MULTILINEAR CALDERÓN–ZYGMUND SINGULAR INTEGRALS

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Contents

This paper contains the material covered in a minicourse given by the author at the Centre de Recerca Matemàtica in Barcelona during the period May 4–9, 2009. The course was an expanded version of a series of three lectures delivered by the author two weeks earlier (April 23–25) at the New Mexico Analysis Seminar held at the University of New Mexico in Albuquerque. The author would like to thank Joan Mateu and Joan Orobitg for coordinating two special minicourses at the Centre de Recerca Matemàtica on Multilinear Harmonic Analysis and Weights presented by the author and Professor Carlos Pérez during the period May 4–8, 2009. These mincourses were part of a special research program for the academic year 2008–2009 entitled *Harmonic Analysis, Geometric Measure Theory, and Quasiconformal Mappings*, coordinated by Xavier Tolsa and Joan Verdera. The author would also like to thank the Centre de Recerca Matemàtica for providing an inspiring environment for research to the participants during these workshops. This research was partially supported by the NSF under grant DMS 0900946.

1. Introduction

It is quite common for linear operators to depend on several functions of which only one is thought of as the main variable and the remaining ones are usually treated as parameters. Examples of such operators are ubiquitous in harmonic analysis: multiplier operators, homogeneous singular integrals associated with functions Ω on the sphere, Littlewood–Paley operators, the Calderón commutators, and the Cauchy integral along Lipschitz curves. Treating the additional functions that arise in these operators as frozen parameters often provides limited results that could be thought analogous to those that one obtains by studying calculus of functions of several variables by freezing variables. In this article, we advocate a more flexible point of view in the study of linear operators, analogous to that employed in pure multivariable calculus. Unfreezing the additional functions and treating them as input variables provides a more robust approach that often yields sharper results in terms of the regularity of the input functions.

We illustrate the power of this idea with a concrete example concerning the Hörmander–Mihlin multiplier theorem [23], [34]. This says that for $\gamma > n/2$ there is a constant $C_{p,n,\gamma}$ such that

$$\begin{aligned} (1) \qquad & \big\| \big(\widehat{f}\sigma\big)^{\vee} \big\|_{L^p(\mathbf{R}^n)} \\ & \le C_{p,n,\gamma} \big\| f \big\|_{L^p(\mathbf{R}^n)} \Big[\big\|\sigma\big\|_{L^\infty(\mathbf{R}^n)} + \sup_{k\in\mathbf{Z}} \big\| \varphi(\xi)\sigma(2^k\xi) \big\|_{L^2_\gamma(\mathbf{R}^n,\, d\xi)} \Big] \end{aligned}$$

where $1 < p < \infty$, while for $p = 1$ the inequality is still valid when the L^p norm on the left is replaced by $L^{1,\infty}$. Here φ is a smooth function supported in the annulus $1/2 < |\xi| < 2$ which is nonvanishing in the smaller annulus $1/\sqrt{2} < |\xi| < \sqrt{2}$, and $L^r_\gamma(\mathbf{R}^n)$ denotes the Sobolev space of functions on $\mathbf{R}^n$ with norm

$$\big\|\sigma\big\|_{L^r_\gamma(\mathbf{R}^n)} = \bigg(\int_{\mathbf{R}^n} \big| \big(\widehat{\sigma}(\xi)(1+|\xi|^2)^{\gamma/2}\big)^{\vee}(x) \big|^r \, dx \bigg)^{1/r}.$$

It makes sense to view the multiplier operator $f \mapsto \big(\widehat{f}\sigma\big)^{\vee}$ as a bilinear operator acting on f and σ, that is,

$$(f,\sigma) \longmapsto B(f,\sigma) = \big(\widehat{f}\sigma\big)^{\vee}.$$

Then, when $\beta > n/q$ and for q near infinity, we have

$$(2) \qquad \big\|B(f,\sigma)\big\|_{L^2} \le \big\|\sigma\big\|_{L^\infty} \big\|f\big\|_{L^2} \le C_{n,q,\beta} \big\|\sigma\big\|_{L^q_\beta} \big\|f\big\|_{L^2},$$

where the last estimate follows by the Sobolev embedding theorem. To make the example more transparent, let us assume that σ is supported in a compact set that does not contain the origin. In view of the assumption on the support of σ and the Sobolev embedding theorem, the estimate (1) with $p = 1$ reduces to

$$(3) \qquad \big\|B(f,\sigma)\big\|_{L^{1,\infty}(\mathbf{R}^n)} \le C_{n,\gamma} \big\|\sigma\big\|_{L^2_\gamma(\mathbf{R}^n)} \big\|f\big\|_{L^1(\mathbf{R}^n)}$$

whenever $\gamma > n/2$. Interpolating bilinearly between (2) and (3) in the complex way, we deduce for $1 \le p < 2$ the inequality

$$(4) \qquad \big\|B(f,\sigma)\big\|_{L^{p,p'}(\mathbf{R}^n)} \le C_{n,p,\delta} \big\|\sigma\big\|_{L^r_\delta(\mathbf{R}^n)} \big\|f\big\|_{L^p(\mathbf{R}^n)},$$

where

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$$

and

$$\delta > \frac{n}{r}.$$

Further use of interpolation provides an improved version of (4) in which $L^{p,p'}$ is replaced by the smaller space L^p whenever $1 < p < 2$. The advantage of (4) versus (1) is that, for p near 2, the number of derivatives required of σ in (4) is proportional to the distance of p from 2, while in (1) this number of derivatives remains constant for all p. This improvement is a result of bilinear interpolation and of the approach of treating the linear multiplier operator $f \mapsto \big(\widehat{f}\sigma\big)^{\vee}$ as a bilinear operator of both functions f and σ.

Having made the point that important information can be extracted by unfreezing parameters and treating them as variables, in the sequel we pursue this idea in a more systematic way. The purpose of these lectures is to present certain fundamental results concerning linear operators of several variables, henceforth called multilinear, that indicate some of the unique challenges that appear in their study, albeit the great similarities they share with their linear counterparts. For the purpose of clarity in the presentation, we only discuss these results in the bilinear case. The proofs that we provide may not contain all necessary details, but references are provided.

If a bilinear operator T commutes with translations, in the sense that

$$T(f,g)(x+t) = T(f(\,\cdot\,+t), g(\,\cdot\,+t))(x) \tag{5}$$

for all $t, x \in \mathbf{R}^n$, then it incorporates a certain amount of homogeneity. Indeed, if it maps $L^{p_1} \times L^{p_2}$ to L^p, then one must necessarily have $1/p_1 + 1/p_2 \geq 1/p$; this was proved in [22] for compactly supported kernels but extended for general kernels in recent work [12]. The situation where $1/p_1 + 1/p_2 = 1/p$ will be referred to as the singular integral case. Bilinear operators that commute with translations as in (5) are exactly the bilinear multiplier operators that have the form

$$T(f_1,f_2)(x) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \sigma(\xi_1,\xi_2)\,\widehat{f_1}(\xi_1)\,\widehat{f_2}(\xi_2)\,e^{2\pi i x\cdot(\xi_1+\xi_2)}\,d\xi_1\,d\xi_2$$

for some bounded function σ. The situation $1/p_1 + 1/p_2 > 1/p$ may also arise. For instance, the fractional integrals

$$J_\alpha(f,g)(x) = \int_{\mathbf{R}^n} f(x-t)\,g(x+t)\,|t|^{\alpha-n}\,dt$$

and

$$J'_\alpha(f,g)(x) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f(x-t)\,g(x-s)\,(|t|+|s|)^{\alpha-2n}\,dt\,ds$$

map $L^{p_1} \times L^{p_2}$ to L^p whenever $1/p_1 + 1/p_2 = 1/p + \alpha/n$. The estimate for J'_α is trivial as this operator is pointwise controlled by the product of two linear fractional integrals, but the corresponding estimate for J_α requires more work; see [15], [27].

Endpoint estimates for linear singular integrals are usually estimates of the form $L^1 \to L^1$ or $L^1 \to L^{1,\infty}$. The analogous bilinear estimates are of the form $L^1 \times L^1 \to L^{1/2,\infty}$. Although one expects some similarities with the linear case, there exist some differences as well. For example, if a linear translation-invariant operator has a positive kernel and it maps L^1 to $L^{1,\infty}$, then it must have an integrable kernel and thus it actually maps L^1 to L^1. In the bilinear case, it is still true that if a bilinear translation-invariant operator has a positive kernel and maps $L^1 \times L^1$ to $L^{1/2,\infty}$, then it must have an integrable kernel, but having an integrable positive kernel does not necessarily imply that the corresponding operator maps $L^1 \times L^1$ to $L^{1/2}$. Results of this type have been studied in [20].

We end this section by discussing certain examples of bilinear operators.

Example 1. The "identity operator" in the bilinear setting is the product operator

$$B_1(f,g)(x) = f(x)\,g(x).$$

In view of Hölder's inequality, B_1 maps $L^p \times L^q \to L^r$ whenever $1/p + 1/q = 1/r$.

Example 2. The action of a linear operator L on the product fg gives rise to a more general "degenerate bilinear operator"

$$B_2(f,g)(x) = L(f\,g)(x),$$

that still maps $L^p \times L^q \to L^r$ whenever $1/p + 1/q = 1/r$, provided L is a bounded operator on L^r.

Example 3. This example captures all interesting bilinear operators. Let $\widetilde{L}$ be a linear operator acting on functions defined on $\mathbf{R}^{2n}$. Then, for functions f, g on $\mathbf{R}^n$, we consider the tensor function $(f \otimes g)(x,y) = f(x)g(y)$ and define

$$B_3(f,g)(x) = \widetilde{L}(f \otimes g)(x,x).$$

In particular, $\widetilde{L}$ could be a singular integral acting on functions on $\mathbf{R}^{2n}$. Boundedness of B_3 from $L^p(\mathbf{R}^n) \times L^q(\mathbf{R}^n) \to L^r(\mathbf{R}^n)$ is a delicate issue and will be investigated in this article for certain classes of linear operators $\widetilde{L}$.

2. Bilinear Calderón–Zygmund operators

It is appropriate to embark on our study with the class of operators that extend the concept of Calderón–Zygmund operators in the multilinear setting. These operators have kernels that satisfy standard estimates and possess boundedness properties analogous to those of the classical linear ones. This class of operators has been previously studied by Coifman and Meyer [6], [7], [8], [9], [33], assuming sufficient smoothness on their symbols and kernels. This area of research is still quite active. Recent developments include the introduction of a new class of multiple weights appropriate for the boundedness of these operators on weighted Lebesgue spaces, see [31].

We will be working on n-dimensional space $\mathbf{R}^n$. We denote by $\mathcal{S}(\mathbf{R}^n)$ the space of all Schwartz functions on $\mathbf{R}^n$ and by $\mathcal{S}'(\mathbf{R}^n)$ its dual space, the set of all tempered distributions on $\mathbf{R}^n$. We use the following definition for the Fourier transform in n-dimensional euclidean space:

$$\widehat{f}(\xi) = \int_{\mathbf{R}^n} f(x)\, e^{-2\pi i x\cdot\xi}\, dx,$$

while $f^\vee(\xi) = \widehat{f}(-\xi)$ denotes the inverse Fourier transform. A bilinear operator $T\colon \mathcal{S}(\mathbf{R}^n) \times \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ is linear in every entry and consequently has two formal transposes. The first transpose T^{*1} of T is defined via

$$\langle T^{*1}(f_1, f_2),\, h\rangle = \langle T(h, f_2),\, f_1\rangle,$$

for all f_1, f_2, h in $\mathcal{S}(\mathbf{R}^n)$. Analogously one defines T^{*2} and we also set $T^{*0} = T$.

Let $K(x, y_1, y_2)$ be a locally integrable function defined away from the *diagonal* $x = y_1 = y_2$ in $(\mathbf{R}^n)^3$, which satisfies the *size estimate*

$$|K(x, y_1, y_2)| \le \frac{A}{(|x-y_1| + |x-y_2|)^{2n}} \tag{6}$$

for some $A > 0$ and all $(x, y_1, y_2) \in (\mathbf{R}^n)^3$ with $x \neq y_j$ for some j. Furthermore, assume that for some $\varepsilon > 0$ we have the smoothness estimates

$$|K(x, y_1, y_2) - K(x', y_1, y_2)| \le \frac{A\,|x-x'|^{\varepsilon}}{(|x-y_1| + \ |x-y_2|)^{2n+\varepsilon}} \tag{7}$$

whenever $|x - x'| \le \frac{1}{2} \max\big(|x-y_1|, |x-y_2|\big)$, and also that

$$|K(x, y_1, y_2) - K(x, y_1', y_2)| \le \frac{A\,|y_j-y_j'|^{\varepsilon}}{(|x-y_1| + \ |x-y_2|)^{2n+\varepsilon}} \tag{8}$$

whenever $|y_1 - y_1'| \le \frac{1}{2} \max\big(|x-y_1|, |x-y_2|\big)$, as well as a similar estimate with the roles of y_1 and y_2 reversed. Kernels satisfying these conditions are called *bilinear Calderón–Zygmund kernels* and are denoted by 2-CZK(A, ε). A bilinear operator T is said to be *associated* with K if

$$T(f_1, f_2)(x) = \int_{(\mathbf{R}^n)^2} K(x, y_1, y_2)\, f_1(y_1)\, f_2(y_2)\, dy_1\, dy_2 \tag{9}$$

whenever f_1, f_2 are smooth functions with compact support and x does not lie in the intersection of the supports of f_1 and f_2.

Certain homogeneous distributions of order $-2n$ are examples of kernels in the class 2-CZK(A, ε). For this reason, boundedness properties of operators T with kernels in 2-CZK(A, ε) from a product $L^{p_1} \times L^{p_2}$ into another L^p space can only hold when

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p},$$

as dictated by homogeneity. If such boundedness holds for a certain triple of Lebesgue spaces, then the corresponding operator is called bilinear Calderón–Zygmund.

The first main result concerning these operators is the bilinear extension of the classical Calderón–Zygmund [3]; the linear result states that, if an operator with smooth enough kernel is bounded on a certain L^r space, then it is of weak type $(1, 1)$ and is also bounded on all L^p spaces for $1 < p < \infty$. A multilinear version of this theorem has been obtained by Grafakos and Torres [22] for operators with kernels in the class 2-CZK(A, ε). A special case of this result was also obtained by Kenig and Stein [27]; all approaches build on previous work by Coifman and Meyer [6].

Theorem 1. *Let T be a bilinear operator with kernel K in* 2-CZK(A, ε). *Assume that, for some $1 \le q_1, q_2 \le \infty$ and some $0 < q < \infty$ with*

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q},$$

T maps $L^{q_1}\times L^{q_2}$ to $L^{q,\infty}$. Then T can be extended to a bounded operator from $L^1\times L^1$ into $L^{1/2,\infty}$. Moreover, for some constant C_n (that depends only on the parameters indicated) we have that

$$\|T\|_{L^1\times L^1\to L^{1/2,\infty}} \le C_n\big(A+\|T\|_{L^{q_1}\times L^{q_2}\to L^{q,\infty}}\big). \tag{10}$$

Proof. Set $B=\|T\|_{L^{q_1}\times L^{q_2}\to L^{q,\infty}}$. Fix an $\alpha>0$ and consider functions $f_j\in L^1$ for $1\le j\le 2$. Without loss of generality we may assume that $\|f_1\|_{L^1}=\|f_2\|_{L^1}=1$. Setting $E_\alpha=\{x\,:\,|T(f_1,f_2)(x)|>\alpha\}$, we need to show that for some constant $C=C_n$ we have

$$|E_\alpha|\le C(A+B)^{1/2}\alpha^{-1/2}. \tag{11}$$

(Once (11) has been established for f_j's with norm one, the general case follows immediately by scaling.) Let γ be a positive real number to be determined later. For each $j=1,2$, apply the Calderón–Zygmund decomposition to the function f_j at height $(\alpha\gamma)^{1/2}$ to obtain 'good' and 'bad' functions g_j and b_j, and families of disjoint cubes $\{Q_{j,k}\}_k$, such that $f_j=g_j+b_j$ and $b_j=\sum_k b_{j,k}$, where

$$\mathrm{support}(b_{j,k})\subset Q_{j,k},$$
$$\int_{Q_{j,k}} b_{j,k}(x)\,dx=0,\qquad \int_{Q_{j,k}}|b_{j,k}(x)|\,dx\le C(\alpha\gamma)^{1/2}|Q_{j,k}|,$$
$$\big|\cup_k Q_{j,k}\big|\le C(\alpha\gamma)^{-1/2},\qquad \|b_j\|_{L^1}\le C,\qquad \|g_j\|_{L^s}\le C(\alpha\gamma)^{1/2s'},$$

for any $1\le s\le\infty$ (here s' is the dual exponent of s). Now let

$$\begin{aligned}
E_1&=\{x\,:\,|T(g_1,g_2)(x)|>\alpha/4\},\\
E_2&=\{x\,:\,|T(b_1,g_2)(x)|>\alpha/4\},\\
E_3&=\{x\,:\,|T(g_1,b_2)(x)|>\alpha/4\},\\
E_4&=\{x\,:\,|T(b_1,b_2)(x)|>\alpha/4\}.
\end{aligned}$$

Since $|\{x\,:\,|T(f_1,f_2)(x)|>\alpha\}|\le\sum_{s=1}^4|E_s|$, it will suffice to prove estimate (11) for each E_s. Chebychev's inequality and $L^{q_1}\times L^{q_2}\to L^{q,\infty}$ boundedness give

$$\begin{aligned}
|E_1|\le&\frac{(4B)^q}{\alpha^q}\big\|g_1\big\|_{L^{q_1}}^q\big\|g_2\big\|_{L^{q_2}}^q\le\frac{CB^q}{\alpha^q}\prod_{j=1}^2(\alpha\gamma)^{\frac{q}{2q_j'}}\\
=&\frac{C'B^q}{\alpha^q}\,(\alpha\gamma)^{(2-\frac1q)\frac q2}=C'B^q\alpha^{-\frac12}\gamma^{q-\frac12}.
\end{aligned}\tag{12}$$

We now show that

$$|E_s|\le C\alpha^{-1/2}\gamma^{1/2}. \tag{13}$$

Let $l(Q)$ denote the side-length of a cube Q and let Q^* be a certain expansion of it with the same center. (This expansion only depends on the dimension.) Fix an $x\notin\cup_{j=1}^2\cup_{k_j}(Q_{j,k_j})^*$. Also fix the cube Q_{1,k_1} and let c_{1,k_1} be its center. For fixed

$y_2 \in \mathbf{R}^n$, the mean value property of the function b_{1,k_1} gives

$$\begin{aligned}
&\left| \int_{Q_{1,k_1}} K(x,y_1,y_2)\, b_{1,k_1}(y_1)\, dy_1 \right| \\
&\qquad\qquad = \left| \int_{Q_{1,k_1}} \big(K(x,y_1,y_2) - K(x,c_{1,k_1},y_2) \big)\, b_{1,k_1}(y_1)\, dy_1 \right| \\
&\qquad\qquad \le \int_{Q_{1,k_1}} |b_{1,k_1}(y_1)| \,\frac{A\,|y_1 - c_{1,k_1}|^{\varepsilon}}{(|x-y_1| + |x-y_2|)^{2n+\varepsilon}}\, dy_1 \\
&\qquad\qquad \le \int_{Q_{1,k_1}} |b_{1,k_1}(y_1)| \,\frac{C\,A\,l(Q_{1,k_1})^{\varepsilon}}{(|x-y_1| + |x-y_2|)^{2n+\varepsilon}}\, dy_1,
\end{aligned}$$

where in the previous to last inequality we used that

$$|y_1 - c_{1,k_1}| \le c_n\, l(Q_{1,k_1}) \le \frac{1}{2}|x-y_1| \le \frac{1}{2}\max\big(|x-y_1|, |x-y_2|\big).$$

Multiplying the derived inequality

$$\left| \int_{Q_{1,k_1}} K(x,y_1,y_2)\, b_{1,k_1}(y_1)\, dy_1 \right| \le \int_{Q_{1,k_1}} \frac{C\,A\,|b_{1,k_1}(y_1)|\, l(Q_{1,k_1})^{\varepsilon}}{(|x-y_1| + |x-y_2|)^{2n+\varepsilon}}\, dy_1$$

by $|g_2(y_2)|$ and integrating over y_2, we obtain the estimate

$$\begin{aligned}
&\int_{\mathbf{R}^n} |g_2(y_2)| \left| \int_{Q_{1,k_1}} K(x,y_1,y_2)\, b_{1,k_1}(y_1)\, dy_1 \right| dy_2 \\
&\qquad \le \big\|g_2\big\|_{L^\infty} \int_{Q_{1,k_1}} |b_{1,k_1}(y_1)| \,\frac{A\,C\,l(Q_{1,k_1})^{\varepsilon}}{(|x-y_1| + |x-y_2|)^{2n-n+\varepsilon}}\, dy_1 \qquad (14) \\
&\qquad \le C\,A\,\big\|g_2\big\|_{L^\infty} \big\|b_{1,k_1}\big\|_{L^1} \frac{l(Q_{1,k_1})^{\varepsilon}}{\big(l(Q_{1,k_1}) + |x - c_{1,k_1}|\big)^{n+\varepsilon}}.
\end{aligned}$$

The last inequality is due to the fact that for $x \notin \cup_{j=1}^{2} \cup_{k_j} (Q_{j,k_j})^*$ and $y_j \in Q_{j,k_j}$ we have that $|x - y_j| \approx l(Q_{j,k_j}) + |x - c_{j,k_j}|$. It is now a simple consequence of (14) that for $x \notin \cup_{j=1}^{2} \cup_{k_j} (Q_{j,k_j})^*$ we have

$$|T(b_1,g_2)(x)| \le C' A\,(\alpha\gamma)^{\frac{1}{2}} \left(\sum_{k_1} \frac{(\alpha\gamma)^{1/2}\, l(Q_{1,k_1})^{n+\varepsilon}}{(l(Q_{1,k_1}) + |x - c_{1,k_1}|)^{n+\varepsilon}} \right) = C'' A\,\alpha\,\gamma M_{\varepsilon}(x), \qquad (15)$$

where

$$M_{\varepsilon}(x) = \sum_{k} \frac{l(Q_{1,k})^{n+\varepsilon}}{(l(Q_{1,k}) + |x - c_{1,k}|)^{n+\varepsilon}}$$

is the Marcinkiewicz function associated with the family of cubes $\{Q_{1,k}\}_k$. It is a known fact [41] that for some constant C there is an estimate

$$\int_{\mathbf{R}^n} M_{\varepsilon}(x)\, dx \le C |\cup_k Q_{1,k}| \le C' (\alpha\gamma)^{-1/2}. \qquad (16)$$

Using (15) and (16), an L^1 estimate outside $\cup_{j=1}^2 \cup_{k_j} (Q_{j,k_j})^*$ gives

$$|\{x \notin \cup_{j=1}^2 \cup_{k_j} (Q_{j,k_j})^* \,:\, |T(b_1,g_2)(x)| > \alpha/4\}| \le C\,\alpha^{-1/2} A\gamma^{1/2}. \tag{17}$$

This estimate, in conjunction with

$$|\cup_{j=1}^2 \cup_{k_j} (Q_{j,k_j})^*| \le C(\alpha\gamma)^{-1/2},$$

yields the required inequality (13).

We have now proved (13) for $\gamma > 0$. Plugging in the value of $\gamma = (A+B)^{-1}$ in both (12) and (13) gives the required estimate (11) for $|E_2|$. The estimate for $|E_3|$ is symmetric, while the analogous estimate for $|E_4|$ requires a variation of the argument for $|E_2|$. Since two bad functions show up in this estimate, a double sum over pairs of cubes appears and one has to use cancellation with respect to the smallest cube in the pair. Then the length of the smaller cube in the numerator is controlled by the square root of the length of the smaller cube times the square root of the length of the larger cube. At the end, the term $|T(b_1,b_2)|$ is pointwise controlled by a product of Marcinkiewicz functions outside the union $\cup_{j=1}^2 \cup_{k_j} (Q_{j,k_j})^*$ and one uses an $L^{1/2}$ estimate over this set (instead of an L^1 estimate) in conjunction with Hölder's inequality. The previous choice of $\gamma = (A+B)^{-1}$ yields the required estimate for $|E_4|$. □

Example 4. Let R_1 be the *bilinear Riesz transform in the first variable*

$$R_1(f_1,f_2)(x) = \text{p.v.} \int_{\mathbf{R}}\int_{\mathbf{R}} \frac{x-y_1}{|(x-y_1,x-y_2)|^3}\, f_1(y_1)\, f_2(y_2)\, dy_1\, dy_2.$$

We will show later that this operator maps $L^{p_1}(\mathbf{R})\times L^{p_2}(\mathbf{R})$ to $L^p(\mathbf{R})$ for $1/p_1+1/p_2 = 1/p$, $1 < p_1, p_2 < \infty$, and $1/2 < p < \infty$. Thus by Theorem 1 it also maps $L^1 \times L^1$ to $L^{1/2,\infty}$. However, it does not map $L^1 \times L^1$ to any Lorentz space $L^{1/2,q}$ for $q < \infty$. In fact, letting $f_1 = f_2 = \chi_{[0,1]}$, an easy computation shows that $R_1(f_1,f_2)(x)$ behaves at infinity like $|x|^{-2}$. This fact indicates that in Theorem 1 the space $L^{1/2,\infty}$ is best possible and cannot be replaced by any smaller space. In particular, it cannot be replaced by $L^{1/2}$.

3. Endpoint estimates and interpolation for bilinear Calderón–Zygmund operators

The real bilinear interpolation is significantly more complicated than the linear one. Early versions appeared in the work of Janson [25] and Strichartz [43]. In this exposition we will use a version of real bilinear interpolation appearing in [15]. This makes use of the notion of *bilinear restricted weak type* (p,q,r) *estimates*. These are estimates of the form

$$\lambda\big|\{x \,:\, |T(\chi_A,\chi_B)(x)| > \lambda\}\big|^{1/r} \le M\,|A|^{1/p}|B|^{1/q}$$

and have a wonderful interpolation property: if an operator T satisfies restricted weak type (p_0,q_0,r_0) and (p_1,q_1,r_1) estimates with constants M_0 and M_1, respectively, then it also satisfies a restricted weak type (p,q,r) estimate with constant $M_0^{1-\theta}M_1^{\theta}$, where

$$\left(\frac{1}{p},\frac{1}{q},\frac{1}{r}\right)=(1-\theta)\left(\frac{1}{p_0},\frac{1}{q_0},\frac{1}{r_0}\right)+\theta\left(\frac{1}{p_1},\frac{1}{q_1},\frac{1}{r_1}\right).$$

We will make use of the following bilinear interpolation result; for a proof see [15].

Theorem 2. *Let $0<p_{ij},p_i\le\infty$, $i=1,2,3$, $j=1,2$, and suppose that the points $(1/p_{11},1/p_{12})$, $(1/p_{21},1/p_{22})$, $(1/p_{31},1/p_{32})$ are the vertices of a nontrivial triangle in $\mathbf{R}^2$. Let $(1/q_1,1/q_2)$ be in the interior of this triangle (i.e., a linear convex combination of the three vertices of the triangle) and suppose that $(1/q_1,1/q_2,1/q)$ is the same linear combination of the points $(1/p_{11},1/p_{12},1/p_1)$, $(1/p_{21},1/p_{22},1/p_2)$, and $(1/p_{31},1/p_{32},1/p_3)$. Suppose that a bilinear operator T satisfies restricted weak type (p_{i1},p_{i2},p_i) estimates for $i=1,2,3$. Then T has a bounded extension from $L^{q_1}\times L^{q_2}$ to L^q whenever $1/q\le 1/q_1+1/q_2$.*

There is an interpolation theorem saying that if a linear operator (that satisfies a mild assumption) and its transpose are of restricted weak type $(1,1)$, then the operator is L^2 bounded. We begin this section by proving a bilinear analogue of this result. For a more detailed version of the result below, see [21].

Theorem 3. *Let $1<p_1,p_2<\infty$ be such that $1/p_1+1/p_2=1/p<1$. Suppose that a bilinear operator has the property that*

$$\sup_{A_0,A_1,A_2}|A_0|^{-1/p'}|A_1|^{-1/p_1}|A_2|^{-1/p_2}\left|\int_{A_0}T(\chi_{A_1},\chi_{A_2})\,dx\right|<\infty, \tag{18}$$

where the supremum is taken over all subsets A_0,A_1,A_2 of finite measure. Also suppose that T, T^{*1}, and T^{*2} are of restricted weak type $(1,1,1/2)$; this means that these operators map $L^1\times L^1$ to $L^{1/2,\infty}$ when restricted to characteristic functions with constants B_0, B_1, B_2, respectively. Then there is a constant C_{p_1,p_2} such that T maps $L^{p_1,1}\times L^{p_2,1}$ to $L^{p,\infty}$ when restricted to characteristic functions with norm at most

$$C_{p_1,p_2}B_0^{1/(2p)}B_1^{1/(2p_1')}B_2^{1/(2p_2')}.$$

Proof. Let M be the supremum in (18). We consider the following cases:

Case 1: Suppose that

$$\frac{|A_0|}{\sqrt{B_0}}\ge\max\left(\frac{|A_1|}{\sqrt{B_1}},\frac{|A_2|}{\sqrt{B_2}}\right).$$

Since T maps $L^1\times L^1$ to weak $L^{1/2}$ when restricted to characteristic functions, there exists a subset A_0' of A_0 of measure $|A_0'|\ge\frac{1}{2}|A_0|$ such that

$$\left|\int_{A_0'}T(\chi_{A_1},\chi_{A_2})\,dx\right|\le C\,B_0\,|A_1|\,|A_2|\,|A_0|^{1-\frac{1}{1/2}}$$

for some constant C. Then

$$\begin{aligned}\left|\int_{A_0} T(\chi_{A_1},\chi_{A_2})\,dx\right| &\le \left|\int_{A_0'} T(\chi_{A_1},\chi_{A_2})\,dx\right| + \left|\int_{A_0\setminus A_0'} T(\chi_{A_1},\chi_{A_2})\,dx\right| \\ &\le C\,B_0\,|A_1|\,|A_2|\,|A_0|^{-1} + M|A_1|^{1/p_1}|A_2|^{1/p_2}\left(\frac{1}{2}|A_0|\right)^{1/p'} \\ &\le C\,B_0\,|A_1|^{1/p_1}\left(\frac{\sqrt{B_1}}{\sqrt{B_0}}\right)^{1/p_1'}|A_2|^{1/p_2}\left(\frac{\sqrt{B_2}}{\sqrt{B_0}}\right)^{1/p_2'}|A_0|^{1/p_1'+1/p_2'-1} \\ &\qquad + M\,2^{-1/p'}|A_1|^{1/p_1}|A_2|^{1/p_2}|A_0|^{1/p'}.\end{aligned}$$

It follows that M has to be less than or equal to

$$C\,B_0\left(\frac{\sqrt{B_1}}{\sqrt{B_0}}\right)^{1/p_1'}\left(\frac{\sqrt{B_2}}{\sqrt{B_0}}\right)^{1/p_2'} + M\,2^{-1/p'}$$

and consequently

$$M \le \frac{C}{1-2^{-1/p'}}B_0^{1/(2p)}B_1^{1/(2p_1')}B_2^{1/(2p_2')}.$$

Case 2: Suppose that

$$\frac{|A_1|}{\sqrt{B_1}} \ge \max\left(\frac{|A_0|}{\sqrt{B_0}},\frac{|A_2|}{\sqrt{B_2}}\right).$$

Here we use that T^{*1} maps $L^1\times L^1$ to weak $L^{1/2}$ when restricted to characteristic functions. Then there exists a subset A_1' of A_1 of measure $|A_1'|\ge\frac{1}{2}|A_1|$ such that

$$\left|\int_{A_1'} T^{*1}(\chi_{A_0},\chi_{A_2})\,dx\right| \le C\,B_1\,|A_0|\,|A_2|\,|A_1|^{-1}$$

for some constant C. Equivalently, we write this statement as

$$\left|\int_{A_0} T(\chi_{A_1'},\chi_{A_2})\,dx\right| \le C\,B_1\,|A_0|\,|A_2|\,|A_1|^{-1}$$

by the definition of the first dual operator T^{*1}. Therefore we obtain

$$\begin{aligned}\left|\int_{A_0} T(\chi_{A_1},\chi_{A_2})\,dx\right| &\le \left|\int_{A_0} T(\chi_{A_1'},\chi_{A_2})\,dx\right| + \left|\int_{A_0} T(\chi_{A_1\setminus A_1'},\chi_{A_2})\,dx\right| \\ &\le C\,B_1\,|A_0|\,|A_2|\,|A_1|^{-1} + M|A_0|^{1/p'}|A_2|^{1/p_2}\left(\frac{1}{2}|A_1|\right)^{1/p_1} \\ &\le C\,B_1\,|A_1|^{-1+1/p+1/p_2'}\left(\frac{\sqrt{B_0}}{\sqrt{B_1}}\right)^{1/p}|A_2|^{1/p_2}\left(\frac{\sqrt{B_2}}{\sqrt{B_1}}\right)^{1/p_2'}|A_0|^{1/p'} \\ &\qquad + M\,2^{-1/p_1}|A_1|^{1/p_1}|A_2|^{1/p_2}|A_0|^{1/p'}.\end{aligned}$$

By the definition of M, it follows that

$$M \le \frac{C}{1-2^{-1/p_1}}B_0^{1/(2p)}B_1^{1/(2p_1')}B_2^{1/(2p_2')}.$$

Case 3: Suppose that

$$\frac{|A_2|}{\sqrt{B_2}} \geq \max\left(\frac{|A_0|}{\sqrt{B_0}}, \frac{|A_1|}{\sqrt{B_1}}\right);$$

here we proceed as in Case 2 with the roles of A_1 and A_2 interchanged.

Then the statement of the theorem follows with

$$C_{p_1,p_2} = C \max\left(\frac{1}{1-2^{-1/p_1}}, \frac{1}{1-2^{-1/p_2}}, \frac{1}{1-2^{-1/p'}}\right).$$

□

Assumption (18) is not as restrictive as it looks. To apply this theorem for bilinear Calderón–Zygmund operators, one needs to consider the family of operators whose kernels are truncated near the origin, i.e.,

$$K_\delta(x, y_1, y_2) = K(x, y_1, y_2)\, \widetilde{\chi}\big((|x-y_1| + |x-y_2|)/\delta\big),$$

where $\widetilde{\chi}$ is a smooth function that is equal to 1 on $[2,\infty)$ and vanishes on $[0,1]$. The kernels K_δ are essentially in the same Calderón–Zygmund kernel class as K, that is, if K lies in 2-CZK(A,ε), then K_ε lie in 2-CZK(A',ε), where A' is a multiple of A. Using Hölder's inequality with exponents p_1, p_2, p', it is easy to see that for the operators T_δ with kernels K_δ, the assumption (18) holds with constants depending on δ.

Theorem 3 provides an interpolation machinery needed to pass from bounds at one point to bounds at every point for bilinear Calderón–Zygmund operators. (An alternative interpolation technique was described in [22].) We have:

Theorem 4. *Suppose that a bilinear operator T with kernel in* 2-CZK(A,δ) *and all of its truncations T_δ map $L^{r_1} \times L^{r_2}$ to L^r for a single triple of indices r_1, r_2, r satisfying $1/r_1 + 1/r_2 = 1/r$ and $1 < r_1, r_2, r < \infty$ uniformly in δ. Then T is bounded from $L^{p_1} \times L^{p_2}$ to L^p for all indices p_1, p_2, p satisfying $1/p_1 + 1/p_2 = 1/p$ and $1 < p_1, p_2 \leq \infty$, $1/2 < p < \infty$.*

Proof. Since T_δ maps $L^{r_1} \times L^{r_2} \to L^r$ and $r > 1$, duality gives that T_δ^{*1} maps $L^{r'} \times L^{r_2} \to L^{r_1'}$ and T_δ^{*2} maps $L^{r_1} \times L^{r'} \to L^{r_2'}$ (uniformly in δ). It follows from Theorems 2 and 3 that T_δ are bounded from $L^{p_1} \times L^{p_2} \to L^p$ for all indices p_1, p_2, p satisfying $1/p_1 + 1/p_2 = 1/p$ and $1 < p_1, p_2 < \infty$, $1/2 < p < \infty$. Passing to the limit, using Fatou's lemma, the same conclusion may be obtained for the nontruncated operator T. The case $p_1 = \infty$ or $p_2 = \infty$ follows by duality from the case $p = 1$. □

The only drawback of this approach is that it is based on the assumption that if T is bounded from $L^{r_1} \times L^{r_2} \to L^r$, then so are all its truncations T_δ (uniformly in $\delta > 0$). This is hardly a problem in concrete applications since the kernels of T and T_δ satisfy equivalent estimates (uniformly in $\delta > 0$) and the method used in the proof of boundedness of the former almost always applies for the latter.

Proposition 1. *Let T be a bilinear operator associated with a kernel of class* 2-CZK(A,ε) *that maps $L^{q_1} \times L^{q_2}$ to L^q for some choice of indices q_1, q_2, q that satisfy $1/q_1 + 1/q_2 = 1/q$. Then T has an extension that maps*

$$L^\infty \times L^\infty \longrightarrow BMO$$

with bound a constant multiple of $A+B$. By duality, T also maps

$$L^\infty \times H^1 \longrightarrow L^1,$$

and also

$$H^1 \times L^\infty \longrightarrow L^1,$$

where H^1 is the Hardy 1*-space.*

The proof is based on a straightforward adaptation of the Peetre, Spanne, and Stein result [35], [39], [40] on boundedness of a linear Calderón–Zygmund operator from L^∞ to BMO, and is omitted.

4. The bilinear $T1$ theorem

In this section we quickly discuss the bilinear version of the $T1$ theorem. The scope of this theorem is to provide a sufficient condition for boundedness of bilinear Calderón–Zygmund operators at some point, i.e., $L^{q_1} \times L^{q_2} \to L^{q,\infty}$ for some choice of points q_1, q_2, q. Once this is known, then the bilinear version of the Calderón–Zygmund theorem (Theorem 1) combined with interpolation yields boundedness of these operators in the entire range of indices where boundedness is possible.

The linear $T1$ theorem was obtained by David and Journé [10]. Its original formulation involves three conditions equivalent to L^2 boundedness. These conditions are that $T1 \in BMO$, $T^*1 \in BMO$, and that a certain weak boundedness property holds. This version of the David–Journé $T1$ theorem was extended to the multilinear setting by Christ and Journé [5]. Another version of the $T1$ theorem using exponentials appears in [10] and is better suited for our purposes. This version is as follows: A linear operator T with kernel in the Calderón–Zygmund class $\mathrm{CZK}(A,\varepsilon)$ maps $L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ if and only if

$$\sup_{\xi\in\mathbf{R}^n} \big(\|T(e^{2\pi i\xi\cdot(\,\cdot\,)})\|_{BMO} + \|T^*(e^{2\pi i\xi\cdot(\,\cdot\,)})\|_{BMO} \big) < \infty.$$

In this section we will state and prove a multilinear version of the $T1$ theorem using the characterization stated above.

Theorem 5. *Fix $1 < q_1, q_2, q < \infty$ with*

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}. \tag{19}$$

Let T be a continuous bilinear operator from $\mathcal{S}(\mathbf{R}^n) \times \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ with kernel K in 2-CZK(A,ε)*. Then T has a bounded extension from $L^{q_1} \times L^{q_2}$ to L^q if and only if*

$$\sup_{\xi_1\in\mathbf{R}^n} \sup_{\xi_2\in\mathbf{R}^n} \|T(e^{2\pi i\xi_1\cdot(\,\cdot\,)}, e^{2\pi i\xi_2\cdot(\,\cdot\,)})\|_{BMO} \le B \tag{20}$$

and also

$$\sup_{\xi_1\in\mathbf{R}^n} \sup_{\xi_2\in\mathbf{R}^n} \|T^{*j}(e^{2\pi i\xi_1\cdot(\,\cdot\,)}, e^{2\pi i\xi_2\cdot(\,\cdot\,)})\|_{BMO} \le B \tag{21}$$

for all $j = 1, 2$. Moreover, if (20) *and* (21) *hold then we have that*

$$\|T\|_{L^{q_1}\times L^{q_2}\to L^q} \le c_{n,q_1,q_2}(A+B),$$

for some constant c_{n,m,q_j} depending only on the parameters indicated.

Proof. Obviously the necessity of conditions (20) and (21) follows from Proposition 1. The thrust of this theorem is provided by the sufficiency of these conditions, i.e., the fact that, if (20) and (21) hold, then T extends to a bounded operator from $L^{q_1} \times L^{q_2}$ to L^q. Although we are not going to be precise in the proof of this result, we make some comments. The outline of the proof is based on another formulation of the $T1$ theorem given by Stein [42]. Let us consider the set of all C^∞ functions supported in the unit ball of $\mathbf{R}^n$ satisfying

$$\|\partial^\alpha \phi\|_{L^\infty} \leq 1$$

for all multiindices $|\alpha| \leq [n/2]+1$. Such functions are called *normalized bumps.* For a normalized bump ϕ, $x_0 \in \mathbf{R}^n$, and $R > 0$, define the function

$$\phi^{R,x_0}(x) = \phi\left(\frac{x-x_0}{R}\right).$$

The formulation in [42, Theorem 3, page 294] says that a necessary and sufficient condition for an operator T with kernel of class $\mathrm{CZK}(A,\varepsilon)$ to be L^2-bounded is that for some constant $B > 0$ we have

$$\|T(\phi^{R,x_0})\|_{L^2} + \|T^*(\phi^{R,x_0})\|_{L^2} \leq B\, R^{n/2}$$

for all normalized bumps ϕ, all $R > 0$ and all $x_0 \in \mathbf{R}^n$. Moreover, the norm of the operator T on L^2 (and therefore on L^p) is bounded by a constant multiple of $A+B$. Adopting this terminology in the bilinear setting, we say that a bilinear operator T is *BMO-restrictedly bounded with bound C* if

$$\|T(\phi_1^{R_1,x_1}, \phi_2^{R_2,x_2})\|_{BMO} \leq C < \infty,$$

and

$$\|T^{*j}(\phi_1^{R_1,x_1}, \phi_2^{R_2,x_2})\|_{BMO} \leq C < \infty$$

for all $j = 1, 2$, all normalized bumps ϕ_j, all $R_j > 0$, and all $x_j \in \mathbf{R}^n$.

The main observation is that if (20) and (21) are satisfied, then T is BMO-restrictedly bounded with bound a multiple of $B > 0$. This observation can be obtained from the corresponding result for linear operators. Consider the linear operator

$$T_{\phi_2^{R_2,x_2}}(f_1) = T(f_1, \phi_2^{R_2,x_2})$$

obtained from T by freezing an arbitrary normalized bump in the second entry. It is easy to see that $T_{\phi_2^{R_2,x_2}}$ satisfies the linear BMO-restrictedly boundedness condition with bound B. It follows from this that $T_{\phi_m^{R_m,x_m}}$ maps the space of bounded functions with compact support to BMO with norm at most a multiple of $A+B$, i.e.,

$$\|T(g, \phi_2^{R_2,x_2})\|_{BMO} \leq c(A+B)\|g\|_{L^\infty} \tag{22}$$

holds for bounded functions g with compact support.

Now consider the operators T_{g_1} defined by

$$T_{g_1}(f_2) = T(g_1, f_2),$$

for a compactly supported and bounded function g_1. The estimate (22) is saying that T_{g_1} satisfies the linear BMO-restrictedly boundedness condition with constant a multiple of $(A+B)\|g_1\|_{L^\infty}$. The corresponding linear result implies that

$$T\colon L^q \times L^\infty_c \longrightarrow L^q,$$

with norm controlled by a multiple of $A+B$. (Here L^∞_c is the space of bounded functions with compact support.) Furthermore, this result can be used as a starting point for the boundedness of T from $L^1 \times L^1$ to $L^{1/2,\infty}$, which in turn implies boundedness of T in the range $L^{q_1} \times L^{q_2} \to L^q$ for all $1 < q_1, q_2 \le \infty$, $1/2 < q < \infty$, in view of Theorem 4. □

As an application, we obtain a bilinear extension of a result of Bourdaud [2].

Example 5. Consider the class of bilinear pseudodifferential operators

$$T(f_1, f_2)(x) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \sigma(x, \xi_1, \xi_2)\, \widehat{f_1}(\xi_1)\, \widehat{f_2}(\xi_2)\, e^{2\pi i x\cdot(\xi_1+\xi_2)}\, d\xi_1\, d\xi_2$$

with symbols σ satisfying

$$|\partial_x^\alpha \partial_{\xi_1}^{\beta_1} \partial_{\xi_2}^{\beta_2} \sigma(x, \xi_1, \xi_2)| \le C_{\alpha,\beta}(1 + |\xi_1| + |\xi_2|)^{|\alpha| - (|\beta_1| + |\beta_2|)}, \tag{23}$$

for all n-tuples α, β_1, β_2 of nonnegative integers. It is easy to see that such operators have kernels in 2-CZK. For these operators we have that

$$T\big(e^{2\pi i \eta_1\cdot(\,\cdot\,)}, e^{2\pi i \eta_2\cdot(\,\cdot\,)}\big) = \sigma(x, \eta_1, \eta_2)\, e^{2\pi i x\cdot(\eta_1+\eta_2)},$$

which is uniformly bounded in $\eta_j \in \mathbf{R}^n$. Theorem 5 implies that a necessary and sufficient condition for T to map a product of L^p spaces into another Lebesgue space with the usual relation on the indices is that $T^{*j}(e^{2\pi i \eta_1\cdot(\,\cdot\,)}, e^{2\pi i \eta_2\cdot(\,\cdot\,)})$ are in BMO uniformly in $\eta_k \in \mathbf{R}^n$. In particular, this is the case if all the transposes of T have symbols that also satisfy (23).

We now look for sufficient conditions on a singular kernel K_0 so that the corresponding translation invariant operator

$$T(f_1, f_2)(x) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} K_0(x - y_1, x - y_2)\, f_1(y_1)\, f_2(y_2)\, dy_1\, dy_2 \tag{24}$$

maps $L^{p_1} \times L^{p_2}$ to L^p when the indices satisfy $1/p_1 + 1/p_2 = 1/p$. We have the following:

Theorem 6. *Let $K_0(u_1, u_2)$ be a locally integrable function on $(\mathbf{R}^n)^2 \setminus \{(0,0)\}$ which satisfies the size estimate*

$$|K_0(u_1, u_2)| \le A|(u_1, u_2)|^{-2n}, \tag{25}$$

the cancellation condition

$$\left| \iint_{R_1 < |(u_1,u_2)| < R_2} K_0(u_1, u_2)\, du_1\, du_2 \right| \le A < \infty \tag{26}$$

for all $0 < R_1 < R_2 < \infty$, and the smoothness condition

$$|K_0(u_1, u_2) - K_0(u_1, u_2')| \le A \frac{|u_2 - u_2'|^\varepsilon}{|(u_1, u_2)|^{2n+\varepsilon}} \tag{27}$$

whenever $|u_2 - u_2'| < \frac{1}{2}|u_2|$. *Then the multilinear operator* T *given by* (24) *maps* $L^{p_1} \times L^{p_2}$ *into* L^p *when* $1 < p_j < \infty$ *and* (29) *is satisfied. In particular, this is the case if* K_0 *has the form*

$$K_0(u_1, u_2) = \frac{\Omega\left(\dfrac{(u_1, u_2)}{|(u_1, u_2)|}\right)}{|(u_1, u_2)|^{2n}}$$

and Ω *is an integrable function with mean value zero on the sphere* $\mathbf{S}^{2n-1}$ *which is Lipschitz of order* $\varepsilon > 0$.

Proof. This theorem is a consequence of Theorem 5. As in the previous application of this theorem (with some formal computations that are easily justified), we have

$$T\big(e^{2\pi i\eta_1\cdot(\,\cdot\,)}, e^{2\pi i\eta_2\cdot(\,\cdot\,)}\big)(x) = e^{2\pi i x\cdot(\eta_1+\eta_2)}\,\widehat{K_0}(\eta_1, \eta_2),$$

which is a bounded function, hence in BMO. The fact that $\widehat{K_0}$ is bounded is a standard fact; see for instance [1]. And certainly the same result is valid for all smooth truncations of K_0, a condition needed in the application of Theorem 4. The calculations with the transposes are similar; for example,

$$T^{*1}\big(e^{2\pi i\eta_1\cdot(\,\cdot\,)}, e^{2\pi i\eta_2\cdot(\,\cdot\,)}\big)(x) = e^{2\pi i x\cdot(\eta_1+\eta_2)}\,\widehat{K_0}(-\eta_1 - \eta_2, \eta_2),$$

which is also in BMO. □

Another application of this result concerns the bilinear extension of the classical Hörmander–Mihlin multiplier theorem discussed in the Introduction. The multilinear analogue of the Hörmander–Mihlin multiplier theorem was obtained by Coifman and Meyer [6], [7] on L^p for $p \geq 1$ and extended to indices $p < 1$ in [22] and [27].

Theorem 7. *Suppose that* $a(\xi_1, \xi_2)$ *is a* C^∞ *function on* $(\mathbf{R}^n)^2\backslash\{(0,0)\}$ *which satisfies*

$$|\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2} a(\xi_1, \xi_2)| \leq C_{\beta_1,\beta_2}(|\xi_1| + |\xi_2|)^{-(|\beta_1|+|\beta_2|)} \tag{28}$$

for all multiindices β_1, β_2. *Then the corresponding bilinear operator* T *with symbol* a *is a bounded operator from* $L^{p_1} \times L^{p_2}$ *into* L^p *when* $1 < p_j < \infty$ *and*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}, \tag{29}$$

and it also maps $L^1 \times L^1$ *to* $L^{1/2,\infty}$.

Proof. Indeed, conditions (28) easily imply that the inverse Fourier transform of a satisfies

$$|\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2} a^\vee(x_1, x_2)| \leq C_{\beta_1,\beta_2}(|x_1| + |x_2|)^{-(2n+|\beta_1|+|\beta_2|)} \tag{30}$$

for all multiindices β_1, β_2. It follows that the kernel

$$K(x, y_1, y_2) = a^\vee(x - y_1, x - y_2)$$

of the operator T satisfies the required size and smoothness conditions (6), (7), and (8). The $L^{q_1} \times L^{q_2} \to L^q$ boundedness of T for a fixed point $(1/q_1, 1/q_2, 1/q)$ follows from the bilinear $T1$ theorem (Theorem 5) once we have verified the required BMO conditions. As in previous examples, we have that

$$T\Big(e^{2\pi i\eta_1\cdot(\,\cdot\,)}, e^{2\pi i\eta_2\cdot(\,\cdot\,)}\Big)(x) = a(\eta_1,\eta_2)\, e^{2\pi i x\cdot(\eta_1+\eta_2)},$$

which is in L^∞ and thus in BMO uniformly in η_1, η_2. The same calculation is valid for the two transposes of T, since their corresponding multipliers also satisfy (28). The weak type results follow from Theorem 1. □

5. Orthogonality properties for bilinear multiplier operators

In view of Plancherel's theorem, the space L^2 turns into a heaven of orthogonality since Fourier multiplier operators turn into multiplication operators. There is no analogous phenomenon in the bilinear setting, but there is a range of indices for which bilinear multiplier operators exhibit properties analogous to those of linear multiplier operators on L^2. The situation where $2 \le p_1, p_2, p' < \infty$, $1/p_1 + 1/p_2 = 1/p$ is called the *local L^2 case*. There is a noteworthy orthogonality lemma for bilinear operators in the local L^2 case. The one-dimensional version of the result below appeared in [18].

Theorem 8. *Suppose that T_j, $j \in \mathbf{Z}$, are bilinear operators whose symbols $m_j(\xi_1, \xi_2)$ are supported in sets $A_j \times B_j$, where $\{A_j\}_j$ is a family of pairwise disjoint rectangles on $\mathbf{R}^n$ and $\{B_j\}_j$ is also a family of pairwise disjoint rectangles on $\mathbf{R}^n$. Assume, furthermore, that $\{A_j + B_j\}_j$ is a family of pairwise disjoint rectangles on $\mathbf{R}^n$ and that for some indices p_1, p_2, p in the local L^2 case one has that the bilinear operators*

$$T_{m_j}(f,g)(x) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} m_j(\xi,\eta)\, \widehat{f}(\xi)\, \widehat{g}(\eta)\, e^{2\pi i x\cdot(\xi+\eta)} d\xi\, d\eta$$

with symbols m_j are uniformly bounded, that is,

$$\sup_{j\in\mathbf{Z}} \big\|T_{m_j}\big\|_{L^{p_1}\times L^{p_2}\to L^p} \le C < \infty$$

whenever p_1, p_2, p are indices in the local L^2 case, i.e., $2 \le p_1, p_2, p' < \infty$. Then there is a finite constant $C_{p_1,p_2,n}$ such that

$$\Big\|\sum_{j\in\mathbf{Z}} T_{m_j}\Big\|_{L^{p_1}\times L^{p_2}\to L^p} \le C_{p_1,p_2,n}\, C, \tag{31}$$

that is, the sum of the T_{m_j} is also a bounded operator in the local L^2 case.

Proof. The key element of the proof of this result is the nth-dimensional version of Rubio de Francia's [36] Littlewood–Paley inequality for arbitrary disjoint intervals. This says that, if R_j is a family of disjoint rectangles in $\mathbf{R}^n$ and if $\Delta_j f = (\widehat{f}\chi_{R_j})^\vee$, then there is a constant $c_{p,n}$ such that for $2 \le p < \infty$ and all functions $f \in L^p(\mathbf{R}^n)$ one has

$$\Big\|\Big(\sum_{j\in\mathbf{Z}} |\Delta_j f|^2\Big)^{1/2}\Big\|_{L^p(\mathbf{R}^n)} \le c_{p,n}\, \big\|f\big\|_{L^p(\mathbf{R}^n)}. \tag{32}$$

This result was proved by Journé [26], but easier proofs of it were later provided by Soria [38] in dimension 2 and Sato [37] in higher dimensions.

We will prove (31) by duality. We denote by $p' = p/(p-1)$ the dual index of p. We introduce Littlewood–Paley operators

$$\begin{aligned}\Delta_j^1 f &= (\widehat{f}\,\chi_{A_j})^\vee\\ \Delta_j^2 g &= (\widehat{g}\,\chi_{A_j})^\vee\\ \Delta_j^3 h &= (\widehat{h}\,\chi_{A_j+B_j})^\vee.\end{aligned}$$

An easy calculation shows that

$$\widehat{T_{m_j}(f,g)}(\xi) = \int_{\mathbf{R}^n} m_j(\eta,\xi-\eta)\,\widehat{f}(\eta)\,\widehat{g}(\xi-\eta)\,d\eta; \tag{33}$$

thus the Fourier transform of $T_{m_j}(f,g)$ is supported in the set $A_j + B_j$. Then we have

$$\langle\, T_{m_j}(f,g), h\rangle = \langle\, T_{m_j}(\Delta_j^1 f, \Delta_j^2 g), \Delta_j^3 h\rangle$$

in view of the hypotheses on the m_j's. Consequently,

$$\begin{aligned}\left|\left\langle\, \sum_j T_{m_j}(f,g), h\right\rangle\right| &= \left|\sum_j \langle T_{m_j}(\Delta_j^1 f, \Delta_j^2 g), \Delta_j^3 h\rangle\right|\\ &\le \int_{\mathbf{R}^n}\Big(\sum_j |T_{m_j}(\Delta_j^1 f, \Delta_j^2 g)|^2\Big)^{1/2}\Big(\sum_j |\Delta_j^3|^2\Big)^{1/2} dx\\ &\le \left\|\Big(\sum_j |T_{m_j}(\Delta_j^1 f, \Delta_j^2 g)|^2\Big)^{1/2}\right\|_{L^p}\left\|\Big(\sum_j |\Delta_j^3 h|^2\Big)^{1/2}\right\|_{L^{p'}}\\ &\le \left\|\Big(\sum_j |T_{m_j}(\Delta_j^1 f, \Delta_j^2 g)|^2\Big)^{1/2}\right\|_{L^p} c_{p',n}\big\|h\big\|_{L^{p'}},\end{aligned}$$

where the last inequality follows from (32) since $p' \ge 2$. It suffices to estimate the square function above. Since $p/2 \le 1$, we have the first inequality below:

$$\begin{aligned}\left\|\Big(\sum_j |T_{m_j}(\Delta_j^1 f,\Delta_j^2 g)|^2\Big)^{1/2}\right\|_{L^p}^p &\le \int_{\mathbf{R}^n}\sum_j |T_{m_j}(\Delta_j^1 f, \Delta_j^2 g)|^p\,dx\\ &= \sum_j \|T_{m_j}(\Delta_j^1 f, \Delta_j^2 g)\|_{L^p}^p\\ &\le C^p \sum_j \|\Delta_j^1 f\|_{L^{p_1}}^p \|\Delta_j^2 g\|_{L^{p_2}}^p\\ &\le C^p\Big(\sum_j \|\Delta_j^1 f\|_{L^{p_1}}^{p_1}\Big)^{p/p_1}\Big(\sum_j \|\Delta_j^2 g\|_{L^{p_2}}^{p_2}\Big)^{p/p_2}\\ &\le C^p\left\|\Big(\sum_j |\Delta_j^1 f|^2\Big)^{1/2}\right\|_{L^{p_1}}^p\left\|\Big(\sum_j |\Delta_j^2 g|^2\Big)^{1/2}\right\|_{L^{p_2}}^p\\ &\le (C c_{p_1,n} c_{p_2,n})^p \|f\|_{L^{p_1}}^p \|g\|_{L^{p_2}}^p,\end{aligned}$$

where we used successively the uniform boundedness of the T_{m_j}'s, Hölder's inequality, and (32). The required conclusion follows with $C_{p_1,p_2,n} = c_{p_1,n}c_{p_2,n}c_{p',n}$. (Recall that $1/p = 1/p_1 + 1/p_2$; thus the dependence of the constant on p is expressed via that of p_1 and p_2.) $\square$

This result was used in [18] in proving that the characteristic function of the unit disc is a bounded bilinear multiplier in the local L^2 case. It also has other applications; below we use this result to provide another proof of the Coifman–Meyer multiplier theorem (Theorem 7) for bilinear operators. In view of the bilinear Calderón–Zygmund theorem, it suffices to prove boundedness at a single point; we pick this point to be $L^{p_1} \times L^{p_2} \to L^2$, where $1/p_1 + 1/p_2 = 1/2$ and we only prove it for a piece of the operator, as the remaining pieces are handled by duality.

We have the following:

Theorem 9. *Suppose that $m(\xi, \eta)$ is a function on $\mathbf{R}^n \times \mathbf{R}^n$ that satisfies*

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)| \le C_{\alpha,\beta}(|\xi| + |\eta|)^{-|\alpha|-|\beta|} \tag{34}$$

for multiindices $|\alpha|, |\beta| \le 2n + 1$. Then the multiplier operator

$$T_m(f_1, f_2)(x) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \widehat{f_1}(\xi_1)\, \widehat{f_2}(\xi_2)\, m(\xi_1, \xi_2)\; e^{2\pi i x\cdot(\xi_1+\xi_2)}\, d\xi_1\, d\xi_2$$

maps $L^{p_1} \times L^{p_2}$ to L^p, where $1/p_1 + 1/p_2 = 1/p$ and $1 < p_1, p_2, p < \infty$.

Proof. We begin by making the observation that T_m^{*1} is a bilinear multiplier operator with symbol

$$m^{*1}(\xi, \eta) = m(-\xi - \eta, \eta)$$

and T_m^{*2} is a bilinear multiplier operator with symbol

$$m^{*2}(\xi, \eta) = m(\xi, -\xi - \eta).$$

Both of these symbols satisfy condition (34) for all multiindices $|\alpha|, |\beta| \le n + 2$ (for some other constants). Moreover, if $m(\xi, \eta)$ is supported near the diagonal $|\xi| \approx |\eta|$, then $m^{*1}(\xi, \eta)$ is supported near the axis $|\xi| \lesssim |\eta|$ and $m^{*2}(\xi, \eta)$ is supported near the axis $|\eta| \lesssim |\xi|$. We introduce a smooth partition of unity on the sphere $\mathbf{S}^{2n-1}$ such that $1 = \phi_0 + \phi_1 + \phi_2 + \phi_3$, where $\phi_0(\xi, \eta)$ is supported in a neighborhood of the diagonal $|\xi| \approx |\eta| \approx |\xi + \eta|$, ϕ_1 is supported in a set of the form $|\xi| \lesssim |\eta|$, ϕ_2 is supported in a set of the form $|\eta| \lesssim |\xi|$, and ϕ_3 is supported in a set of the form $|\xi| \approx |\eta| \approx |\xi - \eta|$. We extend the functions ϕ_j so that they be homogeneous on $\mathbf{R}^{2n}$. We split the multiplier m as

$$m = m\,\phi_0 + m\,\phi_1 + m\,\phi_2 + m\,\phi_3 =: m_0 + m_1 + m_2 + m_3$$

and we say that m_0 is supported in the "good" direction, while m_1, m_2, m_3 are supported in the remaining three "bad" directions. We observe that the three bad directions are "preserved" by duality. By this we mean that $T_{m_3}^{*1}$ is an operator with symbol with the same properties as m_1 and $T_{m_3}^{*2}$ is an operator with symbol with the same properties as m_2.

We introduce a function ψ supported in the annulus $1/4 < |(\xi,\eta)| < 1-10^{-1}$ such that
$$\sum_{j\in\mathbf{Z}} \psi(2^{-j}(\xi,\eta)) = 1$$
for all $(\xi,\eta)\neq 0$. Then we write $m_r^j(\xi,\eta) = m_r(\xi,\eta)\psi(2^{-j}(\xi,\eta))$ and consider the operators $T_{m_r^j}$ with symbols m_r^j. A simple observation is that, for all $r\in\{0,1,2,3\}$ and all $j\in\mathbf{Z}$, the symbols m_r, m_r^j, and m_r^0 satisfy (34) with constants $C'_{\alpha,\beta}$ independent of j and explicitly related to the constants $C_{\alpha,\beta}$ appearing in (34).

We first show that for all $r\in\{0,1,2,3\}$ the operator $T_{m_r^j}$ maps $L^{p_1}\times L^{p_2}$ to L^2 when $1/p_1+1/p_2 = 1/2$ uniformly in j. In fact, it suffices to take $j=0$, noting that a dilation argument reduces the general case to that of $j=0$, since m_r^j satisfy (34) uniformly in j. Let us fix two Schwartz functions f_1 and f_2 on $\mathbf{R}^n$. We have

$$\big\|T_{m_r^0}(f_1,f_2)\big\|_{L^2}^2 = \int_{\mathbf{R}^n}\bigg|\int_{\mathbf{R}^n}\widehat{f}_1(\xi_1)\,\widehat{f}_2(\xi-\xi_1)\,m_r^0(\xi_1,\xi-\xi_1)\,d\xi_1\bigg|^2 d\xi \tag{35}$$

in view of (33). Since m_r^0 is supported in $[-1,1]^{2n}$, we expand it in Fourier series as follows:

$$m_r^0(\xi_1,\xi_2) = \sum_{k,l\in\mathbf{Z}^n} c_{k,l}\, e^{2\pi i k\cdot\xi_1}\, e^{2\pi i l\cdot \xi_2}\,\zeta(\xi_1)\,\zeta(\xi_2), \tag{36}$$

where ζ is a smooth function supported in the annulus $1/8<|\xi|<1$ and is equal to 1 on the support of ψ. Here $c_{k,l}$ are the Fourier coefficients of the expansion given by
$$c_{k,l} = \int_A\int_A m_0^0(t_1,t_2)\,e^{-2\pi i k\cdot t_1}\,e^{-2\pi i l\cdot t_2}\,dt_1\,dt_2,$$
where A is the n-dimensional annulus $1/4<|\xi|<1$. To obtain estimates for $c_{k,l}$, we integrate by parts with respect to the differential operators $(I-\Delta_{t_1})^L(I-\Delta_{t_2})^L$, where Δ_{t_1} is the Laplacian with respect to the variable t_1 and $L=[n/2]+1$. One obtains
$$c_{k,l} = \frac{\int_A\int_A \big\{(I-\Delta_{t_1})^L(I-\Delta_{t_2})^L m_0^0(t_1,t_2)\big\}\,e^{-2\pi i k\cdot t_1}\,e^{-2\pi i l\cdot t_2}\,dt_1\,dt_2}{(1+4\pi^2|k|^2)^L(1+4\pi^2|k|^2)^L}.$$
Condition (34) implies that the integrand above is uniformly bounded in k,l. Consequently we have the estimate
$$|c_{k,l}| \le \frac{C_n\sum_{|\alpha|,|\beta|\le L} C_{\alpha,\beta}}{(1+4\pi^2|k|^2)^L(1+4\pi^2|k|^2)^L}$$
and, since $2L>n$, we deduce that
$$\sum_{k,l\in\mathbf{Z}^n} c_{k,l} < C'_n \sum_{|\alpha|,|\beta|\le L} C_{\alpha,\beta} < \infty.$$
Using this fact and (35) and (36), we obtain
$$\big\|T_{m_r^0}(f_1,f_2)\big\|_{L^2}^2 \le \int_{\mathbf{R}^n}\sum_{k,l\in\mathbf{Z}^n} c_{k,l}\bigg|\int_{\mathbf{R}^n}\widehat{f}_1(\xi_1)\,\widehat{f}_2(\xi-\xi_1)\,\zeta_k(\xi_1)\,\zeta_l(\xi-\xi_1)\,d\xi_1\bigg|^2 d\xi,$$

where $\zeta_k(t) = \zeta(t)e^{2\pi i k\cdot t}$ and $\zeta_l(t) = \zeta(t)e^{2\pi i l\cdot t}$. Let S_{ζ_k} be the linear multiplier operator $S_{\zeta_k} f = \big(\widehat{f}\,\zeta_k\big)^{\vee}$. Clearly the last expression above is equal to

$$\sum_{k,l\in\mathbf{Z}^n} c_{k,l}\int_{\mathbf{R}^n}\big|\widehat{S_{\zeta_k}f_1} * \widehat{S_{\zeta_l}f_2}\big|^2\,d\xi = \sum_{k,l\in\mathbf{Z}^n} c_{k,l}\int_{\mathbf{R}^n}\big|S_{\zeta_k}f_1(x)\,S_{\zeta_l}f_2(x)\big|^2\,dx.$$

By Hölder's inequality, this expression is certainly bounded by

$$\sum_{k,l\in\mathbf{Z}^n} c_{k,l}\big\|S_{\zeta_k}f_1\big\|^2_{L^{p_1}}\big\|S_{\zeta_l}f_2\big\|^2_{L^{p_2}}$$

and the latter is clearly at most a constant multiple of $\|f_1\|^2_{L^{p_1}}\|f_2\|^2_{L^{p_2}}$.

Now we turn to the sum over $j\in\mathbf{Z}$ for each family $\{m_r^j\}_j$. As we pointed out earlier, we have that, for each r, $T_{m_r^j}$ maps $L^{p_1}\times L^{p_2}\to L^2$, where $1/p_1+1/p_2=1/2$ uniformly in j. As a consequence of (34), it follows that the kernel $K(x)$ of each $T_{m_r^j}$ and each T_{m_r} satisfies

$$|K(x_1,x_2)| + |(x_1,x_2)|\,|\nabla K(x_1,x_2)| \le C|(x_1,x_2)|^{-2n}$$

uniformly in j, thus it is of class 2-CZK$(A,1)$. By Theorem 1, it follows that $T_{m_r^j}$ maps $L^1\times L^1\to L^{1/2,\infty}$. It follows from Theorem 2 that $T_{m_r^j}$ maps $L^{p_1}\times L^{p_2}\to L^p$, where $1/p_1+1/p_2=1/p$ and $2\le p_1,p_2,p'\le\infty$. Thus each $T_{m_r^j}$ is bounded on the closure of the local L^2 triangle. We may apply Theorem 8 to obtain boundedness of the part of the operator corresponding to the good part of the symbol m_0, i.e.,

$$m_0(\xi,\eta) = \sum_{j\in\mathbf{Z}} m_0^j(\xi,\eta),$$

since the supports of the projections of the m_0^j's are pairwise disjoint (if they are split up in families of 10 elements indexed by numbers mod 10).

Unfortunately, we may not use orthogonality in the entire local L^2 triangle to obtain the same conclusion for the bad directions corresponding to the symbols m_1^j, m_2^j, and m_3^j. We note, however, that orthogonality can be used to obtain boundedness of T_{m_3} from that of each $T_{m_3^j}$ at the point $L^2\times L^2\to L^1$. This is in view of the observation that $T_{m_3^j}(f_1,f_2) = T_{m_3^j}(\Delta_j f_1,\Delta_j f_2)$ and of the next simple argument:

$$\begin{aligned}
\big\|T_{m_3}(f_1,f_2)\big\|_{L^1} &\le \sum_j \big\|T_{m_3^j}(\Delta_j f_1,\Delta_j f_2)\big\|_{L^1}\\
&\le C\sum_j \big\|T_{m_3^j}(\Delta_j f_1,\Delta_j f_2)\big\|_{L^1}\\
&\le C'\sum_j \big\|\Delta_j f_1\big\|_{L^2}\big\|\Delta_j f_2\big\|_{L^2}\\
&\le C'\Big(\sum_j \big\|\Delta_j f_1\big\|^2_{L^2}\Big)^{\frac12}\Big(\sum_j \big\|\Delta_{kj} f_2\big\|^2_{L^2}\Big)^{\frac12}\\
&\le C''\,\|f_1\|_{L^2}\|f_2\|_{L^2}.
\end{aligned}$$

Here Δ_j is a Littlewood–Paley operator with Fourier transform localization near 2^j. Duality (or a similar orthogonality argument) implies boundedness of T_{m_1} from $L^\infty \times L^2$ to L^2 and analogously boundedness of T_{m_2} from $L^2 \times L^\infty$ to L^2. Bilinear complex interpolation between this estimate and the estimate $L^1 \times L^1 \to L^{1/2,\infty}$ gives that T_{m_2} maps $L^{3/2} \times L^3$ to the Lorentz space $L^{1,4/3}$ (thus to weak $L^{1,\infty}$). But we will need to work with a point near $(2/3, 1/3, 1)$ that still lies on the line that joins $(1/2, 0, 1/2)$ to $(1, 1, 2)$ and has third coordinate strictly smaller than 1, i.e., the target space is $L^{s,\infty}$ for some $s > 1$, which is a dual space. Since T_{m_2} is bounded at this point, it follows by duality that T_{m_3} is bounded from $L^{p,1} \times L^{q,1}$ to $L^{r,\infty}$ for some point $(1/p, 1/q, 1/r)$ near the point $(2/3, 0, 2/3)$ with $q < \infty$. Using Theorem 2 we obtain strong boundedness of T_{m_3} in the open triangle with vertices $(1, 1, 2)$, $(1/2, 1/2, 1)$, and $(2/3, 0, 2/3)$. In particular, it follows that T_{m_3} is bounded from $L^p \times L^q$ to L^r for some point $(1/p, 1/q, 1/r)$ near the point $(3/4, 1/4, 1)$. The preceding argument with $(3/4, 4, 1)$ in place of $(1/2, 1/2, 1)$ yields boundedness of T_{m_3} in yet a bigger region closer to the point $(1, 0, 1)$. Continuing this process indefinitely yields boundedness for T_{m_3} in the open quadrangle with corners $(1, 1, 2)$, $(1, 0, 1)$, $(2/3, 0, 2/3)$, and $(1/2, 1/2, 1)$.

A similar argument (involving the other adjoint) yields boundedness of T_{m_3} in the open quadrangle with corners $(1, 1, 2)$, $(0, 1, 1)$, $(0, 2/3, 2/3)$, and $(1/2, 1/2, 1)$. Further interpolation provides boundedness of T_{m_3} in the pentagon with vertices $(1, 1, 2)$, $(0, 2/3, 2/3)$, $(2/3, 0, 2/3)$, $(1, 1, 0)$, and $(0, 1, 1)$, that is, T_{m_3} is bounded from $L^p \times L^q$ to L^r whenever $r < 3/2$. (Recall that we always have $1/p + 1/q = 1/r$.)

Duality implies boundedness of T_{m_2} from $L^p \times L^q$ to L^r whenever $q > 3$ and that of T_{m_1} from $L^p \times L^q$ to L^r whenever $p > 3$. Further interpolation with the point $(1, 1, 2)$ yields boundedness of T_{m_1} and T_{m_2} in two open (convex) quadrilaterals Q_1 and Q_2 such that $Q_1 \cap Q_2$ has a nontrivial intersection with the region Q_3 described by the condition $r < 3/2$. The intersection $Q_1 \cap Q_2 \cap Q_3$ contains the point $(2/5, 2/5, 4/5)$ and thus T_{m_1}, T_{m_2}, and T_{m_3} are all bounded from $L^{5/2} \times L^{5/2} \to L^{5/4}$. We previously showed that T_{m_0} is bounded in the entire local L^2 triangle, thus T_m is also bounded from $L^{5/2} \times L^{5/2} \to L^{5/4}$. Theorem 4 applies and yields boundedness of T_m in the entire region $L^p \times L^q \to L^r$ with $1 < p, q < \infty$ and $1/2 < r < \infty$. $\square$

We end this section by noting that the Marcinkiewicz multiplier theorem does not hold for bilinear operators; see [16].

6. The bilinear Hilbert transform and the method of rotations

It is also natural to ask whether Theorem 6 is true under less stringent conditions on the function Ω; for instance, instead of assuming that Ω is a Lipschitz function on the sphere, can one assume that it is in some L^q for $q > 1$? It is a classical result obtained by Calderón and Zygmund [4] using the method of rotations that homogeneous linear singular integrals with odd kernels are always L^p bounded for $1 < p < \infty$. We indicate what happens if the method of rotations is used in the multilinear setting. Let Ω be an odd integrable function on $\mathbf{S}^{2n-1}$. Let

$$T_\Omega(f_1,f_2)(x) = \iint_{\mathbf{R}^{2n}} \frac{\Omega((y_1,y_2)/|(y_1,y_2)|)}{|(y_1,y_2)|^{2n}}\, f_1(x-y_1)\, f_2(x-y_2)\, dy_1\, dy_2.$$

Using polar coordinates in $\mathbf{R}^{2n}$, we express

$$T_\Omega(f_1,f_2)(x) = \int_{\mathbf{S}^{2n-1}} \Omega(\theta_1,\theta_2) \left\{ \int_0^{+\infty} f_1(x-t\theta_1)\, f_2(x-t\theta_2)\, \frac{dt}{t} \right\} d(\theta_1,\theta_2).$$

Replacing (θ_1,θ_2) by $-(\theta_1,\theta_2)$, changing variables, and using that Ω is odd, we obtain

$$T_\Omega(f_1,f_2)(x) = \int_{\mathbf{S}^{2n-1}} \Omega(\theta_1,\theta_2) \left\{ \int_0^{+\infty} f_1(x+t\theta_1)\, f_2(x+t\theta_2)\, \frac{dt}{t} \right\} d(\theta_1,\theta_2),$$

and averaging these identities we deduce that

$$T_\Omega(f_1,f_2)(x) = \frac{1}{2}\int_{\mathbf{S}^{2n-1}} \Omega(\theta_1,\theta_2) \left\{ \int_{-\infty}^{+\infty} f_1(x-t\theta_1)\, f_2(x-t\theta_2)\, \frac{dt}{t} \right\} d(\theta_1,\theta_2).$$

The method of rotations gives rise to the operator inside the curly brackets above and one would like to know that this operator is bounded from a product of two Lebesgue spaces into another Lebesgue space (and preferably) uniformly bounded in θ_1, θ_2. Motivated by this calculation, for vectors $u, v \in \mathbf{R}^n$ we introduce the family of operators

$$\mathcal{H}_{u,v}(f_1,f_2)(x) = \text{p.v.} \int_{-\infty}^{+\infty} f_1(x-tu)\, f_2(x-tv)\, \frac{dt}{t}.$$

We call this operator the *directional bilinear Hilbert transform* (in the direction indicated by the vector (u,v) in $\mathbf{R}^{2n}$). In the special case $n=1$, we use the notation

$$H_{\alpha,\beta}(f,g)(x) = \text{p.v.} \int_{-\infty}^{+\infty} f(x-\alpha t)\, g(x-\beta t)\, \frac{dt}{t}$$

for the bilinear Hilbert transform defined for functions f, g on the line and for $x, \alpha, \beta \in \mathbf{R}$.

We mention results concerning boundedness of these operators. The operator $H_{\alpha,\beta}$ was first shown to be bounded by Lacey and Thiele [28], [29] in the range

$$1 < p, q \le \infty, \qquad 2/3 < r < \infty, \qquad 1/p + 1/q = 1/r. \tag{37}$$

Uniform L^r bounds (in α, β) for $H_{\alpha,\beta}$ were obtained by Grafakos and Li [17] in the local L^2 case (i.e., the case when $2 < p, q, r' < \infty$) and extended by Li [32] in the hexagonal region

$$1 < p, q, r < \infty, \qquad \left|\frac{1}{p} - \frac{1}{q}\right| < \frac{1}{2}, \qquad \left|\frac{1}{p} - \frac{1}{r'}\right| < \frac{1}{2}, \qquad \left|\frac{1}{q} - \frac{1}{r'}\right| < \frac{1}{2}. \tag{38}$$

The bilinear Hilbert transforms first appeared in an attempt of Calderón to show that the first commutator

$$\mathcal{C}_1(f;A)(x) = \text{p.v.} \int_{\mathbf{R}} \left(\frac{A(x)-A(y)}{x-y} \right) \frac{f(y)}{x-y}\, dy$$

is L^2 bounded. In fact, in the mid 1960's Calderón observed that the commutator $f \mapsto \mathcal{C}_1(f;A)$ can be written as the average

$$\mathcal{C}_1(f;A)(x) = \int_0^1 H_{1,\alpha}(f,A')(x)\,d\alpha,$$

and the boundedness of $C_1(f;A)$ can therefore be reduced to the uniform (in α) boundedness of $H_{1,\alpha}$. Likewise, the uniform boundedness of the $H_{\alpha,\beta}$ can be used to show that T_Ω is bounded from $L^{p_1}(\mathbf{R}) \times L^{p_2}(\mathbf{R})$ to $L^p(\mathbf{R})$ when Ω is an odd function on the sphere $\mathbf{S}^1$.

We use a similar idea to obtain new bounds for a higher dimensional commutator introduced by Christ and Journé [5]. The n-dimensional commutator is defined as

$$\mathcal{C}_1^{(n)}(f,a)(x) = \text{p.v.}\int_{\mathbf{R}^n} K(x-y)\int_0^1 f(y)\,a((1-t)x+ty)\,dt\,dy, \tag{39}$$

where $K(x)$ is a Calderón–Zygmund kernel in dimension n, and f, a are functions on $\mathbf{R}^n$. Christ and Journé [5] proved that $\mathcal{C}_1^{(n)}$ is bounded from $L^p(\mathbf{R}^n) \times L^\infty(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ for $1 < p < \infty$. Here we discuss some off-diagonal bounds $L^p \times L^q \to L^r$ whenever $1/p + 1/q = 1/r$ and $1 < p, q, r < \infty$.

As the operator $\mathcal{C}_1^{(n)}(f,a)$ is n-dimensional, we will need to "transfer" $H_{\alpha,\beta}$ in higher dimensions. To achieve this we use rotations. We have the following lemma:

Lemma 1. *Suppose that K is a kernel in $\mathbf{R}^{2n}$ (which may be a distribution) and let T_K be the bilinear singular integral operator associated with K,*

$$T_K(f,g)(x) = \iint K(x-y,x-z)\,f(y)\,g(z)\,dy\,dz.$$

Assume that T_K is bounded from $L^p(\mathbf{R}^n) \times L^q(\mathbf{R}^n) \to L^r(\mathbf{R}^n)$ with norm $\|T\|$ when $1/p + 1/q = 1/r$. Let M be a $n \times n$ invertible matrix. Define a $2n \times 2n$ invertible matrix

$$\widetilde{M} = \begin{pmatrix} M & O \\ O & M \end{pmatrix},$$

where O is the zero $n \times n$ matrix. Then the operator $T_{K\circ\widetilde{M}}$ is also bounded from $L^p(\mathbf{R}^n) \times L^q(\mathbf{R}^n) \to L^r(\mathbf{R}^n)$ with norm at most $\|T\|$.

Proof. To prove the lemma we note that

$$T_{K\circ\widetilde{M}}(f,g)(x) = T_K(f\circ M^{-1}, g\circ M^{-1})(Mx),$$

from which it follows that

$$\begin{aligned}
\big\|T_{K\circ\widetilde{M}}(f,g)\big\|_{L^r} &= (\det M)^{-1/r}\big\|T_K(f\circ M^{-1}, g\circ M^{-1})\big\|_{L^r} \\
&\le (\det M)^{-1/r}\|T\|\,\|f\circ M^{-1}\|_{L^p}\|g\circ M^{-1}\|_{L^q} \\
&= \|T\|\,(\det M)^{-1/r}\|T\|\,\|f\|_{L^p}(\det M)^{1/p}\|g\|_{L^q}(\det M)^{1/p} \\
&= \|T\|\,\|f\|_{L^p}\|g\|_{L^q}.
\end{aligned}$$

□

We apply Lemma 1 to the bilinear Hilbert transform. Let $e_1 = (1, 0, \dots, 0)$ be the standard coordinate vector on $\mathbf{R}^n$. We begin with the observation that the operator $\mathcal{H}_{\alpha e_1, \beta e_1}(f, g)$ defined for functions f, g on $\mathbf{R}^n$ is bounded from $L^p(\mathbf{R}^n) \times L^q(\mathbf{R}^n)$ to $L^r(\mathbf{R}^n)$ for the same range of indices as the bilinear Hilbert transform. Indeed, the operator $\mathcal{H}_{\alpha e_1, \beta e_1}$ can be viewed as the classical one-dimensional bilinear Hilbert transform in the coordinate x_1 followed by the identity operator in the remaining coordinates $x_2, \dots, x_n$, where $x = (x_1, \dots, x_n)$. By Lemma 1, for an invertible $n \times n$ matrix M and $x \in \mathbf{R}^n$ we have

$$\mathcal{H}_{\alpha e_1, \beta e_1}(f \circ M^{-1}, g \circ M^{-1})(Mx) = \text{p.v.} \int_{-\infty}^{+\infty} f(x - \alpha t M^{-1} e_1)\, g(x - \beta t M^{-1} e_1)\, \frac{dt}{t}$$

maps $L^p(\mathbf{R}^n) \times L^q(\mathbf{R}^n) \to L^r(\mathbf{R}^n)$ with norm the same as the one-dimensional bilinear Hilbert transform $H_{\alpha,\beta}$ whenever the indices p, q, r satisfy (37). If M is a rotation (i.e., an orthogonal matrix), then $M^{-1}e_1$ can be any unit vector in $\mathbf{S}^{n-1}$. We conclude that the family of operators

$$\mathcal{H}_{\alpha\theta, \beta\theta}(f, g)(x) = \text{p.v.} \int_{-\infty}^{+\infty} f(x - \alpha t\, \theta)\, g(x - \beta t\, \theta)\, \frac{dt}{t}, \qquad x \in \mathbf{R}^n,$$

is bounded from $L^p(\mathbf{R}^n) \times L^q(\mathbf{R}^n)$ to $L^r(\mathbf{R}^n)$ with a bound independent of $\theta \in \mathbf{S}^{n-1}$ whenever the indices p, q, r satisfy (37). This bound is also independent of α, β whenever the indices p, q, r satisfy (38).

It remains to express the higher dimensional commutator $\mathcal{C}_1^{(n)}$ in terms of the operators $\mathcal{H}_{\alpha\theta, \beta\theta}$. Here we make the assumption that K is an odd homogeneous singular integral operator on $\mathbf{R}^n$, such as a Riesz transform. For a fixed $x \in \mathbf{R}^n$ we apply polar coordinates centered at x by writing $y = x - r\theta$. Then we can express the higher dimensional commutator in (39) as

$$\int_{\mathbf{S}^{n-1}} \int_0^\infty \frac{K(\theta)}{r^n} \int_0^1 f(x - r\theta)\, a(x - tr\theta)\, dt\, r^{n-1}\, dr\, d\theta. \tag{40}$$

Changing variables from $\theta \to -\theta$, $r \to -r$, and using that $K(\theta)$ is odd, we write this expression as

$$\int_{\mathbf{S}^{n-1}} \int_{-\infty}^0 K(\theta) \int_0^1 f(x - r\theta)\, a(x - tr\theta)\, dt\, \frac{dr}{r}\, d\theta. \tag{41}$$

Averaging the (40) and (40) we arrive at the identity

$$\mathcal{C}_1^{(n)}(f, a)(x) = \frac{1}{2} \int_{\mathbf{S}^{n-1}} K(\theta) \int_0^1 \mathcal{H}_{\theta, t\theta}(f, a)(x)\, dt\, d\theta.$$

This identity implies boundedness of $\mathcal{C}_1^{(n)}$ from $L^p(\mathbf{R}^n) \times L^q(\mathbf{R}^n)$ to $L^r(\mathbf{R}^n)$ whenever the indices p, q, r satisfy (38). Interpolation with the known $L^p \times L^\infty \to L^p$ bounds yield the following:

Theorem 10. *Let K be an odd homogeneous singular integral on $\mathbf{R}^n$. Then the n-dimensional commutator $\mathcal{C}_1^{(n)}$ associated with K maps $L^p(\mathbf{R}^n) \times L^q(\mathbf{R}^n)$ to $L^r(\mathbf{R}^n)$ whenever $1/p + 1/q = 1/r$ and $(1/p, 1/q, 1/r)$ lies in the open convex hull of the pentagon with vertices* $(0, 1/2, 1/2)$, $(0, 0, 0)$, $(1, 0, 1)$, $(1/2, 1/2, 1)$, *and* $(1/6, 4/6, 5/6)$.

7. Counterexample for the higher dimensional bilinear ball multiplier

In this section we address the question whether the bilinear multiplier operator with symbol the characteristic function of the unit ball B in $\mathbf{R}^{2n}$,

$$T_{\chi_B}(f,g)(x) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \widehat{f}(\xi)\,\widehat{g}(\eta)\,\chi_{|\xi|^2+|\eta|^2<1}\,e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta, \tag{42}$$

is a bounded bilinear operator from $L^p(\mathbf{R}^n)\times L^q(\mathbf{R}^n)$ to $L^r(\mathbf{R}^n)$ for some indices p, q, r related as in Hölder's inequality. We adapt Fefferman's counterexample [13] for the ball multiplier on L^p, $p\neq 2$ to the bilinear setting for indices outside the local L^2 case. We consider this problem only in dimension 2, since it can be extended to higher dimensions via a bilinear adaptation of de Leeuw's theorem [30], proved in [11]. The results in this section can be found in [11].

For a rectangle R in $\mathbf{R}^2$, let R' be the union of the two copies of R adjacent to R in the direction of its longest side. Hence, $R\cup R'$ is a rectangle three times as long as R with the same center. Key to this argument is the following geometric lemma whose proof can be found in [14] or [42].

Lemma 2. *Let $\delta>0$ be given. Then there exists a measurable subset E of $\mathbf{R}^2$ and a finite collection of rectangles R_j in $\mathbf{R}^2$ such that:*

1. *The R_j are pairwise disjoint.*
2. *We have $1/2\le |E|\le 3/2$.*
3. *We have $|E|\le \delta\sum_j |R_j|$.*
4. *For all j we have $|R_j'\cap E|\ge \frac{1}{12}|R_j|$.*

Let $\delta>0$ and let E and R_j be as in Lemma 2. The proof of Lemma 2 implies that there are 2^k rectangles R_j of dimension $2^{-k}\times 3\log(k+2)$. Here, k is chosen so that $k+2\ge e^{1/\delta}$. Let v_j be the unit vector in $\mathbf{R}^2$ parallel to the longest side of R_j and in the direction of the set E indicated by the longest side of R_j.

Lemma 3. *Let R be a rectangle in $\mathbf{R}^2$ and let v be a unit vector in $\mathbf{R}^2$ parallel to the longest side of R. Let R' be as above. Consider the half space $\mathcal{H}_v$ of $\mathbf{R}^4$ defined by*

$$\mathcal{H}_v = \{(\xi,\eta)\in\mathbf{R}^2\times\mathbf{R}^2 \,:\, (\xi+\eta)\cdot v\ge 0\}.$$

Then the following estimate is valid for all $x\in\mathbf{R}^2$:

$$\left|\int_{\mathbf{R}^2}\int_{\mathbf{R}^2} \chi_{\mathcal{H}_v}(\xi,\eta)\,\widehat{\chi_R}(\xi)\,\widehat{\chi_R}(\eta)\,e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta\right| \ge \frac{1}{10}\chi_{R'}(x). \tag{43}$$

Proof. We introduce an orthogonal matrix $\mathcal{O}$ of $\mathbf{R}^2$ such that $\mathcal{O}(v)=(1,0)$. Setting $\xi=(\xi_1,\xi_2)$, $\eta=(\eta_1,\eta_2)$, we write the expression on the left in (43) as

$$\left|\iint_{\mathcal{O}^{-1}(\xi+\eta)\cdot v\ge 0} \widehat{\chi_R}(\mathcal{O}^{-1}\xi)\,\widehat{\chi_R}(\mathcal{O}^{-1}\eta)\,e^{2\pi i x\cdot \mathcal{O}^{-1}(\xi+\eta)}\,d\xi\,d\eta\right|$$

$$= \left|\iint_{\xi_1+\eta_1\ge 0} \widehat{\chi_{\mathcal{O}[R]}}(\xi)\,\widehat{\chi_{\mathcal{O}[R]}}(\eta)\,e^{2\pi i \mathcal{O}x\cdot(\xi+\eta)}\,d\xi\,d\eta\right|.$$

Now the rectangle $\mathcal{O}[R]$ has sides parallel to the axes, say $\mathcal{O}[R] = I_1 \times I_2$. Assume that $|I_1| > |I_2|$, i.e., its longest side is horizontal. Let H be the classical Hilbert transform on the line. Setting $\mathcal{O}x = (y_1, y_2)$, we can write the last displayed expression as

$$\begin{aligned}
&\left|\chi_{I_2}(y_2)^2 \int_{\xi_1 \in \mathbf{R}} \widehat{\chi_{I_1}}(\xi_1)\, e^{2\pi i y_1 \xi_1} \int_{\eta_1 \geq -\xi_1} \widehat{\chi_{I_1}}(\eta_1)\, e^{2\pi i y_1 \eta_1}\, d\eta_1\, d\xi_1\right| \\
&\qquad = \chi_{I_2}(y_2) \left| \int_{\xi_1 \in \mathbf{R}} \widehat{\chi_{I_1}}(\xi_1)\, \frac{1}{2}(I + iH)\big[\chi_{I_1}(\,\cdot\,)\, e^{2\pi i \xi_1 (\,\cdot\,)}\big](y_1)\, d\xi_1 \right| \\
&\qquad = \chi_{I_2}(y_2) \left| \frac{1}{2}(I + iH)\, \chi_{I_1}(y_1) \right| = \left| \big[\chi_{\xi_1 \geq 0}\, \widehat{\chi_{I_1 \times I_2}}(\xi_1, \xi_2)\big]^{\vee}(y_1, y_2) \right|.
\end{aligned}$$

Using the result from [14, Proposition 10.1.2] or [42, estimate (33), page 453], we deduce that the preceding expression is at least

$$\frac{1}{10}\chi_{(I_1 \times I_2)'}(y_1, y_2) = \frac{1}{10}\chi_{(\mathcal{O}[R])'}(\mathcal{O}x) = \frac{1}{10}\chi_{R'}(x).$$

This proves the required conclusion. □

Lemma 4. *Let $v_1, v_2, \dots, v_j, \dots$ be a sequence of unit vectors in $\mathbf{R}^2$. Define a sequence of half-spaces $\mathcal{H}_{v_j}$ in $\mathbf{R}^4$ as in Lemma 3. Let B, B_1, B_2 be the following sets in $\mathbf{R}^4$:*

$$\begin{aligned}
B &= \{(\xi, \eta) \in \mathbf{R}^2 \times \mathbf{R}^2 \,:\, |\xi|^2 + |\eta|^2 \leq 1\} \\
B^{*1} &= \{(\xi, \eta) \in \mathbf{R}^2 \times \mathbf{R}^2 \,:\, |\xi + \eta|^2 + |\eta|^2 \leq 1\} \\
B^{*2} &= \{(\xi, \eta) \in \mathbf{R}^2 \times \mathbf{R}^2 \,:\, |\xi|^2 + |\xi + \eta|^2 \leq 1\}.
\end{aligned}$$

Assume that one of T_{χ_B}, $T_{\chi_{B^{*1}}}$, $T_{\chi_{B^{*2}}}$ is bounded from $L^p(\mathbf{R}^2) \times L^q(\mathbf{R}^2)$ to $L^r(\mathbf{R}^2)$ and has norm $C = C(p, q, r)$. Then the vector-valued inequality

$$\begin{aligned}
&\left\| \left(\sum_j \left| T_{\chi_{\mathcal{H}_{v_j}}}(f_j, g_j) \right|^2 \right)^{1/2} \right\|_{L^r(\mathbf{R}^2)} \\
&\qquad \leq C \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{L^p(\mathbf{R}^2)} \left\| \left(\sum_j |g_j|^2 \right)^{1/2} \right\|_{L^q(\mathbf{R}^2)}
\end{aligned}$$

holds for all functions f_j and g_j.

Proof. Assume that T_{χ_B} is bounded from $L^p(\mathbf{R}^2) \times L^q(\mathbf{R}^2)$ to $L^r(\mathbf{R}^2)$ for some indices $p, q, r > 0$. Set $\xi = (\xi_1, \xi_2)$ and $\eta = (\eta_1, \eta_2) \in \mathbf{R}^2$. For $\rho > 0$ we define sets

$$\begin{aligned}
B_\rho &= \{(\xi, \eta) \in \mathbf{R}^2 \times \mathbf{R}^2 \,:\, |\xi|^2 + |\eta|^2 \leq 2\rho^2\} \\
B_{j,\rho} &= \{(\xi, \eta) \in \mathbf{R}^2 \times \mathbf{R}^2 \,:\, |\xi - \rho v_j|^2 + |\eta - \rho v_j|^2 \leq 2\rho^2\}.
\end{aligned}$$

Note that bilinear multiplier norms are translation and dilation invariant; consequently, we have

$$\|T_{\chi_{B_{j,\rho}}}\|_{L^p \times L^q \to L^r} = \|T_{\chi_{B_\rho}}\|_{L^p \times L^q \to L^r} = \|T_{\chi_B}\|_{L^p \times L^q \to L^r} = C. \tag{44}$$

Moreover, in view of the bilinear version of a theorem of Marcinkiewicz and Zygmund ([19, §9]), we have the following vector-valued extension of our boundedness assumption on $T_{\chi_{B_\rho}}$:

$$\Big\|\Big(\sum_j \big|T_{\chi_{B_\rho}}(f_j,g_j)\big|^2\Big)^{1/2}\Big\|_{L^r} \le C\Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_{L^p}\Big\|\Big(\sum_j |g_j|^2\Big)^{1/2}\Big\|_{L^q}.$$

Since $\chi_{B_{j,\rho}} \to \chi_{\mathcal{H}_{v_j}}$ pointwise as $\rho \to \infty$ for $x \in \mathbf{R}^2$, we deduce that

$$\lim_{\rho\to\infty} T_{\chi_{B_{j,\rho}}}(f,g)(x) = T_{\chi_{\mathcal{H}_{v_j}}}(f,g)(x)$$

for suitable functions f and g. By Fatou's lemma we conclude that

$$\Big\|\Big(\sum_j \big|T_{\chi_{\mathcal{H}_{v_j}}}(f_j,g_j)\big|^2\Big)^{1/2}\Big\|_{L^r} \le \liminf_{\rho\to\infty}\Big\|\Big(\sum_j \big|T_{\chi_{B_{j,\rho}}}(f_j,g_j)\big|^2\Big)^{1/2}\Big\|_{L^r}. \tag{45}$$

Now, observe the following identity:

$$T_{\chi_{B_{j,\rho}}}(f,g)(x) = e^{4\pi i\rho v_j\cdot x}\, T_{\chi_{B_\rho}}(e^{-2\pi i\rho v_j\cdot(\,\cdot\,)}f,\, e^{-2\pi i\rho v_j\cdot(\,\cdot\,)}g)(x). \tag{46}$$

Using (45) and(46), we obtain

$$\begin{aligned}
&\Big\|\Big(\sum_j \big|T_{\chi_{\mathcal{H}_j}}(f_j,g_j)\big|^2\Big)^{1/2}\Big\|_{L^r} \\
&\le \liminf_{\rho\to\infty}\Big\|\Big(\sum_j \big|e^{4\pi i\rho v_j\cdot(\,\cdot\,)}\, T_{\chi_{B_\rho}}(e^{-2\pi i\rho v_j\cdot(\,\cdot\,)}f_j,\, e^{-2\pi i\rho v_j\cdot(\,\cdot\,)}g_j)\big|^2\Big)^{1/2}\Big\|_{L^r} \\
&\le \liminf_{\rho\to\infty}\big\|T_{\chi_{B_\rho}}\big\|_{L^p\times L^q\to L^r}\Big\|\Big(\sum_j \big|e^{-2\pi i\rho v_j\cdot(\,\cdot\,)}f_j\big|^2\Big)^{1/2}\Big\|_{L^p} \\
&\qquad\qquad \Big\|\Big(\sum_j \big|e^{-2\pi i\rho v_j\cdot(\,\cdot\,)}g_j\big|^2\Big)^{1/2}\Big\|_{L^q} \\
&= C\Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_{L^p}\Big\|\Big(\sum_j |g_j|^2\Big)^{1/2}\Big\|_{L^q},
\end{aligned}$$

where in the last step we used (44).

The proof of the analogous statements for $T_{B^{*1}}$ and $T_{B^{*2}}$ is as follows. We introduce sets

$$\begin{aligned}
B^{*1}_\rho &= \{(\xi,\eta)\in\mathbf{R}^2\times\mathbf{R}^2 \,:\, |\xi+\eta|^2+|\eta|^2 \le \rho^2\}\\
B^{*1}_{j,\rho} &= \{(\xi,\eta)\in\mathbf{R}^2\times\mathbf{R}^2 \,:\, |\xi-\rho v_j+\eta|^2+|\eta|^2 \le \rho^2\}\\
B^{*2}_\rho &= \{(\xi,\eta)\in\mathbf{R}^2\times\mathbf{R}^2 \,:\, |\xi|^2+|\xi+\eta|^2 \le \rho^2\}
\end{aligned}$$

$$B^{*2}_{j,\rho} = \{(\xi,\eta) \in \mathbf{R}^2 \times \mathbf{R}^2 \,:\, |\xi|^2 + |\xi+\eta-\rho v_j|^2 \le \rho^2\}$$

and we note that the characteristic functions of $B^{*1}_{j,\rho}$ and $B^{*2}_{j,\rho}$ converge to the characteristic function of $\mathcal{H}_{v_j}$ as $\rho \to \infty$. Using the identities

$$T_{\chi_{B^{*1}_{j,\rho}}}(f,g)(x) = e^{2\pi i \rho v_j \cdot x}\, T_{\chi_{B^{*1}_{\rho}}}(e^{-2\pi i \rho v_j \cdot (\,\cdot\,)} f, g)(x),$$

$$T_{\chi_{B^{*2}_{j,\rho}}}(f,g)(x) = e^{2\pi i \rho v_j \cdot x}\, T_{\chi_{B^{*2}_{\rho}}}(f, e^{-2\pi i \rho v_j \cdot (\,\cdot\,)} g)(x),$$

we obtain the same conclusion assuming boundedness of the bilinear operators $T_{\chi_{B^{*1}}}$ and $T_{\chi_{B^{*2}}}$ from $L^p(\mathbf{R}^2) \times L^q(\mathbf{R}^2)$ to $L^r(\mathbf{R}^2)$. $\square$

The following is the main result of this section.

Theorem 11. *Fix indices p, q, r satisfying $0 < p,q,r, < \infty$ and $1/p + 1/q = 1/r$ in the non-local L^2 region, i.e., in the region where one of p, q, or r' is less than 2. Let B be the unit ball in $\mathbf{R}^4$. Then the operator in (42) (with $n = 2$) is not bounded from $L^p(\mathbf{R}^2) \times L^q(\mathbf{R}^2)$ to $L^r(\mathbf{R}^2)$.*

Proof. First fix p, q, r satisfying $p^{-1} + q^{-1} = r^{-1} < 1/2$ with $r > 2$. To obtain a contradiction, we assume that the operator in (42) (with $n = 2$) is bounded from $L^p(\mathbf{R}^2) \times L^q(\mathbf{R}^2)$ to $L^r(\mathbf{R}^2)$ with norm C.

Suppose that $\delta > 0$ is given. Let E and R_j be as in Lemma 2. Let v_j be the unit vector parallel to the longest side of R_j and pointing in the direction of the set E indicated by the longest side of R_j. We have

$$\begin{aligned}
&\sum_j \int_E \big|T_{\mathcal{H}_{v_j}}(\chi_{R_j},\, \chi_{R_j})(x)\big|^2 dx \\
&\qquad \le |E|^{\frac{r-2}{r}} \bigg\| \Big(\sum_j \big|T_{\mathcal{H}_{v_j}}(\chi_{R_j}, \chi_{R_j})\big|^2 \Big)^{1/2} \bigg\|_{L^r}^2 \\
&\qquad \le C\, |E|^{\frac{r-2}{r}} \bigg\| \Big(\sum_j \big|\chi_{R_j}\big|^2 \Big)^{1/2} \bigg\|_{L^p}^2 \bigg\| \Big(\sum_j \big|\chi_{R_j}\big|^2 \Big)^{1/2} \bigg\|_{L^q}^2 \\
&\qquad = C\, |E|^{\frac{r-2}{r}} \Big(\sum_j |R_j| \Big)^{2/r} \\
&\qquad \le C\, \delta^{\frac{r-2}{r}} \sum_j |R_j|,
\end{aligned}$$

where we used Hölder's inequality with $r > 2$, Lemma 4, the disjointness of the rectangles R_j, and Lemma 2, respectively, in the preceding sequence of estimates.

We also have a reverse inequality:

$$\sum_j \int_E \big|T_{\mathcal{H}_{v_j}}(\chi_{R_j}, \chi_{R_j})(x)\big|^2 dx \ge \sum_j \int_E \Big(\tfrac{1}{10}\chi_{R_j'}(x)\Big)^2 dx$$

$$= \tfrac{1}{100} \sum_j \left|E \cap R_j'\right|$$
$$\geq \tfrac{1}{1200} \sum_j \left|R_j\right|,$$

where we used Lemma 3 and Lemma 2.

Combining the upper and lower estimates for $\sum_j \int_E \left|T_{\mathcal{H}_{v_j}}(\chi_{R_j}, \chi_{R_j})(x)\right|^2 dx$, we obtain the inequality

$$\tfrac{1}{1200} \sum_j |R_j| \leq C\, \delta^{\frac{r-2}{r}} \sum_j |R_j|,$$

and therefore

$$\tfrac{1}{1200} \leq C\, \delta^{\frac{r-2}{r}}$$

for any $\delta > 0$. This is a contradiction since we are assuming that $r > 2$.

The lack of boundedness of the ball multiplier operator (42) in the remaining non-local L^2 regions ($p > 2$, $q < 2$, $r < 2$) and ($p < 2$, $q > 2$, $r < 2$) follows by duality, while in the region ($1 \leq p, q < \infty$, $\frac{1}{2} < r \leq 1$) it is obvious. $\square$

References

[1] A. Benedek, A. Calderón. and R. Panzone, *Convolution operators on Banach-space valued functions*, Proc. Nat. Acad. Sci. USA **48** (1962), 356–365.

[2] G. Bourdaud, *Une algèbre maximale d'opérateurs pseudo-différentiels*, Comm. Partial Diff. Eq. **13** (1988), 1059–1083.

[3] A. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. **88** (1952), 85–139.

[4] A. Calderón and A. Zygmund, *On singular integrals*, Amer. J. Math. **78** (1956), 289–309.

[5] M. Christ and J.-L. Journé, *Polynomial growth estimates for multilinear singular integral operators*, Acta Math. **159** (1987), 51–80.

[6] R. R. Coifman and Y. Meyer, *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. **212** (1975), 315–331.

[7] R. R. Coifman and Y. Meyer, *Commutateurs d'intégrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier, Grenoble **28** (1978), 177–202.

[8] R. R. Coifman and Y. Meyer, *Au-delà des opérateurs pseudo-différentiels*, Astérisque **57**, Société Mathématique de France, Paris, 1978.

[9] R. R. Coifman and Y. Meyer, *Non-linear harmonic analysis, operator theory, and PDE*, in: Beijing Lectures in Harmonic Analysis, E. M. Stein, ed., Ann. of Math. Studies **112**, Princeton University Press, Princeton, NJ, 1986.

[10] G. David and J.-L. Journé, *A boundedness criterion for generalized Calderón–Zygmund operators*, Ann. of Math. **120** (1984), 371–397.

[11] G. Diestel and L. Grafakos, *Unboundedness of the ball bilinear multiplier operator*, Nagoya Math. J. **185** (2007), 151–159.

[12] G. Diestel and N. Kalton, personal communication.

[13] C. Fefferman, *The multiplier problem for the ball*, Ann. of Math. **94** (1971), 330–336.

[14] L. Grafakos, *Modern Fourier Analysis*, 2nd edition, Graduate Texts in Mathematics **250**, Springer, New York, 2008.

[15] L. Grafakos and N. Kalton, *Some remarks on multilinear maps and interpolation*, Math. Ann. **319** (2001), 151–180.

[16] L. Grafakos and N. Kalton, *The Marcinkiewicz multiplier condition for bilinear operators*, Studia Math. **146** (2001), 115–156.

[17] L. Grafakos and X. Li, *Uniform bounds for the bilinear Hilbert transforms I*, Ann. of Math. **159** (2004), 889–933.

[18] L. Grafakos and X. Li, *The disc as a bilinear multiplier*, Amer. J. Math. **128** (2006), 91–119.

[19] L. Grafakos and J. M. Martell, *Extrapolation of weighted norm inequalities for multivariable operators*, J. Geom. Anal. **14** (2004), 19–46.

[20] L. Grafakos and J. Soria, *Translation-invariant bilinear operators with positive kernels*, Integral Equat. Oper. Th. **66** (2010), 253–264.

[21] L. Grafakos and T. Tao, *Multilinear interpolation between adjoint operators*, J. Funct. Anal. **199** (2003), 379–385.

[22] L. Grafakos and R. H. Torres, *Multilinear Calderón–Zygmund theory*, Adv. Math. **165** (2002), 124–164.

[23] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. **104** (1960), 93–140.

[24] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, 2nd edition, Springer, Berlin Heidelberg New York, 1990.

[25] S. Janson, *On interpolation of multilinear operators*, in: Function Spaces and Applications (Lund, 1986), Lecture Notes in Math. **1302**, Springer, Berlin Heidelberg New York, 1988.

[26] J.-L. Journé, *Calderón–Zygmund operators on product spaces*, Rev. Mat. Iber. **1** (1985), 55–91.

[27] C. Kenig and E. M. Stein, *Multilinear estimates and fractional integration*, Math. Res. Lett. **6** (1999), 1–15.

[28] M. T. Lacey and C. M. Thiele, *L^p bounds for the bilinear Hilbert transform,* $2 < p < \infty$, Ann. of Math. **146** (1997), 693–724.

[29] M. T. Lacey and C. M. Thiele, *On Calderón's conjecture*, Ann. of Math. **149** (1999), 475–496.

[30] K. de Leeuw, *On L_p multipliers*, Ann. of Math. **81** (1965), 364–379.

[31] A. Lerner, S. Ombrosi, C. Pérez, R. Torres and R. Trujillo-Gonzalez, *New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory,* Adv. in Math. **220** (2009) 1222-1264.

[32] X. Li, *Uniform bounds for the bilinear Hilbert transforms II*, Rev. Mat. Iber. **22** (2006), 1069–1126.

[33] Y. Meyer and R. R. Coifman, *Wavelets: Calderón–Zygmund and Multilinear Operators*, Cambridge Univeristy Press, Cambridge, UK, 1997.

[34] S. G. Mihlin, *On the multipliers of Fourier integrals* [in Russian], Dokl. Akad. Nauk. **109** (1956), 701–703.

[35] J. Peetre, *On convolution operators leaving $L^{p,\lambda}$ spaces invariant*, Ann. Mat. Pura Appl. **72** (1966), 295–304.

[36] F. Rubio de Francia, *A Littlewood–Paley inequality for arbitrary intervals*, Rev. Mat. Iber. **1** (1985), 1–14.

[37] S. Sato, *Note on Littlewood–Paley operator in higher dimensions*, J. London Math. Soc. **42** (1990), 527–534.

[38] F. Soria, *A note on a Littlewood–Paley inequality for arbitrary intervals in* $\mathbb{R}^2$, J. London Math. Soc. **36** (1987), 137–142.

[39] S. Spanne, *Sur l'interpolation entre les espaces $\mathcal{L}_k{}^{p\Phi}$*, Ann. Scuola Norm. Sup. Pisa **20** (1966), 625–648.

[40] E. M. Stein, *Singular integrals, harmonic functions, and differentiability properties of functions of several variables*, in: Singular Integrals, Proc. Sympos. Pure Math. **10** (1967), 316–335.

[41] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ, 1970.

[42] E. M. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, NJ, 1993.

[43] R. Strichartz, *A multilinear version of the Marcinkiewicz interpolation theorem*, Proc. Amer. Math. Soc. **21** (1969), 441–444.

[44] C. Thiele, *A uniform estimate*, Ann. of Math. **157** (2002), 1–45.