A REMARK ON BILINEAR SQUARE FUNCTIONS

Abstract. We provide some remarks concerning a bilinear square function formed by products of Littlewood-Paley operators over arbitrary intervals. For $1 < p_1, p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$, we show that this square function is bounded from $L^{p_1}(\mathbf{R}) \times L^{p_2}(\mathbf{R})$ to $L^p(\mathbf{R})$ when $p > 2/3$ and unbounded when $p < 2/3$.

Little work is known in the area of bilinear Littlewood-Paley square functions besides the articles of Lacey [6], Diestel [4], and Bernicot [1]. In this note, we study a bilinear square function formed by products of Littlewood-Paley operators over arbitrary intervals.

Given an interval $I = [a, b)$ on $\mathbf{R}$, let Δ_I be the Littlewood-Paley operator defined by multiplication by the characteristic function of I on the Fourier transform side. The Fourier transform of an integrable function g on $\mathbf{R}$ is defined by

$$\widehat{g}(\xi) = \int_{\mathbf{R}} g(x) e^{-2\pi i x \xi}\, dx$$

and its inverse Fourier transform is defined by $g^\vee(\xi) = \widehat{g}(-\xi)$. In terms of these operators we have $\Delta_I(g) = \big(\widehat{g}\chi_I\big)^\vee$.

The Littlewood-Paley square function associated with the function f on $\mathbf{R}$ is given by

$$S(f) = \Big(\sum_{j\in\mathbf{Z}} |\Delta_{I_j}(f)|^2\Big)^{\frac{1}{2}}, \tag{1}$$

where $I_j = [-2^{j+1}, -2^j) \cup [2^j, 2^{j+1})$ and the classical Littlewood-Paley theorem says that

$$\big\|S(f)\big\|_{L^p(\mathbf{R})} \le C_p \|f\|_{L^p(\mathbf{R})}$$

where $1 < p < \infty$ and C_p is a constant independent of the function f in $L^p(\mathbf{R})$ (but depends on p).

In this note, we are interested in estimates for Littlewood-Paley square functions formed by products of Littlewood-Paley operators acting on two functions. To be precise, let a_j and b_j be strictly increasing sequences on the real line with the properties $\lim_{j\to\infty} a_j =$

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$\lim_{j\to\infty} b_j = \infty$ and $\lim_{j\to-\infty} a_j = \lim_{j\to-\infty} b_j = -\infty$ and consider the bilinear square function

$$S_2(f,g) = \Big(\sum_{j\in\mathbf{Z}} |\Delta_{[a_j,a_{j+1})}(f)\Delta_{[b_j,b_{j+1})}(g)|^2\Big)^{\frac{1}{2}}$$

defined for suitable functions f,g on the line. We consider the question whether S_2 satisfies the inequality

$$\big\|S_2(f,g)\big\|_{L^p(\mathbf{R})} \le C_{p_1,p_2}\|f\|_{L^{p_1}(\mathbf{R})}\|g\|_{L^{p_2}(\mathbf{R})} \tag{2}$$

for some constant C_{p_1,p_2} independent of f,g where $1 < p_1,p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$. We have the following result concerning this operator:

Theorem 0.1. *Let $1 < p_1, p_2 < \infty$ be given and define p by setting $1/p = 1/p_1+1/p_2$. Then if $p > 2/3$, there is a constant C_{p_1,p_2} such that* (2) *holds for all functions f, g on the line. Conversely, if* (2) *holds, then we must have $p \ge 2/3$.*

Proof. Consider the maximal function

$$\mathcal{M}(f) = \sup_{-\infty<a<b<\infty} |\Delta_{[a,b)}(f)|$$

and notice that is pointwise controlled by

$$2\sup_{a\in\mathbf{R}} |\Delta_{(-\infty,a)}(f)|$$

and thus is controlled by the following version of the Carleson operator

$$\mathcal{C}(f)(x) = \sup_{N>0}\left|\int_{-\infty}^{N} \widehat{f}(\xi)e^{2\pi i x\xi}d\xi\right|.$$

In view of the Carleson-Hunt theorem [2], [5] we have that $\mathcal{C}$ is bounded on $L^r(\mathbf{R})$ for $1 < r < \infty$.

Consider the case where $2 \le p_1 < \infty$ and $1 < p_2 < \infty$. Then we have that

$$S_2(f,g) \le \Big(\sum_{j\in\mathbf{Z}} |\Delta_{[a_j,a_{j+1})}(f)|^2\Big)^{\frac{1}{2}} \sup_{j\in\mathbf{Z}} |\Delta_{[b_j,b_{j+1})}(g)| = S(f)\mathcal{M}(g)$$

where S is defined as in (1) with $[a_j, a_{j+1})$ in place of I_j. In view of the Rubio de Francia inequality [7] we have that S is bounded on $L^r(\mathbf{R})$ for $2 \le r < \infty$. An application of Hölder's inequality yields the inequality

$$\big\|S_2(f,g)\big\|_{L^p(\mathbf{R})} \le \|S(f)\|_{L^{p_1}(\mathbf{R})}\|\mathcal{M}(g)\|_{L^{p_2}(\mathbf{R})} \tag{3}$$

and this (2) follows from the preceding inequality combined with the boundedness of S on $L^{p_1}(\mathbf{R})$ and $\mathcal{M}$ on $L^{p_2}(\mathbf{R})$.

An analogous argument holds with the roles of p_1 and p_2 are reversed, i.e., when we have $1 < p_1 < \infty$ and $2 \le p_2 < \infty$. Thus boundedness holds for all pairs (p_1, p_2) for which either $p_1 \ge 2$ or $p_2 \ge 2$. But there exist points (p_1, p_2) with $p = (1/p_1 + 1/p_2)^{-1} > 2/3$ for which neither p_1 nor p_2 is at least 2. (For instance $p_1 = p_2 = 7/5$). To deal with these intermediate points we use interpolation.

Given a pair of points (p_1, p_2) with $p = (1/p_1 + 1/p_2)^{-1} > 2/3$ and $1 < p_1, p_2 < 2$, we pick two pairs of points (p_1^1, p_2^1) and (p_1^2, p_2^2) with

$$p > p^1 = (1/p_1^1 + 1/p_2^1)^{-1} = p^2 = (1/p_1^2 + 1/p_2^2)^{-1} > 2/3$$

and $1 < p_2^1 < 2 < p_1^1 < \infty, < 2$ and $1 < p_2^2 < 2 < p_1^2 < \infty$. For instance we take (p_1^1, p_2^1, p^1) near $(1, 2, 2/3)$ and (p_1^2, p_2^2, p^2) near $(2, 1, 2/3)$. Then consider the three points $W_1 = (1/p_1^1, 1/p_2^1, 1/p^1)$, $W_2 = (1/p_1^2, 1/p_2^2, 1/p^2)$, and $W_3 = (1/2, 1/2, 1)$ and notice that the point $(1/p_1, 1/p_2, 1/p)$ lies in the interior of the convex hull of W_1, W_2, and W_3. We consider the bi-sublinear operator

$$(f, g) \mapsto S_2(f, g)$$

which is bounded at the points W_1, W_2, and W_3. Using Corollary 7.2.4 in [3] we obtain that S_2 is bounded from $L^{p_1}(\mathbf{R}) \times L^{p_2}(\mathbf{R})$ to $L^p(\mathbf{R})$. This completes the proof in the remaining case.

Next, we turn to the converse assertion of the theorem. Suppose that for some $1 < p_1, p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$ estimate (2) holds for some constant C_{p_1,p_2} and all suitable functions f, g on the line. Now consider the sequences $a_j = b_j = j$ and the functions

$$f_N = g_N = \chi_{[0,N)}^{\vee} .$$

Then we have

$$f_N(x) = \chi_{[0,N]}^{\vee}(x) = \int_0^N e^{2\pi i x\xi} d\xi = \frac{e^{2\pi i N x} - 1}{2\pi i x}$$

and for $j = 0, 1, \dots, N-1$ we have

$$\Delta_{[j,j+1)}(f_N)(x) = \int_j^{j+1} e^{2\pi i x\xi} d\xi = e^{2\pi i x j} \int_0^1 e^{2\pi i x\xi} d\xi = \frac{e^{2\pi i x j}(e^{2\pi i x} - 1)}{2\pi i x} .$$

Consequently,

$$\Big(\sum_{j=0}^{N-1} \big| \Delta_{[j,j+1)}(f_N)(x) \Delta_{[j,j+1)}(g_N)(x) \big|^2 \Big)^{\frac{1}{2}} = \sqrt{N} \left| \frac{e^{2\pi i x} - 1}{2\pi i x} \right|^2$$

and thus

$$\big\| S_2(f_N, g_N) \big\|_{L^p} \ge \sqrt{N} \left\| \frac{(e^{2\pi i x} - 1)^2}{4\pi^2 x^2} \right\|_{L^p} = c \sqrt{N}$$

as long as $p > 1/2$. On the other hand we have

$$\|f_N\|_{L^{p_1}} = N^{1-\frac{1}{p_1}} \left\| \frac{e^{2\pi ix} - 1}{2\pi ix} \right\|_{L^{p_1}} = c_{p_1} N^{1-\frac{1}{p_1}}$$

whenever $1 < p_1 < \infty$.

Now suppose that (2) holds. Then we must have

$$\big\|S_2(f_N, g_N)\big\|_{L^p(\mathbf{R})} \le C_{p_1,p_2} \|f_N\|_{L^{p_1}(\mathbf{R})} \|g_N\|_{L^{p_2}(\mathbf{R})} \tag{4}$$

and this implies that

$$c\sqrt{N} \le C_{p_1,p_2} c_{p_1} N^{1-\frac{1}{p_1}} c_{p_2} N^{1-\frac{1}{p_2}} = C_{p_1,p_2} c_{p_1} c_{p_2} N^{2-\frac{1}{p}}$$

which forces $p \ge 2/3$ by letting $N \to \infty$. □

It is unclear to us at the moment as to what happens when $p = 2/3$.

We now discuss a related larger square function. Let $1 < p_1, p_2 < \infty$ with $1/p_1 + 1/p_2 = 1/p$. It is not hard to see that the square function

$$S_{22}(f, g) = \Big(\sum_{j\in\mathbf{Z}} \sum_{k\in\mathbf{Z}} |\Delta_{[a_j,a_{j+1})}(f) \Delta_{[b_k,b_{k+1})}(g)|^2 \Big)^{\frac{1}{2}}$$

is bounded from $L^{p_1}(\mathbf{R}) \times L^{p_2}(\mathbf{R})$ to $L^p(\mathbf{R})$ if and only if $p_1, p_2 \ge 2$. Indeed, one direction is a trivial consequence of Hölder's inequality; for the other direction, let

$$f_N(x) = g_N(x) = \chi_{[0,N]}^{\vee}(x) = \int_0^N e^{2\pi ix\xi}\, d\xi = \frac{e^{2\pi iNx} - 1}{2\pi ix}.$$

The preceding argument shows that

$$\big\|S_{22}(f_M, g_N)\big\|_{L^p} \ge c^2 \sqrt{M}\sqrt{N}$$

and we also have

$$\|f_M\|_{L^{p_1}(\mathbf{R})} \|g_N\|_{L^{p_2}(\mathbf{R})} = c_{p_1} c_{p_2} M^{1-\frac{1}{p_1}} N^{1-\frac{1}{p_2}}.$$

Hence, letting $M \to \infty$ with N fixed or $N \to \infty$ with M fixed, we obtain that both p_1 and p_2 satisfy $p_1, p_2 \ge 2$.

I would like to end this note by expressing a few feelings about Cora Sadosky. Although, I have not had a very close personal relationship with her, I have always admired the great dedication and enthusiasm Cora has displayed in mathematics and the sincere love and support she has provided to young people who wished to pursue a research career in harmonic analysis. I warmly recall the personal interest she showed in my search for a permanent position in the USA. Cora's untimely passing away was a big loss for our harmonic analysis community and we are all proud of the strong legacy she has left behind.

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