THE HÖRMANDER MULTIPLIER THEOREM FOR MULTILINEAR OPERATORS

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Abstract. In this paper, we provide a version of the Mihlin-Hörmander multiplier theorem for multilinear operators in the case where the target space is L^p for $p \le 1$. This extends a recent result of Tomita [15] who proved an analogous result for $p > 1$.

1. Introduction

Let $\mathscr{S}(\mathbf{R}^d)$ be the Schwartz space of all rapidly decreasing smooth functions on $\mathbf{R}^d$, for some $d \in \mathbf{Z}^+$. We define the Fourier transform $\mathscr{F}$ and the inverse Fourier transform $\mathscr{F}^{-1}$ of a function $f \in \mathscr{S}(\mathbf{R}^d)$ by

$$\mathscr{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbf{R}^d} e^{-2\pi i x\cdot\xi} f(x)\,dx$$

and

$$\mathscr{F}^{-1}(f)(\xi) = f^{\vee}(\xi) = \int_{\mathbf{R}^d} e^{2\pi i x\cdot\xi} f(x)\,dx$$

The Mihlin multiplier [14] theorem says that if a function σ defined on $\mathbf{R}^d \backslash \{0\}$ has at least $[d/2]+1$ continuous derivatives that satisfy

$$|\partial^\alpha \sigma(\xi)| \le C_\alpha |\xi|^{-|\alpha|} \tag{1}$$

for all $|\alpha| \le [d/2]+1$ ($[t]$ is the integer part of t), then the operator

$$T_\sigma(f)(x) = \int_{\mathbf{R}^d} \widehat{f}(\xi)\sigma(\xi) e^{2\pi i x\cdot\xi}\,d\xi = \mathscr{F}^{-1}(\sigma\,\mathscr{F}(f))(x)\,,$$

initially defined for Schwartz functions, admits a bounded extension on $L^p(\mathbf{R}^d)$ for all $1 < p < \infty$.

An improved version of Mikhlin's theorem was proved by Hörmander. To describe this version, we introduce some notation: the Laplacian on $\mathbf{R}^d$ is $\Delta g = \sum_{j=1}^d \partial^2 g/\partial x_j^2$, i.e., the sum of the second partials of g in every variable. We define the operator $(I-\Delta)^{\gamma/2}(g) = \mathscr{F}^{-1}(w_\gamma \mathscr{F}(g))$, where

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$w_\gamma(\xi) = (1 + 4\pi^2|\xi|^2)^{\gamma/2}$ for $\gamma > 0$. Let $L^r_\gamma(\mathbf{R}^d)$ be the L^r-based Sobolev space with norm

$$\|f\|_{L^r_\gamma} = \|(I - \Delta)^{\gamma/2} f\|_{L^r(\mathbf{R}^d)}, \tag{2}$$

where $1 \le r < \infty$. We also let $\mathscr{S}_1(\mathbf{R}^d)$ be the set of all Schwartz functions Ψ on $\mathbf{R}^d$, whose Fourier transform is supported in an annulus of the form $\{\xi : \ c_1 < |\xi| < c_2\}$, is nonvanishing in a smaller annulus $\{\xi : \ c_1' \le |\xi| \le c_2'\}$ (for some choice of constants $0 < c_1 < c_1' < c_2' < c_2 < \infty$), and satisfies

$$\sum_{j \in \mathbf{Z}} \widehat{\Psi}(2^{-j}\xi) = \text{constant}, \qquad \xi \in \mathbf{R^d} \setminus \{0\}. \tag{3}$$

It is worth observing that a necessary condition on the constants c_1', c_2' such that there exists some function in $\mathscr{S}_1(\mathbf{R}^d)$ whose Fourier transform is nonvanishing in the annulus $\{\xi : \ c_1' \le |\xi| \le c_2'\}$, is that $2c_1' \le c_2'$. In this case, we define Ψ in $\mathscr{S}_1(\mathbf{R}^d)$ to be the inverse Fourier transform of

$$\frac{\widehat{\eta}(\xi)}{\sum_{\ell \in \mathbf{Z}} \widehat{\eta}(2^{-\ell}\xi)} \, ,$$

where η is a function in $\mathscr{S}(\mathbf{R}^d)$ whose Fourier transform is supported in $\{\xi : \ c_1 < |\xi| < c_2\}$ and is nonvanishing in $\{\xi : \ c_1' \le |\xi| \le c_2'\}$.

Hörmander's version (see [9]) of Mikhlin's theorem is the following: Suppose that σ is a bounded function on $\mathbf{R}^d$ that satisfies

$$\sup_{k \in \mathbf{Z}} \|\widehat{\Psi}(\cdot)\, \sigma(2^k(\cdot))\|_{L^r_\gamma(\mathbf{R}^d)} < \infty \tag{4}$$

for some $1 \le r \le 2$, some $\gamma > d/r$, and some $\Psi \in \mathscr{S}_1(\mathbf{R}^d)$. Then σ is a Fourier multiplier on L^p, $1 < p < \infty$, i.e., the operator T_σ admits a bounded extension on $L^p(\mathbf{R}^d)$. We note that condition (4) is weaker than (1) and becomes least restrictive when $r = 2$; we also note that if condition (4) holds for *some* Ψ in $\mathscr{S}_1(\mathbf{R}^d)$, then it holds *for all*[1] Ψ in $\mathscr{S}_1(\mathbf{R}^d)$. Condition (3) can be avoided if the constants c_1, c_2, c_1', c_2' are chosen suitably. However, it appears naturally in many situations and allows one to prove the equivalence of (4) between *one* and *all* functions in $\mathscr{S}_1(\mathbf{R}^d)$. Thus, it provides us with flexibility in the choice of Ψ in (4) and it becomes very useful for the purposes of this article.

In this article, we provide a version of the Hörmander multiplier theorem in the case of multilinear operators. The study of such operators originated in the work of Coifman and Meyer [2], [3], [4] and was later revived by the groundbreaking work of Lacey and Thiele's on the bilinear Hilbert transform [12], [13]. The multilinear Fourier multiplier operator T_σ associated with a

[1]See Lemma 2.3.

symbol σ is defined by

$$T_\sigma(f_1,\dots,f_m)(x) \\ =\int_{(\mathbf{R}^n)^m} e^{2\pi i x\cdot(\xi_1+\cdots+\xi_m)}\sigma(\xi_1,\dots,\xi_m)\widehat{f}_1(\xi_1)\cdots\widehat{f}_m(\xi_m)\,d\xi_1\cdots d\xi_m$$

for $f_i\in\mathscr{S}(\mathbf{R}^n)$, $i=1,\cdots,m$.

Coifman and Meyer [4] proved that if σ is a function on $(\mathbf{R}^n)^m\setminus\{0\}$ that satisfies

$$|\partial_{\xi_1}^{\alpha_1}\cdots\partial_{\xi_m}^{\alpha_m}\sigma(\xi_1,\dots,\xi_m)|\le C_\alpha(|\xi_1|+\cdots+|\xi_m|)^{-(|\alpha_1|+\cdots+|\alpha_m|)} \tag{5}$$

away from the origin for all sufficiently large multiindices α_j, then T_σ is bounded from the product $L^{p_1}(\mathbf{R}^n)\times\cdots\times L^{p_m}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ for all $1<p_1,\dots,p_m,p<\infty$ satisfying $\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p}$. Their proof is based on the idea of writing the Fourier multiplier σ as a rapidly convergent sum of products of functions of the variables ξ_j. The multiplier theorem of Coifman and Meyer was extended to indices $p<1$ (and larger than $1/m$ by Grafakos and Torres [8] and Kenig and Stein [11] (when $m=2$). The approach in these papers is based on a multiple Calderón-Zygmund decomposition which yields weak type estimates for T_σ when at least one index $p_j=1$; in particular, this approach gives a weak type $L^1\times\cdots\times L^1\to L^{1/m,\infty}$ estimate which yields the result for the remaining indices with $p\le 1$, via multilinear interpolation.

It seems that in the proof of Coifman and Meyer [4], the number of derivatives required of σ is at least $2mn$; see Yabuta [16]. On the other hand, by using the m-linear $T1$ theorem Grafakos and Torres [8], it follows that $mn+1$ derivatives of σ are sufficient to imply the boundedness of T_σ. However, even this number of derivatives is too big from the viewpoint of the linear case. Exploiting the idea of the proof of the Hörmander multiplier theorem in [5], Tomita [15] proved the following result in the m-linear case:

Theorem A. [15] *Let $\sigma\in L^\infty((\mathbf{R}^n)^m)$. Let Ψ be a Schwartz function whose Fourier transform is supported in the set $\{\vec{\xi}\in(\mathbf{R}^n)^m : 1/2\le|\vec{\xi}|\le 2\}$ and satisfies*

$$\sum_{j\in\mathbf{Z}}\widehat{\Psi}(\vec{\xi}/2^j)=1 \tag{6}$$

for all $\vec{\xi}\in(\mathbf{R}^n)^m\setminus\{0\}$. Suppose that for some $s>mn/2$, the function $\sigma\in L^\infty((\mathbf{R}^n)^m)$ satisfies

$$\sup_{k\in\mathbf{Z}}\|\sigma^k\,\widehat{\Psi}\|_{L^2_s}<\infty.$$

where for $k\in\mathbf{Z}$, σ^k is defined as

$$\sigma^k(\xi_1,\dots,\xi_m)=\sigma(2^k\xi_1,\dots,2^k\xi_m)\,. \tag{7}$$

Then T_σ is bounded from $L^{p_1}(\mathbf{R}^n)\times\cdots\times L^{p_m}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$, where $1<p_1,p_2,\dots,p_m,p<\infty$ and $1/p_1+\cdots+1/p_m=1/p$.

In this work we extend Theorem A to the case where the target space is L^p for $p \le 1$. The following is our main result:

Theorem 1.1. *Let $1 < r \le 2$. Suppose that σ is a function on $(\mathbf{R}^n)^m$ and Ψ is a function in $\mathscr{S}_1((\mathbf{R}^n)^m)$ that satisfies for some $\gamma > \frac{mn}{r}$*

$$\sup_{k\in\mathbf{Z}} \|\sigma^k\, \widehat{\Psi}\|_{L^r_\gamma((\mathbf{R}^n)^m)} = K < \infty, \tag{8}$$

where σ^k is defined in (7). Then there is a number $\delta = \delta(mn, \gamma, r)$ satisfying $0 < \delta \le r-1$, such that the m-linear operator T_σ, associated with the multiplier σ, is bounded from $L^{p_1}(\mathbf{R}^n) \times \cdots \times L^{p_m}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$, whenever $r - \delta < p_j < \infty$ for all $j = 1, \dots, m$, and p is given by

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}\,. \tag{9}$$

Corollary 1.1. *Assume that $r = 2$ in Theorem 1.1. Then T_σ is bounded from $L^{p_1}(\mathbf{R}^n) \times \cdots \times L^{p_m}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$, whenever $1 < p_1, \dots, p_m, p' \le \infty$, and exactly one of the numbers $p_1, \dots, p_m, p'$ is equal to infinity.*

2. Preliminaries

We begin this section by proving Corollary 1.1 assuming Theorem 1.1.

Proof. We first prove that condition (8) is invariant under the adjoints, that is, it is also valid for the symbols of the dual operators. Indeed, the symbol of the kth dual operator is

$$\sigma^{*k}(\xi_1, \dots, \xi_m) = \sigma(\xi_1, \dots, \xi_{k-1}, -(\xi_1 + \cdots + \xi_m), \xi_{k+1}, \dots, \xi_m)\,,$$

with the obvious modification if $k = 1$ or $k = m$. This is equal to $\sigma(A_k\vec{\xi})$, where $\vec{\xi}$ is the column vector $(\xi_1, \dots, \xi_m)$ and A_k is a modified $m\times m$ identity matrix whose kth row has been replaced by the row $(-1, \dots, -1)$. Notice that $A_k^{-1} = A_k$. Condition (8) for σ^{*k} is

$$\sup_{j\in\mathbf{Z}} \int_{(\mathbf{R}^n)^m} \big|\big[\sigma(2^j A_k\vec{\xi})\widehat{\Psi}(\vec{\xi})\big]\widehat{\ }(\vec{y})\big|^2 w_\gamma(\vec{y})\, d\vec{y} < \infty\,, \tag{10}$$

where the hat denotes Fourier transform in the $\vec{\xi}$ variable. We note that the function Ψ_k whose Fourier transform is the function $\vec{\xi} \to \widehat{\Psi}(A_k\vec{\xi})$ lies in $\mathscr{S}_1((\mathbf{R}^n)^m)$, since it satisfies (3).

By a change of variables inside the Fourier transform, (10) transforms to

$$\sup_{j\in\mathbf{Z}} \int_{(\mathbf{R}^n)^m} \big|\big[\sigma(2^j\vec{\xi})\widehat{\Psi_k}(\xi)\big]\widehat{\ }(A_k^t\vec{y})\big|^2 w_\gamma(\vec{y})\, d\vec{y} < \infty\,, \tag{11}$$

where A_k^t is the transpose of A_k. But $(A_k^t)^{-1} = A_k^t$ and $|A_k^t\vec{y}\,| \approx |\vec{y}\,|$, thus $w_\gamma(A_k^t\vec{y}\,) \approx w_\gamma(\vec{y}\,)$. Therefore by another change of variables, condition (11) is equivalent to

$$\sup_{j\in\mathbf{Z}} \int_{(\mathbf{R}^n)^m} \big|\big[\sigma(2^j\vec{\xi})\widehat{\Psi_k}(\xi)\big]\widehat{\ }(\vec{y}\,)\big|^2 w_\gamma(\vec{y}\,)\, d\vec{y} < \infty\,, \tag{12}$$

which is valid in view of Lemma 2.3. Thus condition (8) for σ^{*k} holds.

We now have that (8) holds for σ^{*k} for all Ψ in $\mathscr{S}_1((\mathbf{R}^n)^m)$. Theorem 1.1 implies that T_σ^{*k}, the kth adjoint of T_σ, is bounded from the product $L^{p_1}(\mathbf{R}^n)\times\cdots\times L^{p_m}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ whenever $2 < p_j < \infty$. Multilinear interpolation ([7], [1]) yields that T_σ is bounded from $L^{p_1}(\mathbf{R}^n)\times\cdots\times L^{p_m}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ for all indices p_j satisfying $1 < p_1, \dots, p_m, p < \infty$, i.e., in the interior of the "Banach case". Thus boundedness holds in this case.

Theorem 1.1 also gives that T_σ is bounded from $L^{p_1}(\mathbf{R}^n)\times\cdots\times L^{p_m}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ for indices p_j satisfying $2-\delta < p_j < \infty$, for some $\delta > 0$. In particular, T_σ is bounded from $L^{q_1}(\mathbf{R}^n)\times\cdots\times L^{q_m}(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$, where $q_1 = \cdots = q_m = 2-\delta/2$ and $q = (2-\delta/2)/m < 1$. Interpolating with the interior of the Banach case, yields boundedness from $L^{p_1}(\mathbf{R}^n)\times\cdots\times L^{p_m}(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n)$, whenever $1 < p_j < \infty$ and $1/p_1 + \cdots + 1/p_m = 1$. Duality allows one (but not all) of the indices p_j to be equal to 1. □

Remark 2.1. *It is unclear to us at this time, if the result of Corollary 1.1 can be improved so that more than one index p_j be equal to infinity.*

Definition 2.1. *The Hardy-Littlewood maximal operator $\mathcal{M}$ is defined by*

$$\mathcal{M}(f)(x) = \sup_{r>0} \frac{1}{v_n r^n} \int_{|x-y|\le r} |f(y)| dy,$$

where f is a locally integrable function on $\mathbf{R}^n$ and v_n is the volume of the unit ball on $\mathbf{R}^n$. It is well known that $\mathcal{M}$ is bounded on $L^p(\mathbf{R}^n)$ for all $1 < p < \infty$.

A fundamental property of the Hardy-Littlewood maximal operator is the following. For any $\epsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\sup_{r>0} \int_{\mathbf{R}^n} \frac{r^n\,|f(y)|}{(1+r|x-y|)^{n+\epsilon}}\, dy \le C_\varepsilon\, \mathcal{M}(f)(x)$$

for all locally integrable functions $f \in \mathbf{R}^n$ and all $x \in \mathbf{R}^n$.

Recall that for $s \in \mathbf{R}$, w_s denotes the weight

$$w_s(x) = (1 + 4\pi^2|x|^2)^{s/2}\,.$$

Definition 2.2. *For $1 \le p < \infty$, the weighted Lebesgue space $L^p(w_s)$ is defined as the set of all measurable functions f on $\mathbf{R}^d$ such that*

$$\|f\|_{L^p(w_s)} = \left(\int_{\mathbf{R}^d} |f(x)|^p\, w_s(x)\, dx\right)^{1/p} < \infty.$$

We note that for $1 < r \le 2$ one has

$$
\begin{aligned}
\|\widehat{g}\,\|_{L^{r'}(w_s)} &= \left(\int_{\mathbf{R}^d} |\widehat{g}\,|^{r'} w_s \, d\xi \right)^{\frac{1}{r'}} \\
&= \left(\int_{\mathbf{R}^d} |\widehat{g}\, w_{s/r'}|^{r'} \, d\xi \right)^{\frac{1}{r'}} \\
&= \left(\int_{\mathbf{R}^d} \big| \big[(I-\Delta)^{\frac{s}{2r'}} g \big]\widehat{\ } \big|^{r'} \, d\xi \right)^{\frac{1}{r'}} \qquad (13) \\
&\le \left(\int_{\mathbf{R}^d} \big| (I-\Delta)^{\frac{s}{2r'}} g \big|^{r} \, dx \right)^{\frac{1}{r}} \\
&= \|g\,\|_{L^r_{s/r'}} \, ,
\end{aligned}
$$

via the Hausdorff-Young inequality.

Lemma 2.1. *Let $1 \le p < q < \infty$. Then for every $s \ge 0$ there exists a constant $C = C(p, q, s, d) > 0$ such that for all functions g supported in a ball of a fixed finite radius in $\mathbf{R}^d$ we have*

$$\|g\|_{L^p_s(\mathbf{R}^d)} \le C \, \|g\|_{L^q_s(\mathbf{R}^d)} \, .$$

Proof. Since g is supported in a ball of finite fixed radius, then $g = g\,\varphi$ for some compactly supported smooth function φ that is equal to one on the support of g. Pick r such that

$$1/p = 1/q + 1/r \, .$$

The Kato-Ponce rule [10] gives the estimate

$$
\begin{aligned}
\|g\|_{L^p_s(\mathbf{R}^d)} &= \big\| (I-\Delta)^{s/2} (g\,\varphi) \big\|_{L^p} \\
&\le C \big[\big\| (I-\Delta)^{s/2} g \big\|_{L^q} \|\varphi\|_{L^r} + \|g\|_{L^q} \big\| (I-\Delta)^{s/2} \varphi \big\|_{L^r} \big] \\
&= C_\varphi \big[\big\| (I-\Delta)^{s/2} g \big\|_{L^q} + \|g\|_{L^q} \big] \, .
\end{aligned}
$$

Now the Bessel potential operator $J_s = (I-\Delta)^{-s/2}$ is bounded from L^q to itself for all $s > 0$. This implies that

$$\|g\|_{L^q} \le C' \big\| (I-\Delta)^{s/2} g \big\|_{L^q}$$

Combining this estimate with the one previously obtained, we deduce that

$$\|g\|_{L^p_s(\mathbf{R}^d)} \le 2\, C_\varphi \, C' \big\| (I-\Delta)^{s/2} g \big\|_{L^q(\mathbf{R}^d)} = C \, \|g\|_{L^q_s(\mathbf{R}^d)} \, .$$

□

Lemma 2.2. *Suppose that $s \ge 0$ and $1 < r < \infty$. Assume that φ lies in $\mathscr{S}(\mathbf{R}^d)$. Then there is a constant c_φ such that for all $g \in L^r_s(\mathbf{R}^d)$ we have*

$$\|g\,\varphi\|_{L^r_s} \le c_\varphi \, \|g\|_{L^r_s} \, .$$

Proof. We write

$$(I-\Delta)^{s/2}(g\,\varphi) = \int_{\mathbf{R}^d} \widehat{\varphi}(\tau)(I-\Delta)^{s/2}(g\,e^{2\pi i\tau\cdot(\cdot)})\,d\tau\,.$$

It will suffice to show that the L^r norm of $(I-\Delta)^{s/2}(g\,e^{2\pi i\tau\cdot(\cdot)})$ is controlled by $C_M\,(1+|\tau|)^M$ times the L^r norm of $(I-\Delta)^{s/2}g$, for some $M>0$. This statement is equivalent to showing that the function

$$\left(\frac{1+|\xi-\tau|^2}{1+|\xi|^2}\right)^{\frac{s}{2}}$$

is an L^r Fourier multiplier with norm at most a multiple of $(1+|\tau|)^M$. But this is an easy consequence of the Mihlin multiplier theorem. □

Lemma 2.3. *Let $1<r\le 2$. If condition* (4) *holds for some function in $\mathscr{S}_1(\mathbf{R}^d)$, then it holds for all functions Ψ in $\mathscr{S}_1(\mathbf{R}^d)$.*

Proof. Suppose that condition (4) holds for some function Ψ in $\mathscr{S}_1(\mathbf{R}^d)$. Let Θ be another function in $\mathscr{S}_1(\mathbf{R}^d)$. Then using (3) we write

$$\widehat{\Theta}(\xi) = \frac{1}{\text{const}}\sum_{j\in\mathbf{Z}}\widehat{\Psi}(2^{-j}\xi)\widehat{\Theta}(\xi)\,. \tag{14}$$

Since $\widehat{\Theta}$ and $\widehat{\Psi}$ are supported in fixed annuli, only a finite number of terms in the previous sum is nonzero, that is, there is a constant c_0 such that $\widehat{\Psi}(2^{-j}\xi)\widehat{\Theta}(\xi)=0$ for all ξ whenever $|j|>c_0$. Since $\widehat{\Theta}$ is a smooth function with compact support, it follows from (14) and Lemma 2.2 that

$$\begin{aligned}
\sup_{k\in\mathbf{Z}}\|\widehat{\Theta}(\cdot)\,\sigma(2^k(\cdot))\|_{L^r_\gamma(\mathbf{R}^d)} &\le \frac{1}{\text{const}}\sum_{|j|\le c_0}\sup_{k\in\mathbf{Z}}\|\widehat{\Psi}(2^{-j}(\cdot))\widehat{\Theta}(\cdot)\,\sigma(2^k(\cdot))\|_{L^r_\gamma(\mathbf{R}^d)}\\
&\le \frac{C_\Theta}{\text{const}}\sum_{|j|\le c_0}\sup_{k\in\mathbf{Z}}\|\widehat{\Psi}(2^{-j}(\cdot))\,\sigma(2^k(\cdot))\|_{L^r_\gamma(\mathbf{R}^d)}\\
&\le \frac{C_\Theta}{\text{const}}\sum_{|j|\le c_0}\sup_{k\in\mathbf{Z}}\|\widehat{\Psi}(\cdot)\,\sigma(2^{k+j}(\cdot))\|_{L^r_\gamma(\mathbf{R}^d)}\\
&\le \frac{C_\Theta}{\text{const}}(2c_0+1)\sup_{k'\in\mathbf{Z}}\|\widehat{\Psi}(\cdot)\,\sigma(2^{k'}(\cdot))\|_{L^r_\gamma(\mathbf{R}^d)}\\
&<\infty\,.
\end{aligned}$$

□

Finally, we will need the following classical result of Fefferman and Stein

Lemma B [6]. *Let $1<p,q<\infty$. Then there exist positive finite constants $C(p,q)$ such that*

$$\Big\|\Big\{\sum_{k\in\mathbf{Z}}|\mathcal{M}(f_k)|^q\Big\}^{1/q}\Big\|_{L^p(\mathbf{R}^n)} \le C(p,q)\Big\|\Big\{\sum_{k\in\mathbf{Z}}|f_k|^q\Big\}^{1/q}\Big\|_{L^p(\mathbf{R}^n)}$$

for all sequences $\{f_k\}_{k\in\mathbf{Z}}$ of locally integrable functions on $\mathbf{R}^n$.

Lemma 2.4. *Let Δ_k be the Littlewood-Paley operator given by $\Delta_k(g)\widehat{\ }(\xi) = \widehat{g}(\xi)\widehat{\Psi}(2^{-k}\xi)$, $k \in \mathbf{Z}$, where Ψ is a Schwartz function whose Fourier transform is supported in the annulus $\{\xi : 2^{-b} < |\xi| < 2^b\}$, for some $b \in \mathbf{Z}^+$ and satisfies $\sum_{k\in\mathbf{Z}} \widehat{\Psi}(2^{-k}\xi) = c_0$, for some constant c_0. Let $0 < p < \infty$. Then there is a constant $c = c(n, p, c_0, \Psi)$, such that for L^p functions f we have*

$$\|f\|_{L^p} \le c \Big\| \Big(\sum_{k\in\mathbf{Z}} |\Delta_k(f)|^2 \Big)^{1/2} \Big\|_{L^p}.$$

Proof. Let Φ be a Schwartz function with integral one. Then the following quantity provides a characterization of the H^p norm:

$$\|f\|_{H^p} \approx \big\| \sup_{t>0} |f * \Phi_t| \big\|_{L^p}.$$

It follows that for f in $H^p \cap L^2$, which is a dense subclass of H^p, one has the estimate

$$|f| \le \sup_{t>0} |f * \Phi_t|,$$

since the family $\{\Phi_t\}_{t>0}$ is an approximate identity. Thus

$$\|f\|_{L^p} \le c\, \|f\|_{H^p}$$

whenever f is a function in H^p.

Keeping this observation in mind we can write:

$$\begin{aligned}
\|f\|_{L^p} \le\ & c\, \|f\|_{H^p} \\
\le\ & \Big\| \Big(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2 \Big)^{1/2} \Big\|_{L^p} \\
=\ & c \Big\| \Big(\sum_{j\in\mathbf{Z}} \Big| \Delta_j \Big(\sum_{k\in\mathbf{Z}} \Delta_k(f) \Big) \Big|^2 \Big)^{1/2} \Big\|_{L^p} \\
\le\ & c' \Big\| \Big(\sum_{k\in\mathbf{Z}} |\Delta_k(f)|^2 \Big)^{1/2} \Big\|_{L^p}
\end{aligned}$$

in view of the fact that $\Delta_j\Delta_k = 0$ unless $|j - k| \le b$. $\square$

3. The proof of the main result

In this section we discuss the proof of the main theorem.

Proof. For each $j = 1, \dots, m$, we let R_j be the set of points $(\xi_1, \dots, \xi_m)$ in $(\mathbf{R}^n)^m$ such that $|\xi_j| = \max\{|\xi_1|, \dots, |\xi_m|\}$. For $j = 1, \dots, m$, we introduce nonnegative smooth functions ϕ_j on $[0, \infty)^{m-1}$ that are supported in $[0, \frac{11}{10}]^{m-1}$ such that

$$1 = \sum_{j=1}^{m} \phi_j \Big(\frac{|\xi_1|}{|\xi_j|}, \dots, \widehat{\frac{|\xi_j|}{|\xi_j|}}, \dots, \frac{|\xi_m|}{|\xi_j|} \Big)$$

for all $(\xi_1,\dots,\xi_m)\neq 0$, with the understanding that the variable with the hat is missing. These functions introduce a partition of unity of $(\mathbf{R}^n)^m\setminus\{0\}$ subordinate to a conical neighborhood of the region R_j.

Each region R_j can be written as the union of sets

$$R_{j,k}=\big\{(\xi_1,\dots,\xi_m)\in R_j:\ |\xi_k|\geq|\xi_s|\quad\text{for all } s\neq j\big\}$$

over $k=1,\dots,m$. We need to work with a finer partition of unity, subordinate to each $R_{j,k}$. To achieve this, for each j, we introduce smooth functions $\phi_{j,k}$ on $[0,\infty)^{m-2}$ supported in $[0,\frac{11}{10}]^{m-2}$ such that

$$1=\sum_{\substack{k=1\\k\neq j}}^m \phi_{j,k}\Big(\frac{|\xi_1|}{|\xi_k|},\dots,\widehat{\frac{|\xi_k|}{|\xi_k|}},\dots,\widehat{\frac{|\xi_j|}{|\xi_k|}},\dots,\frac{|\xi_m|}{|\xi_k|}\Big)$$

for all $(\xi_1,\dots,\xi_m)$ in the support of ϕ_j with $\xi_k\neq 0$.

We now have obtained the following partition of unity of $(\mathbf{R}^n)^m\setminus\{0\}$:

$$1=\sum_{j=1}^m\sum_{\substack{k=1\\k\neq j}}^m \phi_j(\dots)\,\phi_{j,k}(\dots)\,,$$

where the dots indicate the variables of each function.

We now introduce a nonnegative smooth bump ψ supported in the interval $[(10m)^{-1},2]$ and equal to 1 on the interval $[(5m)^{-1},\frac{12}{10}]$, and we decompose the identity on $(\mathbf{R}^n)^m\setminus\{0\}$ as follows

$$1=\sum_{j=1}^m\sum_{\substack{k=1\\k\neq j}}^m \big[\Phi_{j,k}+\Psi_{j,k}\big]\,,$$

where

$$\Phi_{j,k}(\xi_1,\dots,\xi_m)=\phi_j(\dots)\,\phi_{j,k}(\dots)\bigg(1-\psi\Big(\frac{|\xi_k|}{|\xi_j|}\Big)\bigg)$$

and

$$\Psi_{j,k}(\xi_1,\dots,\xi_m)=\phi_j(\dots)\,\phi_{j,k}(\dots)\psi\Big(\frac{|\xi_k|}{|\xi_j|}\Big)\,.$$

This partition of unity induces the following decomposition of σ:

$$\sigma=\sum_{j=1}^m\sum_{\substack{k=1\\k\neq j}}^m \big[\sigma\,\Phi_{j,k}+\sigma\,\Psi_{j,k}\big]\,.$$

We will prove the required assertion for each piece of this decomposition, i.e., for the multipliers $\sigma\,\Phi_{j,k}$ and $\sigma\,\Psi_{j,k}$ for each pair (j,k) in the previous sum. In view of the symmetry of the decomposition, it suffices to consider the case of a *fixed* pair (j,k) in the previous sum. To simplify notation, we fix the pair $(m,m-1)$, thus, for the rest of the proof we fix $j=m$ and $k=m-1$ and we prove boundedness for the m-linear operators whose symbols are $\sigma_1=\sigma\,\Phi_{m,m-1}$ and $\sigma_2=\sigma\,\Psi_{m,m-1}$. These correspond to the

m-linear operators T_{σ_1} and T_{σ_2}, respectively. The important thing to keep in mind is that σ_1 is supported in the set where

$$\max(|\xi_1|,\dots,|\xi_{m-2}|)\le \tfrac{11}{10}\,|\xi_{m-1}|\le \tfrac{11}{10}\cdot\tfrac{1}{5m}\,|\xi_m|$$

and σ_2 is supported in the set where

$$\max(|\xi_1|,\dots,|\xi_{m-2}|)\le \tfrac{11}{10}\,|\xi_{m-1}|$$

and

$$\tfrac{1}{10m}\le \tfrac{|\xi_{m-1}|}{|\xi_m|}\le 2\,.$$

We first consider $T_{\sigma_1}(f_1,\dots,f_m)$, where f_j are fixed Schwartz functions. We fix a Schwartz radial function η whose Fourier transform is supported in the annulus $1-\frac{1}{25}\le|\xi|\le 2$ and satisfies

$$\sum_{j\in\mathbf{Z}}\widehat{\eta}(2^{-j}\xi)=1,\qquad \xi\in\mathbf{R}^n\backslash\{0\}.$$

Associated with η we define the Littlewood-Paley operator $\Delta_j(f)=f*\eta_{2^{-j}}$, where $\eta_t(x)=t^{-n}\eta(t^{-1}x)$ for $t>0$. We decompose the function f_m as $\sum_{j\in\mathbf{Z}}\Delta_j(f_m)$ and we note that the spectrum (i.e. the Fourier transform) of $T_{\sigma_1}(f_1,\dots,f_{m-1},\Delta_j(f_m))$ is contained in the set

$$\big\{\xi_1:\,|\xi_1|\le\tfrac{3\cdot 2^j}{5m}\big\}+\dots+\big\{\xi_{m-1}:\,|\xi_{m-1}|\le\tfrac{3\cdot 2^j}{5m}\big\}+\big\{\xi_m:\,\tfrac{24}{25}\cdot 2^j\le|\xi_m|\le 2\cdot 2^j\big\}$$

This algebraic sum of these sets is contained in the annulus

$$\{z\in\mathbf{R}^n:\ \tfrac{9}{25}\cdot 2^j\le|z|\le\tfrac{65}{25}\cdot 2^j\}\,.$$

We now introduce another bump that is equal to 1 on the annulus $\{z\in\mathbf{R}^n:\ \frac{9}{25}\le|z|\le\frac{65}{25}\}$ and vanishes in the complement of the larger annulus $\{z\in\mathbf{R}^n:\ \frac{8}{25}<|z|<\frac{66}{25}\}$. We call $\widetilde{\Delta}_j$ the Littlewood-Paley operators associated with this bump and we note that

$$\widetilde{\Delta}_j(T_{\sigma_1}(f_1,\dots,\Delta_j(f_m)))=T_{\sigma_1}(f_1,\dots,\Delta_j(f_m)).$$

Finally, we define an operator S_j by setting

$$S_j(g)=g*\zeta_{2^{-j}}\,,$$

where ζ is a smooth function whose Fourier transform is equal to 1 on the ball $|z|<3/5m$ and vanishes outside the double of this ball. Using this notation, we may write

$$\begin{aligned}T_{\sigma_1}(f_1,\dots,f_{m-1},f_m)&=\sum_j T_{\sigma_1}\big(f_1,\dots,f_{m-1},\Delta_j(f_m)\big)\\&=\sum_j T_{\sigma_1}\big(S_j(f_1),\dots,S_j(f_{m-1}),\Delta_j(f_m)\big)\\&=\sum_j \widetilde{\Delta}_j\big(T_{\sigma_1}(S_j(f_1),\dots,S_j(f_{m-1}),\Delta_j(f_m))\big)\,.\end{aligned}$$

Since the Fourier transforms of $\widetilde{\Delta}_j\big(T_{\sigma_1}(S_j(f_1),\dots,S_j(f_{m-1}),\Delta_j(f_m))\big)$ have bounded overlap, Lemma 2.4 yields that

$$\|T_{\sigma_1}(f_1,\dots,f_{m-1},f_m)\|_{L^p}\le C\Big\|\Big[\sum_j \big|T_{\sigma_1}\big(S_j(f_1),\dots,S_j(f_{m-1}),\Delta_j(f_m)\big)\big|^2\Big]^{\frac12}\Big\|_{L^p}$$

Obviously, we have

$$\begin{aligned}&T_{\sigma_1}\big(S_j(f_1),\dots,S_j(f_{m-1}),\Delta_j(f_m)\big)(x)\\ &=\int_{(\mathbf{R}^n)^m} e^{2\pi i x\cdot(\xi_1+\dots+\xi_m)}\sigma_1(\xi_1,\dots,\xi_m)\prod_{k=1}^{m-1}\widehat{S_j(f_k)}(\xi_k)\ \widehat{\Delta_j(f_m)}(\xi_m)\,d\xi_1\cdots d\xi_m\,.\end{aligned}$$

A simple calculation yields that the support of the integrand in the previous integral is contained in the annulus

$$\Big\{(\xi_1,\dots,\xi_m)\in(\mathbf{R}^n)^m:\ \tfrac{7}{10}\cdot 2^j<|(\xi_1,\dots,\xi_m)|<\tfrac{21}{10}\cdot 2^j\Big\},$$

so one may introduce in the previous integral the factor $\widehat{\Psi}(2^{-j}\xi_1,\dots,2^{-j}\xi_m)$, where Ψ is a radial function in $\mathscr{S}_1((\mathbf{R}^n)^m)$ whose Fourier transform is supported in some annulus and is equal to 1 on the annulus

$$\Big\{(z_1,\dots,z_m)\in(\mathbf{R}^n)^m:\ \tfrac{7}{10}\le|(z_1,\dots,z_m)|\le\tfrac{21}{10}\Big\}.$$

Inserting this factor and taking the inverse Fourier transform, we obtain that

$$T_{\sigma_1}\big(S_j(f_1),\dots,S_j(f_{m-1}),\Delta_j(f_m)\big)(x)$$

is equal to

$$\int_{(\mathbf{R}^n)^m} 2^{mnj}(\sigma_1^j\,\widehat{\Psi})^\vee(2^j(x-y_1),\dots,2^j(x-y_m))\prod_{i=1}^{m-1}S_j(f_i)(y_i)\,\Delta_j(f_m)(y_m)\,d\vec{y},$$

where $d\vec{y}=dy_1\dots dy_m$, the check indicates the inverse Fourier transform in all variables, and

$$\sigma_1^j(\xi_1,\xi_2,\dots,\xi_m)=\sigma_1(2^j\xi_1,\dots,2^j\xi_m)\,.$$

We pick a ρ such that $1<\rho<r\le 2$ and $\gamma>mn/\rho$. This is possible since $\gamma>mn/r$; for instance

$$\rho=\frac{mn}{\gamma}+\frac{1}{1000}(r-\frac{mn}{\gamma})$$

is a good choice if this number is bigger than 1; otherwise we set $\rho=\frac{1+r}{2}$. We define $\delta=r-\rho$. We now have:

$$\begin{aligned}&|T_{\sigma_1}(S_j(f_1),\dots,S_j(f_{m-1}),\Delta_j(f_m))(x)|\\ &\le\int_{(\mathbf{R}^n)^m} w_\gamma\big(2^j(x-y_1),\dots,2^j(x-y_m)\big)\ |(\sigma_1^j\,\widehat{\Psi})^\vee(2^j(x-y_1),\dots,2^j(x-y_m))|\\ &\qquad\qquad\times\frac{2^{mnj}|S_j(f_1)(y_1)\cdots S_j(f_{m-1})(y_{m-1})\Delta_j(f_m)(y_m)|}{w_\gamma\big(2^j(x-y_1),\dots,2^j(x-y_m)\big)}\,d\vec{y}\end{aligned}$$

$$
\begin{aligned}
\leq & \Big[\int_{(\mathbf{R}^n)^m} \big| \big(w_\gamma\, (\sigma_1^j\, \widehat{\Psi})^\vee \big) (2^j(x-y_1), \dots, 2^j(x-y_m)) \big|^{\rho'} d\vec{y} \Big]^{\frac{1}{\rho'}} \\
& \qquad \times 2^{mnj} \left(\int_{(\mathbf{R}^n)^m} \frac{|S_j(f_1)(y_1) \cdots S_j(f_{m-1})(y_{m-1}) \Delta_j(f_m)(y_m)|^\rho}{w_{\gamma\rho}\big(2^j(x-y_1), \dots, 2^j(x-y_m)\big)} \, d\vec{y} \right)^{\frac{1}{\rho}} \\
\leq & C \left(\int_{(\mathbf{R}^n)^m} w_{\gamma\rho'}(y_1, \dots, y_m) |(\sigma_1^j\, \widehat{\Psi})^\vee (y_1, \dots, y_m)|^{\rho'} d\vec{y} \right)^{\frac{1}{\rho'}} \\
& \qquad \times \left(\int_{(\mathbf{R}^n)^m} \frac{2^{mnj} |S_j(f_1)(y_1) \cdots S_j(f_{m-1})(y_{m-1}) \Delta_j(f_m)(y_m)|^\rho}{(1+2^j|x-y_1|)^{\gamma\rho/m} \cdots (1+2^j|x-y_m|)^{\gamma\rho/m}} \, d\vec{y} \right)^{\frac{1}{\rho}} \\
\leq & \|(\sigma_1^j\, \widehat{\Psi})^\vee\|_{L^{\rho'}(w_{\gamma\rho'})} \prod_{i=1}^{m-1} \left(\int_{\mathbf{R}^n} \frac{2^{jn} |S_j(f_i)(y_i)|^\rho}{(1+2^j|x-y_i|)^{\gamma\rho/m}} \, dy_i \right)^{\frac{1}{\rho}} \\
& \qquad \qquad \times \left(\int_{\mathbf{R}^n} \frac{2^{jn} |\Delta_j(f_m)(y_m)|^\rho}{(1+2^j|x-y_m|)^{\gamma\rho/m}} \, dy_m \right)^{\frac{1}{\rho}} \\
\leq & \|(\sigma_1^j\, \widehat{\Psi})^\vee\|_{L^{\rho'}(w_{\gamma\rho'})} c^{m/\rho} \prod_{i=1}^{m-1} \left(\mathcal{M}(\mathcal{M}(f_i)^\rho)(x) \right)^{\frac{1}{\rho}} \left(\mathcal{M}(|\Delta_j(f_m)|^\rho)(x) \right)^{\frac{1}{\rho}} ,
\end{aligned}
$$

where we used that

$$
\int_{\mathbf{R}^n} \frac{2^{jn} |h(y)|}{(1+2^j|x-y|)^{\gamma\rho/m}} \, dy \leq c\, \mathcal{M}(h)(x) ,
$$

a consequence of the fact that $\gamma\rho/m > n$.

We now have the sequence of inequalities:

$$
\|(\sigma_1^j\, \widehat{\Psi})^\vee\|_{L^{\rho'}(w_{\gamma\rho'})} \leq \|\sigma_1^j\, \widehat{\Psi}\|_{L^\rho_\gamma} \leq C'' \, \|\sigma_1^j\, \widehat{\Psi}\|_{L^r_\gamma} \leq C' \, \|\sigma^j\, \widehat{\Psi}\|_{L^r_\gamma} < C\, K ,
$$

justified by the result in the calculation (13) for the first, Lemma 2.1 together with the facts that $1 < \rho < r$ and σ_1^j is supported in a ball of a fixed radius for the second inequality, Lemma 2.2 for the third, and the hypothesis of Theorem 1.1 for the last inequality.

Thus we have obtained the estimate:

$$
\begin{aligned}
& |T_{\sigma_1}(S_j(f_1), \dots, S_j(f_{m-1}), \Delta_j(f_m))| \\
& \qquad \qquad \leq C\, K \prod_{i=1}^{m-1} \big(\mathcal{M}(\mathcal{M}(f_i)^\rho) \big)^{\frac{1}{\rho}} \left(\mathcal{M}(|\Delta_j(f_m)|^\rho) \right)^{\frac{1}{\rho}} .
\end{aligned}
$$

We now square the previous expression, we sum over $j \in \mathbf{Z}$ and we take square roots. Since $r - \delta = \rho$, the hypothesis $p_j > r - \delta$ implies $p_j > \rho$, and thus each term $(\mathcal{M}(\mathcal{M}(f_i)^\rho))^{\frac{1}{\rho}}$ is bounded on $L^{p_j}(\mathbf{R}^n)$. We obtain

$$
\begin{aligned}
&\big\|T_{\sigma_1}(f_1,\dots,f_{m-1},f_m)\big\|_{L^p(\mathbf{R}^n)}\\
\le\;& C\,K\Big\|\Big\{\sum_j |T_{\sigma_1}(S_j(f_1),\dots,S_j(f_{m-1}),\Delta_j(f_m))|^2\Big\}^{\frac12}\Big\|_{L^p(\mathbf{R}^n)}\\
\le\;& C'\,K\Big\|\Big\{\sum_j \mathcal{M}(|\Delta_j(f_m)|^\rho)^{\frac{2}{\rho}}\Big\}^{\frac12}\Big\|_{L^{p_m}(\mathbf{R}^n)}\prod_{i=1}^{m-1}\big\|\big(\mathcal{M}\big(\mathcal{M}(f_i)^\rho\big)\big)^{\frac1\rho}\big\|_{L^{p_i}(\mathbf{R}^n)}\\
\le\;& C''\,K\Big\|\Big\{\sum_j \mathcal{M}(|\Delta_j(f_m)|^\rho)^{\frac{2}{\rho}}\Big\}^{\frac{\rho}{2}}\Big\|^{\frac1\rho}_{L^{p_m/\rho}(\mathbf{R}^n)}\prod_{i=1}^{m-1}\|f_i\|_{L^{p_i}(\mathbf{R}^n)}
\end{aligned}
$$

and this is bounded by

$$
C''\,K\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(\mathbf{R}^n)}
$$

in view of Lemma B with $q=2/\rho$ and the Littlewood-Paley theorem.

Next we deal with σ_2. Using the notation introduced earlier, we write

$$
T_{\sigma_2}(f_1,\dots,f_{m-1},f_m)=\sum_{j\in\mathbf{Z}}T_{\sigma_2}(f_1,\dots,f_{m-1},\Delta_j(f_m))\,.
$$

The key observation in this case is that

$$
T_{\sigma_2}(f_1,\dots,f_{m-1},\Delta_j(f_m))=T_{\sigma_2}\big(S_j'(f_1),\dots,S_j'(f_{m-2}),\Delta_j'(f_{m-1}),\Delta_j(f_m)\big)
$$

for some other Littlewood-Paley operator Δ_j' which is given on the Fourier transform by multiplication with a bump $\widehat{\Theta}(2^{-j}\xi)$, where $\widehat{\Theta}$ is equal to one on the annulus $\{\xi\in\mathbf{R}^n:\ \frac{24}{25}\cdot\frac{1}{10m}\le|\xi|\le 4\}$ and vanishes on a larger annulus. Also, S_j' is given by convolution with $\zeta'_{2^{-j}}$, where ζ' is a smooth function whose Fourier transform is equal to 1 on the ball $|z|<\frac{22}{10}$ and vanishes outside the double of this ball.

As in the previous case, one has that in the support of the integral

$$
\begin{aligned}
&T_{\sigma_2}\big(S_j'(f_1),\dots,S_j'(f_{m-2}),\Delta_j'(f_{m-1}),\Delta_j(f_m)\big)(x)\\
&=\int_{(\mathbf{R}^n)^m}e^{2\pi i x\cdot(\xi_1+\cdots+\xi_m)}\sigma_2(\vec{\xi}\,)\prod_{t=1}^{m-2}\widehat{S_j'(f_t)}(\xi_t)\ \widehat{\Delta_j'(f_{m-1})}(\xi_{m-1})\widehat{\Delta_j(f_m)}(\xi_m)\,d\vec{\xi}
\end{aligned}
$$

we have that

$$
|\xi_1|+\cdots+|\xi_m|\approx 2^j\,,
$$

thus one may insert in the integrand the factor $\widehat{\Psi}(2^{-j}\xi_1,\dots,2^{-j}\xi_m)$, for some Ψ in $\mathscr{S}_1((\mathbf{R}^n)^m)$ that is equal to one on a sufficiently wide annulus.

A calculation similar to the one in the case for σ_1 yields the estimate

$$
\begin{aligned}
&|T_{\sigma_2}(S_j'(f_1),\dots,S_j'(f_{m-2}),\Delta_j'(f_{m-1}),\Delta_j(f_m))|\\
&\le C\,K\prod_{i=1}^{m-2}(\mathcal{M}(\mathcal{M}(f_i)^\rho))^{\frac1\rho}\big(\mathcal{M}(|\Delta_j'(f_{m-1})|^\rho)\big)^{\frac1\rho}(\mathcal{M}(|\Delta_j(f_m)|^\rho))^{\frac1\rho}\ .
\end{aligned}
$$

Summing over j and taking L^p norms yields

$$
\begin{aligned}
&\|T_{\sigma_2}(f_1,\dots,f_{m-1},f_m)\|_{L^p(\mathbf{R}^n)}\\
\le& C\,K\Big\|\prod_{i=1}^{m-2}\big(\mathcal{M}(\mathcal{M}(f_i)^\rho)\big)^{\frac{1}{\rho}}\sum_{j\in\mathbf{Z}}\big(\mathcal{M}\left(|\Delta_j'(f_{m-1})|^\rho\right)\big)^{\frac{1}{\rho}}\left(\mathcal{M}\left(|\Delta_j(f_m)|^\rho\right)\right)^{\frac{1}{\rho}}\Big\|_{L^p}\\
\le& C\,K\Big\|\prod_{i=1}^{m-2}\big(\mathcal{M}(\mathcal{M}(f_i)^\rho)\big)^{\frac{1}{\rho}}\Big\{\prod_{i=m-1}^{m}\sum_{j\in\mathbf{Z}}|\mathcal{M}\left(|\Delta_j(f_i)|^\rho\right)|^{\frac{2}{\rho}}\Big\}^{\frac{1}{2}}\Big\|_{L^p(\mathbf{R}^n)}
\end{aligned}
$$

where the last step follows by the Cauchy-Schwarz inequality and we omitted the prime from the term with $i=m-1$ for matters of simplicity. Applying Hölder's inequality and using that $\rho<2$ and Lemma B we obtain the conclusion that the expression above is bounded by

$$C'\,K\,\|f_1\|_{L^{p_1}(\mathbf{R}^n)}\cdots\|f_m\|_{L^{p_m}(\mathbf{R}^n)}\,.$$

This concluses the proof of the theorem. □

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