2)$ i.e. $s/n > 1/r$.
(ii) $2 \le q_0 < \infty$ and $s > n(1/p_0 - 1/q_0)$ i.e. $s/n > 1/r + 1/2 - 1/q_0$.
Then, there is a constant $C(p_0, r, s, q_0, \alpha, \gamma)$ such that

$$\|T_\sigma\|_{\mathrm{Sp}_\gamma(p_0,q_0')} \le C(p_0, r, s, q_0, \alpha, \gamma) \sup_{j\in\mathbb{Z}} \|(I-\Delta)^{s/2}[\hat{\Psi}\sigma(2^j\cdot)]\|_{L^r}.$$

Remark : (i) and (ii) both imply that $s/n > 1/r$. In addition, on the critical line

$$\frac{s}{n} = \frac{1}{p_0} - \frac{1}{q_0}$$

the sparse bound $T_\sigma \in \mathrm{Sp}_\gamma(p_0, q_0')$ is established under stronger Besov-type assumptions on the multiplier [4, Theorems 1.1, 1.2 and 1.3].

Proposition 4.2 ([5, Proposition 6.4])**.** *Let $1 < p_0 < p < q_0 < \infty$. Suppose a linear operator T on L^p is bounded and*

$$\|T\|_{\mathrm{Sp}_\gamma\left(p_0,q_0'\right)} < \infty.$$

Then one has

$$\|T\|_{L^p(w)\to L^p(w)} \le C(p,p_0,q_0)\|T\|_{\mathrm{Sp}_\gamma\left(p_0,q_0'\right)}\Big([w]_{A_{\frac{p}{p_0}}}[w]_{RH_{(\frac{q_0}{p})'}}\Big)^\alpha,$$

where $\alpha = \max(\frac{1}{p-p_0}, \frac{q_0-1}{q_0-p})$.

Combing Proposition 4.1 and Theorem 4.2 yields the following theorem.

Theorem 4.3 ([3] [5])**.** *Let $n > s > 0$, $s/n > 1/r$, $s/n > |1/2 - 1/p|$, and suppose that the multiplier σ satisfies*

$$\sup_{j\in\mathbb{Z}} \Big\|(I-\Delta)^{s/2}\big[\hat{\Psi}\,\sigma(2^j\cdot)\big]\Big\|_{L^r} < \infty.$$

Then there exists a constant $\gamma \in (0,1)$, independent of the weight w, such that

$$\|T_\sigma\|_{L^p(w)\to L^p(w)} \le C(p,q_0^*,r,s,\gamma)\sup_{j\in\mathbb{Z}}\Big\|(I-\Delta)^{s/2}\big[\hat{\Psi}\,\sigma(2^j\cdot)\big]\Big\|_{L^r}\Big([w]_{A_{p(\frac{1}{r}+\frac{1}{2})}}[w]_{RH_{(\frac{q_0^*}{p})'}}\Big)^\alpha, \tag{28}$$

where

$$\alpha = \max\left(\frac{1}{p-\frac{1}{\frac{1}{r}+\frac{1}{2}}}, \frac{q_0^*-1}{q_0^*-p}\right)$$

in the following three cases:
Case 1: *$p = 2$. If $q_0^* > 2$ satisfies*

$$\frac{s}{n} - \frac{1}{r} > \frac{1}{2} - \frac{1}{q_0^*},$$

then (28) holds for any weight

$$w \in A_{\frac{2}{r}+1} \cap RH_{(\frac{q_0^*}{2})'}.$$

Case 2: *$p > 2$ and $\frac{s}{n} - \frac{1}{r} > \frac{1}{2} - \frac{1}{p}$. If $q_0^* > p$ satisfies*

$$\frac{s}{n} - \frac{1}{r} > \frac{1}{2} - \frac{1}{q_0^*},$$

then (28) holds for any weight

$$w \in A_{\frac{p}{r}+\frac{p}{2}} \cap RH_{(\frac{q_0^*}{p})'}.$$

Case 3: *$p < 2$. Then (28) holds for any weight*

$$w \in A_{\frac{p}{r}+\frac{p}{2}} \cap RH_{(\frac{q_0^*}{p})'},$$

provided that one of the following holds:

(i) $1 \le \frac{1}{\frac{1}{r}+\frac{1}{2}} < p < q_0^* \le 2$;

(ii) $\frac{1}{\frac{1}{r}+\frac{1}{2}} < p < 2 \le q_0^*$ *and* $\frac{s}{n} - \frac{1}{r} > \frac{1}{2} - \frac{1}{q_0^*}$.

In the statement of Theorem 4.3, the condition $|\frac{1}{2} - \frac{1}{p}| < \frac{s}{n}$ follows from (i) and (ii) in Proposition 4.1 with $\frac{1}{p_0} = \frac{1}{r} + \frac{1}{2}$. We notice that when $1 < p < 2$, letting $\frac{1}{r}$ approach $\frac{s}{n}$ implies that the maximum of q_0^* approaches 2. In this case, Case 3 of Theorem 4.3 yields that the operator T_σ is bounded on $L^p(w)$ for the class of weights

$$w \in A_{p(\frac{1}{r}+\frac{1}{2})} \cap RH_{\frac{2}{2-p}}$$

which approaches the class

$$A_{p(\frac{s}{n}+\frac{1}{2})} \cap RH_{\frac{2}{2-p}}$$

as $r \to \frac{n}{s}$. Then we may use a duality argument to obtain a new class of weights in the case $p > 2$.

Theorem 4.4 (The case $p > 2$ by duality)**.** *Let $p > 2$, $0 < s < \frac{n}{2}$, $\frac{1}{2} - \frac{1}{p} < \frac{s}{n}$, $p'(\frac{1}{r} + \frac{1}{2}) > 1$ and $rs > n$. Define*

$$\bar{\ell} = \frac{p}{2}, \quad \bar{q} = \frac{p'-1}{p'(\frac{1}{r}+\frac{1}{2})-1}. \tag{29}$$

Let $w \in A_{\bar{\ell}} \cap RH_{\bar{q}}$ and let σ satisfy

$$\sup_{j\in\mathbb{Z}} \left\| (I-\Delta)^{\frac{s}{2}}[\hat{\Psi}\sigma(2^j\cdot)] \right\|_{L^r} < \infty.$$

Then for any bounded compactly supported function f on $\mathbb{R}^n$ we have

$$\|T_\sigma f\|_{L^p(w)} \lesssim \sup_{j\in\mathbb{Z}} \left\| (I-\Delta)^{\frac{s}{2}}[\hat{\Psi}\sigma(2^j\cdot)] \right\|_{L^r} \|f\|_{L^p(w)}.$$

Proof. Fix $p > 2$. Let

$$\tilde{w} = w^{-\frac{1}{p-1}} \in A_{p'(\frac{1}{r}+\frac{1}{2})} \cap RH_{\frac{2}{2-p'}}.$$

We use a duality argument to prove that

$$\|T\|_{L^p(w)\to L^p(w)} = \|T^*\|_{L^{p'}(\tilde{w})\to L^{p'}(\tilde{w})}. \tag{30}$$

Assume $\|T\|_{L^p(w)\to L^p(w)} = C(T) < \infty$. For any $g \in L^{p'}(\mathbb{R}^n, \tilde{w})$, we have

$$\|T^*g\|_{L^{p'}(\mathbb{R}^n,\tilde{w})} = \sup_{\|f\|_{L^p(\tilde{w})}\le 1} \left| \int (T^*g) f \tilde{w}\, dx \right| \tag{31}$$

$$= \sup_{\|f\|_{L^p(\tilde{w})}\le 1} \left| \int g w^{-\frac{1}{p}} T(f\tilde{w}) w^{\frac{1}{p}}\, dx \right| \tag{32}$$

$$= \sup_{\|f\|_{L^p(\tilde{w})}\le 1} \left(\int |T(f\tilde{w})|^p w\, dx \right)^{\frac{1}{p}} \left(\int |g|^{p'} w^{-\frac{p'}{p}}\, dx \right)^{\frac{1}{p'}} \tag{33}$$

$$\le C(T) \sup_{\|f\|_{L^p(\tilde{w})}\le 1} \left(\int |f\tilde{w}|^p w\, dx \right)^{\frac{1}{p}} \left(\int |g|^{p'} w^{-\frac{p'}{p}}\, dx \right)^{\frac{1}{p'}} \tag{34}$$

$$= C(T) \sup_{\|f\|_{L^p(\tilde{w})}\le 1} \|f\|_{L^p(\tilde{w})} \|g\|_{L^{p'}(\tilde{w})} \tag{35}$$

$$\leq C(T)\|g\|_{L^{p'}(\tilde{w})}. \tag{36}$$

Thus one obtains

$$\|T^*\|_{L^{p'}(\tilde{w})\to L^{p'}(\tilde{w})} \leq C(T).$$

Applying the same argument with p replaced by p' and w replaced by $\tilde{w}$, we deduce the reverse inequality, and therefore (30) holds.

In particular, if $T = T_\sigma$ for some multiplier σ, then we have

$$\|T_\sigma\|_{L^p(w)\to L^p(w)} = \|T_\sigma^*\|_{L^{p'}(\tilde{w})\to L^{p'}(\tilde{w})}.$$

(Notice that if σ is real-valued, then $T_\sigma^* = T_\sigma$.) Moreover, we observe that

$$\begin{aligned}
w^{-\frac{1}{p-1}} \in A_{p'(\frac{1}{r}+\frac{1}{2})} \cap RH_{\frac{2}{2-p'}} \quad &\iff \quad w^{(1-p')(\frac{2}{2-p'})} \in A_{\frac{2}{2-p'}\left(p'(\frac{1}{r}+\frac{1}{2})-1\right)+1} = A_P,\\
&\qquad \text{where } P = \frac{2}{2-p'}\left(p'\left(\frac{1}{r}+\frac{1}{2}\right)-1\right)+1,\\
&\iff \quad w^{(1-p')(\frac{2}{2-p'})(1-P')} \in A_{P'},\\
&\iff \quad w \in A_{\frac{P'-1}{S}+1} \cap RH_S,\\
&\qquad \text{where } S = (1-p')\left(\frac{2}{2-p'}\right)(1-P').
\end{aligned}$$

Simplifying these parameters, we obtain

$$\bar{q} = S = \frac{p'-1}{p'\left(\frac{1}{r}+\frac{1}{2}\right)-1}, \qquad \bar{\ell} = \frac{P'-1}{S}+1 = \frac{p}{2}.$$

These are exactly the relationships (29) on the indices appearing in the statement of the theorem. Moreover, $\tilde{q}$ is finite, so $w \in A_{\bar{\ell}} \cap RH_{\bar{q}}$ is not covered in the weight classes in Case 2 of Theorem 4.3. $\square$

5. Weighted boundedness for higher order of smoothness

In this section we study weighted estimates for multipliers with orders of smoothness bigger than half the dimension of the ambient space. The dependence of these estimates on the Muckenhoupt characteristic constant of the weight is crucial for interpolation and for this reason we keep track of it. Unfortunately, this leads to some cumbersome expressions in the proof of the next theorem.

Theorem 5.1. *Let $\frac{n}{2} < s < n$ and $\frac{n}{s} < p < \infty$ and $w \in A_{\frac{ps}{n}}$. If $\sigma \in L^\infty(\mathbb{R}^n)$ satisfies*

$$\sup_{j\in\mathbb{Z}} \left\|(I-\Delta)^{\frac{s}{2}}[\hat{\Psi}\sigma(2^j\cdot)]\right\|_{L^{\frac{n}{s},1}} < \infty,$$

then, for any bounded compactly supported function f on $\mathbb{R}^n$ we have

$$\|T_\sigma f\|_{L^p(w)} \leq \frac{C(p,s,n)}{q([w]_{A_{\frac{ps}{n}}}) - \frac{n}{s}} [w]^\alpha_{A_{\frac{ps}{n}}} \sup_{j\in\mathbb{Z}} \left\|(I-\Delta)^{\frac{s}{2}}[\hat{\Psi}\sigma(2^j\cdot)]\right\|_{L^{\frac{n}{s},1}} \|f\|_{L^p(w)}, \tag{37}$$

where for $t > 0$,

$$q(t) = p\left(\frac{\frac{ps}{n}-1}{1+\left(c(s,n)t^{(\frac{ps}{n})'-1}\right)^{-1}}+1\right)^{-1} \in \left(\frac{n}{s}, p\right),$$

$$\alpha = 6\max\left(\frac{1}{\frac{ps}{n}-1}, \frac{1}{\frac{2s}{n}-1}\right) > 0,$$

and $c(s,n)$, $C(p,s,n)$ are constants depending only on the indicated parameters.

Remark 5.2. The conclusion of Theorem 5.1 is also valid under condition (3) on σ.

Proof. Let

$$K = \sup_{j\in\mathbb{Z}} \left\| (I-\Delta)^{\frac{s}{2}}[\hat{\Psi}\sigma(2^j\cdot)] \right\|_{L^{\frac{n}{s},1}}.$$

Introduce the function Θ defined by

$$\hat{\Theta} = \hat{\Psi}\left(\frac{\xi}{2}\right) + \hat{\Psi}(\xi) + \hat{\Psi}(2\xi),$$

and observe that

$$\hat{\Theta}(\xi) = 1 \quad \text{on } \operatorname{supp}\hat{\Psi}.$$

Let Δ_j and Δ_j^Θ denote the Littlewood–Paley operators associated with Ψ and Θ, respectively, i.e.,

$$\Delta_j f(x) \equiv \int_{\mathbb{R}^n} \hat{\Psi}(2^{-j}\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,\mathrm{d}\xi, \quad \Delta_j^\Theta f(x) \equiv \int_{\mathbb{R}^n} \hat{\Theta}(2^{-j}\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,\mathrm{d}\xi.$$

Then, we have

$$\begin{aligned}
\Delta_j T_\sigma(f)(x) &= \int_{\mathbb{R}^n} \widehat{f}(\xi)\widehat{\Psi}\left(2^{-j}\xi\right)\sigma(\xi)e^{2\pi i x\cdot\xi}d\xi \\
&= \int_{\mathbb{R}^n} \left(\Delta_j^\Theta f\right)^\wedge(\xi)\widehat{\Psi}\left(2^{-j}\xi\right)\sigma(\xi)e^{2\pi i x\cdot\xi}d\xi \\
&= 2^{jn}\int_{\mathbb{R}^n} \left(\Delta_j^\Theta f\right)^\wedge\left(2^j\xi'\right)\widehat{\Psi}\left(\xi'\right)\sigma\left(2^j\xi'\right)e^{2\pi i x\cdot 2^j\xi'}d\xi' \\
&= 2^{jn}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\Delta_j^\Theta f\right)(y)e^{-2\pi i y\cdot 2^j\xi'}dy\widehat{\Psi}\left(\xi'\right)\sigma\left(2^j\xi'\right)e^{2\pi i x\cdot 2^j\xi'}d\xi' \\
&= 2^{jn}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} e^{-2\pi i y\cdot 2^j\xi'}\widehat{\Psi}\left(\xi'\right)\sigma\left(2^j\xi'\right)e^{2\pi i x\cdot 2^j\xi'}d\xi'\right)\left(\Delta_j^\Theta f\right)(y)dy \\
&= \int_{\mathbb{R}^n}\left(\Delta_j^\Theta f\right)\left(x+2^{-j}y\right)\left[\widehat{\Psi}\sigma\left(2^j\cdot\right)\right]^\wedge(y)dy \\
&= \int_{\mathbb{R}^n}\frac{\left(\Delta_j^\Theta f\right)\left(x+2^{-j}y\right)}{(1+|y|)^s}(1+|y|)^s\left[\widehat{\Psi}\sigma\left(2^j\cdot\right)\right]^\wedge(y)dy.
\end{aligned}$$

By Hölder's and Hausdorff–Young's inequalities on Lorentz spaces we obtain

$$\begin{aligned}
|\Delta_j T_\sigma(f)(x)| &\le \left\|\frac{\left(\Delta_j^\Theta f\right)\left(x+2^{-j}y\right)}{(1+|y|)^s}\right\|_{L^{\frac{n}{s},\infty}(\mathbb{R}^n)}\left\|(1+|y|)^s\left[\widehat{\Psi}\sigma\left(2^j\cdot\right)\right]^\wedge(y)\right\|_{L^{(\frac{n}{s})',1}(\mathbb{R}^n)} \\
&\le C'\left\|\frac{\left(\Delta_j^\Theta f\right)\left(x+2^{-j}y\right)}{(1+|y|)^s}\right\|_{L^{\frac{n}{s},\infty}(\mathbb{R}^n)}\left\|(I-\Delta)^{\frac{s}{2}}\left[\widehat{\Psi}\sigma\left(2^j\cdot\right)\right]\right\|_{L^{\frac{n}{s},1}(\mathbb{R}^n)} \qquad (38)\\
&\le C'K\left\|\frac{\left(\Delta_j^\Theta f\right)\left(x+2^{-j}y\right)}{(1+|y|)^s}\right\|_{L^{\frac{n}{s},\infty}(\mathbb{R}^n)}.
\end{aligned}$$

Note that by Lemma 3.1, there exists $q \in (n/s, \min(2,p))$ which is close to $\frac{n}{s}$ such that $w \in A_{p/q}$. Then for this q, applying Lemma 3.16, we get

$$\left\| \frac{\Delta_j^\Theta f(x+2^{-j}\cdot)}{(1+|\cdot|)^s} \right\|_{L^{\frac{n}{s},\infty}} \le \frac{C_0(s,n)}{1-2^{n-sq}} M_q[\Delta_j^\Theta f](x). \tag{39}$$

Combining (38) and (39) we deduce for any $x \in \mathbb{R}^n$

$$|\Delta_j T_\sigma(f)(x)| \le C' K \frac{C_0(s,n)}{1-2^{n-sq}} M_q[\Delta_j^\Theta f](x). \tag{40}$$

Applying (24) in Theorem 3.13, (40), Theorem 3.14, Theorem 3.15, and (23) in Theorem 3.13 we obtain

$$\begin{aligned}
&\|T_\sigma f\|_{L^p(w)} \\
&\le 3 c_n^{p^2 p'} [w]_{A_p}^{\max(1,\frac{1}{p-1})} \left\| \left(\sum_{j\in\mathbb{Z}} |\Delta_j T_\sigma f|^2 \right)^{\frac12} \right\|_{L^p(w)} \\
&\le 3 \frac{C_0(s,n)}{1-2^{n-sq}} c_n^{p^2 p'} [w]_{A_p}^{\max(1,\frac{1}{p-1})} K \left\| \left(\sum_{j\in\mathbb{Z}} (M_q[\Delta_j^\Theta f])^2 \right)^{\frac12} \right\|_{L^p(w)} \\
&\le \frac{C(s,n)2^{sp}}{q-\frac{n}{s}} c_n^{p^2 p'} [w]_{A_p}^{\max(1,\frac{1}{p-1})} K \left\| \left(\sum_{j\in\mathbb{Z}} (M|\Delta_j^\Theta f|^q)^{\frac{2}{q}} \right)^{\frac{q}{2}} \right\|_{L^{\frac{p}{q}}(w)}^{\frac{1}{q}} \\
&\le \left(c_n^{\frac{p}{q}(\frac{p}{q})'\frac{2}{q}(\frac{2}{q})'} \right)^{\max\left(\frac{\frac{2}{q}-\frac{p}{q}}{\frac{p}{q}-1}+1,1\right)} \frac{C(s,n)2^{sp}}{q-\frac{n}{s}} \; c_n^{p^2p'} [w]_{A_{\frac{p}{q}}}^{2\max\left(\frac{1}{\frac{p}{q}-1},\frac{1}{\frac{2}{q}-1}\right)} K \left\| \left(\sum_{j\in\mathbb{Z}} |\Delta_j^\Theta f|^{q\cdot\frac{2}{q}} \right)^{\frac{q}{2}} \right\|_{L^{\frac{p}{q}}(w)}^{\frac{1}{q}} \\
&= \left(c_n^{\frac{p}{q}(\frac{p}{q})'\frac{2}{q}(\frac{2}{q})'} \right)^{\max\left(\frac{\frac{2}{q}-1}{\frac{p}{q}-1},1\right)} \frac{C(s,n)2^{sp}}{q-\frac{n}{s}} \; c_n^{p^2p'} [w]_{A_{\frac{p}{q}}}^{2\max\left(\frac{1}{\frac{p}{q}-1},\frac{1}{\frac{2}{q}-1}\right)} K \left\| \left(\sum_{j\in\mathbb{Z}} |\Delta_j^\Theta f|^2 \right)^{\frac12} \right\|_{L^p(w)} \\
&\le \left(c_n^{\frac{p}{q}(\frac{p}{q})'\frac{2}{q}(\frac{2}{q})'} \right)^{\max\left(\frac{\frac{2}{q}-1}{\frac{p}{q}-1},1\right)} \frac{C(s,n)2^{sp}}{q-\frac{n}{s}} c_n^{2p^2p'} [w]_{A_{\frac{p}{q}}}^{3\max\left(\frac{1}{\frac{p}{q}-1},\frac{1}{\frac{2}{q}-1}\right)} K \|f\|_{L^p(w)} \\
&\le \left(c_n^{\frac{ps}{n}\cdot 2(\frac{ps}{n})'\frac{2s}{n}\cdot 2(\frac{2s}{n})'} \right)^{\max\left(2(\frac{2-\frac{n}{s}}{p-\frac{n}{s}}),1\right)} \frac{C(s,n)2^{sp}}{q-\frac{n}{s}} c_n^{2p^2p'} \left(2^{\frac{ps}{n}-1} [w]_{A_{\frac{ps}{n}}} \right)^{6\max\left(\frac{1}{\frac{ps}{n}-1},\frac{1}{\frac{2s}{n}-1}\right)} K \|f\|_{L^p(w)},
\end{aligned}$$

where in the third inequality, since $q \in (n/s, \min(2,p))$ which is close to $\frac{n}{s}$ such that $w \in A_{p/q}$, which implies $\frac{p}{q} > 1$ and $\frac{2}{q} > 1$. In the fourth inequality, we applied Theorem 3.15 with the corresponding constant

$$\left(c_n^{\frac{p}{q}(\frac{p}{q})'\frac{2}{q}(\frac{2}{q})'} \right)^{\max\left(\frac{\frac{2}{q}-\frac{p}{q}}{\frac{p}{q}-1}+1,1\right)} [w]_{A_{\frac{p}{q}}}^{\max\left(1,\frac{1}{\frac{p}{q}-1},\frac{1}{\frac{2}{q}-1},\frac{\frac{2}{q}-1}{\frac{p}{q}-1}\right)}.$$

Note that $\max\left(1, \frac{1}{\frac{p}{q}-1}, \frac{1}{\frac{2}{q}-1}, \frac{\frac{2}{q}-1}{\frac{p}{q}-1}\right) \leq \max\left(\frac{1}{\frac{p}{q}-1}, \frac{1}{\frac{2}{q}-1}\right)$. Moreover, by Theorem 3.14, q can be chosen as follows:

$$q = p\left(\frac{\frac{ps}{n}-1}{1+\left(c(s,n)[w]_{A_{\frac{ps}{n}}}^{(\frac{ps}{n})'-1}\right)^{-1}}+1\right)^{-1}$$

with sufficiently large $c(s,n)$ so that

$$\frac{1}{\frac{p}{q}-1} \leq \frac{2}{\frac{ps}{n}-1}, \qquad \frac{1}{\frac{2}{q}-1} \leq \frac{2}{\frac{2s}{n}-1},$$

which imply

$$\frac{p}{q} \leq \frac{ps}{n}, \qquad \left(\frac{p}{q}\right)' \leq 2\left(\frac{ps}{n}\right)', \qquad \frac{2}{q} \leq \frac{2s}{n}, \qquad \left(\frac{2}{q}\right)' \leq 2\left(\frac{2s}{n}\right)'.$$

These estimates justify the inequalities in the preceding alignment. Now letting

$$q(t) = p\left(\frac{\frac{ps}{n}-1}{1+\left(c(s,n)t^{(\frac{ps}{n})'-1}\right)^{-1}}+1\right)^{-1},$$

we deduce the constant in the claimed bound in (37). □

6. Interpolation theorem for Fourier multipliers

The main result of this section is an interpolation theorem for Fourier multipliers on spaces of weights whose powers lie in intersections of A_p and RH_s classes. This is based on an extension of the 3-lines lemma due to Hirschmann [16].

For the purposes of the following theorem, we define

$$\mathcal{B} = \{f \in L^\infty : \mathbb{R}^n \to \mathbb{C} |\ \operatorname{supp} f \subset \overline{B(0,R)} \setminus B(0,\epsilon), \quad \text{for some } 0<\epsilon<R<\infty\}$$

the space of all bounded functions whose support is compact and misses the origin. Note that for any fixed $p \in (1,\infty)$, $\mathcal{B}$ is a dense subset of L^p. Moreover, we have

$$\int_{\mathbb{R}^n} |f|^p w\, dx \leq \|f\|_{L^\infty}^p \int_{\operatorname{supp} f} w\, dx < \infty,$$

and $\mathcal{B}$ is dense in $L^p(w)$ for any weight w by [12, Proposition 8.2.2], so $\mathcal{B}$ is a dense subset of $L^p(w)$ for any weight w. In addition, for a fixed weight w, we set

$$w_{\delta,M} := \delta\chi_{\{w\leq\delta\}} + w\chi_{\{\delta<w<M\}} + M\chi_{\{w\geq M\}}$$

for $0<\delta<M<\infty$. Then notice that

$$\lim_{\delta\to0, M\to\infty} w_{\delta,M} = w \quad \text{pointwise.}$$

Theorem 6.1. *Let* $0<\theta<1, 1<p,p_i<\infty$, $1<q,q_i<\infty$, $0<s,s_i,s^*,s_i^*<n$, $\frac{s_i^*}{n} \geq \frac{1}{p_i}$ *and* $r_i > \frac{n}{s_i}$, $i\in\{0,1\}$. *Suppose that*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad s = (1-\theta)s_0 + \theta s_1, \quad s^* = (1-\theta)s_0^* + \theta s_1^*,$$

$$\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \quad \frac{1}{pq} = \frac{1-\theta}{p_0q_0} + \frac{\theta}{p_1q_1}.$$

Assume that for each $i \in \{0,1\}$ there exists increasing functions $M_i(\cdot)$ such that for all $w_i \in A_{\frac{p_i s_i^*}{n}} \cap RH_{q_i} = \mathcal{A}_{\rho_i}^{q_i}$, where $\rho_i = (\frac{p_i s_i^*}{n} - 1)q_i + 1$, for all σ satisfying

$$\sup_{j\in\mathbb{Z}} \|(I-\Delta)^{\frac{s_i}{2}}(\hat{\Psi}\sigma(2^j\cdot))\|_{L^{r_i}} < \infty$$

and all $f \in \mathcal{B}$ we have

$$\|T_\sigma f\|_{L^{p_i}(w_i)} \le M_i([w_i^{q_i}]_{A_{\rho_i}}) \sup_{j\in\mathbb{Z}} \|(I-\Delta)^{\frac{s_i}{2}}(\hat{\Psi}\sigma(2^j\cdot))\|_{L^{r_i}} \|f\|_{L^{p_i}(w_i)}. \tag{41}$$

Then for every $w \in A_{\frac{ps^*}{n}} \cap RH_q = \mathcal{A}_\rho^q$, where $\rho = (\frac{ps^*}{n} - 1)q + 1$, there is an increasing function M_θ such that for all $f \in \mathcal{B}$ one has

$$\|T_\sigma f\|_{L^p(w)} \le C\, M_\theta([w^q]_{A_\rho}) \sup_{j\in\mathbb{Z}} \|(I-\Delta)^{\frac{s}{2}}(\hat{\Psi}\sigma(2^j\cdot))\|_{L^r} \|f\|_{L^p(w)} \tag{42}$$

for all σ satisfying $\sup_{j\in\mathbb{Z}} \|(I-\Delta)^{\frac{s}{2}}(\hat{\Psi}\sigma(2^j\cdot))\|_{L^r} < \infty$ with $C = C(s_0, s_1, r_0, r_1, n, \theta, \widehat{\Psi})$.

Proof. For a bounded function σ satisfying

$$\sup_{j\in\mathbb{Z}} \left\|(I-\Delta)^{\frac{s}{2}}[\hat{\Psi}\sigma(2^j\cdot)\right\|_{L^r} < \infty \quad \text{with} \quad r > \frac{n}{s}, \tag{43}$$

we denote

$$\varphi_j = (I-\Delta)^{\frac{s}{2}}[\hat{\Psi}\sigma(2^j\cdot)], \quad j \in \mathbb{Z}.$$

For a complex number z with $0 \le \Re(z) \le 1$, we define

$$\sigma_z(\xi) = \sum_{k\in\mathbb{Z}} (I-\Delta)^{-\frac{(1-z)s_0+zs_1}{2}} \left[|\varphi_k|^{\frac{r}{r_0}(1-z)+\frac{r}{r_1}z} e^{iArg\varphi_k}\right](2^{-k}\xi)\hat{\Phi}(2^{-k}\xi), \tag{44}$$

where $\hat{\Phi}$ is a Schwartz function supported in the set $\{\xi \in \mathbb{R}^n : 1/4 \le |\xi| \le 4\}$ and $\hat{\Phi} \equiv 1$ in the support of $\hat{\Psi}$.

Fix $f, g \in \mathcal{B}$, Given $\varepsilon > 0$, by the density we can choose $f_*, g_* \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that

$$\|f_* - f\|_{L^p} < \varepsilon, \quad \|g_* - g\|_{L^{p'}} < \varepsilon, \quad \text{and } \|f_* - f\|_{L^2} < \varepsilon, \quad \|g_* - g\|_{L^2} < \varepsilon \tag{45}$$

for any fixed $p > 1$. Let f_z^ε and g_z^ε be functions having the form (27), satisfying

$$\|f_\theta^\varepsilon - f_*\|_{L^2} < \varepsilon, \quad \|g_\theta^\varepsilon - g_*\|_{L^2} < \varepsilon, \tag{46}$$

and

$$\begin{aligned} \|f_{it}^\varepsilon\|_{L^{p_0}} &\le (\|f_*\|_{L^p} + \varepsilon)^{\frac{p}{p_0}}, \quad \|f_{1+it}^\varepsilon\|_{L^{p_1}} \le (\|f_*\|_{L^p} + \varepsilon)^{\frac{p}{p_1}}, \\ \|g_{it}^\varepsilon\|_{L^{p_0}} &\le (\|g_*\|_{L^{p'}} + \varepsilon)^{\frac{p'}{p_0'}}, \quad \|g_{1+it}^\varepsilon\|_{L^{p_1'}} \le (\|g_*\|_{L^{p'}} + \varepsilon)^{\frac{p'}{p_1'}}. \end{aligned} \tag{47}$$

Combining (45), (46) and (47), we obtain

$$\|f_\theta^\varepsilon - f\|_{L^2} < 2\varepsilon, \quad \|g_\theta^\varepsilon - g\|_{L^2} < 2\varepsilon, \tag{48}$$

and

$$\begin{aligned} \|f_{it}^\varepsilon\|_{L^{p_0}} &\le (\|f\|_{L^p} + 2\varepsilon)^{\frac{p}{p_0}}, \quad \|f_{1+it}^\varepsilon\|_{L^{p_1}} \le (\|f\|_{L^p} + 2\varepsilon)^{\frac{p}{p_1}}, \\ \|g_{it}^\varepsilon\|_{L^{p_0}} &\le (\|g\|_{L^{p'}} + 2\varepsilon)^{\frac{p'}{p_0'}}, \quad \|g_{1+it}^\varepsilon\|_{L^{p_1'}} \le (\|g\|_{L^{p'}} + 2\varepsilon)^{\frac{p'}{p_1'}}. \end{aligned} \tag{49}$$

Define

$$F(z) = \int_{\mathbb{R}^n} [\widetilde{T}_z f_z^\varepsilon(x)] g_z^\varepsilon(x) \mathrm{d}x,$$

where

$$\widetilde{T}_z f := T_{\sigma_z}\left[f\, w_{0,\delta,M}^{-\frac{1-z}{p_0}} w_{1,\delta,M}^{-\frac{z}{p_1}}\right]\ w_{0,\delta,M}^{\frac{1-z}{p_0}} w_{1,\delta,M}^{\frac{z}{p_1}}$$

and z is a complex number with $0 \le \Re(z) \le 1$.

Next we verify that F is analytic on the strip $S = \{z \in \mathbb{C} : 0 < \Re(z) < 1\}$ and continuous on its closure. Let $z \in \tau + it$ satisfy $0 \le \tau < 1$ and $t \in \mathbb{R}$. set $s_\tau = (1-\tau)s_0 + \tau s_1$ and $\frac{1}{r_\tau} = \frac{1-\tau}{r_0} + \frac{\tau}{r_1}$. By applying [12, Theorem 5.4.9 (c) and Corollary 5.3.3] we obtain

$$\begin{aligned}
\|\sigma_z\|_{L^\infty} &\le \sup_{k\in\mathbb{Z}} \left\|(I-\Delta)^{-\frac{(1-z)s_0+zs_1}{2}}\left[|\varphi_k|^{\frac{r}{r_0}(1-z)+\frac{r}{r_1}z} e^{iArg\varphi_k}\right]\right\|_{L^\infty} \\
&\le C(n,r_0,r_1,s_0,s_1) \sup_{k\in\mathbb{Z}} \left\|(I-\Delta)^{-\frac{(1-z)s_0+zs_1}{2}}\left[|\varphi_k|^{\frac{r}{r_0}(1-z)+\frac{r}{r_1}z} e^{iArg\varphi_k}\right]\right\|_{L^{r_\tau}_{s_\tau}} \\
&\le C(n,r_0,r_1,s_0,s_1) \sup_{k\in\mathbb{Z}} \left\|(I-\Delta)^{it\frac{s_0-s_1}{2}}\left[|\varphi_k|^{\frac{r}{r_0}(1-z)+\frac{r}{r_1}z} e^{iArg\varphi_k}\right]\right\|_{L^{r_\tau}} \\
&\le C'(n,r_0,r_1,s_0,s_1)(1+|t|)^{[\frac{n}{2}]+1} \sup_{k\in\mathbb{Z}} \left\||\varphi_k|^{\frac{r}{r_0}(1-z)+\frac{r}{r_1}z} e^{iArg\varphi_k}\right\|_{L^{r_\tau}} \\
&= C'(n,r_0,r_1,s_0,s_1)(1+|t|)^{[\frac{n}{2}]+1} \sup_{k\in\mathbb{Z}} \|\varphi_k\|_{L^r}^{\frac{r}{r_\tau}}.
\end{aligned}$$

Here the constant $C(n,r_0,r_1,s_0,s_1)$ is obtained from [12, Theorem 5.4.9 (c) and Proposition 5.2.3]. Additionally, we have the estimate

$$\|f_z^\varepsilon\|_{L^2}^2 = \sum_{j=1}^{N_\varepsilon} |c_j^\varepsilon|^{(1-\Re(z))\frac{2p}{p_0}+\Re(z)\frac{2p}{p_1}} \|h_j^\varepsilon\|_{L^2} \le C(p_0,p_1,\varepsilon,f) < \infty.$$

Similarly, one obtains

$$\|g_z^\varepsilon\|_{L^2}^2 \le C(p_0,p_1,\varepsilon,g) < \infty.$$

Then we have

$$\begin{aligned}
|F(z)| &= \left|\int_{\mathbb{R}^n} \sigma_z \left[f_z^\varepsilon \cdot w_{0,\delta,M}^{-\frac{1-z}{p_0}} w_{1,\delta,M}^{-\frac{z}{p_1}}\right]^\wedge(\xi) \left[g_z^\varepsilon \cdot w_{0,\delta,M}^{\frac{1-z}{p_0}} w_{1,\delta,M}^{\frac{z}{p_1}}\right]^\vee(\xi)\, \mathrm{d}\xi\right| \\
&\le \|\sigma_z\|_{L^\infty} \int_{\mathbb{R}^n} \left|\left[f_z^\varepsilon \cdot w_{0,\delta,M}^{-\frac{1-z}{p_0}} w_{1,\delta,M}^{-\frac{z}{p_1}}\right]^\wedge(\xi) \left[g_z^\varepsilon \cdot w_{0,\delta,M}^{\frac{1-z}{p_0}} w_{1,\delta,M}^{\frac{z}{p_1}}\right]^\vee(\xi)\right| \mathrm{d}\xi \\
&\le \|\sigma_z\|_{L^\infty} \left\|f_z^\varepsilon \cdot w_{0,\delta,M}^{-\frac{1-\Re(z)}{p_0}}\ w_{1,\delta,M}^{-\frac{\Re(z)}{p_1}}\right\|_{L^2} \left\|g_z^\varepsilon \cdot w_{0,\delta,M}^{\frac{1-\Re(z)}{p_0}}\ w_{1,\delta,M}^{\frac{\Re(z)}{p_1}}\right\|_{L^2} \\
&\le C(n,r_0,r_1,s_0,s_1,\varepsilon,f,g,\delta,M)(1+|t|)^{[\frac{n}{2}]+1} \sup_{j\in\mathbb{Z}} \|\varphi_j\|_{L^r}^{\frac{r}{r_\tau}}.
\end{aligned} \tag{50}$$

Here the Fourier and inverse Fourier transform above are well defined. In fact,

$$f_z^\varepsilon \cdot w_{0,\delta,M}^{-\frac{1-z}{p_0}} w_{1,\delta,M}^{-\frac{z}{p_1}}$$

is a bounded and compactly supported function, so

$$f_z^\varepsilon \cdot w_{0,\delta,M}^{-\frac{1-z}{p_0}} w_{1,\delta,M}^{-\frac{z}{p_1}}$$

lies in $L^2(\mathbb{R}^n)$. This implies

$$\left[f_z^{\varepsilon}\cdot w_{0,\delta,M}^{-\frac{1-z}{p_0}} w_{1,\delta,M}^{-\frac{z}{p_1}}\right]^{\wedge} \in L^2(\mathbb{R}^n).$$

Moreover, $\sigma_z \in L^\infty$, then we have

$$\sigma_z\left[f_z^{\varepsilon}\cdot w_{0,\delta,M}^{-\frac{1-z}{p_0}} w_{1,\delta,M}^{-\frac{z}{p_1}}\right]^{\wedge} \in L^2(\mathbb{R}^n).$$

Hence we can get the first equality of (50) by [12, Prop 2.3.7].

For any point z_0 in S, pick $\delta > 0$ such that

$$2\delta < \Re\Big(\frac{z_0}{p_1}\Big) < 1.$$

By [12, lemma 2.7.5] for $0 < \frac{z}{p_1} < \delta$, we obtain

$$\left|\frac{w_{1,\delta,M}^{\frac{z}{p_1}} - 1}{z}\right| \le \frac{p_1}{\delta}\max\{w_{1,\delta,M}^{2\delta}, w_{1,\delta,M}^{-2\delta}\},$$

which gives

$$\left|\frac{w_{1,\delta,M}^{\frac{z+z_0}{p_1}} - w_{1,\delta,M}^{\frac{z_0}{p_1}}}{z}\right| \le \frac{p_1}{\delta}\max\{w_{1,\delta,M}^{2}, 1\} \le \frac{p_1}{\delta}\max\{M^2, 1\}.$$

In addition, from the argument above of (50), the Fourier transform and the inverse Fourier transform at the first line is in L^∞, so we can apply the Lebesgue dominated convergence theorem when the z derivative hits the term $w_{1,\delta,M}^{\frac{z}{p_1}}$. Also we use the similar way to deal with other terms in $F(z)$ involving with z. This yields the analyticity of F on S, while its continuity on $\bar{S}$ is simpler and omitted.

Then, we consider $F(1+it)$ and $F(it)$. For the function $\widehat{\Psi}\,\sigma_{1+it}(2^j\cdot)$, only finitely many indices k near j contribute nonzero terms in (44). Therefore, for each $j \in \mathbb{Z}$ we have

$$\begin{aligned}
&\left\|(I-\Delta)^{\frac{s_1}{2}}[\hat{\Psi}\sigma_{1+it}(2^j\cdot)]\right\|_{L^{r_1}}\\
&\le \left\|(I-\Delta)^{\frac{s_1}{2}}\left[\widehat{\Psi}(I-\Delta)^{-\frac{s_1+it(s_1-s_0)}{2}}\left[|\varphi_j|^{\frac{r}{r_0}(-it)+\frac{r}{r_1}(1+it)}e^{iArg\varphi_j}\right]\right]\right\|_{L^{r_1}}\\
&\le C_1'(s_1,r_1,\widehat{\Psi})\left\|(I-\Delta)^{\frac{s_1}{2}}\left[(I-\Delta)^{-\frac{s_1+it(s_1-s_0)}{2}}\left[|\varphi_j|^{\frac{r}{r_0}(-it)+\frac{r}{r_1}(1+it)}e^{iArg\varphi_j}\right]\right]\right\|_{L^{r_1}}\\
&\le C_1'(s_1,r_1,\widehat{\Psi})\left\|(I-\Delta)^{-\frac{it(s_1-s_0)}{2}}\left[|\varphi_j|^{\frac{r}{r_0}(-it)+\frac{r}{r_1}(1+it)}e^{iArg\varphi_j}\right]\right\|_{L^{r_1}}\\
&\le C_1(s_0,s_1,r_1,n,\widehat{\Psi})(1+|t|)^{\frac{n}{2}+1}\left\||\varphi_j|^{\frac{r}{r_0}(-it)+\frac{r}{r_1}(1+it)}\right\|_{L^{r_1}}\\
&= C_1(s_0,s_1,r_1,n,\widehat{\Psi})(1+|t|)^{\frac{n}{2}+1}\left\|\varphi_j\right\|_{L^r}^{\frac{r}{r_1}},
\end{aligned} \tag{51}$$

where we have used [12, Corollary 5.3.3] for the fourth inequality. By Hölder's inequality and (41) we obtain

$$\begin{aligned} |F(1+it)| &\le \|\widetilde{T}_{1+it} f^{\varepsilon}_{1+it}\|_{L^{p_1}} \|g^{\varepsilon}_{1+it}\|_{L^{p_1'}} \\ &= \|T_{\sigma_{1+it}}(f^{\varepsilon}_{1+it} w_{0,\delta,M}^{-\frac{it}{p_0}} w_{1,\delta,M}^{-\frac{1+it}{p_1}})\|_{L^{p_1}(w_{1,\delta,M})} \|g^{\varepsilon}_{1+it}\|_{L^{p_1'}} \\ &\le M_1\big([w^{q_1}_{1,\delta,M}]_{A_{\rho_1}}\big) \sup_{j\in\mathbb{Z}} \left\|(I-\Delta)^{\frac{s_1}{2}}[\hat{\Psi}\sigma_{1+it}(2^j\cdot)]\right\|_{L^{r_1}} \|f^{\varepsilon}_{1+it}\|_{L^{p_1}} \|g^{\varepsilon}_{1+it}\|_{L^{p_1'}}. \end{aligned} \tag{52}$$

Note that $f^{\varepsilon}_{1+it} \in \mathcal{C}^{\infty}_0(\mathbb{R}^n)$ implies $f^{\varepsilon}_{1+it} w_{0,\delta,M}^{-\frac{it}{p_0}} w_{1,\delta,M}^{-\frac{1+it}{p_1}} \in \mathcal{B}$, and $w_{1,\delta,M} \in A_{\frac{p_1 s_1^*}{n}} \cap RH_{q_1}$. Combining (49), (51) and (52) we deduce

$$\begin{aligned} |F(1+it)| \le\, & M_1\big([w^{q_1}_{1,\delta,M}]_{A_{\rho_1}}\big) C_1(s_0,s_1,r_1,n,\widehat{\Psi})(1+|t|)^{\frac{n}{2}+1} \\ & \sup_{j\in\mathbb{Z}} \|\varphi_j\|_{L^r}^{\frac{r}{r_1}} (\|f\|_{L^p}+\varepsilon)^{\frac{p}{p_1}} (\|g\|_{L^{p'}}+\varepsilon)^{\frac{p'}{p_1'}}. \end{aligned} \tag{53}$$

Similarly, for $\hat{\Psi}\sigma_{it}(2^j\cdot)$, only finitely many indices k near j contribute in (44). Therefore, for each $j \in \mathbb{Z}$, we have

$$\begin{aligned} &\left\|(I-\Delta)^{\frac{s_0}{2}}[\hat{\Psi}\sigma_{it}(2^j\cdot)]\right\|_{L^{r_0}} \\ &\le \left\|(I-\Delta)^{\frac{s_0}{2}}\left[\widehat{\Psi}(I-\Delta)^{-\frac{s_0+it(s_1-s_0)}{2}}\left[|\varphi_j|^{\frac{r}{r_0}(1-it)+\frac{r}{r_1}it} e^{iArg\varphi_j}\right]\right]\right\|_{L^{r_0}} \\ &\le C_2'(s_0,r_0,\widehat{\Psi}) \left\|(I-\Delta)^{\frac{s_0}{2}}\left[(I-\Delta)^{-\frac{s_0+it(s_1-s_0)}{2}}\left[|\varphi_j|^{\frac{r}{r_0}(1-it)+\frac{r}{r_1}it} e^{iArg\varphi_j}\right]\right]\right\|_{L^{r_0}} \\ &= C_2'(s_0,r_0,\widehat{\Psi}) \left\|(I-\Delta)^{-\frac{it(s_1-s_0)}{2}}\left[|\varphi_j|^{\frac{r}{r_0}(1-it)+\frac{r}{r_1}it} e^{iArg\varphi_j}\right]\right\|_{L^{r_0}} \\ &\le C_2(s_0,r_0,s_1,n,\widehat{\Psi})(1+|t|)^{\frac{n}{2}+1} \left\||\varphi_j|^{\frac{r}{r_0}(1-it)+\frac{r}{r_1}it} e^{iArg\varphi_j}\right\|_{L^{r_0}} \\ &= C_2(s_0,r_0,s_1,n,\widehat{\Psi})(1+|t|)^{\frac{n}{2}+1} \|\varphi_j\|_{L^r}^{\frac{r}{r_0}}. \end{aligned} \tag{54}$$

Now we can apply the Hölder's inequality and (41) to obtain

$$\begin{aligned} |F(it)| &\le \|\widetilde{T}_{it} f^{\varepsilon}_{it}\|_{L^{p_0}} \|g^{\varepsilon}_{it}\|_{L^{p_0'}} \\ &\le M_0\big([w^{q_0}_{0,\delta,M}]_{A_{\rho_0}}\big) \sup_{j\in\mathbb{Z}} \left\|(I-\Delta)^{\frac{s_0}{2}}[\hat{\Psi}\sigma_{it}(2^j\cdot)]\right\|_{L^{r_0}} \|f^{\varepsilon}_{it}\|_{L^{p_0}} \|g^{\varepsilon}_{it}\|_{L^{p_0'}}. \end{aligned}$$

Here $f^{\varepsilon}_{it} \in \mathcal{C}^{\infty}_0(\mathbb{R}^n)$ implies $f^{\varepsilon}_{it} w_{0,\delta,M}^{-\frac{1-it}{p_0}} w_{1,\delta,M}^{-\frac{it}{p_1}} \in \mathcal{B}$ and the truncation ensures that $w_{0,\delta,M}$ lies in $A_{\frac{p_0 s_0^*}{n}} \cap RH_{q_0}$. Therefore,

$$\begin{aligned} |F(it)| \le\, & M_0\big([w^{q_0}_{0,\delta,M}]_{A_{\rho_0}}\big) C_2(s_0,r_0,s_1,n,\widehat{\Psi}) \\ & (1+|t|)^{\frac{n}{2}+1} \sup_{j\in\mathbb{Z}} \|\varphi_j\|_{L^r}^{\frac{r}{r_0}} (\|f\|_{L^p}+\varepsilon)^{\frac{p}{p_0}} (\|g\|_{L^{p'}}+\varepsilon)^{\frac{p'}{p_0'}}. \end{aligned} \tag{55}$$

A combination of (53), (55) and [12, Corollary C.0.3] yields

$$\begin{aligned} |F(\theta)| \le\, & C(s_0,s_1,r_0,r_1,n,\theta,\widehat{\Psi}) M_0\big([w^{q_0}_{0,\delta,M}]_{A_{\rho_0}}\big)^{1-\theta} M_1\big([w^{q_1}_{1,\delta,M}]_{A_{\rho_1}}\big)^{\theta} \\ & \sup_{j\in\mathbb{Z}} \|\varphi_j\|_{L^r} (\|f\|_{L^p}+\varepsilon)(\|g\|_{L^{p'}}+\varepsilon) \end{aligned} \tag{56}$$

for all $f,g\in\mathcal{B}$ where $C(s_0,s_1,r_0,r_1,n,\theta,\widehat{\Psi}):=C_2(s_0,r_0,s_1,n,\widehat{\Psi})^{1-\theta}C_1(s_0,s_1,r_1,n,\widehat{\Psi})^{\theta}$

Consider the following inequality

$$\left|\int_{\mathbb{R}^n}(\widetilde{T}_\theta f)g\,\mathrm{d}x\right|\le|F(\theta)|+\left|\int_{\mathbb{R}^n}(\widetilde{T}_\theta f)g\,\mathrm{d}x-\int_{\mathbb{R}^n}(\widetilde{T}_\theta f_\theta^\varepsilon)g_\theta^\varepsilon\,\mathrm{d}x\right|, \tag{57}$$

where $f,g\in\mathcal{B}$ and $f_\theta^\varepsilon,g_\theta^\varepsilon\in\mathcal{C}_0^\infty(\mathbb{R}^n)$. Set

$$w_{\theta,\delta,M}=w_{0,\delta,M}^{\frac{1-\theta}{p_0}}w_{1,\delta,M}^{\frac{\theta}{p_1}}\in(\delta^{\frac{1-\theta}{p_0}+\frac{\theta}{p_1}},M^{\frac{1-\theta}{p_0}+\frac{\theta}{p_1}})$$

and estimate the second term in (57) as follows

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}(\widetilde{T}_\theta f)g\,\mathrm{d}x-\int_{\mathbb{R}^n}(\widetilde{T}_\theta f_\theta^\varepsilon)g_\theta^\varepsilon\,\mathrm{d}x\right|\\
&=\left|\int_{\mathbb{R}^n}(\sigma_\theta\widehat{fw_{\theta,\delta,M}^{-1}})^\vee\cdot gw_{\theta,\delta,M}-(\sigma_\theta\widehat{f_\theta^\varepsilon w_{\theta,\delta,M}^{-1}})^\vee\cdot g_\theta^\varepsilon w_{\theta,\delta,M}\,\mathrm{d}x\right|\\
&=\left|\int_{\mathbb{R}^n}\sigma_\theta\widehat{fw_{\theta,\delta,M}^{-1}}\cdot(gw_{\theta,\delta,M})^\vee-\sigma_\theta\widehat{f_\theta^\varepsilon w_{\theta,\delta,M}^{-1}}\cdot(g_\theta^\varepsilon w_{\theta,\delta,M})^\vee\,\mathrm{d}x\right|\\
&=\left|\int_{\mathbb{R}^n}\sigma_\theta\left[\widehat{f_\theta^\varepsilon w_{\theta,\delta,M}^{-1}}(g_\theta^\varepsilon w_{\theta,\delta,M}-gw_{\theta,\delta,M})^\vee+(gw_{\theta,\delta,M})^\vee(\widehat{f_\theta^\varepsilon w_{\theta,\delta,M}^{-1}}-\widehat{fw_{\theta,\delta,M}^{-1}})\right]\,\mathrm{d}x\right|\\
&\le\|\sigma_\theta\|_\infty\Big(\|f_\theta^\varepsilon w_{\theta,\delta,M}^{-1}\|_{L^2}\|g_\theta^\varepsilon w_{\theta,\delta,M}-gw_{\theta,\delta,M}\|_{L^2}+\|gw_{\theta,\delta,M}\|_{L^2}\|f_\theta^\varepsilon w_{\theta,\delta,M}^{-1}-fw_{\theta,\delta,M}^{-1}\|_{L^2}\Big)\\
&\le C(\delta,M,p_0,p_1,q_0,q_1,q)\|\sigma_\theta\|_\infty(\|f_\theta^\varepsilon\|_{L^2}\|g_\theta^\varepsilon-g\|_{L^2}+\|g\|_{L^2}\|f_\theta^\varepsilon-f\|_{L^2}).
\end{aligned}$$

In the calculation above, $\widehat{fw_{\theta,\delta,M}^{-1}}$, $\widehat{f_\theta^\varepsilon w_{\theta,\delta,M}^{-1}}$, $\sigma_\theta\widehat{fw_\theta^{-1}}$, $\sigma_\theta\widehat{f_\theta^\varepsilon w_{\theta,\delta,M}^{-1}}$ $gw_{\theta,\delta,M}$ and $g_\theta^\varepsilon w_{\theta,\delta,M}$ are all in L^2, so the corresponding Fourier transform and the inverse Fourier transform are well defined. Letting $\varepsilon\to0$ the second term of (57) goes to 0, by using the duality argument and the fact that $\mathcal{B}$ is dense in $L^p(\mathbb{R}^n)$ we have

$$\|\widetilde{T}_\theta f\|_{L^p}\le C(s_0,s_1,r_0,r_1,n,\theta,\widehat{\Psi})M_0\big([w_{0,\delta,M}^{q_0}]_{A_{\rho_0}}\big)^{1-\theta}M_1\big([w_{1,\delta,M}^{q_1}]_{A_{\rho_1}}\big)^{\theta}\sup_{j\in\mathbb{Z}}\|\varphi_j\|_{L^r}\|f\|_{L^p}$$

for all $f\in\mathcal{B}$, equivalently, replacing $f\mapsto f\,w_{0,\delta,M}^{\frac{1-\theta}{p_0}}w_{1,\delta,M}^{\frac{\theta}{p_1}}\in\mathcal{B}$, we have

$$\begin{aligned}
\|T_\sigma f\|_{L^p(w_{0,\delta,M}^{\frac{1-\theta}{p_0}p}w_{1,\delta,M}^{\frac{\theta}{p_1}p})}&=\|T_{\sigma_\theta}f\|_{L^p(w_{0,\delta,M}^{\frac{1-\theta}{p_0}p}w_{1,\delta,M}^{\frac{\theta}{p_1}p})}\\
&\le C(s_0,s_1,r_0,r_1,n,\theta,\widehat{\Psi})M_0\big([w_{0,\delta,M}^{q_0}]_{A_{\rho_0}}\big)^{1-\theta}M_1\big([w_{1,\delta,M}^{q_1}]_{A_{\rho_1}}\big)^{\theta}\\
&\qquad\sup_{j\in\mathbb{Z}}\|\varphi_j\|_{L^r}\|f\|_{L^p(w_{0,\delta,M}^{\frac{1-\theta}{p_0}p}w_{1,\delta,M}^{\frac{\theta}{p_1}p})},
\end{aligned}$$

where $f\in\mathcal{B}\subset L^p(w_{0,\delta,M}^{\frac{1-\theta}{p_0}p}w_{1,\delta,M}^{\frac{\theta}{p_1}p})$. Now we prove that

$$\lim_{M\to\infty}\lim_{\delta\to0}\|f\|_{L^p(w_{\theta,\delta,M}^p)}=\|f\|_{L^p(w_\theta^p)}\quad\text{for all } f\in\mathcal{B}$$

where

$$w_{\theta,\delta,M}:=w_{0,\delta,M}^{\frac{1-\theta}{p_0}}w_{1,\delta,M}^{\frac{\theta}{p_1}}\qquad w_\theta:=w_0^{\frac{1-\theta}{p_0}}w_1^{\frac{\theta}{p_1}}.$$

First, fix M, we have that $|fw_{\theta,\delta,M}|^p$ is bounded by an integrable function, $|fM^{\frac{1-\theta}{p_0}+\frac{\theta}{p_1}}|^p$, so we can apply the Lebesgue dominated convergence theorem to get

$$\lim_{\delta\to 0}\|f\|_{L^p(w^p_{\theta,\delta,M})}=\|f\|_{L^p(w^p_{\theta,M})}\quad\text{with}\quad w_{\theta,M}:=w_{0,M}^{\frac{1-\theta}{p_0}}w_{1,M}^{\frac{\theta}{p_1}}.$$

Secondly, since $|fw_{\theta,M}|^p$ converges to $|fw_\theta|^p$ increasingly as $M\to\infty$, applying the Lebesgue monotone convergence theorem we have

$$\lim_{M\to\infty}\|f\|_{L^p(w^p_{\theta,M})}=\|f\|_{L^p(w^p_\theta)}.$$

By Fatou's lemma we obtain

$$\begin{aligned}\|T_\sigma f\|_{L^p(w_0^{\frac{1-\theta}{p_0}p}w_1^{\frac{\theta}{p_1}p})}&\leq C(s_0,s_1,r_0,r_1,n,\theta,\widehat{\Psi})\\&\liminf_{\delta\to 0,M\to\infty}M_0\big([w_{0,\delta,M}^{q_0}]_{A_{\rho_0}}\big)^{1-\theta}M_1\big([w_{1,\delta,M}^{q_1}]_{A_{\rho_1}}\big)^{\theta}\sup_{j\in\mathbb{Z}}\|\varphi_j\|_{L^r}\|f\|_{L^p(w_{0,\delta,M}^{\frac{1-\theta}{p_0}p}w_{1,\delta,M}^{\frac{\theta}{p_1}p})}.\end{aligned}$$

Consequently, we obtain for all $f\in\mathcal{B}$

$$\begin{aligned}\|T_\sigma f\|_{L^p(w_0^{\frac{1-\theta}{p_0}p}w_1^{\frac{\theta}{p_1}p})}&\leq C(s_0,s_1,r_0,r_1,n,\theta,\widehat{\Psi})\\&M_0\big([w_0^{q_0}]_{A_{\rho_0}}\big)^{1-\theta}M_1\big([w_1^{q_1}]_{A_{\rho_1}}\big)^{\theta}\sup_{j\in\mathbb{Z}}\|\varphi_j\|_{L^r}\|f\|_{L^p(w_0^{\frac{1-\theta}{p_0}p}w_1^{\frac{\theta}{p_1}p})}.\end{aligned}$$

Here the weight $w_0^{\frac{1-\theta}{p_0}p}w_1^{\frac{\theta}{p_1}p}$ lies in $A_{\frac{ps^*}{n}}\cap RH_q$ by Lemma 3.7.

Finally, any given $w\in A_{\frac{ps^*}{n}}\cap RH_q$ can be factored by Lemma 3.8 as $w=w_0^{\frac{1-\theta}{p_0}p}w_1^{\frac{\theta}{p_1}p}$ with $w_0\in A_{\frac{p_0s_0^*}{n}}\cap RH_{q_0}$ and $w_1\in A_{\frac{p_1s_1^*}{n}}\cap RH_{q_1}$ with (20) being valid. Setting

$$M_\theta(t):=M_0\Big(c_0t^{1+\frac{\rho_0-1}{\rho-1}}\Big)^{1-\theta}M_1\Big(c_1t^{1+\frac{\rho_1-1}{\rho-1}}\Big)^{\theta},\qquad t\geq 1,$$

and using that M_0 and M_1 are increasing functions, we obtain

$$M_0\big([w_0^{q_0}]_{A_{\rho_0}}\big)^{1-\theta}M_1\big([w_1^{q_1}]_{A_{\rho_1}}\big)^{\theta}\leq M_\theta\big([w^q]_{A_\rho}\big)$$

concluding the proof of (42) for w. Notice that M_θ is also an increasing function. □

Remark 6.2. Theorem 6.1 also holds for multipliers satisfying a Lorentz-type condition:

$$\sup_{j\in\mathbb{Z}}\|(I-\Delta)^{\frac{s_i}{2}}(\widehat{\Psi}\,\sigma(2^j\cdot))\|_{L^{r_i,1}}<\infty,$$

where $r_is_i>n$, or even the less restrictive conditions

$$\sup_{j\in\mathbb{Z}}\|(I-\Delta)^{\frac{s_i}{2}}(\widehat{\Psi}\,\sigma(2^j\cdot))\|_{L^{n/s_i,1}}<\infty.$$

Then conclusion (42) contains the corresponding Lorenz-type norm for σ on the right-hand side. The proof follows from an adaptation of the given argument to this setting.

7. Applications

In this section we use Theorem 6.1 to obtain results concerning weighted bounds for Fourier multipliers whose symbols satisfy Hörmander's condition. Our main application concerns the characterization of weights in classes of the form $A_\ell \cap RH_q$ for which such Fourier multipliers are bounded on the associated weighted Lebesgue spaces.

Theorem 7.1. *Let $1 < p < \infty$, $0 < s \le \frac{n}{2}$, $|\frac{1}{2} - \frac{1}{p}| < \frac{s}{n}$, $rs > n$ and σ be a bounded function on $\mathbb{R}^n$ which satisfies*

$$\sup_{j\in\mathbb{Z}} \left\| (I-\Delta)^{\frac{s}{2}}[\hat{\Psi}\sigma(2^j\cdot)] \right\|_{L^r} < \infty. \tag{58}$$

Define the functions

$$\bar{\ell}(\alpha) := p\Big(\frac{s}{n} + \alpha\Big), \quad \bar{q}(\alpha) := \begin{cases} (\frac{1}{p\alpha})' & 0 < \alpha \le \frac{1}{p}, \\ 1 & \alpha = 0 \end{cases} \tag{59}$$

for

$$\alpha \in \begin{cases} \left[\frac{1}{2} - \frac{s}{n}, \frac{1}{p}\right] & \text{when } 2 \le p < \infty, \\ \left[\frac{1}{p} - \frac{s}{n}, \frac{1}{2}\right] & \text{when } 1 < p < 2. \end{cases} \tag{60}$$

Then, for every weight $w \in A_{\bar{\ell}(\alpha)} \cap RH_{\bar{q}(\alpha)}$, there exists a constant C_w such that

$$\|T_\sigma f\|_{L^p(w)} \le C_w \sup_{j\in\mathbb{Z}} \left\| (I-\Delta)^{\frac{s}{2}}[\hat{\Psi}\sigma(2^j\cdot)] \right\|_{L^r} \|f\|_{L^p(w)} \qquad \text{for all } f \in \mathcal{B}. \tag{61}$$

The relevant pairs $(\frac{1}{p}, \alpha)$ of Theorem 7.1 are displayed in Figure 1.

Remark 7.2. We note that the weight classes in Theorem 7.1 are independent of the parameter r that appears in (58). Moreover, the constant C_w in (61) depends only on the Muckenhoupt characteristic constant of a suitable power of the weight w.

Proof. **Case $2 \le p < \infty$.** By the change of variable $\theta = 1 - 2\alpha$, we have $\theta \in \left[1 - \frac{2}{p}, \frac{2s}{n}\right]$. We define

$$\ell(\theta) := \frac{ps}{n} + \frac{p}{2}(1-\theta),$$

$$q(\theta) := \begin{cases} \dfrac{1}{1 - \frac{p}{2}(1-\theta)} = \left(\dfrac{2}{p(1-\theta)}\right)' & \text{if } 0 \le \theta < 1, \\ 1 & \text{if } \theta = 1. \end{cases}$$

For fixed p and s, it suffices to show that (61) holds for all $f \in \mathcal{B}$ and all $w \in A_{\ell(\theta)} \cap RH_{q(\theta)}$ whenever $\theta \in [1 - \frac{2}{p}, \frac{2s}{n}]$.

When $\theta \in (1 - \frac{2}{p}, \frac{2s}{n})$, we first choose $s_0 \in (0, s)$, $s_1 \in (\frac{n}{2}, n)$, $p_0 = 2$, and $p_1 \in [\frac{p}{2}, \infty]$ such that

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad s = (1-\theta)s_0 + \theta s_1. \tag{62}$$

Then we define the following parameters for small $\varepsilon_0, \varepsilon_1 > 0$,

$$\frac{1}{r_0} = \frac{s_0}{n} - \varepsilon_0, \quad \frac{s_0^*}{n} = \frac{s_0}{n} - \varepsilon_0 + \frac{1}{2}, \quad q_0^* = \frac{4}{2-\varepsilon_0}, \quad q_0 = \Big(\frac{q_0^*}{2}\Big)' = \frac{2}{\varepsilon_0},$$
$$\frac{1}{r_1} = \frac{s_1}{n} - \varepsilon_1, \quad s_1^* = s_1, \quad q_1 = 1.$$

Next we choose $s^*(\varepsilon_0, \varepsilon_1)$, $r(\varepsilon_0, \varepsilon_1)$, and $q(\varepsilon_0, \varepsilon_1)$ such that

$$s^*(\varepsilon_0,\varepsilon_1) = (1-\theta)s_0^* + \theta s_1^*, \quad \frac{1}{r(\varepsilon_0,\varepsilon_1)} = (1-\theta)\frac{1}{r_0} + \theta\frac{1}{r_1}, \quad \frac{1}{pq(\varepsilon_0,\varepsilon_1)} = \frac{1-\theta}{p_0 q_0} + \frac{\theta}{p_1 q_1}.$$

We observe that as $\varepsilon_0, \varepsilon_1 \to 0$ we have

$$\frac{s^*(\varepsilon_0,\varepsilon_1)}{n} = (1-\theta)\Big(\frac{s_0}{n} - \varepsilon_0 + \frac{1}{2}\Big) + \theta\frac{s_1}{n} \longrightarrow \frac{s}{n} + \frac{1-\theta}{2} = \frac{\ell(\theta)}{p},$$

$$\frac{1}{r(\varepsilon_0,\varepsilon_1)} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1} \longrightarrow \frac{s}{n},$$

$$q(\varepsilon_0,\varepsilon_1) = \frac{1}{\frac{p}{2}(1-\theta)\frac{\varepsilon_0}{2} + (\frac{1}{p} - \frac{1-\theta}{2})p} \longrightarrow \frac{1}{1-\frac{p}{2}(1-\theta)} = q(\theta).$$

For any given $w \in A_{\ell(\theta)} \cap RH_{q(\theta)}$ and given $r \in (\frac{n}{s}, \infty)$, we choose sufficiently small $\varepsilon_0, \varepsilon_1$ by Lemma 3.6, such that for $q = q(\varepsilon_0, \varepsilon_1)$, $s^* = s^*(\varepsilon_0, \varepsilon_1)$ we have

$$w \in A_{\frac{ps^*}{n}} \cap RH_q$$
$$r > r(\varepsilon_0,\varepsilon_1) > \frac{n}{s}.$$

Then we apply Lemma 3.8 with weights

$$w_0 \in A_{\frac{p_0 s_0^*}{n}} \cap RH_{q_0}, \qquad w_1 \in A_{\frac{p_1 s_1^*}{n}} \cap RH_{q_1},$$

such that

$$w = w_0^{\frac{(1-\theta)p}{p_0}} w_1^{\frac{\theta p}{p_1}}.$$

We explain why there are increasing functions M_i, $i \in \{0,1\}$, such that one has

$$\|T_\sigma f\|_{L^{p_i}(w_i)} \le M_i([w_i^{q_i}]_{A_{\rho_i}}) \sup_{j\in\mathbb{Z}} \big\|(I-\Delta)^{\frac{s_i}{2}}\big[\hat{\Psi}\sigma(2^j\cdot)\big]\big\|_{L^{r_i}} \|f\|_{L^{p_i}(w_i)} \tag{63}$$

for all σ satisfying $\sup_{j\in\mathbb{Z}} \|(I-\Delta)^{\frac{s_i}{2}}(\hat{\Psi}\sigma(2^j\cdot))\|_{L^{r_i}} < \infty$, all $w_i \in A_{\frac{p_i s_i^*}{n}} \cap RH_{q_i}$ and $i \in \{0,1\}$. In particular, when $i = 0$, one verifies that

$$\frac{1}{r_0} < \frac{s_0}{n}, \qquad \frac{s_0}{n} - \frac{1}{r_0} > \frac{1}{2} - \frac{1}{q_0^*}.$$

Hence, (63) follows from Theorem 4.3 (Case 1 or Case 2), where

$$M_0(t) = C(p_0, q_0^*, r_0, s_0, \gamma)t^{\frac{2\alpha_0}{q_0}}$$

with

$$\alpha_0 = \max\Bigg(\frac{1}{p_0 - \frac{1}{\frac{1}{r_0}+\frac{1}{2}}}, \frac{q_0^*-1}{q_0^*-p_0}\Bigg).$$

For $i=1$, Theorem 5.1 yields the bound

$$M_1(t)=\frac{C(p_1,s_1,n)}{s_1q(t)-n}t^{\alpha_1}$$

$$q(t)=p_1\Bigg(\frac{\frac{p_1s_1}{n}-1}{1+\big(c(s_1,n)t^{(\frac{p_1s_1}{n})'-1}\big)^{-1}}+1\Bigg)^{-1},\quad \alpha_1=6\max\Big(\frac{1}{\frac{p_1s_1}{n}-1},\frac{1}{\frac{2s_1}{n}-1}\Big)$$

obtained in (37). Finally, applying Theorem 6.1, we deduce

$$\|T_\sigma f\|_{L^p(w)}\le C(s_0,s_1,r_0,r_1,n,\theta,\widehat{\Psi})\,C_w\sup_{j\in\mathbb{Z}}\big\|(I-\Delta)^{\frac{s}{2}}\big[\hat{\Psi}\sigma(2^j\cdot)\big]\big\|_{L^{r(\varepsilon_0,\varepsilon_1)}}\|f\|_{L^p(w)}$$

for all σ satisfying

$$\sup_{j\in\mathbb{Z}}\big\|(I-\Delta)^{\frac{s}{2}}\big[\hat{\Psi}\sigma(2^j\cdot)\big]\big\|_{L^{r(\varepsilon_0,\varepsilon_1)}}<\infty. \tag{64}$$

Here we set

$$C_w=\ M_0\Big(c_0[w^q]_{A_\rho}^{1+\frac{\rho_0-1}{\rho-1}}\Big)^{1-\theta}M_1\Big(c_1[w^q]_{A_\rho}^{1+\frac{\rho_1-1}{\rho-1}}\Big)^{\theta}.$$

Finally, we notice that all σ that satisfy (58) must also satisfy (64) by Lemma 3.10, then (61) holds.

It remains to treat the endpoint cases for θ. When $\theta=1-\frac{2}{p}$, for any $w\in A_{\frac{ps}{n}+1}\cap RH_\infty$, we can choose $\bar{\theta}>1-\frac{2}{p}$ sufficiently close to $1-\frac{2}{p}$ such that

$$\ell(\bar{\theta})<\frac{ps}{n}+1\quad\text{and}\quad w\in A_{\ell(\bar{\theta})}.$$

Since $RH_\infty=\bigcap_{q>1}RH_q$, it follows that $w\in RH_{q(\bar{\theta})}$. Hence we have

$$w\in A_{\ell(\bar{\theta})}\cap RH_{q(\bar{\theta})}.$$

Similarly, when $\theta=\frac{2s}{n}$, assume first that $s<\frac{n}{2}$. For any $w\in A_{\frac{p}{2}}\cap RH_{(\frac{2n}{p(n-2s)})'}$, there exists $\tilde{\theta}<\frac{2s}{n}$, sufficiently close to $\frac{2s}{n}$, such that

$$w\in A_{\ell(\tilde{\theta})}\cap RH_{q(\tilde{\theta})},$$

by Lemma 3.6.

Now suppose $s=\frac{n}{2}$. For any $w\in A_{\frac{p}{2}}$, Lemma 3.5 implies that there exists $r>1$ such that $w\in RH_r$. Hence, we can choose $\tilde{\theta}$ with $1-\frac{2}{p}<\tilde{\theta}<1$ such that $q(\tilde{\theta})=r$. Moreover, since $\ell(\tilde{\theta})>\frac{p}{2}$, we obtain $w\in A_{\frac{p}{2}}\subset A_{\ell(\tilde{\theta})}\cap RH_{q(\tilde{\theta})}$.

Case $1<p<2$. In this case, using the same change of variable $\theta=2\alpha-1+\frac{2s}{n}$, we have $\theta\in\big[1-\frac{2}{p'},\frac{2s}{n}\big]$. We define

$$\tilde{\ell}(\theta):=\frac{p}{2}(1+\theta),\quad \tilde{q}(\theta):=\frac{1}{p\left(\frac{s}{n}-\frac{1+\theta}{2}\right)+1}=\Bigg(\frac{2n}{p((1+\theta)n-2s)}\Bigg)'. \tag{65}$$

For fixed $1<p<2$ and $0<s\le\frac{n}{2}$, it enough to prove (61) holds for any $f\in\mathcal{B}$ and any $w\in A_{\tilde{\ell}(\theta)}\cap RH_{\tilde{q}(\theta)}$ whenever $\theta\in[1-\frac{2}{p'},\frac{2s}{n}]$. We let

$$\tilde{w}=w^{-\frac{1}{p-1}}\in A_{\frac{p's}{n}+\frac{p'}{2}(1-\theta).}\cap RH_{\frac{1}{1-\frac{p'}{2}(1-\theta)}}.$$

From the duality argument in Theorem 4.4 we have that, if

$$T_\sigma : L^{p'}(\mathbb{R}^n, \tilde{w}) \to L^{p'}(\mathbb{R}^n, \tilde{w})$$

then

$$T_\sigma : L^{p}(\mathbb{R}^n, w) \to L^{p}(\mathbb{R}^n, w).$$

with $\|T_\sigma\|_{L^p(w)\to L^p(w)} = \|T_\sigma\|_{L^{p'}(\tilde{w})\to L^{p'}(\tilde{w})}$. Moreover, we observe that

$$\begin{aligned}
\tilde{w} = w^{-\frac{1}{p-1}} \in A_{\frac{p's}{n}+\frac{p'}{2}(1-\theta)} \cap RH_{\frac{1}{1-\frac{p'}{2}(1-\theta)}} \\
&\iff w^{-\frac{1}{p-1}\bar{q}_1} \in A_{\bar{q}_1\left(\frac{p's}{n}+\frac{p'}{2}(1-\theta)\right)+1} =: A_P \\
&\iff w^{-\frac{1-p'}{P-1}\bar{q}_1} \in A_{P'} \\
&\iff w^{\frac{p'-1}{P-1}\bar{q}_1} \in A_{P'} \\
&\iff w \in A_{\tilde{\ell}(\theta)} \cap RH_{\tilde{q}(\theta)},
\end{aligned}$$

where

$$\bar{q}_1 = \frac{1}{1-\frac{p'}{2}(1-\theta)}, \quad \tilde{q}(\theta) = \frac{p'-1}{P-1}\bar{q}_1, \quad \tilde{\ell}(\theta) = \frac{1}{\tilde{q}}(P'-1)+1.$$

Eliminating the auxiliary parameter P, we obtain

$$\tilde{\ell}(\theta) = \frac{p}{2}(1+\theta), \quad \tilde{q}(\theta) = \frac{1}{p(\frac{s}{n}-\frac{1+\theta}{2})+1} = \left(\frac{2n}{p((1+\theta)n-2s)}\right)'.$$

These are exactly the relationships (65) of the indices in the statement of the theorem. □

We now show that the weight classes in Theorem 7.1 are optimal in the following sense.

Theorem 7.3. *Let $1 < p < \infty$, $0 < s \le \frac{n}{2}$, $\left|\frac{1}{2}-\frac{1}{p}\right| < \frac{s}{n}$, and $rs > n$. Let $\bar{\ell}(\alpha)$ and $\bar{q}(\alpha)$ be as in (59). Given any α as in (60), suppose that a pair $(\ell, q) \in [1,\infty]^2$ satisfies either $\ell > \bar{\ell}(\alpha)$ or $q < \bar{q}(\alpha)$. Then there exist a weight $w \in A_\ell \cap RH_q$ and a multiplier σ satisfying (58) such that T_σ is unbounded from $L^p(w)$ to $L^p(w)$.*

Proof. When $2 \le p < \infty$, if $\ell > \frac{ps}{n}+1$ or $q < (\frac{2n}{p(n-2s)})'$ then the conslusion is obtained by Theorem 2.3.

If $1 \le \ell \le \frac{ps}{n}+1$ and $(\frac{2n}{p(n-2s)})' \le q \le \infty$, then there exsits $\alpha_0 \in \left[\frac{1}{2}-\frac{s}{n}, \frac{1}{p}\right]$ such that $q = \bar{q}(\alpha_0) = (\frac{1}{p\alpha_0})'$ since the range of $\bar{q}(\alpha)$ is $\left[(\frac{2n}{p(n-2s)})', \infty\right]$. Under the assumption, there exists $\varepsilon_0 > 0$ such that $\ell = \bar{\ell}(\alpha_0+\varepsilon_0) = p(\frac{s}{n}+\alpha_0+\varepsilon_0)$. Now suppose (61) holds for all $f \in \mathcal{B}$ and all $w \in A_\ell \cap RH_q$. Let

$$\frac{1}{p_0} = \frac{s}{n}+\alpha_0+\varepsilon_0, \quad q_0 = \frac{1}{\alpha_0},$$

which implies that $p_0 \le p \le q_0$ and $p_0 < q_0$. By the relevant extrapolation theorem for such spaces [2, Theorem 4.9], it follows that for any $p_0 < h < q_0$ we have

$$\|T_\sigma f\|_{L^h(w)} \lesssim \sup_{j\in\mathbb{Z}} \left\|(I-\Delta)^{\frac{s}{2}}[\hat{\Psi}\sigma(2^j\cdot)]\right\|_{L^r} \|f\|_{L^h(w)}$$

for any $f \in \mathcal{B}$ and any $w \in A_{\frac{h}{p_0}} \cap RH_{(\frac{q_0}{h})'}$. Moreover, by Theorem 2.3, we have $\frac{h}{p_0} \le \frac{hs}{n} + 1$, which implies $\frac{1}{p_0} \le \frac{s}{n} + \frac{1}{h}$ for all $p_0 < h < q_0$. Letting $h \to q_0^-$, we obtain

$$\frac{1}{p_0} \le \frac{s}{n} + \frac{1}{q_0}\,.$$

Substituting the expressions for p_0 and q_0, this is equivalent to

$$\frac{s}{n} + \alpha_0 + \varepsilon_0 \le \frac{s}{n} + \alpha_0\,.$$

However, this inequality cannot hold whenever $\varepsilon_0 > 0$, which yields a contradiction.

When $1 < p < 2$, if $\ell > \frac{ps}{n} + \frac{p}{2}$ or $q < \frac{n}{ps}$ then the conslusion is obtained by Theorem 2.3. If $1 \le \ell \le \frac{ps}{n} + \frac{p}{2}$, then there exists $\alpha_1 \in \left[\frac{1}{p} - \frac{s}{n}, \frac{1}{2}\right]$ such that $\ell = \bar{\ell}(\alpha_1) = p(\frac{s}{n} + \alpha_1)$. Under the assumption, we can choose $\varepsilon_1 > 0$ such that $q = \bar{q}(\alpha_1 - \varepsilon_1)$. Now suppose (61) holds for any $f \in \mathcal{B}$ and any $w \in A_\ell \cap RH_q$. Let

$$\frac{1}{p_0} = \frac{s}{n} + \alpha_1, \quad q_0 = \frac{1}{\alpha_1 - \varepsilon_1}\,,$$

which implies that $p_0 \le p \le q_0$ and $p_0 < q_0$. By the extrapolation theorem [2, Theorem 4.9] it follows that for any $p_0 < h < q_0$ we get

$$\|T_\sigma f\|_{L^h(w)} \lesssim \sup_{j \in \mathbb{Z}} \left\| (I - \Delta)^{\frac{s}{2}} [\hat{\Psi} \sigma(2^j \cdot)] \right\|_{L^r} \|f\|_{L^h(w)}$$

for any $f \in \mathcal{B}$ and any $w \in A_{\frac{h}{p_0}} \cap RH_{(\frac{q_0}{h})'}$. Moreover, by Theorem 2.3, we have $(\frac{q_0}{h})' \ge \frac{n}{hs}$, which implies

$$q_0 \le \frac{n}{\frac{n}{h} - s}\,.$$

Letting $h \to p_0^+$, we obtain

$$q_0 \le \frac{n}{\frac{n}{p_0} - s} \quad \Longleftrightarrow \quad \frac{1}{\alpha_1 - \varepsilon_1} \le \frac{1}{\alpha_1}\,.$$

However, the last inequality cannot hold whenever $\varepsilon_1 > 0$, which yields a contradiction. □

Remark 7.4. The chart below lists the spaces claimed by Theorems 7.1, 4.3, and 4.4. The spaces given by Theorem 7.1 provide the optimal classes of weights in all cases.

Case	Thm 7.1	Thm 4.3 ([3], [5])	Thm 4.4
$p > 2$	$\bigcup_{\alpha \in \left[\frac{1}{2} - \frac{s}{n}, \frac{1}{p}\right]} \left(A_{p(\frac{s}{n} + \alpha)} \cap RH_{(\frac{1}{p\alpha})'} \right)$	$A_{p(\frac{1}{r} + \frac{1}{2})} \cap RH_{(\frac{q_0^*}{p})'}$ where $\frac{s}{n} - \frac{1}{r} > \frac{1}{2} - \frac{1}{q_0^*}$	$A_{\frac{p}{2}} \cap RH_{\bar{q}}$
$p = 2$	$\bigcup_{\alpha \in \left[\frac{1}{2} - \frac{s}{n}, \frac{1}{2}\right]} \left(A_{2(\frac{s}{n} + \alpha)} \cap RH_{(\frac{1}{2\alpha})'} \right)$	$A_{2(\frac{1}{r} + \frac{1}{2})} \cap RH_{(\frac{q_0^*}{2})'}$ where $\frac{s}{n} - \frac{1}{r} > \frac{1}{2} - \frac{1}{q_0^*}$	—
$1 < p < 2$	$\bigcup_{\alpha \in \left[\frac{1}{p} - \frac{s}{n}, \frac{1}{2}\right]} \left(A_{p(\frac{s}{n} + \alpha)} \cap RH_{(\frac{1}{p\alpha})'} \right)$	$A_{p(\frac{1}{r} + \frac{1}{2})} \cap RH_{\frac{2}{2-p}}$	—

TABLE 1. Outline of results in the case $0 < s < \frac{n}{2}$

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