On sharp Olsen's and trace inequalities for multilinear fractional integrals

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Abstract

We establish a sharp Olsen type inequality

$$\big\|g\,\mathcal{I}_\alpha(f_1,\dots,f_m)\big\|_{L^q_r}\le C\big\|g\big\|_{L^q_\ell}\prod_{j=1}^m\big\|f_j\big\|_{L^{p_j}_{s_j}}$$

for multilinear fractional integrals $\mathcal{I}_\alpha(\vec{f})(x)=\int\limits_{(\mathbb{R}^n)^m}\frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}}d\vec{y},\ \ x\in\mathbb{R}^n$, $0<\alpha<mn$, where L^q_r, L^q_ℓ, $L^{p_j}_{s_j}$, $j=1,\dots,m$, are Morrey space with indices satisfying certain homogeneity conditions. This inequality is sharp because it gives necessary and sufficient condition on weights function V for which the inequality

$$\big\|\mathcal{I}_\alpha(f_1,\dots,f_m)\big\|_{L^q_r(V)}\le C\prod_{j=1}^m\big\|f_j\big\|_{L^{p_j}_{s_j}}$$

holds.

Morrey spaces play an important role in relation to regularity problems of solutions of partial differential equations. They describe the integrability more precisely than Lebesgue spaces.

We also derive a characterization of the trace inequality

$$\big\|B_\alpha(f_1,f_2)\big\|_{L^q_r(d\mu)}\le C\prod_{j=1}^2\big\|f_j\big\|_{L^{p_j}_{s_j}(\mathbb{R}^n)},$$

in terms of a Borel measure μ, where B_α is the bilinear fractional integral operator given by the formula $B_\alpha(f_1,f_2)(x)=\int\limits_{\mathbb{R}^n}\frac{f_1(x+t)f_2(x-t)}{|t|^{n-\alpha}}dt,\quad 0<\alpha<n,$

Some of our results are new even in the linear case, i.e. when $m=1$.

1 Introduction

Let $0<\alpha<n$. The fractional integral operator

$$I_\alpha(f)(x)=\int\limits_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}\,dy,\ \ x\in\mathbb{R}^n,$$

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plays a fundamental role in Harmonic Analysis; it also finds applications in PDEs, such as in the theory of Sobolev embeddings, for instance see Maz'ya [22].

A variant of this operator is the bilinear fractional integral operator

$$B_\alpha(f_1, f_2)(x) = \int\limits_{\mathbb{R}^n} \frac{f_1(x+t) f_2(x-t)}{|t|^{n-\alpha}} dt, \quad 0 < \alpha < n,$$

introduced in [6]. The complete Lebesgue space boundedness properties of this operator were independently obtained by Kenig and Stein [13] and Grafakos and Kalton [7]. These say that B_α maps $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ exactly when $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}$.

A very natural intermediate operator between $(I_{\alpha_1} f_1)(I_{\alpha_2} f_2)$ and $B_{\alpha_1+\alpha_2}(f_1, f_2)$ is

$$\mathcal{I}_\alpha(\vec{f})(x) = \int\limits_{(\mathbb{R}^n)^m} \frac{f_1(y_1) \cdots f_m(y_m)}{(|x-y_1| + \cdots + |x-y_m|)^{mn-\alpha}} d\vec{y}, \quad x \in \mathbb{R}^n,$$

(expressed in its multilinear form) where $0 < \alpha < nm$, $\vec{f} := (f_1, \ldots, f_m)$, $\vec{y} := (y_1, \ldots, y_m)$, $d\vec{y} = dy_1 \cdots dy_m$. The introduction of this operator is also motivated by its corresponding well-known (fractional) maximal analogue

$$\mathcal{M}_\alpha(\vec{f})(x) = \sup_{Q \ni x} \prod_{i=1}^m \frac{1}{|Q|^{1-\alpha/(nm)}} \int\limits_Q |f_i(y_i)| dy_i, \quad 0 \le \alpha < mn,$$

where $|Q|$ denotes the volume of the cube Q in $\mathbb{R}^n$ with sides parallel to the coordinate axes. When $\alpha = 0$, the multisublinear Hardy–Littlewood maximal operator $\mathcal{M}_0$ appears naturally in connection with the multilinear Calderón–Zygmund theory; on this see the work of Lerner, Ombrosi, Pérez, Torres, and Trujillo–González [21].

In this article we study the behavior of the operator $\mathcal{I}_\alpha$ on Morrey spaces (Theorem 3.1). As a consequence, we establish a sharp Olsen type inequality for these spaces. Using the definition of these spaces given in (5), the Olsen inequality is the following estimate:

$$\|g\, \mathcal{I}_\alpha(\vec{f})\|_{L^q_r} \le C \|g\|_{L^q_\ell} \prod_{j=1}^m \|f_j\|_{L^{p_j}_{s_j}}, \tag{1}$$

where $1 < q \le r < \infty$, $1 < p_j \le s_j < \infty$, $j = 1, \ldots, m$, $p < q < \infty$, $0 < \alpha < \frac{n}{s}$, $\frac{1}{p} - \frac{1}{q} = \frac{1}{s} - \frac{1}{r} = \frac{\alpha}{n} - \frac{1}{\ell}$. Here and throughout the paper we assume the following relationship on the preceding indices:

$$\frac{1}{p} := \sum_{i=1}^m \frac{1}{p_i}, \quad \frac{1}{s} := \sum_{i=1}^m \frac{1}{s_i}, \quad m \ge 2. \tag{2}$$

In the linear case ($m = 1$), inequalities of type (1) play an important role in the study of perturbed Schrödinger equation; see Olsen [27]. We refer to [31] and [32] for subsequent improvements of Olsen's original inequality and applications.

Estimate (1) is crucial in obtaining a complete characterization of weight functions V such that the estimate (trace inequality) below is valid:

$$\|\mathcal{I}_\alpha(\vec{f})\|_{L^q_r(V)} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}_{s_j}}. \tag{3}$$

As a consequence of our work, we show that if $1 < p < q < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{1}{s} - \frac{1}{r}$, $0 < \alpha < \frac{n}{s}$, then (3) holds for all $f_j \in L^{p_j}_{s_j}$, $j = 1, \dots, m$, if and only if the Adams' type condition

$$[V]_{\alpha,p,q} := \sup_{Q\in\mathcal{Q}} \left(\int\limits_Q V(x)dx \right)^{\frac{1}{q}} |Q|^{\frac{\alpha}{n} - \frac{1}{p}} < \infty \tag{4}$$

is satisfied. Here $\mathcal{Q}$ denotes the class of all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axis.

In the linear case (when $m = 1$ and $\mathcal{I}_\alpha = I_\alpha$), the aforementioned characterization goes back to Adams [1] on Lebesgue spaces, i.e., when $p = s$ and $q = r$. This result was later extended to the multilinear setting by Kokilashvili, Mastyło and Meskhi [14] for Lebesgue spaces ($q = r$, $p_i = s_i$, $i = 1, \dots, m$). In the linear case it was also extended by Eridani, Kokilashvili and Meskhi [5] to the more general setting of quasi-metric measure spaces.

We end this introductory section by recalling a few historical facts concerning Morrey spaces and multilinear fractional operators.

Morrey spaces were introduced in 1938 by C. Morrey in relation to regularity problems of solutions of partial differential equations.

Weighted Morrey spaces first appeared in Komori and Shirai [18] in 2009. In that paper, the authors studied the boundedness of singular integral operators in those spaces. In the definition of weighted Morrey space introduced in [18], the weighted norm $\|\chi_B f\|_{L^p(W)}$ is divided by $W(B)^\lambda$, where W is weight function. For weighted results regarding linear fractional integrals I_α and corresponding fractional maximal operators M_α in Morrey spaces we refer to the papers: [30], [25], [28], [26]. The unweighted and weighted problems for multilinear fractional integrals in Morrey spaces were studied in [10], [11], [12], [16], [8] (see also the references cited in [16]). In particular, in [10] and [11] Olsen's type inequalities for multilinear fractional integrals have been derived.

For the multilinear fractional operators $\mathcal{I}_\alpha$ and $\mathcal{M}_\alpha$ Moen [23] obtained one-weight criteria, as well as "power bump" conditions for the two–weight inequalities. Various type of one and two–weight multilinear problems for these operators in Lebesgue spaces were also studied in [4], [9], [14], [15], [16], [17], [19], [20], [24], [29], [33], et. al.

Notation: the relation $A \approx B$ between two variable quantities A and B indicates the two-sided estimate: $\frac{1}{c}A \le B \le cA$, for some positive constant c.

2 Background, Preliminaries, and Known Results

Let $1 \le q \le r < \infty$ and let $d\mu$ be a Borel measure on $\mathbb{R}^n$. We denote by $L^q_r(d\mu)$ the Morrey space of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^q_r(d\mu)} := \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{\frac{1}{q}-\frac{1}{r}}} \left(\int\limits_Q |f(x)|^q d\mu(x) \right)^{1/q} < \infty. \tag{5}$$

In this definition cubes can be replaced by balls and the supremum will then be over all balls B in $\mathbb{R}^n$. This yields a norm equivalent to $\|\cdot\|_{L^q_r(d\mu)}$. If V is a locally integrable a.e. positive function on $\mathbb{R}^n$, i.e. a weight on $\mathbb{R}^n$, then we denote $L^q_r(d\mu)$ by $L^q_r(V)$.

The following equivalent form of Morrey space norm appears in the literature

$$\|f\|_{L^{q,\lambda}(V)} := \sup_{Q\in\mathcal{Q}} \left(\frac{1}{|Q|^\lambda} \int\limits_Q |f(x)|^q V(x)dx \right)^{1/q}. \tag{6}$$

Note that $\|\cdot\|_{L^q_r(V)}$ coincides with $\|\cdot\|_{L^{q,\lambda/q}(V)}$ given in (6) when $\lambda = 1-\frac{q}{r}$.

The weak weighted Morrey space $WL^q_r(V)$ is defined as the space of all measurable functions f such that

$$\|f\|_{WL^q_r(V)} := \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{\frac{1}{q}-\frac{1}{r}}} \sup_{\lambda>0} \lambda \Big(\int_{\{x\in Q: |f(x|>\lambda\}} V(x)dx \Big)^{1/q} < \infty.$$

Obviously, one has $WL^q_r(V) \hookrightarrow L^q_r(V)$.

If V is a constant function, then we denote $L^q_r(V)$ and $WL^q_r(V)$ by L^q_r and WL^q_r respectively. In the case $q=r$ we have weighted Lebesgue spaces $L^q(V)$ and $WL^q(V)$, respectively.

Two boundedness results for fractional integral operator on Morrey spaces are known:

Proposition A. (Spanne, unpublished) *Let $0<\alpha<n$, $1<p_0\le s_0<\infty$, $1<q_0\le r_0<\infty$. Suppose that $\frac{1}{s_0}-\frac{1}{r_0}=\frac{1}{p_0}-\frac{1}{q_0}=\frac{\alpha}{n}$. Then I_α is bounded from $L^{p_0}_{s_0}$ to $L^{q_0}_{r_0}$.*

Proposition B. (Adams [3]) *Let $0<\alpha<n$, $1<p_0\le s_0<\infty$, $1<q_0\le r_0<\infty$. Suppose that $\frac{1}{r_0}=\frac{1}{s_0}-\frac{\alpha}{n}$, $\frac{q_0}{r_0}=\frac{p_0}{s_0}$. Then I_α is bounded from $L^{p_0}_{s_0}$ to $L^{q_0}_{r_0}$.*

In the unweighted case the following multilinear result is also known.

Proposition C. ([34]) *Let $0<\alpha<mn$, $1<q\le r<\infty$, $1<p_i\le s_i<\infty$, $i=1,\dots,m$ be such that*

$$\frac{1}{s}-\frac{1}{r}=\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n},$$

where p and s are defined by (2). Then there exists a positive constant C such that for all $f_j\in L^{p_j}_{s_j}$, $j=1,\dots,m$, we have

$$\|\mathcal{I}_\alpha(\vec{f})\|_{L^q_r} \le C\prod_{j=1}^m \|f_j\|_{L^{p_j}_{s_j}}.$$

Adams [1] (see also [2]) proved the trace inequality for the Riesz Potentials I_α.

Theorem A. *Let $1<p<q<\infty$ and let $0<\alpha<n/p$. Suppose that μ is a Borel measure on $\mathbb{R}^n$. Then the inequality*

$$\|I_\alpha(f)\|_{L^q(\mu)} \le C\|f\|_{L^p}$$

holds if and only if

$$[\mu] := \sup_Q (\mu(Q))^{\frac{1}{q}}|Q|^{\frac{\alpha}{n}-\frac{1}{p}} < \infty. \tag{7}$$

Moreover, $\|I_\alpha\|_{L^p\mapsto L^q_{d\mu}} \approx [\mu]$.

An analogous multilinear characterization is the following.

Theorem B. ([14]) *Let $1<p_i<\infty$, $i=1,\dots,m$. Assume that $0<\alpha<n/p$ and $p<q<\infty$. Then the following assertions are equivalent:*

(i) *For all f_i in L^{p_j} we have*

$$\big\|\mathcal{I}_\alpha(\vec{f})\big\|_{L^q(V)} \le C\prod_{i=1}^m \big\|f_i\big\|_{L^{p_j}}; \tag{8}$$

(ii) *the weak type inequality below is valid*

$$V\big(\{x\in\mathbb{R}^n : |\mathcal{I}_\alpha(\vec{f})(x)|>\lambda\}\big)^{1/q}\le \frac{C}{\lambda}\prod_{i=1}^m \big\|f_j\big\|_{L^{p_j}} \tag{9}$$

(iii) *condition* (4) *is satisfied.*

The proof of Theorem B is based on the following statements:

Lemma A. ([14]) *Let $1<p_i<\infty$, $i=1,\cdots,m$. Suppose that $0<\alpha,\beta<n/p$ with the condition $\beta<\alpha$. There is a positive constant $C=C_{\alpha,\beta,p}$ such that for all non-negative $f_i\in L^{p_i}$, $i=1,\dots,m$, the pointwise estimate*

$$\mathcal{I}_\alpha(\vec{f})(x)\le C\Big[\Big(\mathcal{M}_{\alpha-\beta}(\vec{f})(x)\Big)^{\frac{\alpha-n/p}{\alpha-\beta-n/p}}\Big(\prod_{i=1}^m\|f_i\|_{L^{p_i}}\Big)^{\frac{\beta}{\beta-\alpha+n/p}}\Big]$$

holds for all $x\in\mathbb{R}^n$.

Proposition D. ([23]) *Let $1<p_i<\infty$, $i=1,\dots,m$. Assume that $0<\alpha<n/p$ and $p<q<\infty$. Then the inequality*

$$\|\mathcal{M}_\alpha(\overrightarrow{f}\,)\|_{L^q(V)}\le C\prod_{i=1}^m\bigg(\int\limits_{\mathbb{R}^n}\big|f_i(x)\big|^{p_i}dx\bigg)^{1/p_i}, \tag{10}$$

holds for the multilinear fractional maximal operator $\mathcal{M}_\alpha$ if and only if (4) *is satisfied. Moreover, if C is the best possible constant in* (10)*, then $C\approx[V]_{\alpha,p,q}$.*

Proposition D is proved in [23] in the two–weighted setting under the power-bump condition on weights but here we need that result only in a special case. Finally, for the purposes of this paper we need the following sharpening of Theorem B.

Proposition 2.1. *Let $1<p_i<\infty$, $i=1,\dots,m$. Assume that $\alpha<n/p$ and $p<q<\infty$. Then the following estimate holds:*

$$\big\|\mathcal{I}_\alpha(\vec{f})\big\|_{L^q(V)}\le C\,[V]_{\alpha,p,q}\prod_{j=1}^m\big\|f_j\big\|_{L^{p_j}}. \tag{11}$$

Proof. We adapt the arguments in [14]. Let β be as in Lemma A. We set

$$q_1:=q\frac{\alpha-\frac{n}{p}}{\alpha-\beta-\frac{n}{p}}=q\frac{\frac{\alpha}{n}-\frac{1}{p}}{\frac{\alpha-\beta}{n}-\frac{1}{p}}. \tag{12}$$

Then taking condition (4) and identity (12) into account we see that the following relations hold:

$$[V]^{q_1}_{\alpha-\beta,p,q_1}=\sup_{Q\in\mathcal{Q}}v(Q)|Q|^{((\alpha-\beta)/n-1/p)q_1}=[V]^q_{\alpha,p,q}=\sup_{Q\in\mathcal{Q}}v(Q)|Q|^{(\alpha/n-1/p)q}<\infty.$$

Applying Lemma A and Proposition D we write

$$
\begin{aligned}
\|\mathcal{I}_\alpha(\vec{f})\|_{L^q(V)} &\le c_{\alpha,\beta,p}\Big\|\mathcal{M}_{\alpha-\beta}(\vec{f})^{\frac{\alpha-n/p}{\alpha-\beta-n/p}}\Big\|_{L^q(V)}\prod_{i=1}^m\|f_i\|_{L^{p_i}}^{\frac{\beta}{\beta-\alpha+n/p}}\\
&= c_{\alpha,\beta,p}\|\mathcal{M}_{\alpha-\beta}(\vec{f})\|_{L^{q_1}(V)}^{q_1/q}\prod_{i=1}^m\|f_i\|_{L^{p_i}}^{\frac{\beta}{\beta-\alpha+n/p}}\\
&\le c[V]_{\alpha-\beta,p,q_1}^{q_1/q}\prod_{i=1}^m\|f_i\|_{L^{p_i}}^{q_1/q}\prod_{i=1}^m\|f_i\|_{L^{p_i}}^{\frac{\beta}{\beta-\alpha+n/p}}\\
&= c[V]_{\alpha,p,q}\prod_{i=1}^m\|f_i\|_{L^{p_i}}.
\end{aligned}
$$

In the two equalities we used that $\frac{q_1}{q}=\frac{\alpha-\frac{n}{p}}{\alpha-\beta-\frac{n}{p}}$, which is a consequence of (12). □

3 Main Results

The main results of this paper are as follows:

Theorem 3.1. *Let $1<q\le r<\infty$, $1<p_i\le s_i<\infty$, $i=1,\dots,m$, $1<p<q$, $0<\alpha<\frac{n}{s}$. Let $\frac{1}{p}-\frac{1}{q}=\frac{1}{s}-\frac{1}{r}=\frac{\alpha}{n}-\frac{1}{\ell}$, where $\frac{1}{s}=\sum_{j=1}^m\frac{1}{s_j}$, $\frac{1}{p}=\sum_{j=1}^m\frac{1}{p_j}$. Then there exists a positive constant C depending only on n, α, q, r, p_i, s_i, $i=1,\dots,m$, such that for all $f_j\in L_{s_j}^{p_j}$, $j=1,\dots,m$, inequality (1) holds.*

Theorem 3.2. *Let $1<q\le r<\infty$, $1<p_i\le s_i<\infty$, $i=1,\dots,m$, $1<p<q$, $0<\alpha<\frac{n}{s}$. Let $\frac{1}{p}-\frac{1}{q}=\frac{1}{s}-\frac{1}{r}$, where $\frac{1}{s}=\sum_{j=1}^m\frac{1}{s_j}$, $\frac{1}{p}=\sum_{j=1}^m\frac{1}{p_j}$. Suppose that V is a weight function on $\mathbb{R}^n$. Then the following statements are equivalent:*

(i) *there is a positive constant C such that for all measurable $\vec{f}$ we have*

$$
\|\mathcal{I}_\alpha(\vec{f})\|_{L_r^q(V)}\le C\prod_{j=1}^m\|f_j\|_{L_{s_j}^{p_j}}. \tag{13}
$$

(ii) *there is a positive constant C such that for all measurable $\vec{f}$ we have*

$$
\|\mathcal{I}_\alpha(\vec{f})\|_{WL_r^q(V)}\le C\prod_{j=1}^m\|f_j\|_{L_{s_j}^{p_j}}. \tag{14}
$$

(iii) *condition (4) is satisfied.*

Moreover, under either assumption, we have the norm equivalence $\|\mathcal{I}_\alpha\|\approx[V]_{\alpha,p,q}$.

In the linear case, i.e., when $m=1$, we have:

Corollary 3.1. *Let $1<q\le r<\infty$, $1<p\le s<\infty$, $1<p<q$ and $0<\alpha<\frac{n}{s}$. Let $\frac{1}{p}-\frac{1}{q}=\frac{1}{s}-\frac{1}{r}=\frac{\alpha}{n}-\frac{1}{\ell}$. Then there is a positive constant C depending only on n, α, q, r, p, s such that for all $f\in L_s^p$ and $g\in L_\ell^q$ we have*

$$
\big\|g\,I_\alpha(f)\big\|_{L_r^q}\le C\big\|g\big\|_{L_\ell^q}\big\|f\big\|_{L_s^p}
$$

We also have a result for the bilinear fractional integral operator B_α.

Theorem 3.3. *Let $1 < q \le r$, $1 < p_i \le s_i < \infty$, $i = 1, 2$. Let $1 < p < q < \infty$ and $0 < \alpha < \min\{\frac{1}{s_1}, \frac{1}{s_2}\}$, $\frac{1}{p} - \frac{1}{q} = \frac{1}{s} - \frac{1}{r} = \frac{\alpha}{n} - \frac{1}{\ell}$, where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}$. Then there is a positive constant C depending only on n, α, q, r, r, p_1, p_2, s_1, s_2 such that for all $f_1, f_2, g \ge 0$ we have*

$$\|g\, B_\alpha(f_1, f_2)\|_{L^q_r} \le C\|g\|_{L^q_\ell}\|f_1\|_{L^{p_1}_{s_1}}\|f_2\|_{L^{p_2}_{s_2}}. \tag{15}$$

Furthermore, we have the trace inequality for B_α which analogous to that of Adams [1]; see also [5] in the linear case.

Theorem 3.4. *Let $1 < q \le r$, $1 < p_i \le s_i < \infty$, $i = 1, 2$, and let $1 < p < q < \infty$. Let $0 < \alpha < \min\{\frac{1}{s_1}, \frac{1}{s_2}\}$, $\frac{1}{p} - \frac{1}{q} = \frac{1}{s} - \frac{1}{r}$, where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}$. Then there is a positive constant C depending on n, α, q, r, p_1, p_2, s_1, s_2 such that for all $f_1, f_2 \ge 0$,*

$$\|B_\alpha(f_1, f_2)\|_{L^q_r(d\mu)} \le C[\mu]\|f_1\|_{L^{p_1}_{s_1}}\|f_2\|_{L^{p_2}_{s_2}}, \tag{16}$$

holds, where $[\mu]$ is defined in (7).

As a corollary we have the trace inequality for classical Lebesgue spaces.

Corollary 3.2. *Let $1 < p_i < \infty$, $1 < p < q < \infty$ and let $0 < \alpha < \min\{\frac{1}{p_1}, \frac{1}{p_2}\}$. Suppose that μ is a Borel measure on $\mathbb{R}^n$. Then there is a positive constant C such that for all $f_1, f_2 \ge 0$,*

$$\|B_\alpha(f_1, f_2)\|_{L^q(d\mu)} \le C[\mu]\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}},$$

where $[\mu]$ is defined in (7).

4 Proofs

Proof of Theorem 3.1. First observe that $p < q < \ell$ and $\frac{n}{\ell} < \alpha < \frac{n}{s} < \frac{n}{p}$. Without loss of generality we assume that $g \ge 0$, $f_j \ge 0$, $j = 1, \dots, m$. For any ball $B := B(a, r)$, let $2B := B(a, 2r)$ be the ball with center a and radius $2r$. We write $f_j = f_j^0 + f_j^\infty$, where

$$f_j^0 = f_j\chi_{2B}, \quad f_j^\infty = f_j\chi_{(2B)^c}, \quad j = 1, \dots, m.$$

Let $f_j \ge 0$, $j = 1, \dots, m$. In view of this representation we write

$$\mathcal{I}_\alpha\vec{f}(x) \le \mathcal{I}_\alpha(f_1^0, \dots, f_m^0)(x) + \mathcal{I}_\alpha(f_1^\infty, \dots, f_m^\infty)(x) + \sum_{j=1}^m \mathcal{I}_\alpha(f_1^{\beta_1}, \dots, f_m^{\beta_m})(x),$$

where $\beta_1, \dots, \beta_m \in \{0, \infty\}$ and the sum contains at least one $\beta_j = 0$ and $\beta_j = \infty$. Consequently,

$$\begin{aligned} \big\|g\mathcal{I}_\alpha(\vec{f})\big\|_{L^q(B)} &\le \big\|g\mathcal{I}_\alpha(f_1^0, \dots, f_m^0)\big\|_{L^q(B)} + \big\|gI_\alpha(f_1^\infty, \dots, f_m^\infty)\big\|_{L^q(B)} \\ &\quad + \sum_{\beta_1, \dots, \beta_m} \big\|\mathcal{I}_\alpha(f_1^{\beta_1}, \dots, f_m^{\beta_m})\big\|_{L^q(B)} := N_1 + N_2 + \sum. \end{aligned}$$

Using Proposition 2.1 for $V = |g|^q$, we write

$$\begin{aligned} N_1 &\le C\big\|g\|_{L^q_\ell} \prod_{j=1}^m \big\|\chi_{2B}f_j\|_{L^{p_j}} \le C\big\|g\big\|_{L^q_\ell} \prod_{j=1}^m \big\|\chi_{2B}f_j\big\|_{L^{p_j}_{s_j}} r^{n\sum_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{s_j}\right)} \\ &= C\big\|g\big\|_{L^q_\ell} \prod_{j=1}^m \big\|\chi_{2B}f_j\big\|_{M^{p_j}_{s_j}} r^{n\left(\frac{1}{p} - \frac{1}{s}\right)} = C\big\|g\big\|_{L^q_\ell} \prod_{j=1}^m \big\|\chi_{2B}f_j\big\|_{M^{p_j}_{s_j}} r^{n\left(\frac{1}{q} - \frac{1}{r}\right)}. \end{aligned}$$

Let us estimate N_2. First observe that if $x \in B$ and $y_j \in (2B)^c$, then by simple geometric observations we find that $\frac{1}{2}|a-y_j| \le |x-y_j| \le \frac{3}{2}|a-y_j|$. Thus, we get

$$\begin{aligned}
\mathcal{I}_\alpha(f_1^\infty,\dots,f_m^\infty)(x) &\le C\int_{2r}^{\infty} s^{\alpha-mn-1}\Big(\prod_{j=1}^{m}\int_{\{y_j:|x-y_j|<s\}} f_j^\infty(y_j)dy_j\Big)ds\\
&\le C\int_{2r}^{\infty} s^{\alpha-mn-1}\Big(\prod_{j=1}^{m}\int_{\{y_j:|a-y_j|<2s\}} f_j^\infty(y_j)dy_j\Big)ds\\
&\le C\int_{2r}^{\infty} s^{\alpha-mn-1}\Big(\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(B(a,2s))}\Big)\prod_{j=1}^{m}(s^{n/p_j'})ds\\
&\le C\prod_{j=1}^{m}\|f_j\|_{L^{p_j}_{s_j}}\int_{2r}^{\infty} s^{\alpha-1-\sum_{j=1}^m \frac{n}{p_j}+n\sum_{j=1}^m\left(\frac{1}{p_j}-\frac{1}{s_j}\right)}ds\\
&= C\prod_{j=1}^{m}\|f_j\|_{L^{p_j}_{s_j}}\int_{2r}^{\infty} s^{\alpha-1-\frac{n}{p}+n\left[\frac{1}{q}-\frac{1}{r}\right]}ds\\
&= C\prod_{j=1}^{m}\|f_j\|_{L^{p_j}_{s_j}} r^{\alpha-\frac{n}{p}+n\left[\frac{1}{q}-\frac{1}{r}\right]}.
\end{aligned}$$

Here we used the fact that $\alpha < \frac{n}{s} = n\left[\frac{1}{p}-\frac{1}{q}+\frac{1}{r}\right] < \frac{n}{p}$.

Hence,

$$\begin{aligned}
&\Big(\int_B \mathcal{I}_\alpha(f_1^\infty,\dots,f_m^\infty)^q(x)g^q(x)dx\Big)^{1/q} \le Cr^{\alpha-\frac{n}{p}+n\left[\frac{1}{q}-\frac{1}{r}\right]}\Big(\int_B g^q(x)dx\Big)^{1/q}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}_{s_j}}\\
&\le Cr^{n\left[\frac{1}{q}-\frac{1}{r}\right]}\|g\|_{L^q_\ell}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}_{s_j}}.
\end{aligned}$$

In the last equality we used the condition $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}-\frac{1}{\ell}$.

It remains to estimate $\sum$. For simplicity we take $m \ge 3$, $\beta_1=\beta_2=\infty$ and $\beta_3=\cdots=\beta_m=0$. Recall that $|x-y_j|\approx|a-y_j|$ for all $x\in B$ and $y_j\in(2B)^c$, $j=1,2$. Thus, without loss of generality, we have that one of the terms of Σ can be estimated as follows:

$$\begin{aligned}
&\mathcal{I}_\alpha(f_1^\infty,f_2^\infty,f_3^0,\dots,f_m^0)(x)\\
&\quad= \int_{(2B)^c\times(2B)^c\times 2B\times\ldots\times 2B}\frac{f_1(y_1)f_2(y_2)f_3(y_3)\cdots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}}d\vec{y}\\
&\quad\le C\Big(\int_{(2B)^c\times(2B)^c}\frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|a-y_1|+|a-y_2|)^{mn-\alpha}}\Big)\Big(\int_{(2B)\times\cdots\times(2B)} f_3(y_3)\cdots f_m(y_m)dy_3\cdots dy_m\Big)\\
&\quad:= CI_1\cdot I_2.
\end{aligned}$$

Now we estimate I_1 and I_2 separately. By Hölder's inequality and simple observations we obtain:

$$I_1 = C \int\limits_{(2B)^c\times(2B)^c} \Bigg(\int\limits_{|a-y_1|+|a-y_2|}^{\infty} s^{-mn+\alpha-1} ds \Bigg) f_1(y_1) f_2(y_2) dy_1 dy_2$$

$$\le C \int\limits_{2r}^{\infty} \Bigg(\int\limits_{\{y_1,y_2:|a-y_1|+|a-y_2|<s\}} f_1(y_1) f_2(y_2) dy_1 dy_2 \Bigg) s^{-mn+\alpha-1} ds$$

$$\le C \int_{2r}^{\infty} \prod_{i=1}^{2} \Bigg(\int_{B(a,s)} f_i^{p_i}(y_i) dy_i \Bigg)^{1/p_i} s^{\frac{n}{p_1'}+\frac{n}{p_2'}-mn+\alpha-1} ds$$

$$\le C \int_{2r}^{\infty} \prod_{i=1}^{2} \Bigg(\frac{1}{|B(a,s)|^{1-\frac{p_i}{s_i}}} \int\limits_{B(a,s)} f_i^{p_i}(y_i) dy_i \Bigg)^{1/p_i} s^{\alpha-mn-1+n\left(\frac{1}{p_1'}+\frac{1}{p_2'}\right)+n\left[\frac{1}{p_1}+\frac{1}{p_1}-\left(\frac{1}{s_1}+\frac{1}{s_2}\right)\right]} ds$$

$$\le C \prod_{i=1}^{2} \|f_i\|_{L^{p_i}_{s_i}} r^{\alpha-mn+2n-n\left[\frac{1}{p_1}+\frac{1}{p_2}\right]+n\left[\frac{1}{p_1}-\frac{1}{s_1}+\frac{1}{p_2}-\frac{1}{s_2}\right]}.$$

In the latter estimate we used fact that

$$\alpha - mn + 2n - n\Big(\frac{1}{p_1}+\frac{1}{p_2}\Big) + n\Big[\frac{1}{p_1}+\frac{1}{p_2}-\Big(\frac{1}{s_1}+\frac{1}{s_2}\Big)\Big]$$
$$= \alpha - mn + 2n - n\Big[\frac{1}{s_1}+\frac{1}{s_2}\Big] < 0$$

which is a consequence of the condition $\alpha < \frac{n}{s}$. Further, by using Hölder's inequality again, we find that

$$I_2 \le C \prod_{i=3}^{m} \|f_i\|_{L^{p_i}_{s_i}} r^{n\left(\sum_{k=3}^{m}\frac{1}{p_k'}\right)+n\left(\sum_{k=3}^{m}\left[\frac{1}{p_k}-\frac{1}{s_k}\right]\right)}$$
$$= C \prod_{i=3}^{m} \|f_i\|_{L^{p_i}_{s_i}} r^{n(m-2)-n\sum_{k=3}^{m}\frac{1}{p_k}} r^{n\left(\sum_{k=3}^{m}\left[\frac{1}{p_k}-\frac{1}{s_k}\right]\right)}.$$

Consequently, summarizing estimates for I_1 and I_2 we find that

$$\Bigg(\int\limits_B \mathcal{I}_\alpha(f_1^\infty, f_2^\infty f_3^0, \cdots, f_m^0)^q(x) g^q(x) dx \Bigg)^{1/q} \le C r^{n\left[\frac{\alpha}{n}-\frac{1}{p}\right]} r^{n\left[\frac{1}{p}-\frac{1}{s}\right]+n\left[\frac{1}{q}-\frac{1}{\ell}\right]} \|g\|_{L^q_\ell} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}_{s_j}}$$
$$= C r^{n\left[\frac{1}{q}-\frac{1}{r}\right]} \|g\|_{L^q_\ell} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}_{s_j}}.$$

In the last equality we again used the condition: $\frac{1}{p}-\frac{1}{q}=\frac{1}{s}-\frac{1}{r}=\frac{\alpha}{n}-\frac{1}{\ell}$.

This completes the proof. □

Proof of Theorem 3.2. The implication $(iii) \Rightarrow (i)$ follows from Theorem 3.1 taking $V = |g|^q$ there and observing that $\|g\|_{L^q_\ell} = [V]_{\alpha,p,q}$, where $\frac{1}{q}-\frac{1}{\ell}=\frac{1}{p}-\frac{\alpha}{n}$. The implication $(ii) \Rightarrow (iii)$ is a consequence of taking the test functions $f_j = \chi_B$, $j = 1, \ldots, m$ in (ii). Since $(i) \Rightarrow (ii)$, we are done. □

Taking $m = 1$ in Theorem 3.1, we deduce Corollary 3.1.

Next we prove Theorem 3.4 and we note that Theorem 3.3 will be its consequence.

Proof of Theorem 3.4. Let us fix a ball $B := B(a, r)$. Suppose that $f_1, f_2 \geq 0$. Using Hölder's inequality twice with exponents $\frac{p_1}{p}$ and $\frac{p_2}{p}$ we find that

$$\begin{aligned}
&\|\chi_B B_\alpha(f_1, f_2)\|_{L^q(d\mu)} \leq \|\chi_B \big[I_\alpha(f_1^{p_1/p})\big]^{p/p_1} \big[I_\alpha(f_2^{p_2/p})\big]^{p/p_2} \|_{L^q(d\mu)} \\
&\leq \|\chi_B I_\alpha(f_1^{p_1/p})\|^{p/p_1}_{L^q(d\mu)} \|\chi_B I_\alpha(f_2^{p_2/p})\|^{p/p_2}_{L^q(d\mu)} := N_1 \cdot N_2,
\end{aligned}$$

where I_α is the Riesz potential defined on $\mathbb{R}^n$.

Now we estimate N_1 and N_2 separately. Representing f_1 as $f_{1,1} + f_{1,2}$, where $f_{1,1} = f_1 \cdot \chi_{2B}$, $f_{1,2} = f_1 - f_{1,1}$, we find that

$$\begin{aligned}
N_1 &\leq \Big(\int\limits_B \big[I_\alpha(f_1^{p_1/p})(x)\big]^q d\mu(x) \Big)^{p/(qp_1)} \\
&\leq C \Big[\Big(\int\limits_B \big[I_\alpha(f_{1,1}^{p_1/p})(x)\big]^q d\mu(x) \Big)^{p/(qp_1)} + \Big(\int\limits_B \big[I_\alpha(f_{1,2}^{p_1/p})(x)\big]^q d\mu(x) \Big)^{p/(qp_1)} \Big] \\
&:= N_{1,1} + N_{1,2}.
\end{aligned}$$

In view of Theorem A we have that

$$N_{1,1} \leq C[\mu]^{\frac{p}{p_1}} \|f_1\|_{L^{p_1}_{s_1}} |B|^{\frac{1}{p_1} - \frac{1}{s_1}}.$$

Now we estimate $N_{1,2}$. First observe that if $x \in B$ and $y \in (2B)^c$, then $|y - a| \leq 2|x - y|$. Consequently, by Hölder's inequality with respect to the exponents p and p', and the condition $0 < \alpha < \frac{1}{s_1}$ we get:

$$\begin{aligned}
&N_{2,1} \leq C(\mu(B))^{p/(qp_1)} \Big(\int_{(2B)^c} \frac{(f_{1,2})^{p_1/p}(y)}{|y-a|^{n-\alpha}} dy \Big)^{\frac{p}{p_1}} \\
&= C(\mu(B))^{p/(qp_1)} \Big[\sum_{k=1}^{\infty} \int_{(2^{k+1}B)\setminus(2^k B)} \frac{(f_{1,2}(y))^{p_1/p}}{|y-a|^{n-\alpha}} dy \Big]^{\frac{p}{p_1}} \\
&\leq C(\mu(B))^{p/(qp_1)} \Big[\sum_{k=1}^{\infty} \Big(\int_{(2^{k+1}B)\setminus(2^k B)} (f_{1,2}(y))^{p_1} dy \Big)^{\frac{1}{p}} \Big(\int_{(2^{k+1}B)\setminus(2^k B)} |y-a|^{(\alpha-n)p'} \Big)^{\frac{1}{p'}} \Big]^{\frac{p}{p_1}} \\
&\leq C(\mu(B))^{p/(qp_1)} \Big[\sum_{k=1}^{\infty} \Big(\int_{(2^{k+1}B)\setminus(2^k B)} (f_{1,2}(y))^{p_1} dy \Big)^{\frac{1}{p}} |2^k B|^{\frac{\alpha}{n} - \frac{1}{p}} \Big]^{\frac{p}{p_1}} \\
&= C(\mu(B))^{p/(qp_1)} \Big[\sum_{k=1}^{\infty} \Big(\Big(\int_{(2^{k+1}B)\setminus(2^k B)} (f_{1,2}(y))^{p_1} dy \Big)^{\frac{1}{p_1}} |2^{k+1}B|^{\frac{1}{s_1} - \frac{1}{p_1}} \Big)^{\frac{p_1}{p}} |2^k B|^{\frac{\alpha}{n} - \frac{p_1}{s_1 p}} \Big]^{\frac{p}{p_1}} \\
&\leq C(\mu(B))^{p/(qp_1)} \|f_1\|_{L^{p_1}_{s_1}} |B|^{\frac{\alpha p}{n p_1} - \frac{1}{s_1}} \leq C[\mu]^{p/p_1} \|f_1\|_{L^{p_1}_{s_1}} |B|^{\frac{1}{p_1} - \frac{1}{s_1}}.
\end{aligned}$$

Summarizing estimates for $N_{1,1}$ and $N_{1,2}$ we find that

$$N_1 \leq C[\mu]^{\frac{p}{p_1}} \|f_1\|_{L^{p_1}_{s_1}} |B|^{\frac{1}{p_1} - \frac{1}{s_1}}.$$

Analogously for N_2 we have that

$$N_2 \le C[\mu]^{\frac{p}{p_2}} \|f_2\|_{L^{p_2}_{s_2}} |B|^{\frac{1}{p_2}-\frac{1}{s_2}}.$$

These estimates give

$$\|\chi_B B_\alpha(f_1,f_2)\|_{L^q(\mu)} \le C[\mu] \|f_1\|_{L^{p_1}_{s_1}} \|f_2\|_{L^{p_2}_{s_2}} |B|^{\frac{1}{p}-\frac{1}{s}},$$

which implies the desired estimate. □

Proof of Theorem 3.3. Taking $d\mu(x) = g^q(x)dx$ in Theorem 3.4, setting $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n} - \frac{1}{\ell}$, and observing that $[\mu] = \|g\|_{L^q_\ell}$, we derive the claimed conclusion.

To conclude, taking $p_1 = s_1$, $p_2 = s_2$, $q = r$ we deduce Corollary 3.2 as a consequence of Theorem 3.4.

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Declarations

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References

[1] D. R. Adams, Traces of potentials arising from translation invariant operators, *Ann. Scuola Norm. Sup. Pisa* **25** (1971), 203-217.

[2] D. R. Adams, A trace inequality for generalized potentials, *Studia Math.* **48** (1973), 99–105.

[3] D. R. Adams, A note on Riesz potentials, *Duke Math. J.* **42** (1975), 765–778.

[4] X. Chen and Q. Xue, Weighted estimates for a class of multilinear fractional type operators, *J. Math. Anal. Appl.* **362** (2010) 355-373

[5] A. Eridani, V. Kokilashvili, and A. Meskhi, Morrey spaces and fractional integral operators, *Expo. Math.* **27** (2009), 227–239.

[6] L. Grafakos, On multilinear fractional integrals, *Studia Math.* **102** (1992), 49–56.

[7] L. Grafakos and N. Kalton, Some remarks on multilinear maps and interpolation, *Math. Ann.* **319** (2001), 151–180.

[8] Q. He and D. Yan, Bilinear fractional integral operators on Morrey spaces, *Positivity*, **25** (2021), 399–429.

[9] T. Iida and E. Sato, A note on multilinear fractional integrals, *Anal. Theory Appl.* **26** (2010), 301–307.

[10] T. Iida, E. Sato, Y. Sawano, and H. Tanaka, Weighted norm inequalities for multilinear fractional operators on Morrey spaces, *Studia Math.* **205** (2011), 139–170.

[11] T. Iida, E. Sato, Y. Sawano, and H. Tanaka, Multilinear fractional integrals on Morrey spaces, *Acta Math. Sinica (English Series)* **28** (2012), 1375–1384.

[12] T. Iida, E. Sato, Y. Sawano, and H. Tanaka, Sharp bounds for multilinear fractional integral operators on Morrey type spaces, *Positivity* **16** (2012), 339–358.

[13] C. Kenig and E. Stein, Multilinear estimates and fractional integration, *Math. Res Lett.* **6** (1999), 1–15.

[14] V. Kokilashvili, M. Mastyło, and A. Meskhi, On the boundedness of the multilinear fractional integral operators, *Nonlinear Analysis, Theory, Methods and Applications* **94** (2014), 142–147.

[15] V. Kokilashvili, M. Mastyło, and A. Meskhi, Two-weight norm estimates for multilinear fractional integrals in classical Lebesgue spaces, *Frac. Calc. Appl. Anal.* **18** (2015), 1146–1163.

[16] V. Kokilashvili, M. Mastyło, and A.Meskhi, On the boundedness of multilinear fractional integral operators, *J. Geom. Anal.* **30** (2020), 667–679.

[17] Y. Komori-Furuya, Weighted estimates for bilinear fractional integral operators: a necessary and sufficient condition for power weights, *Collect. Math.* **71** (2020), 25–37.

[18] Y. Komori and S. Shirai, Weighted Morrey spaces and a singular integral operator, *Math. Nachr.* **282** (2009), 219–231.

[19] K. Li, K. Moen and W. Sun, Sharp weighted inequalities for multilinear fractional maximal operators and fractional integrals, *Math. Nachr.* **288** (2015), 619–632.

[20] K. Li and W. Sun, Two weight norm inequalities for the bilinear fractional integrals, *Manuscripta Math.* **150** (2016), 159–175.

[21] A. K. Lerner, S. Ombrosi, C. Pérez, R. H. Torres, and R. Trujillo–González, New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory, *Adv. Math.* **220** (2009), 1222–1264.

[22] V. Maz'ya, Sobolev spaces, *Springer-Verlag, Berlin and New York*, 1985.

[23] K. Moen, Weighted inequalities for multilinear fractional integral operators, *Collect. Math.* **60** (2009), 213–238.

[24] K. Moen, New weighted estimates for bilinear fractional integral operators, *Trans. Amer. Math. Soc.* **366** (2014), 627–646.

[25] Sh. Nakamura, Generalized weighted Morrey spaces and classical operators, *Math. Nachr.* **289** (2016), 2235–2262.

[26] Sh. Nakamura, Y. Sawano, and H. Tanaka, The fractional operators on weighted Morrey spaces, *J. Geom. Anal.* **28** (2018), 1502–1524.

[27] P. A. Olsen, Fractional integration, Morrey spaces and a Schrödinger equation, *Comm. PDE* **20** (1995), 2005–2055.

[28] J. Pan and W. Sun, Two-weight norm inequalities for fractional maximal functions and fractional integral operators on weighted Morrey spaces, *Math. Nachr.* **293** (2020), 970–982.

[29] G. Pradolini, Weighted inequalities and pointwise estimates for the multilinear fractional integral and maximal operators, *J. Math. Anal. Appl.* **367** (2010), 640–656.

[30] N. Samko, Weighted Hardy and potential operators in Morrey spaces, *J. Funct. Spaces Appl.* 2012, Art. ID 678171, 21 pp.

[31] Y. Sawano, S. Sugano, and H. Tanaka, Generalized fractional integral operators and fractional maximal operators in the framework of Morrey spaces, *Trans. Amer. Math. Soc.* **363** (2011), 6481–6503.

[32] Y. Sawano, S. Sugano, and H. Tanaka, Olsen's inequality and its applications to Schrödinger equations, *RIMS Kôkyûroku Bessatsu* **B26** (2011), 51–80.

[33] Y. Shi and X. Tao, Weighted L^p boundedness for multilinear fractional integral on product spaces, *Anal. Theory Appl.* **24** (2008), 280–291.

[34] L. Tang, Endpoint estimates for multilinear fractional integrals. *J. Austral. Math. Soc.* **84** (2008), 419–429.

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