

A.E. CONVERGENCE VERSUS BOUNDEDNESS

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ABSTRACT. In this paper, we extend Stein’s maximal theorem [26] to the bilinear setting; precisely, let M be a homogeneous space on which a compact abelian group acts transitively and let $1 \leq p, q \leq 2$, $1/2 \leq r \leq 1$ be related by $1/p + 1/q = 1/r$. Given a family of bounded bilinear operators $T_m : L^p(M) \times L^q(M) \rightarrow L^r(M)$, $m = 1, 2, \dots$, that commute with translations and converge a.e. as $m \rightarrow \infty$, we show that the associated maximal operator $T^*(f, g) = \sup_m |T_m(f, g)|$ satisfies a weak-type $L^p(M) \times L^q(M) \rightarrow L^{r, \infty}(M)$ estimate. Our proof is based on probabilistic arguments, properties of the Rademacher functions, and measure-theoretic constructions. The main obstacle we overcome is an extension of Stein’s lemma that provides an estimate for the L^2 norm of a tail of a double Rademacher series in terms of the L^∞ norm of the series over a measurable set.

We also obtain a bilinear analogue of Sawyer’s [25] extension of Stein’s theorem to the wider range $1 \leq p, q < \infty$. This result is valid for positive bilinear operators that commute with a mixing family of measure-preserving transformations.

Our main application concerns the boundedness of a maximal bilinear tail operator associated with an ergodic measure-preserving transformation on a finite measure space, which was shown by Assani and Buczolich [2] to be finite a.e.. Our results imply that this maximal operator is bounded from $L^p \times L^q$ to L^r for the natural exponent $r = (\frac{1}{p} + \frac{1}{q})^{-1}$ when $p, q > 1$; this extends the integrability of the bilinear maximal tail operator from the subcritical region to the “natural” boundedness region. We discuss an additional application related to the a.e. convergence of bilinear Bochner-Riesz means.

1. INTRODUCTION

A well-known classical fact is that the L^p boundedness of a maximal family of linear operators together with pointwise convergence for a dense subspace of the domain implies a.e. convergence for all functions on the domain. In contrast to this classical direction, our results show that a.e. convergence of a bilinear sequence forces weak-type bounds for the associated maximal operator. Motivated by the work of Stein [26] and Sawyer [25], we obtain bilinear analogues of their results and we also study some of their consequences in terms of applications. The main results of this paper are Theorems 1 and 3, while our main application is Theorem 5.

In Stein’s celebrated work [26], it is shown that if a sequence of translation-invariant operators $\{T_m\}$ on a compact group acts boundedly on $L^p(M)$, $1 \leq p \leq 2$, and if $T_m f \rightarrow T f$ pointwise for every $f \in L^p(M)$, then the maximal operator

$$T^* f = \sup_{m \geq 1} |T_m f|$$

satisfies a weak-type (p, p) inequality. The argument is based on probabilistic techniques with Rademacher functions and provides a powerful connection between a.e. convergence and weak-type (p, p) inequalities. In this work we develop an analogue for sequences of

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bilinear operators $T_m : L^p(M) \times L^q(M) \rightarrow L^r(M)$ with $1/p + 1/q = 1/r$ and $1 \leq p, q \leq 2$. It should be noted that the opposite direction also holds: if a sequence of multilinear operators satisfies a maximal weak-type estimate $L^p \times L^q \rightarrow L^{r,\infty}$, then it converges a.e. for all $L^p \times L^q$ functions, assuming it does so for a dense subclass; for a proof of this fact see for instance [9, Proposition 2].

Interest in bilinear operators originated in the pioneering work of Coifman and Meyer [4] [5] in the seventies and by the celebrated work of Lacey and Thiele [19, 20] on the bilinear Hilbert transform in the nineties. These works have spurred a resurgence of activity on adaptations of many classical linear results to multilinear analogues, such as the Calderón-Zygmund theory [12] and many other topics. In relation to a.e. convergence of multilinear singular operators and boundedness of the associated maximal operators (which naturally implies a.e. convergence) we refer the reader to [18] [22], [13] [23], [21], [7], [10], [8], [17]. This list is by no means exhaustive, but is representative of the work in this area.

Another contribution of this work is the bilinear analogue of Sawyer's extension of Stein's theorem to the range $1 \leq p, q < \infty$ for positive bilinear operators under certain mild mixing conditions. The main application of our results concerns improved weak-type bounds for a bilinear tail operator associated with an ergodic measure-preserving transformation on a finite measure space.

We begin by introducing some preliminaries and then develop the probabilistic and measure-theoretic framework necessary for establishing Theorem 1.

Let G be a topological group and M a Hausdorff topological space. We say that G acts continuously on M if there is a continuous map

$$G \times M \rightarrow M, \quad (g, x) \mapsto g(x),$$

such that

$$e(x) = x$$

for every $x \in M$, where e denotes the identity element of G , and

$$(g_1 g_2)(x) = g_1(g_2(x))$$

for every $g_1, g_2 \in G$ and $x \in M$.

In the measure-theoretic arguments below, we additionally assume that the action map is jointly measurable with respect to the relevant product measure spaces.

We call M a *homogeneous space* of G if G acts continuously on M and transitively; the latter means for every pair of points $x, y \in M$ there exists an element $g \in G$ such that

$$g(x) = y.$$

We also define the translation operator τ_g acting on functions f on M by

$$(\tau_g f)(x) = f(g^{-1}x), \quad g \in G.$$

Now suppose G is a compact group and M is a homogeneous space of G . Then the homogeneous space M inherits a unique normalized G -invariant measure $d\mu$ from G that satisfies

$$\int_M f(g(x)) d\mu(x) = \int_M f(x) d\mu(x), \quad \text{for all } f \in L^1(M) \text{ and } g \in G.$$

If we normalize the measure $d\mu$ on M such that

$$\int_M d\mu = 1,$$

then G also has a finite Haar measure $d\omega_G$, that satisfies

$$\int_G d\omega_G = 1.$$

It is useful to recall the relation between the measures $d\mu$ on M and $d\omega_G$ on G . For a fixed point $x_0 \in M$ and a Borel subset E of M we define the *fiber of E over the point x_0 under the action of G* as follows:

$$\hat{E} = \{g \in G : g(x_0) \in E\}.$$

Then, by translation invariance, for any $x_0 \in M$ we have

$$(1) \quad \mu(E) = \omega_G(\hat{E}).$$

Some classical examples that the reader may keep in mind are the following. First, when both the group and the space coincide with the k -torus, $G = M = \mathbb{T}^k$, the action is simply translation modulo 1:

$$g \cdot x = g + x \pmod{1}, \quad g, x \in \mathbb{T}^k,$$

that is, addition is taken componentwise on the torus.

A second, closely related example arises when $G = M$ is the *dyadic group*, in which case the action is given by componentwise dyadic addition:

$$g \cdot x = g + x \pmod{2}.$$

Finally, one may think of the rotational setting: let $G = \text{SO}(n)$, the group of rotations in $\mathbb{R}^n$, and $M = S^{n-1}$, the unit sphere. Here the natural action is the usual rotation of vectors,

$$g \cdot x = g(x), \quad g \in \text{SO}(n), \quad x \in S^{n-1}.$$

These three cases serve as canonical examples of compact group actions and motivate the general framework discussed below.

We now discuss the setup in the bilinear setting. Let M and G be as above.

Definition 1. For each $m = 1, 2, \dots$, let T_m be a bilinear operator defined on $L^p(M, d\mu) \times L^q(M, d\mu)$, where $1 \leq p, q \leq 2$.

- We say that each T_m is bounded if for each $m = 1, 2, \dots$ there is a constant C_m such that

$$\|T_m(f, h)\|_{L^r(M)} \leq C_m \|f\|_{L^p(M)} \|h\|_{L^q(M)},$$

for all $f \in L^p(M)$ and $h \in L^q(M)$. We do not assume the constants C_m are uniform in m .

- We say that each T_m commutes with simultaneous translations if for every $g \in G$ we have

$$T_m(\tau_g f, \tau_g h)(x) = \tau_g T_m(f, h)(x), \quad \text{a.e. } x \in M.$$

- For $f \in L^p(M)$ and $h \in L^q(M)$ we define the associated maximal operator:

$$T^*(f, h) := \sup_{m \geq 1} |T_m(f, h)|.$$

The first main result of this work is the following theorem:

Theorem 1. *Let $1 \leq p, q \leq 2$ and $\frac{1}{2} \leq r \leq 1$, such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Let T_m be a sequence of bounded operators from $L^p(M) \times L^q(M)$ to $L^r(M)$ that commute with simultaneous translations. Suppose that for every $f \in L^p(M)$, $h \in L^q(M)$, the pointwise limit*

$$\lim_{m \rightarrow \infty} T_m(f, h)(x)$$

exists for almost every $x \in M$.

Then, there exists a constant $C > 0$ such that for all $f \in L^p(M)$, $h \in L^q(M)$, and all $\alpha > 0$,

$$(2) \quad \mu(\{x \in M : T^*(f, h)(x) > \alpha\}) \leq \frac{C}{\alpha^r} \|f\|_{L^p(M)}^r \|h\|_{L^q(M)}^r.$$

In Section 5 we also prove a version of this result for positive operators. In Section 6 we discuss applications. Naturally, our results extend to the case of multilinear operators, but for the sake of notational simplicity we only focus on the bilinear case.

Notation: We denote Lebesgue measure on Euclidean space or the torus by $|\cdot|$. The indicator function of a set S is denoted by $\mathbf{1}_S$. We use the notation $A \lesssim B$ to indicate that the quantity A is controlled by a constant multiple of B , for some inessential constant.

2. SOME PRELIMINARY RESULTS

We begin the proof with some lemmas. The following summability lemma allows us to balance size and frequency of occurrence of the terms of a series.

Lemma 1.1. *Let $(a_n)_{n=1}^\infty$ be a sequence of numbers satisfying $0 < a_n < \frac{A}{n}$ for some $A > 0$. Then we can find a sequence of natural numbers $n_1 \leq n_2 \leq n_3 \leq \dots$ such that:*

$$(3) \quad \sum_{k=1}^\infty a_{n_k} < \infty,$$

$$(4) \quad \sum_{k=1}^\infty n_k a_{n_k} = \infty.$$

Proof. If $\limsup_{n \rightarrow \infty} n a_n = a > 0$, we define $n_1 = 1$ and for $k \geq 2$ we define inductively

$$n_k = \min \left\{ n > \max(k^2, n_{k-1}) : n a_n > \frac{a}{k+1} \right\}.$$

Then we have

$$\sum_{k=1}^\infty a_{n_k} \leq \sum_{k=1}^\infty \frac{A}{k^2} < \infty,$$

while, for every $K \geq 2$,

$$\sum_{k=1}^K n_k a_{n_k} \geq a \sum_{k=2}^K \frac{1}{k+1}.$$

Since the series on the right diverges as $K \rightarrow \infty$, we obtain the desired divergence.

Now suppose that

$$\limsup_{n \rightarrow \infty} n a_n = 0.$$

For each $t \in \mathbb{N}$, set

$$b_t = \left\lfloor \frac{1}{2^t a_{2^t}} \right\rfloor,$$

and define the sequence (a_{n_k}) by repeating a_{2^t} exactly b_t times for each t ; that is,

$$\underbrace{a_2, \dots, a_2}_{b_1 \text{ times}}, \underbrace{a_{2^2}, \dots, a_{2^2}}_{b_2 \text{ times}}, \dots, \underbrace{a_{2^t}, \dots, a_{2^t}}_{b_t \text{ times}}, \dots$$

Here

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$$

denotes the floor of x , i.e., the greatest integer less than or equal to x .

Since

$$2^t a_{2^t} \longrightarrow 0,$$

we have

$$b_t 2^t a_{2^t} = \left\lfloor \frac{1}{2^t a_{2^t}} \right\rfloor 2^t a_{2^t} \longrightarrow 1.$$

Moreover,

$$b_t a_{2^t} = \left\lfloor \frac{1}{2^t a_{2^t}} \right\rfloor a_{2^t} \leq 2^{-t}.$$

Therefore,

$$\sum_{k=1}^{\infty} a_{n_k} = \sum_{t=1}^{\infty} b_t a_{2^t} \leq \sum_{t=1}^{\infty} 2^{-t} < \infty,$$

whereas

$$\sum_{k=1}^{\infty} n_k a_{n_k} = \sum_{t=1}^{\infty} b_t 2^t a_{2^t} = \infty,$$

because

$$b_t 2^t a_{2^t} \longrightarrow 1.$$

Thus (3) and (4) hold. □

Lemma 1.2. *For a positive numerical series $\sum_{n=1}^{\infty} a_n < \infty$, we can find a sequence of positive numbers R_n such that $R_n \rightarrow \infty$ as $n \rightarrow \infty$, but also such that $\sum_{n=1}^{\infty} R_n a_n < \infty$.*

Proof. Denote by

$$T_n = \sum_{k=n+1}^{\infty} a_k$$

the n th tail of $\sum_{n=1}^{\infty} a_n$. Since

$$\sum_{n=1}^{\infty} a_n < \infty,$$

we have $T_n \rightarrow 0$ as $n \rightarrow \infty$.

Assume first that $T_n > 0$ for every n . Define

$$R_n = \frac{1}{\sqrt{T_{n-1}}}.$$

Then $R_n \rightarrow \infty$, since $T_{n-1} \rightarrow 0$. Moreover, using

$$a_n = T_{n-1} - T_n,$$

we obtain

$$\begin{aligned}
 \sum_{n=1}^N R_n a_n &= \sum_{n=1}^N \frac{T_{n-1} - T_n}{\sqrt{T_{n-1}}} \\
 &= \sum_{n=1}^N (\sqrt{T_{n-1}} - \sqrt{T_n}) \frac{\sqrt{T_{n-1}} + \sqrt{T_n}}{\sqrt{T_{n-1}}} \\
 &\leq 2 \sum_{n=1}^N (\sqrt{T_{n-1}} - \sqrt{T_n}) \\
 &= 2(\sqrt{T_0} - \sqrt{T_N}) \\
 &\leq 2\sqrt{T_0}.
 \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} R_n a_n < \infty.$$

If $T_n = 0$ for some n , then $a_k = 0$ for all sufficiently large k . In this case, we may choose any sequence $R_n \rightarrow \infty$, and the series

$$\sum_{n=1}^{\infty} R_n a_n$$

is automatically finite.

Thus, in all cases, there exists a sequence $R_n \rightarrow \infty$ such that

$$\sum_{n=1}^{\infty} R_n a_n < \infty.$$

□

Then by the above lemma, we have the following

Lemma 1.3. *Let $0 < p, q < \infty$. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be sequences of positive numbers such that*

$$\sum_{n=1}^{\infty} a_n^p < \infty, \quad \sum_{n=1}^{\infty} b_n^q < \infty.$$

Then, there exists a sequence $\{R_n\}_{n=1}^{\infty}$ with $R_n \rightarrow \infty$ such that

$$\sum_{n=1}^{\infty} R_n^p a_n^p < \infty, \quad \sum_{n=1}^{\infty} R_n^q b_n^q < \infty.$$

Proof. Take $c_n = a_n^p + b_n^q$, then by Lemma 1.2, there exists a sequence $\{R'_n\}_{n=1}^{\infty}$ with $R'_n \rightarrow \infty$ such that

$$\sum_{n=1}^{\infty} R'_n c_n < \infty.$$

Note here we can choose $R'_n \geq 1$ by replacing it by $\max\{1, R'_n\}$. Setting $R'_n = R_n^{\max\{p, q\}}$, we rewrite the preceding as

$$\sum_{n=1}^{\infty} R_n^{\max\{p, q\}} c_n < \infty.$$

Then we have that

$$\begin{aligned}\sum_{n=1}^{\infty} R_n^p a_n^p &< \sum_{n=1}^{\infty} R_n^{\max\{p,q\}} c_n < \infty, \\ \sum_{n=1}^{\infty} R_n^q b_n^q &< \sum_{n=1}^{\infty} R_n^{\max\{p,q\}} c_n < \infty.\end{aligned}$$

This proves our claim. □

Lemma 1.4. *Let $(t_n)_{n \geq 1}$ be a sequence of real numbers such that*

$$0 < t_n < 1 \quad \text{for all } n = 1, 2, \dots$$

and

$$\sum_{n=1}^{\infty} t_n = \infty.$$

Then, for every fixed natural number N , we have

$$\prod_{n=N}^M (1 - t_n) \longrightarrow 0 \quad \text{as } M \rightarrow \infty.$$

Proof. Fix $N \in \mathbb{N}$ and set

$$P_M := \prod_{n=N}^M (1 - t_n), \quad M \geq N.$$

Since $0 < t_n < 1$, each factor $1 - t_n$ lies in $(0, 1)$, so $P_M > 0$ and

$$\ln P_M = \sum_{n=N}^M \ln(1 - t_n).$$

We now estimate the logarithms. For $x \in (0, 1)$, we have the standard inequality

$$\ln(1 - x) \leq -x.$$

Applying this with $x = t_n$, we obtain

$$\ln(1 - t_n) \leq -t_n \quad \text{for all } n.$$

Therefore, one has

$$\ln P_M = \sum_{n=N}^M \ln(1 - t_n) \leq - \sum_{n=N}^M t_n,$$

thus

$$P_M = e^{\sum_{n=N}^M \ln(1 - t_n)} \leq e^{-\sum_{n=N}^M t_n}.$$

By hypothesis, $\sum_{n=1}^{\infty} t_n = \infty$, hence also $\sum_{n=N}^{\infty} t_n = \infty$. Thus

$$\sum_{n=N}^M t_n \xrightarrow{M \rightarrow \infty} \infty,$$

which implies $P_M \rightarrow 0$ as $M \rightarrow \infty$. This is exactly the desired conclusion. □

Lemma 1.5. *Let $E_1, E_2, \dots, E_n, \dots$ be a collection of sets in M , with the property that $\sum_{n=1}^{\infty} \mu(E_n) = \infty$. Then there exists a sequence of elements $g_1, g_2, \dots, g_n, \dots$ of G , so that the translated sets $F_1, F_2, \dots$, defined by $F_n = g_n[E_n]$, have the property that almost every point of M belongs to infinitely many sets F_n . Precisely, let*

$$F_0 = \limsup F_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} F_n.$$

Then $\mu(F_0) = 1$.

Proof. We consider two infinite product spaces. First, let

$$\mathcal{M} = \prod_{k=1}^{\infty} M_k,$$

where each M_k is a copy of M . Thus the points of $\mathcal{M}$ are sequences $\{x_n\}$, where $x_n \in M$. We impose the usual product measure of the measures $d\mu$ on each M_k on $\mathcal{M}$. We call this product measure $d\mu^*$.

Next we consider the infinite product group

$$\Gamma = \prod_{k=1}^{\infty} G_k,$$

where each G_k is a copy of G . The elements of Γ are sequences $\{g_n\}$, where $g_n \in G$. On Γ we also consider the usual product measure.

We notice that Γ is a compact group which acts on $\mathcal{M}$ in a the following way

$$\gamma \cdot (x_1, x_2, \dots) = (g_1(x_1), g_2(x_2), \dots), \quad \gamma = (g_1, g_2, \dots) \in \Gamma.$$

Hence $(\mathcal{M}, \mu^*)$ is a homogeneous space of Γ .

Consider now the collection of sets $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_n, \dots$ on $\mathcal{M}$ defined as follows

$$\mathcal{E}_n = \{x = (x_1, x_2, \dots) \in \mathcal{M} : x_n \in E_n\}, \quad n = 1, 2, \dots$$

Then $\mu^*(\mathcal{E}_n) = \mu(E_n)$. Let

$$\mathcal{E}_0 = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \mathcal{E}_n.$$

We claim that $\mu^*(\mathcal{E}_0) = 1$.

Recall that

$$\mathcal{E}_0^c = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \mathcal{E}_n^c.$$

Hence, by the subadditivity of the outer measure μ^* we have

$$(5) \quad \mu^*(\mathcal{E}_0^c) \leq \sum_{k=1}^{\infty} \mu^*\left(\bigcap_{n=k}^{\infty} \mathcal{E}_n^c\right).$$

Fix $k \geq 1$. For any $m > k$,

$$\bigcap_{n=k}^m \mathcal{E}_n^c \subseteq \bigcup_{n=k}^m \mathcal{E}_n^c,$$

so by the independence of the coordinate events, one obtains

$$\mu^* \left(\bigcap_{n=k}^m \mathcal{E}_n^c \right) = \prod_{n=k}^m \mu(E_n^c) = \prod_{n=k}^m (1 - \mu(E_n)).$$

Since, we can assume that $\mu^*(\mathcal{E}_n) = \mu(E_n) > 0$, we have $\mu^*(\mathcal{E}_n^c) = 1 - \mu(E_n)$ by the definition of the product measure. Thus,

$$\mu^* \left(\bigcap_{n=k}^m \mathcal{E}_n^c \right) \leq \prod_{n=k}^m (1 - \mu(E_n)).$$

Letting $m \rightarrow \infty$, we obtain

$$(6) \quad \mu^* \left(\bigcap_{n=k}^{\infty} \mathcal{E}_n^c \right) \leq \lim_{m \rightarrow \infty} \prod_{n=k}^m (1 - \mu(E_n)) = 0,$$

by Lemma 1.4.

Now summing over k and substituting (6) into (5), we find that

$$\mu^*(\mathcal{E}_0^c) \leq \sum_{k=1}^{\infty} \mu^* \left(\bigcap_{n=k}^{\infty} \mathcal{E}_n^c \right) = 0.$$

Hence, it follows that

$$\mu^*(\mathcal{E}_0) = 1 - \mu^*(\mathcal{E}_0^c) = 1.$$

Therefore we have $\mu(\mathcal{E}_0^c) = 0$ and thus $\mu(\mathcal{E}_0) = 1$.

Let now $\psi(x_1, x_2, \dots, x_n, \dots)$ be the characteristic function of the set $\mathcal{E}_0$. Consider the function $f(\gamma, x)$ defined on $\Gamma \times M$ as follows

$$f(\gamma, x) = \psi(g_1^{-1}(x), g_2^{-1}(x), \dots, g_n^{-1}(x), \dots),$$

where $\gamma = \{g_n\} \in \Gamma$, and $x \in M$. For $x \in M$, let

$$p_x = (x, x, \dots, x, \dots).$$

Define the set

$$A_x := \{\gamma \in \Gamma : f(\gamma, x) = 1\} = \{\gamma \in \Gamma : \gamma^{-1} \cdot p_x \in \mathcal{E}_0\}.$$

We apply (1) with G replaced by Γ and $E = \mathcal{E}_0$ to obtain

$$\mu^*(\mathcal{E}_0) = \omega_{\Gamma}(A_x),$$

where ω_{Γ} is the product Haar measure induced by ω_G . Since $\mu^*(\mathcal{E}_0) = 1$, it follows that for every fixed $x \in M$,

$$\omega_{\Gamma}(A_x) = 1, \quad \text{i.e.} \quad f(\gamma, p_x) = 1 \text{ for almost every } \gamma \in \Gamma.$$

We then have that for each x , $f(\gamma, p_x) = 1$ for almost every γ . Hence, by Fubini's theorem

$$\int_M \int_{\Gamma} f(\gamma, p_x) d\omega_{\Gamma} d\mu^* = 1 = \int_{\Gamma} \int_M f(\gamma, p_x) d\mu^* d\omega_{\Gamma}.$$

Thus for almost every $\gamma = \{g_n\} \in \Gamma$, we have $f(\gamma, p_x) = 1$ for almost every $x \in M$.

Therefore for almost every $x \in M$, $\gamma^{-1}(p_x) \in \mathcal{E}_0$. That is, for almost every $x \in M$, $g_n^{-1}(x) \in E_n$ for infinitely many n . Hence, for almost every $x \in M$, we have $x \in g_n[E_n]$ for infinitely many n . This proves the lemma. $\square$

Next we consider the **Rademacher** functions $r_n(t)$ defined by

$$r_n(t) = r_1(2^{n-1}t), \quad n = 1, 2, \dots,$$

for $t \in I := [0, 1]$, where $r_0(t) = 1$ and

$$r_1(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq 1/2, \\ -1, & \text{if } 1/2 < t < 1. \end{cases}$$

These functions are orthonormal over $[0, 1]$ and their importance lies in the fact that they can be thought of as mutually independent random variables. We shall not make explicit use of this fact, but instead we shall use a property that, in effect, follows from this.

Our next lemma is an analogue of Lemma 2 in [26] (see also [24]) for double Rademacher series. The approach in the proof below is based on the work of [1]. Let $r_n(s)$, $r_k(t)$ be the Rademacher functions indexed by $k, n \geq 0$, and for numbers $a_{n,k}$ with $\sum_{n,k=0}^{\infty} |a_{n,k}|^2 < \infty$, we consider the double Rademacher series

$$F(s, t) = \sum_{n,k=0}^{\infty} a_{n,k} r_n(s) r_k(t).$$

The following lemmas concern F .

Lemma 1.6. *Let $r_1, r_2, \dots$ denote the Rademacher functions on $[0, 1]$, and set*

$$r_0 \equiv 1.$$

For every finite collection of complex numbers

$$\{c_{j,k}\}_{0 \leq j \leq m, 0 \leq k \leq n},$$

let

$$P(s, t) = \sum_{j=0}^m \sum_{k=0}^n c_{j,k} r_j(s) r_k(t), \quad (s, t) \in [0, 1]^2.$$

Then one has

$$(7) \quad \int_{[0,1]^2} |P(s, t)|^4 ds dt \leq 9 \left(\int_{[0,1]^2} |P(s, t)|^2 ds dt \right)^2.$$

In particular,

$$\int_{[0,1]^2} |P(s, t)|^4 ds dt \leq 9 \left(\sum_{j=0}^m \sum_{k=0}^n |c_{j,k}|^2 \right)^2.$$

Proof. We first recall the following one-dimensional fourth-moment estimate. If

$$Q(x) = b_0 + \sum_{j=1}^m b_j r_j(x), \quad b_j \in \mathbb{C},$$

then

$$(8) \quad \int_0^1 |Q(x)|^4 dx \leq 3 \left(\sum_{j=0}^m |b_j|^2 \right)^2.$$

This is the $p = 4$ case of Khintchine's inequality.

For completeness, (8) also follows directly. Let $\varepsilon_1, \dots, \varepsilon_m$ be independent Rademacher random variables. Since

$$\int_0^1 |Q(x)|^4 dx = \mathbb{E} \left| b_0 + \sum_{j=1}^m b_j \varepsilon_j \right|^4,$$

introduce an additional independent Rademacher variable ε_0 . Multiplication by ε_0 does not change the absolute value, and hence

$$\mathbb{E} \left| b_0 + \sum_{j=1}^m b_j \varepsilon_j \right|^4 = \mathbb{E} \left| b_0 \varepsilon_0 + \sum_{j=1}^m b_j \varepsilon_0 \varepsilon_j \right|^4.$$

The random variables

$$\varepsilon_0, \varepsilon_0 \varepsilon_1, \dots, \varepsilon_0 \varepsilon_m$$

are again independent Rademacher random variables. Thus it suffices to estimate

$$S = \sum_{j=0}^m b_j \varepsilon_j.$$

Expanding the fourth moment and using independence gives

$$\mathbb{E}|S|^4 = 2 \left(\sum_{j=0}^m |b_j|^2 \right)^2 + \left| \sum_{j=0}^m b_j^2 \right|^2 - 2 \sum_{j=0}^m |b_j|^4.$$

Since

$$\left| \sum_{j=0}^m b_j^2 \right| \leq \sum_{j=0}^m |b_j|^2,$$

we obtain

$$\mathbb{E}|S|^4 \leq 3 \left(\sum_{j=0}^m |b_j|^2 \right)^2,$$

which proves (8).

We now prove the two-parameter estimate. For $0 \leq t, s \leq 1$ write

$$P(s, t) = \sum_{j=0}^m A_j(t) r_j(s),$$

where

$$A_j(t) = \sum_{k=0}^n c_{j,k} r_k(t).$$

For each fixed t in $[0, 1]$, applying (8) in the s -variable gives

$$\int_0^1 |P(s, t)|^4 ds \leq 3 \left(\sum_{j=0}^m |A_j(t)|^2 \right)^2.$$

Integrating with respect to t , we obtain

$$(9) \quad \int_{[0,1]^2} |P(s, t)|^4 ds dt \leq 3 \int_0^1 \left(\sum_{j=0}^m |A_j(t)|^2 \right)^2 dt.$$

By Minkowski's inequality on $L^2([0, 1])$ we have

$$\begin{aligned} \left[\int_0^1 \left(\sum_{j=0}^m |A_j(t)|^2 \right)^2 dt \right]^{1/2} &= \left\| \sum_{j=0}^m |A_j|^2 \right\|_{L^2}^{1/2} \\ &\leq \sum_{j=0}^m \| |A_j|^2 \|_{L^2}^{1/2} \\ &= \sum_{j=0}^m \left(\int_0^1 |A_j(t)|^4 dt \right)^{1/2}. \end{aligned}$$

Applying (8) to each A_j , we obtain

$$\left(\int_0^1 |A_j(t)|^4 dt \right)^{1/2} \leq \sqrt{3} \sum_{k=0}^n |c_{j,k}|^2.$$

Consequently,

$$\left[\int_0^1 \left(\sum_{j=0}^m |A_j(t)|^2 \right)^2 dt \right]^{1/2} \leq \sqrt{3} \sum_{j=0}^m \sum_{k=0}^n |c_{j,k}|^2,$$

and hence

$$\int_0^1 \left(\sum_{j=0}^m |A_j(t)|^2 \right)^2 dt \leq 3 \left(\sum_{j=0}^m \sum_{k=0}^n |c_{j,k}|^2 \right)^2.$$

Combining this with (9) yields

$$\int_{[0,1]^2} |P(s, t)|^4 ds dt \leq 9 \left(\sum_{j=0}^m \sum_{k=0}^n |c_{j,k}|^2 \right)^2.$$

Finally, orthonormality of the Rademacher products gives

$$\int_{[0,1]^2} |P(s, t)|^2 ds dt = \sum_{j=0}^m \sum_{k=0}^n |c_{j,k}|^2,$$

and thus (7) follows. $\square$

Corollary 1. *Let $R = I \times J$ be a dyadic rectangle, and suppose that, after the affine rescaling of I and J onto $[0, 1]$, a function P is a double Rademacher series as in Lemma 1.6. Then*

$$(10) \quad \frac{1}{|R|} \int_R |P(s, t)|^4 ds dt \leq 9 \left(\frac{1}{|R|} \int_R |P(s, t)|^2 ds dt \right)^2.$$

Equivalently,

$$\int_R |P(s, t)|^4 ds dt \leq \frac{9}{|R|} \left(\int_R |P(s, t)|^2 ds dt \right)^2.$$

The same estimate holds for square-summable double Rademacher series.

Proof. The finite-series estimate follows immediately from Lemma 1.6 by an affine change of variables.

For a square-summable series, let P_ν be its rectangular partial sums. Then $P_\nu \rightarrow P$ in $L^2(R)$, and hence a subsequence ν_j converges almost everywhere on R . Moreover,

$$\int_R |P_\nu(s, t)|^4 ds dt \leq \frac{9}{|R|} \left(\int_R |P_\nu(s, t)|^2 \right)^2 ds dt.$$

Since

$$\int_R |P_{\nu_j}(s, t)|^2 ds dt \longrightarrow \int_R |P(s, t)|^2 ds dt,$$

as $j \rightarrow \infty$, Fatou's lemma gives

$$\int_R |P(s, t)|^4 ds dt \leq \liminf_{j \rightarrow \infty} \int_R |P_{\nu_j}(s, t)|^4 ds dt \leq \frac{9}{|R|} \left(\int_R |P(s, t)|^2 \right)^2 ds dt$$

which proves (10). $\square$

Lemma 1.7. *Let $E \subset [0, 1]^2$ be a measurable set with $|E| > 0$, and let*

$$F(s, t) = \sum_{n,k=0}^{\infty} a_{n,k} r_n(s) r_k(t), \quad \sum_{n,k \geq 0} |a_{n,k}|^2 < \infty,$$

where $\{a_{n,k}\}_{n,k=0}^{\infty}$ is a sequence of complex numbers. Then there exists an integer $N = N(E)$ not depending on the coefficients $a_{n,k}$, such that

$$\left(\sum_{\substack{n > N(E) \\ k > N(E)}} |a_{n,k}|^2 \right)^{1/2} \leq \sqrt{2} \operatorname{ess\,sup}_{(s,t) \in E} |F(s, t)|.$$

Proof. Fix $\varepsilon = \frac{1}{36}$. By a standard measure theory theorem, there exists a dyadic rectangle

$$R_N = I_N \times J_N, \quad |I_N| = |J_N| = 2^{-N},$$

such that

$$|E \cap R_N| \geq (1 - \varepsilon)|R_N|.$$

Equivalently, we have

$$(11) \quad |R_N \setminus E| \leq \varepsilon |R_N|.$$

Since ε is fixed, the integer N depends only on E . We write $R = R_N$.

On R , the functions r_n with $n \leq N$ are constant on I_N , and the functions r_k with $k \leq N$ are constant on J_N . Decompose

$$F = H_{00} + F^{(1)} + F^{(2)} + F^{(3)},$$

where

$$\begin{aligned} H_{00}(s, t) &= \sum_{\substack{0 \leq n \leq N \\ 0 \leq k \leq N}} a_{n,k} r_n(s) r_k(t), \\ F^{(1)}(s, t) &= \sum_{\substack{n > N \\ k > N}} a_{n,k} r_n(s) r_k(t), \\ F^{(2)}(s, t) &= \sum_{\substack{n > N \\ 0 \leq k \leq N}} a_{n,k} r_n(s) r_k(t), \\ F^{(3)}(s, t) &= \sum_{\substack{0 \leq n \leq N \\ k > N}} a_{n,k} r_n(s) r_k(t), \end{aligned}$$

and

$$F^{(3)}(s, t) = \sum_{\substack{0 \leq n \leq N \\ k > N}} a_{n,k} r_n(s) r_k(t).$$

The function H_{00} is constant on R . Moreover, for $n > N$,

$$\int_{I_N} r_n(s) ds = 0,$$

and, for $k > N$,

$$\int_{J_N} r_k(t) dt = 0.$$

It follows that the four functions H_{00} , $F^{(1)}$, $F^{(2)}$, $F^{(3)}$ are pairwise orthogonal in $L^2(R)$.

Also, the family

$$\{r_n(s) r_k(t) : n > N, k > N\}$$

is orthogonal on R . Hence

$$\int_R |F^{(1)}(s, t)|^2 ds dt = |R| \sum_{\substack{n > N \\ k > N}} |a_{n,k}|^2.$$

By the orthogonality of the four pieces,

$$(12) \quad |R| \sum_{\substack{n > N \\ k > N}} |a_{n,k}|^2 \leq \int_R |F(s, t)|^2 ds dt.$$

By Corollary 1 we have

$$(13) \quad \int_R |F(s, t)|^4 ds dt \leq \frac{9}{|R|} \left(\int_R |F(s, t)|^2 ds dt \right)^2.$$

It follows from the Cauchy–Schwarz inequality, (13), and (11) that

$$\begin{aligned} \int_{R \setminus E} |F(s, t)|^2 ds dt &\leq |R \setminus E|^{1/2} \left(\int_R |F(s, t)|^4 ds dt \right)^{1/2} \\ &\leq 3 \sqrt{\frac{|R \setminus E|}{|R|}} \int_R |F(s, t)|^2 ds dt \\ &\leq 3\sqrt{\varepsilon} \int_R |F(s, t)|^2 ds dt \\ &= \frac{1}{2} \int_R |F(s, t)|^2 ds dt, \end{aligned}$$

where in the last line we used $\varepsilon = 1/36$.

Since

$$\int_R |F(s, t)|^2 ds dt = \int_{R \cap E} |F(s, t)|^2 ds dt + \int_{R \setminus E} |F(s, t)|^2 ds dt,$$

we obtain

$$(14) \quad \frac{1}{2} \int_R |F(s, t)|^2 ds dt \leq \int_{R \cap E} |F(s, t)|^2 ds dt.$$

If

$$\operatorname{ess\,sup}_{(s,t) \in E} |F(s, t)| = \infty,$$

then the desired inequality is immediate. Thus assume that this essential supremum is finite, and set

$$M = \operatorname{ess\,sup}_{(s,t) \in E} |F(s, t)|.$$

Then

$$\int_{R \cap E} |F(s, t)|^2 ds dt \leq |R \cap E| M^2 \leq |R| M^2.$$

Combining this with (14), we obtain

$$\int_R |F(s, t)|^2 ds dt \leq 2|R| M^2.$$

Using (12), we therefore have

$$|R| \sum_{\substack{n > N \\ k > N}} |a_{n,k}|^2 \leq 2|R| M^2$$

which gives

$$\sum_{\substack{n > N \\ k > N}} |a_{n,k}|^2 \leq 2 \left(\operatorname{ess\,sup}_{(s,t) \in E} |F(s, t)| \right)^2.$$

Taking square roots yields

$$\left(\sum_{\substack{n > N \\ k > N}} |a_{n,k}|^2 \right)^{1/2} \leq \sqrt{2} \operatorname{ess\,sup}_{(s,t) \in E} |F(s, t)|.$$

Thus the conclusion holds. $\square$

3. THE PROOF OF THEOREM 1

We now have the tools required to prove Theorem 1. In this section, we provide the proof.

Proof of Theorem 1. We assume that conclusion (2) is false and we will reach a contradiction. Then for each $n \in \mathbb{N}$, there exist non-zero a.e. functions $f'_n \in L^p(M)$, $h'_n \in L^q(M)$, and $\alpha_n > 0$ such that:

$$\mu \left(\left\{ x \in M : T^* \left(\frac{f'_n}{\alpha_n}, \frac{h'_n}{\alpha_n} \right) (x) > 1 \right\} \right) \geq n \left\| \frac{f'_n}{\alpha_n} \right\|_{L^p}^r \left\| \frac{h'_n}{\alpha_n} \right\|_{L^q}^r.$$

Thus we can reduce matters to the following situation: $f_n = \frac{f'_n}{\alpha_n} \in L^p(M)$, $h_n = \frac{h'_n}{\alpha_n} \in L^q(M)$, $E_n := \{x : T^*(f_n, h_n)(x) > 1\}$. Then:

$$\mu(E_n) \geq n \|f_n\|_{L^p}^r \|h_n\|_{L^q}^r.$$

Apply Lemma 1.1 to extract a subsequence n_k of the natural numbers such that the triple $(f_{n_k}, h_{n_k}, E_{n_k})$ satisfies

$$(15) \quad \begin{aligned} \sum_k \|f_{n_k}\|_{L^p}^r \|h_{n_k}\|_{L^q}^r &< \infty, \\ \sum_k \mu(E_{n_k}) &= \infty. \end{aligned}$$

By bilinearity, it is not hard to show that for any $\alpha > 0$, condition (15) for $(f_{n_k}, h_{n_k}, E_{n_k})$ and for $(\alpha f_{n_k}, \alpha^{-1} h_{n_k}, E_{n_k})$ are equivalent. Now we take $\alpha = \alpha_{n_k} = \|f_{n_k}\|_{L^p}^{-\frac{r}{p}} \|h_{n_k}\|_{L^q}^{\frac{r}{p}}$, then we have that:

$$\|\alpha_{n_k} f_{n_k}\|_{L^p}^p = \alpha_{n_k}^p \|f_{n_k}\|_{L^p}^p = \|f_{n_k}\|_{L^p}^{-\frac{rp}{p}} \|h_{n_k}\|_{L^q}^{\frac{rp}{p}} \|f_{n_k}\|_{L^p}^p = \|f_{n_k}\|_{L^p}^r \|h_{n_k}\|_{L^q}^r.$$

Similarly, we have

$$\|\alpha_{n_k}^{-1} h_{n_k}\|_{L^q}^q = \|f_{n_k}\|_{L^p}^r \|h_{n_k}\|_{L^q}^r.$$

For notational convenience, we replace $(\alpha_{n_k} f_{n_k}, \alpha_{n_k}^{-1} h_{n_k}, E_{n_k})$ by (f_n, h_n, E_n) . Then we have the following situation:

For $n \in \mathbb{N}$, there are $f_n \in L^p(M)$, $h_n \in L^q(M)$, and $E_n = \{x : T^*(f_n, h_n)(x) > 1\}$ such that

$$\begin{aligned} \sum_n \mu(E_n) &= \infty, \\ \sum_n \|f_n\|_{L^p}^p &= \sum_k \|f_{n_k}\|_{L^p}^r \|h_{n_k}\|_{L^q}^r < \infty, \\ \sum_n \|h_n\|_{L^q}^q &= \sum_k \|f_{n_k}\|_{L^p}^r \|h_{n_k}\|_{L^q}^r < \infty. \end{aligned}$$

Apply Lemma 1.3 to choose $R_n \rightarrow \infty$ such that:

$$\begin{aligned} \sum_n \|R_n f_n\|_{L^p}^p &< \infty, \\ \sum_n \|R_n h_n\|_{L^q}^q &< \infty. \end{aligned}$$

Apply Lemma 1.5 to the sets

$$E_n = \{x \in M : T^*(f_n, h_n)(x) > 1\}.$$

Since $\sum_n \mu(E_n) = \infty$, choose $g_n \in G$ such that

$$F_0 := \limsup_{n \rightarrow \infty} g_n[E_n], \quad \mu(F_0) = 1.$$

We record explicitly the null sets appearing below. For every $m, n \in \mathbb{N}$, set

$$Z_{m,n}^{\text{tr}} := \left\{x \in M : T_m(\tau_{g_n} f_n, \tau_{g_n} h_n)(x) \neq T_m(f_n, h_n)(g_n^{-1} x)\right\}.$$

By the hypothesis of commuting with simultaneous translations we obtain $\mu(Z_{m,n}^{\text{tr}}) = 0$ for every m, n . Hence

$$Z^{\text{tr}} := \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} Z_{m,n}^{\text{tr}}$$

is a null subset of M . Thus, for every $x \in M \setminus Z^{\text{tr}}$, the above translation identity holds simultaneously for every pair (m, n) .

For a function F on $M \times [0, 1]$ we introduce the slices

$$F_x(t) = F^t(x) = F(x, t), \quad H_x(s) = H^s(x) = H(x, s).$$

We claim that there exists a pair of functions $F(x, t)$, $H(x, s)$ both defined on $M \times [0, 1]$ with the properties:

- (i) For almost every $(t, s) \in [0, 1]^2$, we have $F^t \in L^p(M)$, $H^s \in L^q(M)$, and hence by assumption we have $T_m(F^t, H^s) \in L^r(M)$ for each m .
- (ii) For almost every $(t, s) \in [0, 1]^2$, the maximal function

$$T^*(F^t, H^s)(x) := \sup_{m \geq 1} |T_m(F^t, H^s)(x)| = \infty$$

for almost every $x \in M$.

Note that the validity of (i) and (ii) leads to a contradiction since the assumption is that for every $f \in L^p(M)$, $h \in L^q(M)$, the limit

$$\lim_{m \rightarrow \infty} T_m(f, h)(x)$$

exists for almost every $x \in M$. Thus, the desired inequality (2) must hold.

Let $I = [0, 1]$. We define measurable functions on $M \times I$ by setting

$$F_N(x, t) = \sum_{n=1}^N r_n(t) R_n \tau_{g_n}(f_n)(x),$$

$$H_N(x, s) = \sum_{k=1}^N r_k(s) R_k \tau_{g_k}(h_k)(x),$$

for $x \in M$. For $N_2 > N_1$ natural numbers we consider

$$\int_I \int_M |F_{N_1}(x, t) - F_{N_2}(x, t)|^p dx dt.$$

We need the following observation. Let

$$F(t) = \sum_{n=N_1}^{N_2} b_n r_n(t),$$

then for $1 \leq p \leq 2$

$$\int_0^1 |F(t)|^p dt \leq \left(\int_0^1 |F(t)|^2 dt \right)^{\frac{p}{2}} = \left(\sum_{n=N_1}^{N_2} |b_n|^2 \right)^{\frac{p}{2}} \leq \sum_{n=N_1}^{N_2} |b_n|^p.$$

Notice $\|f_n\|_{L^p(M)} = \|\tau_{g_n}(f_n)\|_{L^p(M)}$ by translation invariance, then by Fubini's theorem, we obtain

$$\int_I \int_M |F_{N_1}(x, t) - F_{N_2}(x, t)|^p dx dt \leq \sum_{n=N_1}^{N_2} \|R_n f_n\|_{L^p}^p,$$

and similarly

$$\int_I \int_M |H_{N_1}(x, s) - H_{N_2}(x, s)|^q dx ds \leq \sum_{n=N_1}^{N_2} \|R_n h_n\|_{L^q}^q.$$

Then the sequence $F_N(x, t)$ is Cauchy in $L^p(M \times I)$, since the series $\sum_{n=1}^{\infty} \|R_n f_n\|_{L^p}^p$ is convergent. Hence there exists a function $F(x, t)$ in $L^p(M \times I)$ such that

$$\|F_N(\cdot, \cdot) - F(\cdot, \cdot)\|_{L^p(M \times I)} \rightarrow 0$$

as $N \rightarrow \infty$. Define

$$G_N(t) := \|F_N(\cdot, t) - F(\cdot, t)\|_{L^p(M)}^p = \int_M |F_N(x, t) - F(x, t)|^p d\mu(x).$$

By Fubini's theorem we have

$$\|F_N - F\|_{L^p(M \times I)}^p = \int_I G_N(t) dt.$$

Since $F_N \rightarrow F$ in $L^p(M \times I)$, the right-hand side converges to 0. Thus $G_N \rightarrow 0$ in $L^1(I)$.

Convergence in $L^1(I)$ implies the existence of a subsequence G_{N_i} of G_N such that $G_{N_i}(t) \rightarrow 0$ for $t \in I$, except for a subset of measure zero which we denote by $\mathcal{A}$. Equivalently, we have

$$(16) \quad \|F_{N_i}(\cdot, t) - F(\cdot, t)\|_{L^p(M)} \rightarrow 0 \quad \text{for } t \in I \setminus \mathcal{A}, \quad (\text{as } i \rightarrow \infty).$$

Similarly, there exists an $H(x, s) \in L^q(M \times I)$ such that

$$\|H_{N_i}(\cdot, \cdot) - H(\cdot, \cdot)\|_{L^q(M \times I)} \rightarrow 0,$$

and there is a subsequence N_{i_j} of N_i and a subset $\mathcal{B}$ of I of measure zero such that

$$(17) \quad \|H_{N_{i_j}}(\cdot, s) - H(\cdot, s)\|_{L^q(M)} \rightarrow 0 \quad \text{for } s \in I \setminus \mathcal{B} \quad (\text{as } j \rightarrow \infty).$$

Keep the original partial sums F_N and H_N indexed by their actual cutoffs. Let

$$\nu_j := N_{i_j}$$

denote the common subsequence obtained above. Since ν_j is a subsequence of N_i , equations (16) and (17) imply

$$F_{\nu_j}(\cdot, t) \rightarrow F(\cdot, t) \quad \text{in } L^p(M) \quad \text{for every } t \in I \setminus \mathcal{A},$$

and

$$H_{\nu_j}(\cdot, s) \rightarrow H(\cdot, s) \quad \text{in } L^q(M) \quad \text{for every } s \in I \setminus \mathcal{B}.$$

In particular,

$$F^t := F(\cdot, t) \in L^p(M) \quad (t \in I \setminus \mathcal{A}), \quad H^s := H(\cdot, s) \in L^q(M) \quad (s \in I \setminus \mathcal{B}),$$

so property (i) holds. We do not rename F_{ν_j} or H_{ν_j} as F_N, H_N ; below F_N, H_N always denote the original partial sums with cutoff N .

Let us now fix $m \in \mathbb{N}$. For $n, k, N \in \mathbb{N}$, set

$$a_{n,k}^{(m)}(x) := R_n R_k T_m(\tau_{g_n}(f_n), \tau_{g_k}(h_k))(x)$$

and

$$P_{m,N}(x, t, s) := \sum_{n=1}^N \sum_{k=1}^N a_{n,k}^{(m)}(x) r_n(t) r_k(s).$$

By bilinearity we write

$$P_{m,N}(x, t, s) = T_m(F_N^t, H_N^s)(x).$$

We first consider the limit as $N \rightarrow \infty$. If $N > L$, then

$$P_{m,N} - P_{m,L} = T_m(F_N - F_L, H_N) + T_m(F_L, H_N - H_L).$$

Since $1/2 \leq r \leq 1$, the r -triangle inequality and the boundedness of $T_m : L^p(M) \times L^q(M) \rightarrow L^r(M)$ give

$$\begin{aligned} \|P_{m,N} - P_{m,L}\|_{L^r(M \times I^2)}^r &\leq C_m^r \left(\|F_N - F_L\|_{L^p(M \times I)}^r \|H_N\|_{L^q(M \times I)}^r \right. \\ &\quad \left. + \|F_L\|_{L^p(M \times I)}^r \|H_N - H_L\|_{L^q(M \times I)}^r \right). \end{aligned}$$

Here we used $r \leq p, q$ and $|I| = 1$. Hence $P_{m,N}$ is Cauchy in $L^r(M \times I^2)$, and there is a jointly measurable representative Q_m such that

$$P_{m,N} \longrightarrow Q_m \quad \text{in } L^r(M \times I^2) \quad \text{as } N \rightarrow \infty.$$

Recall the common subsequence ν_j above. For every $(t, s) \in (I \setminus \mathcal{A}) \times (I \setminus \mathcal{B})$, the continuity of T_m gives

$$P_{m,\nu_j}(\cdot, t, s) = T_m(F_{\nu_j}^t, H_{\nu_j}^s) \longrightarrow T_m(F^t, H^s) \quad \text{in } L^r(M).$$

On the other hand, define

$$d_{m,j}(t, s) := \|P_{m,\nu_j}(\cdot, t, s) - Q_m(\cdot, t, s)\|_{L^r(M)}^r.$$

Since $P_{m,\nu_j} \rightarrow Q_m$ in $L^r(M \times I^2)$, we have

$$\int_{I^2} d_{m,j}(t, s) dt ds \longrightarrow 0.$$

Choose a subsequence $\nu_{j_\ell}^{(m)}$ such that $d_{m,j_\ell}(t, s) \rightarrow 0$ outside a null set $\mathcal{C}_m \subset I^2$. Enlarging $\mathcal{C}_m$, if necessary, by the null set

$$(\mathcal{A} \times I) \cup (I \times \mathcal{B}),$$

we may assume that for every $(t, s) \in I^2 \setminus \mathcal{C}_m$,

$$Q_m(\cdot, t, s) = T_m(F^t, H^s) \quad \text{in } L^r(M).$$

Define the common parameter-null set

$$\mathcal{C} := \bigcup_{m=1}^{\infty} \mathcal{C}_m.$$

Then $|\mathcal{C}| = 0$, and for every $(t, s) \in I^2 \setminus \mathcal{C}$ the preceding identity holds simultaneously for all m .

For later use, for every $(t, s) \in I^2 \setminus \mathcal{C}$ and every m , choose

$$Z_m^{\text{id}}(t, s) := \left\{ x \in M : Q_m(x, t, s) \neq T_m(F^t, H^s)(x) \right\}.$$

Then $\mu(Z_m^{\text{id}}(t, s)) = 0$. Thus

$$Z^{\text{id}}(t, s) := \bigcup_{m=1}^{\infty} Z_m^{\text{id}}(t, s)$$

is null and, whenever $x \notin Z^{\text{id}}(t, s)$,

$$Q_m(x, t, s) = T_m(F^t, H^s)(x) \quad \text{for every } m.$$

For purposes of bookkeeping of null sets, define the product-space null set

$$\mathfrak{Z}^{\text{id}} := (M \times \mathcal{C}) \cup \left\{ (x, t, s) : (t, s) \in I^2 \setminus \mathcal{C}, x \in Z^{\text{id}}(t, s) \right\}.$$

By Fubini's theorem, $(\mu \times dt \times ds)(\mathfrak{Z}^{\text{id}}) = 0$, and on $(M \times I^2) \setminus \mathfrak{Z}^{\text{id}}$ one has

$$Q_m(x, t, s) = T_m(F^t, H^s)(x) \quad \text{simultaneously for every } m.$$

We now prove square summability of the coefficients. Let $(c_{n,k})$ be finitely supported and set

$$\Phi(t, s) := \sum_{n,k} c_{n,k} r_n(t) r_k(s), \quad S := \sum_{n,k} |c_{n,k}|^2.$$

If $S = 0$, the desired estimate is trivial. Assume $S > 0$. By orthogonality,

$$\int_{I^2} |\Phi(t, s)|^2 dt ds = S,$$

and Lemma 1.6 gives

$$\int_{I^2} |\Phi(t, s)|^4 dt ds \leq 9S^2.$$

Applying the Paley–Zygmund inequality to $|\Phi|^2$ with $\theta = 1/2$ yields

$$\left| \left\{ (t, s) \in [0, 1]^2 : |\Phi(t, s)|^2 \geq \frac{S}{2} \right\} \right| \geq \frac{1}{36}.$$

Consequently, for every $r > 0$,

$$S^{r/2} \leq 36 \cdot 2^{r/2} \int_{I^2} |\Phi(t, s)|^r dt ds.$$

Thus, with $C_r = 36 \cdot 2^{r/2}$,

$$\left(\sum_{n,k} |c_{n,k}|^2 \right)^{r/2} \leq C_r \int_{I^2} \left| \sum_{n,k} c_{n,k} r_n(t) r_k(s) \right|^r dt ds.$$

Apply this finite-polynomial inequality to $P_{m,N}(x, \cdot, \cdot)$ and integrate in x . We obtain

$$\begin{aligned} \int_M \left(\sum_{n=1}^N \sum_{k=1}^N |a_{n,k}^{(m)}(x)|^2 \right)^{r/2} d\mu(x) &\leq C_r \|P_{m,N}\|_{L^r(M \times I^2)}^r \\ &\leq C_r C_m^r \|F_N\|_{L^p(M \times I)}^r \|H_N\|_{L^q(M \times I)}^r. \end{aligned}$$

Since $F_N \rightarrow F$ in $L^p(M \times I)$ and $H_N \rightarrow H$ in $L^q(M \times I)$, the right-hand side is bounded uniformly in N . By monotone convergence we deduce

$$\int_M \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |a_{n,k}^{(m)}(x)|^2 \right)^{r/2} d\mu(x) < \infty.$$

Define

$$Z_m^{\ell^2} := \left\{ x \in M : \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |a_{n,k}^{(m)}(x)|^2 = \infty \right\}.$$

Then $\mu(Z_m^{\ell^2}) = 0$.

Put

$$e_{m,N}(x) := \|P_{m,N}(x, \cdot, \cdot) - Q_m(x, \cdot, \cdot)\|_{L^r(I^2)}^r.$$

Again,

$$\int_M e_{m,N}(x) d\mu(x) = \|P_{m,N} - Q_m\|_{L^r(M \times I^2)}^r \longrightarrow 0.$$

Choose a subsequence $N_j^{(m)}$ such that $e_{m, N_j^{(m)}}(x) \rightarrow 0$ for all $x \in M \setminus Z_m^{\text{fib}}$, where $\mu(Z_m^{\text{fib}}) = 0$.

Fix $x \in M \setminus (Z_m^{\ell^2} \cup Z_m^{\text{fib}})$. Since the coefficients $a_{n,k}^{(m)}(x)$ are square summable, the full sequence $P_{m,N}(x, \cdot, \cdot)$ converges in $L^2(I^2)$, hence in $L^r(I^2)$, to the double Rademacher series

$$\mathcal{R}_m(x, t, s) := \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{n,k}^{(m)}(x) r_n(t) r_k(s).$$

The subsequence $P_{m, N_j^{(m)}}(x, \cdot, \cdot)$ also converges in $L^r(I^2)$ to $Q_m(x, \cdot, \cdot)$. Uniqueness of the $L^r(I^2)$ -limit therefore gives

$$Q_m(x, \cdot, \cdot) = \mathcal{R}_m(x, \cdot, \cdot) \quad \text{in } L^r(I^2).$$

Accordingly, for each such x , define

$$\mathcal{R}_{m,x} := \left\{ (t, s) \in I^2 : Q_m(x, t, s) \neq \mathcal{R}_m(x, t, s) \right\}.$$

Then $|\mathcal{R}_{m,x}| = 0$ whenever $x \in M \setminus (Z_m^{\ell^2} \cup Z_m^{\text{fib}})$.

Now define the remaining common null sets in M :

$$Z^{\ell^2} := \bigcup_{m=1}^{\infty} Z_m^{\ell^2}, \quad Z^{\text{fib}} := \bigcup_{m=1}^{\infty} Z_m^{\text{fib}}, \quad Z^{\text{limsup}} := M \setminus F_0.$$

Define the common full-measure set

$$M_1 := M \setminus (Z^{\text{limsup}} \cup Z^{\text{tr}} \cup Z^{\ell^2} \cup Z^{\text{fib}}).$$

Then $\mu(M_1) = 1$. For every $x \in M_1$, set

$$\mathcal{R}_x := \bigcup_{m=1}^{\infty} \mathcal{R}_{m,x}.$$

Since this is a countable union of null sets, $|\mathcal{R}_x| = 0$. For every $x \in M_1$, every $(t, s) \in I^2 \setminus \mathcal{R}_x$, and every $m \in \mathbb{N}$, we therefore have the simultaneous Rademacher representation

$$Q_m(x, t, s) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} r_n(t) r_k(s) R_n R_k T_m(\tau_{g_n} f_n, \tau_{g_k} h_k)(x).$$

Moreover, because $x \notin Z^{\text{tr}}$, for every $m, n \in \mathbb{N}$,

$$T_m(\tau_{g_n} f_n, \tau_{g_n} h_n)(x) = T_m(f_n, h_n)(g_n^{-1} x).$$

Finally, because $x \in F_0$,

$$x \in g_n[E_n] \quad \text{for infinitely many } n.$$

Choose the jointly measurable representatives Q_m constructed above and define

$$Q^*(x, t, s) := \sup_{m \geq 1} |Q_m(x, t, s)|.$$

Suppose, towards a contradiction, that $Q^* < \infty$ on a set of positive measure in $M \times I^2$. Since

$$\{Q^* < \infty\} = \bigcup_{j=1}^{\infty} \{Q^* \leq j\},$$

there exists a finite constant $A > 0$ such that

$$S_A := \{(x, t, s) \in M \times I^2 : Q^*(x, t, s) \leq A\}$$

has positive product measure.

Since $\mu(M \setminus M_1) = 0$, the set

$$S'_A := S_A \cap (M_1 \times I^2)$$

still has positive product measure. For $x \in M_1$, define

$$E_x := \{(t, s) \in I^2 : (x, t, s) \in S'_A\}.$$

By Fubini's theorem,

$$M_0 := \{x \in M_1 : |E_x| > 0\}$$

has positive measure.

Fix $x \in M_0$. Recall the common parameter-null set

$$\mathcal{R}_x = \bigcup_{m=1}^{\infty} \mathcal{R}_{m,x}, \quad |\mathcal{R}_x| = 0,$$

and define

$$E_x^\sharp := E_x \setminus \mathcal{R}_x.$$

Then $|E_x^\sharp| = |E_x| > 0$. On $E_x^\sharp$, for every m ,

$$Q_m(x, t, s) = \sum_{n,k \geq 1} a_{n,k}^{(m)}(x) r_n(t) r_k(s),$$

and, because $(x, t, s) \in S_A$,

$$|Q_m(x, t, s)| \leq A \quad \text{for every } m.$$

Apply Lemma 1.7 to the positive-measure set $E_x^\sharp$. Its cutoff depends only on the set, so there exists

$$N_x := N(E_x^\sharp)$$

which is independent of m , such that for every m we have

$$\left(\sum_{n > N_x} \sum_{k > N_x} |a_{n,k}^{(m)}(x)|^2 \right)^{1/2} \leq \sqrt{2} \operatorname{ess\,sup}_{(t,s) \in E_x^\sharp} |Q_m(x, t, s)| \leq \sqrt{2} A.$$

Therefore, for every m and every $n > N_x$, the diagonal coefficient satisfies

$$R_n^2 |T_m(\tau_{g_n} f_n, \tau_{g_n} h_n)(x)| = |a_{n,n}^{(m)}(x)| \leq \sqrt{2} A.$$

Since $x \in M_1 \subset M \setminus Z^{\text{tr}}$, the simultaneous translation property yields

$$R_n^2 |T_m(f_n, h_n)(g_n^{-1}x)| \leq \sqrt{2} A, \quad m \geq 1, \quad n > N_x.$$

Taking the supremum over m gives

$$(18) \quad R_n^2 T^*(f_n, h_n)(g_n^{-1}x) \leq \sqrt{2} A, \quad n > N_x.$$

On the other hand, $x \in M_1 \subset F_0$, so $x \in g_n[E_n]$ for infinitely many n . For every such n , $g_n^{-1}x \in E_n$, and therefore

$$T^*(f_n, h_n)(g_n^{-1}x) > 1.$$

There are infinitely many such $n > N_x$, and consequently

$$R_n^2 T^*(f_n, h_n)(g_n^{-1}x) > R_n^2$$

for infinitely many $n > N_x$. Since $R_n \rightarrow \infty$, this contradicts the bound in (18). Hence

$$Q^*(x, t, s) = \infty \quad \text{for almost every } (x, t, s) \in M \times I^2.$$

We now return to the original operators. For every $(t, s) \in I^2 \setminus \mathcal{C}$, recall that

$$Z^{\text{id}}(t, s) = \bigcup_{m=1}^{\infty} Z_m^{\text{id}}(t, s)$$

is a null subset of M , and for every $x \in M \setminus Z^{\text{id}}(t, s)$,

$$Q_m(x, t, s) = T_m(F^t, H^s)(x) \quad \text{for all } m.$$

Thus

$$Q^*(x, t, s) = T^*(F^t, H^s)(x)$$

for all such x . Since $|\mathcal{C}| = 0$, Fubini's theorem and the fact that Q^* equals infinity a.e. on $M \times I^2$ imply that for almost every $(t, s) \in I^2$,

$$F^t \in L^p(M), \quad H^s \in L^q(M), \quad T^*(F^t, H^s)(x) = \infty \quad \text{for almost every } x \in M.$$

Choose a pair (t, s) such that $T^*(F^t, H^s)(x) = \infty$ for almost every $x \in M$. For every m , define

$$Z_m^{\text{fin}}(t, s) := \{x \in M : |T_m(F^t, H^s)(x)| = \infty\}.$$

Since $T_m(F^t, H^s) \in L^r(M)$, $\mu(Z_m^{\text{fin}}(t, s)) = 0$. Hence

$$Z^{\text{fin}}(t, s) := \bigcup_{m=1}^{\infty} Z_m^{\text{fin}}(t, s)$$

is null, and all the individual terms are finite on $M \setminus Z^{\text{fin}}(t, s)$. But

$$T^*(F^t, H^s)(x) = \infty$$

for almost every x , and therefore

$$\limsup_{m \rightarrow \infty} |T_m(F^t, H^s)(x)| = \infty \quad \text{for almost every } x \in M.$$

This contradicts the hypothesis that $T_m(f, h)(x)$ converges almost everywhere for every $(f, h) \in L^p(M) \times L^q(M)$. Hence the weak-type inequality (2) must hold. $\square$

Under the structural and exponent hypotheses of Theorem 1, its weak-type conclusion remains valid if the almost-everywhere convergence assumption is replaced by

$$(19) \quad \mu(\{x : T^*(f, h)(x) < \infty\}) > 0 \quad \text{for every } (f, h) \in L^p(M) \times L^q(M).$$

Equivalently, one may assume

$$\mu(\{x : \limsup_{m \rightarrow \infty} |T_m(f, h)(x)| < \infty\}) > 0 \quad \text{for every } (f, h) \in L^p(M) \times L^q(M).$$

Indeed, the proof above shows that failure of the weak-type estimate produces one pair with $T^*(f, h) = \infty$ almost everywhere, contradicting the first assumption. The two assumptions are equivalent because, outside a common null set, every individual $T_m(f, h)(x)$ is finite: a sequence of finite numbers is bounded in absolute value if and only if the limsup of its absolute values is finite. This is the consequence used in Theorem 2.

4. THE EXTENSION TO BOREL MEASURES

In this section, we focus on the case where $p = q = 1$ and $r = 1/2$ in which we extend Theorem 1 to Borel measures. This extension could be helpful in certain applications; see Section 6.

Throughout this section, let $(M, d\mu)$ be a compact metrizable abelian group equipped with Haar measure μ . We denote by $\mathcal{C}(M)$ the space of continuous functions on M , equipped with the supremum norm.

For each $m \geq 1$, let

$$T_m : L^1(M) \times L^1(M) \longrightarrow \mathcal{C}(M)$$

be a bounded bilinear operator. We assume that T_m is represented by a specified bounded Borel kernel

$$K_m : M \times M \longrightarrow \mathbb{C}$$

in the sense that

$$T_m(f, h)(x) = \iint_{M \times M} K_m(x - y, x - z) f(y) h(z) d\mu(y) d\mu(z), \quad f, h \in L^1(M).$$

Here K_m denotes a fixed pointwise-defined representative, rather than merely an element of $L^\infty(M \times M)$.

For finite complex Borel measures ν_1, ν_2 on M , we define

$$T_m(\nu_1, \nu_2)(x) = \iint_{M \times M} K_m(x - y, x - z) d\nu_1(y) d\nu_2(z).$$

Since K_m is bounded and Borel measurable, this integral is well-defined and satisfies

$$|T_m(\nu_1, \nu_2)(x)| \leq \|K_m\|_\infty \|\nu_1\| \|\nu_2\|,$$

where $\|\nu_1\|$ and $\|\nu_2\|$ denote the total variations of ν_1 and ν_2 , respectively. This definition agrees with the original operator whenever

$$d\nu_1 = f d\mu, \quad d\nu_2 = h d\mu.$$

We use the same notation T_m for this extension.

We additionally assume that, for each $m \geq 1$, the kernel K_m admits a finite tensor-product decomposition

$$K_m(a, b) = \sum_{j=1}^{J_m} A_{mj}(a) B_{mj}(b), \quad (a, b) \in M \times M,$$

where $J_m < \infty$ and

$$A_{mj}, B_{mj} \in L^\infty(M)$$

are bounded measurable functions. Then we have the following.

Theorem 2. *Suppose that for every $f, g \in L^1(M)$, we have*

$$\limsup_{m \rightarrow \infty} |T_m(f, g)(x)| < \infty$$

on a set of positive measure (which may depend on f, g).

Then there exists a constant $A > 0$ such that for all bounded Borel measures ν_1, ν_2 on M , and all $\alpha > 0$,

$$\mu(\{x \in M : T^*(\nu_1, \nu_2)(x) > \alpha\}) \leq \frac{A}{\sqrt{\alpha}} \|\nu_1\|^{1/2} \|\nu_2\|^{1/2}.$$

We begin with a useful lemma.

Lemma 2.1. *Let $T_1, \dots, T_N$ be a finite collection of operators satisfying the conditions above. Let ν_1, ν_2 be finite Borel measures with finite total variation $\|\nu_1\|, \|\nu_2\|$ and set $h_m(x) := T_m(\nu_1, \nu_2)(x)$. Then there exist sequences $\{f_k\}, \{g_k\}$ where $f_k, g_k \in L^1(M)$ with $\|f_k\|_{L^1} \leq \|\nu_1\|$ and $\|g_k\|_{L^1} \leq \|\nu_2\|$ such that for all $m = 1, \dots, N$ we have*

$$\lim_{k \rightarrow \infty} T_m(f_k, g_k)(x) \rightarrow h_m(x)$$

for almost all $x \in M$.

Proof. Let

$$f_k = \phi_k * \nu_1, \quad g_k = \phi_k * \nu_2,$$

where $\{\phi_k\}$ is an approximate identity on M . As established above,

$$\|f_k\|_{L^1(M)} \leq \|\nu_1\|, \quad \|g_k\|_{L^1(M)} \leq \|\nu_2\|.$$

Fix $m = 1, \dots, N$. By hypothesis, the kernel of T_m has the form

$$K_m(a, b) = \sum_{j=1}^{J_m} A_{mj}(a) B_{mj}(b),$$

where A_{mj} and B_{mj} are bounded measurable functions on M . Define

$$a_{mj}(x) = \int_M A_{mj}(x - u) d\nu_1(u)$$

and

$$b_{mj}(x) = \int_M B_{mj}(x - v) d\nu_2(v).$$

Then $a_{mj}, b_{mj} \in L^\infty(M)$, with

$$\|a_{mj}\|_{L^\infty(M)} \leq \|A_{mj}\|_{L^\infty(M)} \|\nu_1\|, \quad \|b_{mj}\|_{L^\infty(M)} \leq \|B_{mj}\|_{L^\infty(M)} \|\nu_2\|.$$

Moreover, by the definition of the extension of T_m to finite Borel measures, we have

$$T_m(\nu_1, \nu_2)(x) = \sum_{j=1}^{J_m} a_{mj}(x) b_{mj}(x).$$

On the other hand, using Fubini's theorem and the definition of $f_k = \phi_k * \nu_1$, we have

$$\begin{aligned} \int_M A_{mj}(x - y) f_k(y) d\mu(y) &= \int_M \int_M A_{mj}(x - y) \phi_k(y - u) d\nu_1(u) d\mu(y) \\ &= (\phi_k * a_{mj})(x). \end{aligned}$$

Similarly,

$$\int_M B_{mj}(x - z) g_k(z) d\mu(z) = (\phi_k * b_{mj})(x).$$

Consequently, we have

$$(20) \quad T_m(f_k, g_k)(x) = \sum_{j=1}^{J_m} (\phi_k * a_{mj})(x) (\phi_k * b_{mj})(x).$$

Since $a_{mj}, b_{mj} \in L^1(M)$ and $\{\phi_k\}$ is an approximate identity,

$$\phi_k * a_{mj} \longrightarrow a_{mj} \quad \text{in } L^1(M),$$

and

$$\phi_k * b_{mj} \longrightarrow b_{mj} \quad \text{in } L^1(M).$$

For the finitely many functions

$$\{a_{mj}, b_{mj} : 1 \leq m \leq N, 1 \leq j \leq J_m\},$$

we may therefore choose a single subsequence $\{k_i\}_{i=1}^\infty$ such that, simultaneously for every $m = 1, \dots, N$ and every $j = 1, \dots, J_m$,

$$(\phi_{k_i} * a_{mj})(x) \longrightarrow a_{mj}(x)$$

and

$$(\phi_{k_i} * b_{mj})(x) \longrightarrow b_{mj}(x)$$

for almost every $x \in M$.

Since all the factors are uniformly bounded, it follows from (20) that, for almost every $x \in M$,

$$T_m(f_{k_i}, g_{k_i})(x) \longrightarrow \sum_{j=1}^{J_m} a_{mj}(x) b_{mj}(x) = T_m(\nu_1, \nu_2)(x)$$

simultaneously as k_i tends to infinity for every $m = 1, \dots, N$.

Thus, after passing to one common subsequence,

$$T_m(f_{k_i}, g_{k_i})(x) \longrightarrow T_m(\nu_1, \nu_2)(x) \quad \text{for almost every } x \in M$$

for all $m = 1, \dots, N$. This completes the proof. $\square$

Proof of Theorem 2. The kernel representation makes each T_m commute with simultaneous translations. Since $\mu(M) = 1$ and $T_m : L^1(M) \times L^1(M) \rightarrow \mathcal{C}(M)$ is bounded, it is also bounded into $L^{1/2}(M)$. Then condition (19) therefore applies with $p = q = 1$ and $r = 1/2$. Consequently, there exists $A < \infty$ such that

$$\mu(\{x : T^*(f, g)(x) > \alpha\}) \leq \frac{A}{\sqrt{\alpha}} \|f\|_{L^1}^{1/2} \|g\|_{L^1}^{1/2}, \quad f, g \in L^1(M), \quad \alpha > 0.$$

Fix finite complex Borel measures ν_1, ν_2 , an integer $N \geq 1$, and $\alpha > 0$. If either measure has zero total variation, the desired estimate is immediate. By Lemma 2.1, using the pointwise finite-tensor kernel hypothesis, choose sequences $f_k, g_k \in L^1(M)$ such that

$$\|f_k\|_{L^1} \leq \|\nu_1\|, \quad \|g_k\|_{L^1} \leq \|\nu_2\|,$$

and, simultaneously for $1 \leq m \leq N$,

$$T_m(f_k, g_k)(x) \longrightarrow T_m(\nu_1, \nu_2)(x) \quad \text{for almost every } x \in M.$$

These sequences may depend on N ; the constant A does not. Writing

$$T_N^*(u, v) = \max_{1 \leq m \leq N} |T_m(u, v)|,$$

finiteness of this maximum implies

$$T_N^*(f_k, g_k)(x) \longrightarrow T_N^*(\nu_1, \nu_2)(x) \quad \text{a.e.}$$

Hence

$$\mathbf{1}_{\{T_N^*(\nu_1, \nu_2) > \alpha\}} \leq \liminf_{k \rightarrow \infty} \mathbf{1}_{\{T_N^*(f_k, g_k) > \alpha\}} \quad \text{a.e.}$$

Fatou's lemma and the preceding weak-type estimate give

$$\begin{aligned} \mu\{T_N^*(\nu_1, \nu_2) > \alpha\} &\leq \liminf_{k \rightarrow \infty} \mu(\{T_N^*(f_k, g_k) > \alpha\}) \\ &\leq \liminf_{k \rightarrow \infty} \mu(\{T^*(f_k, g_k) > \alpha\}) \\ &\leq \frac{A}{\sqrt{\alpha}} \|\nu_1\|^{1/2} \|\nu_2\|^{1/2}. \end{aligned}$$

Finally, as $N \rightarrow \infty$ we obtain for any $\alpha > 0$

$$\{T_N^*(\nu_1, \nu_2) > \alpha\} \uparrow \{T^*(\nu_1, \nu_2) > \alpha\}.$$

Continuity of the measure therefore yields

$$\mu(\{T^*(\nu_1, \nu_2) > \alpha\}) \leq \frac{A}{\sqrt{\alpha}} \|\nu_1\|^{1/2} \|\nu_2\|^{1/2},$$

as required. □

5. THE EXTENSION TO POSITIVE OPERATORS

Sawyer [25] removed the restriction $p \leq 2$ in Stein's theorem [26] under the assumption of positivity on the linear operators. Theorem 3 below extends Sawyer's result to positive bilinear operators which commute with an ergodic family of measure-preserving transformations; precisely, we prove that for such operators a.e. finiteness implies a weak-type estimate from $L^p \times L^q$ to weak L^r in the range $1 \leq p, q < \infty$ where $1/r = 1/p + 1/q$. The proof uses a delicate inductive construction, analogous to that in [25], and relates to Stein's probabilistic argument described in the proof of Theorem 1.

Let $(X, \mathcal{B}, \mu)$ be a finite measure space normalized so that $\mu(X) = 1$.

Definition 2 (Continuity-in-measure). *Let*

$$\mathcal{B}_m : L^p(X) \times L^q(X) \rightarrow \{\text{measurable functions on } X\}$$

be a sequence of bilinear operators. We say that each $\mathcal{B}_m$ is continuous-in-measure if the following holds: whenever

$$f_k \rightarrow f \quad \text{in } L^p(X), \quad g_k \rightarrow g \quad \text{in } L^q(X),$$

we have, for every fixed m ,

$$\mathcal{B}_m(f_k, g_k)(x) \longrightarrow \mathcal{B}_m(f, g)(x) \quad \text{in measure.}$$

Equivalently, for every $\varepsilon > 0$,

$$\mu\left(x : |\mathcal{B}_m(f_k, g_k)(x) - \mathcal{B}_m(f, g)(x)| > \varepsilon\right) \longrightarrow 0, \quad \text{as } k \rightarrow \infty.$$

Definition 3 (Positive bilinear operator). *A bilinear operator $\mathcal{B}$ defined on $L^p(X) \times L^q(X)$ and taking values in the set of measurable functions of another measure space is called positive if $f \geq 0$ and $h \geq 0$ a.e. imply $\mathcal{B}(f, h) \geq 0$ a.e.. Precisely, for any nonnegative f in $L^p(X)$ and nonnegative h in $L^q(X)$ there is a set of measure zero $\mathcal{E}^{\mathcal{B}(f, h)}$ such that $\mathcal{B}(f, h)(x) \geq 0$ for $x \in X \setminus \mathcal{E}^{\mathcal{B}(f, h)}$.*

Lemma 2.2. *Let $\mathcal{B}$ be a positive bilinear operator, meaning that*

$$\mathcal{B}(f, h)(x) \geq 0 \quad \text{for almost every } x \in X \quad \text{whenever } f, h \geq 0.$$

Then the following properties hold.

(i) Monotonicity. *If $f_1, f_2 \in L^p(X)$ and $h_1, h_2 \in L^q(X)$ satisfy*

$$0 \leq f_1 \leq f_2, \quad 0 \leq h_1 \leq h_2 \quad \text{a.e. on } X,$$

then

$$\mathcal{B}(f_1, h_1)(x) \leq \mathcal{B}(f_2, h_2)(x) \quad \text{for almost every } x \in X,$$

more precisely, for all

$$x \in X \setminus \left(\mathcal{E}^{\mathcal{B}(f_2 - f_1, h_1)} \cup \mathcal{E}^{\mathcal{B}(f_1, h_2 - h_1)} \cup \mathcal{E}^{\mathcal{B}(f_2 - f_1, h_2 - h_1)} \right).$$

(ii) Domination by absolute values. *For all $f \in L^p(X)$ and $h \in L^q(X)$,*

$$|\mathcal{B}(f, h)(x)| \leq \mathcal{B}(|f|, |h|)(x) \quad \text{for almost every } x \in X,$$

more precisely, for all

$$x \in X \setminus \bigcup_{\varepsilon, \delta \in \{+, -\}} \mathcal{E}^{\mathcal{B}(f^\varepsilon, h^\delta)},$$

where $f = f^+ - f^-$ and $h = h^+ - h^-$ and $f^+, f^-, h^+, h^- \geq 0$.

Proof. Proof of (i). Since $f_1 \leq f_2$ and $h_1 \leq h_2$, write

$$f_2 = f_1 + u, \quad h_2 = h_1 + v, \quad u, v \geq 0.$$

By bilinearity we have

$$\mathcal{B}(f_2, h_2) = \mathcal{B}(f_1 + u, h_1 + v) = \mathcal{B}(f_1, h_1) + \mathcal{B}(u, h_1) + \mathcal{B}(f_1, v) + \mathcal{B}(u, v).$$

By positivity, each of the additional terms is non-negative a.e.. Therefore,

$$\mathcal{B}(f_2, h_2)(x) \geq \mathcal{B}(f_1, h_1)(x) \quad \text{for all } x \notin \mathcal{E}^{\mathcal{B}(f_2 - f_1, h_1)} \cup \mathcal{E}^{\mathcal{B}(f_1, h_2 - h_1)} \cup \mathcal{E}^{\mathcal{B}(f_2 - f_1, h_2 - h_1)}.$$

This proves (i).

Proof of (ii). Write the sign decompositions

$$f = f^+ - f^-, \quad h = h^+ - h^-.$$

By bilinearity,

$$\mathcal{B}(f, h) = \mathcal{B}(f^+, h^+) - \mathcal{B}(f^+, h^-) - \mathcal{B}(f^-, h^+) + \mathcal{B}(f^-, h^-).$$

Set

$$A := \mathcal{B}(f^+, h^+) + \mathcal{B}(f^-, h^-), \quad B := \mathcal{B}(f^+, h^-) + \mathcal{B}(f^-, h^+).$$

By positivity,

$$A(x) \geq 0, \quad B(x) \geq 0,$$

for all

$$x \notin \bigcup_{\varepsilon, \delta \in \{+, -\}} \mathcal{E}^{\mathcal{B}(f^\varepsilon, h^\delta)}.$$

Then

$$\mathcal{B}(f, h) = A - B, \quad |\mathcal{B}(f, h)| \leq A + B.$$

Finally,

$$\begin{aligned} A + B &= \mathcal{B}(f^+, h^+) + \mathcal{B}(f^+, h^-) + \mathcal{B}(f^-, h^+) + \mathcal{B}(f^-, h^-) \\ &= \mathcal{B}(f^+ + f^-, h^+ + h^-) \\ &= \mathcal{B}(|f|, |h|), \end{aligned}$$

since $|f| = f^+ + f^-$ and $|h| = h^+ + h^-$.

Thus

$$|\mathcal{B}(f, h)(x)| \leq \mathcal{B}(|f|, |h|)(x)$$

outside the stated exceptional set. This completes the proof. $\square$

Definition 4 (Measure-preserving transformation). *A measurable map $\tau : X \rightarrow X$ is called measure-preserving if*

$$\mu(\tau^{-1}[A]) = \mu(A) \quad \text{for all } A \in \mathcal{B}.$$

Equivalently, τ is measure-preserving if for every integrable $h : X \rightarrow \mathbb{C}$ one has

$$\int_X h \circ \tau \, d\mu = \int_X h \, d\mu.$$

Definition 5 (Ergodic family). Let $\mathcal{T}$ be a collection of measure-preserving transformations on a measure space $(X, \mathcal{B}, \mu)$. We say that $\mathcal{T}$ is an *ergodic family* on X if whenever a measurable set $A \in \mathcal{B}$, up to a set of measure zero, satisfies

$$\tau^{-1}[A] = A \quad \text{for every } \tau \in \mathcal{T},$$

then

$$\mu(A) = 0 \quad \text{or} \quad \mu(A) = 1.$$

Definition 6 (Bilinear distributive sequence). Let

$$\mathcal{B}_m : L^p(X) \times L^q(X) \rightarrow \{\text{measurable functions on } X\}, \quad m \geq 1,$$

be a sequence of bilinear operators. Following Sawyer [25], we say that $\{\mathcal{B}_m\}_m$ is *distributive on X* if there exists an ergodic family $\mathcal{T}$ of measure-preserving transformations on X such that, for every $\tau \in \mathcal{T}$ and every $f \in L^p(X)$, $h \in L^q(X)$, the relationship holds for each m

$$\mathcal{B}_m(f, h) \circ \tau(x) = \mathcal{B}_m(f \circ \tau, h \circ \tau)(x) \quad \text{for almost all } x \in X.$$

One can show that if $\{\mathcal{B}_m\}_m$ is distributive, then the maximal operator

$$\mathcal{B}^*(f, h) := \sup_{m \geq 1} |\mathcal{B}_m(f, h)|.$$

is also distributive, i.e., it satisfies

$$\mathcal{B}^*(f, h) \circ \tau(x) = \mathcal{B}^*(f \circ \tau, h \circ \tau)(x) \quad \text{for almost all } x \in X.$$

Precisely, the preceding holds for $X \setminus \mathcal{C}^{\mathcal{B}^*(f, h) \circ \tau}$, where $\mathcal{C}^{\mathcal{B}^*(f, h) \circ \tau}$ has measure zero.

After introducing all the necessary definitions, we are now ready to state the theorem.

Theorem 3. *Let $1 \leq p, q \leq \infty$ and $\frac{1}{2} \leq r \leq \infty$, such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Let $\mathcal{B}_m$ be a distributive sequence with the ergodic family $\mathcal{T}$ of continuous-in-measure positive bilinear transformations of $L^p(X) \times L^q(X)$ to measurable functions on X . Suppose that for every $f \in L^p(X)$ and $h \in L^q(X)$ we have*

$$\mathcal{B}^*(f, h)(x) < \infty$$

for almost every $x \in X$.

Then, there exists a constant $C > 0$ such that for all $f \in L^p(X)$, $h \in L^q(X)$, and all $\alpha > 0$, for $r < \infty$,

$$(21) \quad \mu(\{x \in X : \mathcal{B}^*(f, h)(x) > \alpha\}) \leq \frac{C}{\alpha^r} \|f\|_{L^p(X)}^r \|h\|_{L^q(X)}^r.$$

Additionally, if $p = q = \infty$, there exists C_∞ such that

$$\|\mathcal{B}^*(f, h)\|_{L^\infty(X)} \leq C_\infty \|f\|_{L^\infty(X)} \|h\|_{L^\infty(X)}.$$

To prove Theorem 3 we need the following two lemmas.

Lemma 3.1 (cf. Sawyer [25]). *Let $\mathcal{T}$ be an ergodic family of measure-preserving transformations on a probability space $(X, \mathcal{B}, \mu)$, and let $\mathcal{S}$ denote the semigroup generated by $\mathcal{T}$ and the identity transformation. Then for every $A, B \in \mathcal{B}$ and every $\theta > 1$, there exists $\sigma \in \mathcal{S}$ such that*

$$\mu(B \cap \sigma^{-1}(A)) \leq \theta \mu(A) \mu(B).$$

Lemma 3.2. *Let $\mathcal{S}$ be the semigroup generated by the ergodic family $\mathcal{T}$ and the identity. If $\sum_n \mu(A_n) = \infty$, then there exist transformations $\tau_n \in \mathcal{S}$ such that*

$$\mu\left(\limsup_{n \rightarrow \infty} \tau_n^{-1}(A_n)\right) = 1.$$

Equivalently,

$$\mu\left(\bigcup_{n \geq M} \tau_n^{-1}(A_n)\right) = 1 \quad \text{for every } M \geq 1.$$

Proof. Since

$$\sum_{n=1}^{\infty} \mu(A_n) = \infty,$$

every tail also diverges:

$$\sum_{n=N}^{\infty} \mu(A_n) = \infty \quad \text{for every } N \geq 1.$$

We first partition the positive integers into successive finite blocks

$$J_j = \{N_{j-1} + 1, \dots, N_j\}, \quad j \geq 1,$$

with $N_0 = 0$, so that

$$(22) \quad \prod_{n \in J_j} \mu(A_n^c) \leq 2^{-j-1}.$$

Let us explain why this is possible. Since

$$\mu(A_n^c) = 1 - \mu(A_n)$$

and $1 - t \leq e^{-t}$ for $0 \leq t \leq 1$, we have

$$\prod_{n=p}^q \mu(A_n^c) = \prod_{n=p}^q (1 - \mu(A_n)) \leq \exp \left(- \sum_{n=p}^q \mu(A_n) \right).$$

Now suppose that N_{j-1} has already been chosen. Since

$$\sum_{n=N_{j-1}+1}^{\infty} \mu(A_n) = \infty,$$

we can choose $N_j > N_{j-1}$ sufficiently large that

$$\sum_{n=N_{j-1}+1}^{N_j} \mu(A_n) \geq (j+1) \log 2.$$

Consequently,

$$\prod_{n=N_{j-1}+1}^{N_j} \mu(A_n^c) \leq e^{-(j+1) \log 2} = 2^{-j-1},$$

which proves (22). Write

$$J_j = \{n_1, \dots, n_{L_j}\}, \quad L_j = |J_j|.$$

For $n \in J_j$, choose

$$\theta_n := 2^{1/L_j} > 1.$$

Then

$$\prod_{n \in J_j} \theta_n = 2.$$

We choose the transformations τ_n successively after choosing θ_n . For the first index n_1 , apply Lemma 3.1 with

$$B = X, \quad A = A_{n_1}^c.$$

This gives a transformation τ_{n_1} such that

$$\mu(\tau_{n_1}^{-1}(A_{n_1}^c)) \leq \theta_{n_1} \mu(A_{n_1}^c).$$

Suppose now that $\tau_{n_1}, \dots, \tau_{n_{\ell-1}}$ have been chosen. Set

$$B_{\ell-1} := \bigcap_{r=1}^{\ell-1} \tau_{n_r}^{-1}(A_{n_r}^c).$$

Applying Lemma 3.1 with

$$B = B_{\ell-1}, \quad A = A_{n_\ell}^c, \quad \theta = \theta_{n_\ell},$$

we may choose τ_{n_ℓ} such that

$$\mu(B_{\ell-1} \cap \tau_{n_\ell}^{-1}(A_{n_\ell}^c)) \leq \theta_{n_\ell} \mu(A_{n_\ell}^c) \mu(B_{\ell-1}).$$

Iterating this estimate throughout the block gives

$$\mu \left(\bigcap_{n \in J_j} \tau_n^{-1}(A_n^c) \right) \leq \left(\prod_{n \in J_j} \theta_n \right) \prod_{n \in J_j} \mu(A_n^c).$$

By the choice of the block and of the numbers θ_n ,

$$\mu \left(\bigcap_{n \in J_j} \tau_n^{-1}(A_n^c) \right) \leq 2 \cdot 2^{-j-1} = 2^{-j}.$$

Define

$$F_j := \bigcap_{n \in J_j} \tau_n^{-1}(A_n^c).$$

and notice that

$$\sum_{j=1}^{\infty} \mu(F_j) \leq \sum_{j=1}^{\infty} 2^{-j} < \infty.$$

Therefore, by the first Borel–Cantelli lemma we have

$$\mu \left(\limsup_{j \rightarrow \infty} F_j \right) = 0.$$

Hence, for almost every $x \in X$, there exists an index $j_0(x)$ such that

$$x \notin F_j \quad \text{for every } j \geq j_0(x).$$

Since

$$x \notin F_j = \bigcap_{n \in J_j} \tau_n^{-1}(A_n^c)$$

there exists some $n \in J_j$ for which

$$x \in \tau_n^{-1}(A_n).$$

We conclude that, for almost every x , there is at least one such n in every sufficiently large block J_j . In particular,

$$x \in \tau_n^{-1}(A_n)$$

for infinitely many n .

It follows that for every $M \geq 1$,

$$\mu \left(\bigcup_{n \geq M} \tau_n^{-1}(A_n) \right) = 1.$$

Equivalently,

$$\tau_n(x) \in A_n$$

for infinitely many n , for almost every $x \in X$. $\square$

Proof of Theorem 3. Let $\mathcal{S}$ denote the semigroup generated by $\mathcal{T}$ and the identity transformation, as in Lemma 3.2. Since every $\mathcal{B}_m$ commutes with every transformation in $\mathcal{T}$, the commutation relation also holds for every transformation in $\mathcal{S}$.

We first consider the case $p, q < \infty$. Arguing as in the proof of Theorem 1, suppose, towards a contradiction, that the weak-type estimate fails. Then there exist sequences

$$\{\tilde{f}_n\}_{n \geq 1} \subset L^p(X), \quad \{\tilde{h}_n\}_{n \geq 1} \subset L^q(X),$$

and positive numbers $R_n \rightarrow \infty$ such that

$$\sum_{n=1}^{\infty} \mu(\{x \in X : \mathcal{B}^*(\tilde{f}_n, \tilde{h}_n)(x) > 1\}) = \infty,$$

while

$$\sum_{n=1}^{\infty} \|R_n \tilde{f}_n\|_{L^p}^p < \infty, \quad \sum_{n=1}^{\infty} \|R_n \tilde{h}_n\|_{L^q}^q < \infty.$$

Set

$$f_n = R_n \tilde{f}_n, \quad h_n = R_n \tilde{h}_n.$$

Since each $\mathcal{B}_m$ is bilinear,

$$\mathcal{B}^*(f_n, h_n) = R_n^2 \mathcal{B}^*(\tilde{f}_n, \tilde{h}_n),$$

and therefore

$$\sum_{n=1}^{\infty} \mu(\{x \in X : \mathcal{B}^*(f_n, h_n)(x) > R_n^2\}) = \infty.$$

Moreover,

$$\sum_{n=1}^{\infty} \|f_n\|_{L^p}^p < \infty, \quad \sum_{n=1}^{\infty} \|h_n\|_{L^q}^q < \infty.$$

Since each $\mathcal{B}_m$ is positive, Lemma 2.2 gives

$$|\mathcal{B}_m(f, h)(x)| \leq \mathcal{B}_m(|f|, |h|)(x)$$

for almost every $x \in X$. Since the family $\{\mathcal{B}_m\}_{m \geq 1}$ is countable, taking the supremum over m yields

$$\mathcal{B}^*(f, h)(x) \leq \mathcal{B}^*(|f|, |h|)(x)$$

for almost every $x \in X$.

Hence, replacing f_n by $|f_n|$ and h_n by $|h_n|$, if necessary, we may assume that

$$f_n, h_n \geq 0 \quad \text{for every } n.$$

Define

$$A_n := \{x \in X : \mathcal{B}^*(f_n, h_n)(x) > R_n^2\}.$$

Then

$$\sum_{n=1}^{\infty} \mu(A_n) = \infty, \quad \sum_{n=1}^{\infty} \|f_n\|_{L^p}^p < \infty, \quad \sum_{n=1}^{\infty} \|h_n\|_{L^q}^q < \infty.$$

Apply Lemma 3.2 to the sequence $\{A_n\}$. Since $\sum_n \mu(A_n) = \infty$, there exists a sequence

$$\{\tau_n\}_{n \geq 1} \subset \mathcal{S}$$

such that

$$x \in \tau_n^{-1}(A_n) \quad \text{for infinitely many } n$$

for almost every $x \in X$. Equivalently,

$$\mathcal{B}^*(f_n, h_n)(\tau_n(x)) > R_n^2 \quad \text{for infinitely many } n$$

for almost every $x \in X$. Since $R_n \rightarrow \infty$, it follows that

$$(23) \quad \sup_{n \geq 1} (\mathcal{B}^*(f_n, h_n) \circ \tau_n)(x) = \infty \quad \text{for almost every } x \in X.$$

Because each τ_n is measure preserving,

$$\|f_n \circ \tau_n\|_{L^p}^p = \|f_n\|_{L^p}^p, \quad \|h_n \circ \tau_n\|_{L^q}^q = \|h_n\|_{L^q}^q.$$

Furthermore, by the simultaneous commutativity hypothesis, for every fixed n and every m ,

$$\mathcal{B}_m(f_n \circ \tau_n, h_n \circ \tau_n) = \mathcal{B}_m(f_n, h_n) \circ \tau_n$$

almost everywhere. Taking absolute values and then the supremum over m , and using the fact that the family $\{\mathcal{B}_m\}$ is countable, we obtain

$$\mathcal{B}^*(f_n \circ \tau_n, h_n \circ \tau_n) = \mathcal{B}^*(f_n, h_n) \circ \tau_n$$

almost everywhere. Taking a further countable union of the exceptional sets over n , (23) therefore implies

$$(24) \quad \sup_{n \geq 1} \mathcal{B}^*(f_n \circ \tau_n, h_n \circ \tau_n)(x) = \infty \quad \text{for almost every } x \in X.$$

Now set

$$g_n := f_n \circ \tau_n, \quad k_n := h_n \circ \tau_n.$$

Then $g_n, k_n \geq 0$ and

$$\sum_{n=1}^{\infty} \|g_n\|_{L^p}^p < \infty, \quad \sum_{n=1}^{\infty} \|k_n\|_{L^q}^q < \infty.$$

For $x \in X$ define

$$F(x) := \left(\sum_{n=1}^{\infty} g_n(x)^p \right)^{1/p}, \quad H(x) := \left(\sum_{n=1}^{\infty} k_n(x)^q \right)^{1/q}.$$

By Tonelli's theorem we have

$$\|F\|_{L^p}^p = \sum_{n=1}^{\infty} \|g_n\|_{L^p}^p < \infty,$$

and similarly

$$\|H\|_{L^q}^q = \sum_{n=1}^{\infty} \|k_n\|_{L^q}^q < \infty.$$

Thus $F \in L^p(X)$ and $H \in L^q(X)$.

For every n ,

$$0 \leq g_n \leq F, \quad 0 \leq k_n \leq H$$

almost everywhere. Since $\mathcal{B}_m$ is positive and bilinear, we have

$$\mathcal{B}_m(F, H) - \mathcal{B}_m(g_n, k_n) = \mathcal{B}_m(F - g_n, H) + \mathcal{B}_m(g_n, H - k_n) \geq 0$$

almost everywhere. Hence, for every fixed n and every m ,

$$\mathcal{B}_m(F, H) \geq \mathcal{B}_m(g_n, k_n) \geq 0$$

almost everywhere.

After removing one null set obtained by taking the countable union over m and n , we therefore have

$$\begin{aligned} \mathcal{B}^*(F, H)(x) &= \sup_m |\mathcal{B}_m(F, H)(x)| \\ &= \sup_m \mathcal{B}_m(F, H)(x) \\ &\geq \sup_m \mathcal{B}_m(g_n, k_n)(x) \\ &= \mathcal{B}^*(g_n, k_n)(x) \end{aligned}$$

for every n for almost every $x \in X$. Taking the supremum over n gives

$$\mathcal{B}^*(F, H)(x) \geq \sup_{n \geq 1} \mathcal{B}^*(g_n, k_n)(x)$$

for almost every $x \in X$. By (24),

$$\mathcal{B}^*(F, H)(x) = \infty \quad \text{for almost every } x \in X.$$

This contradicts the hypothesis that

$$\mathcal{B}^*(F, H)(x) < \infty \quad \text{for almost every } x \in X$$

whenever $F \in L^p(X)$ and $H \in L^q(X)$. Therefore $\mathcal{B}^*$ satisfies the weak-type (p, q, r) estimate (21) whenever $p, q < \infty$.

We next consider the endpoint case $p = \infty$ and $q < \infty$. Define

$$L_m h := \mathcal{B}_m(1, h), \quad L^* h := \sup_m |L_m h|.$$

Then each L_m is a positive linear operator, and

$$L^* h = \mathcal{B}^*(1, h) < \infty \quad \text{almost everywhere}$$

for every $h \in L^q(X)$. Moreover, the commutativity with translations for $\mathcal{B}_m$ implies the corresponding relation for L_m .

We claim that L^* is of weak type (q, q) . Indeed, if this were false, the one-input version of the preceding bad-sequence argument would give nonnegative functions $v_n \in L^q(X)$ and numbers $W_n \rightarrow \infty$ such that

$$\sum_{n=1}^{\infty} \|v_n\|_{L^q}^q < \infty$$

and

$$\sum_{n=1}^{\infty} \mu(\{x \in X : L^* v_n(x) > W_n\}) = \infty.$$

Applying Lemma 3.2 to these level sets, we can choose $\tau_n \in \mathcal{S}$ such that, by commutativity,

$$\sup_{n \geq 1} L^*(v_n \circ \tau_n)(x) = \infty \quad \text{for almost every } x \in X.$$

Set

$$V(x) = \left(\sum_{n=1}^{\infty} (v_n \circ \tau_n(x))^q \right)^{1/q}.$$

Then $V \in L^q(X)$ and

$$0 \leq v_n \circ \tau_n \leq V.$$

By positivity,

$$L_m V \geq L_m(v_n \circ \tau_n)$$

for every m and n , outside a null set. Consequently,

$$L^* V \geq \sup_{n \geq 1} L^*(v_n \circ \tau_n) = \infty$$

almost everywhere, contradicting the almost-everywhere finiteness of $L^* V$. Thus L^* is of weak type (q, q) .

Now, for $f \in L^\infty(X)$ and $h \in L^q(X)$, Lemma 2.2 and positivity give

$$\begin{aligned} |\mathcal{B}_m(f, h)| &\leq \mathcal{B}_m(|f|, |h|) \\ &\leq \|f\|_{L^\infty} \mathcal{B}_m(1, |h|) \\ &= \|f\|_{L^\infty} L_m(|h|). \end{aligned}$$

Taking the supremum over m ,

$$\mathcal{B}^*(f, h) \leq \|f\|_{L^\infty} L^*(|h|).$$

The weak-type (q, q) estimate for L^* therefore yields the desired weak-type estimate for $\mathcal{B}^*$ when $p = \infty$ and $q < \infty$.

The case $p < \infty$ and $q = \infty$ is symmetric, using

$$\tilde{L}_m f := \mathcal{B}_m(f, 1).$$

Finally, suppose $p = q = \infty$. Set

$$V(x) := \mathcal{B}^*(1, 1)(x).$$

By hypothesis, $V < \infty$ almost everywhere. For every $\tau \in \mathcal{T}$, the commutation relation gives

$$V \circ \tau = V \quad \text{almost everywhere.}$$

Hence V is an invariant measurable function. By ergodicity, V is constant almost everywhere; say

$$V(x) = c \quad \text{for almost every } x \in X,$$

where $c < \infty$.

For arbitrary $f, h \in L^\infty(X)$, positivity gives

$$\begin{aligned} \mathcal{B}^*(f, h) &\leq \mathcal{B}^*(|f|, |h|) \\ &\leq \|f\|_{L^\infty} \|h\|_{L^\infty} \mathcal{B}^*(1, 1) \\ &= c \|f\|_{L^\infty} \|h\|_{L^\infty} \end{aligned}$$

almost everywhere. This proves the desired estimate in the case $p = q = \infty$.

Therefore $\mathcal{B}^*$ satisfies the weak-type (p, q, r) condition (21) in all the stated cases, completing the proof. $\square$

6. AN APPLICATION CONCERNING MAXIMAL BILINEAR AVERAGES

Let $(X, \mathcal{B}, \mu)$ be a finite measure space and let $\tau : X \rightarrow X$ be a measure-preserving transformation. Then the quadruple $(X, \mathcal{B}, \mu, \tau)$ forms a measure-preserving dynamical system. In this section we study the family of bilinear averages

$$\mathcal{B}_m(f, h)(x) := \frac{f(\tau^m x) h(\tau^{2m} x)}{|m| + 1}, \quad m \in \mathbb{N}_0 := \{0, 1, 2, \dots\},$$

defined for $(f, h) \in L^p(X) \times L^q(X)$, and the associated maximal operator

$$\mathcal{B}^*(f, h)(x) := \sup_{m \in \mathbb{Z}} |\mathcal{B}_m(f, h)(x)|, \quad x \in X.$$

This operator is commonly referred to as the *bilinear tail maximal operator* and was introduced and studied by Assani and Buczolic [2]. We recall the following result.

Theorem 4 (Assani–Buczolic [2]). *Let $(X, \mathcal{B}, \mu, \tau)$ be a measure-preserving dynamical system on a probability space X . If $1 \leq p, q \leq \infty$ satisfy*

$$\frac{1}{p} + \frac{1}{q} < 2,$$

then the maximal operator

$$\mathcal{B}^*(f, h)(x) = \sup_{m \in \mathbb{N}_0} \left| \frac{f(\tau^m x) h(\tau^{2m} x)}{|m| + 1} \right|$$

maps $L^p(X) \times L^q(X)$ into $L^r(X)$ for every exponent

$$0 < r < \frac{1}{2}.$$

Assuming that τ is ergodic, we are able to obtain a substantially stronger conclusion. More precisely, we show that $\mathcal{B}^*$ satisfies a strong-type estimate at the critical exponent $r = (\frac{1}{p} + \frac{1}{q})^{-1}$. This extends the integrability range of $\mathcal{B}^*(f, h)$ beyond that obtained in [2].

The ergodicity hypothesis is mild in this context, used in our application of Theorem 3: it allows the family $\{\mathcal{B}_m\}$ to be placed within Sawyer's abstract framework for maximal operators generated by ergodic families of transformations, while all other structural features of the bilinear averages remain unchanged.

Our results allow us to improve the target space L^r in Theorem 4.

Theorem 5. *Let $(X, \mathcal{B}, \mu, \tau)$ be a measure-preserving dynamical system on a probability space X , and suppose that τ is ergodic. Let $1 \leq p, q \leq \infty$ and r satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then there exists a constant $C = C(p, q) > 0$ such that:*

If $1 < p, q \leq \infty$, $f \in L^p(X)$, $h \in L^q(X)$ then

$$(25) \quad \|\mathcal{B}^*(f, h)\|_{L^r(X)} \leq C \|f\|_{L^p(X)} \|h\|_{L^q(X)}.$$

If $p = 1$ and $1 < q \leq \infty$, then for $\alpha > 0$ we have

$$(26) \quad \mu(\{\mathcal{B}^*(f, h) > \alpha\}) \leq C \alpha^{-r} \|f\|_{L^1}^r \|h\|_{L^q}^r \quad (f, h) \in L^1 \times L^q.$$

Analogously, if $q = 1$ and $1 < p \leq \infty$, then for any $\alpha > 0$ we have

$$(27) \quad \mu(\{\mathcal{B}^*(f, h) > \alpha\}) \leq C \alpha^{-r} \|f\|_{L^p}^r \|h\|_{L^1}^r \quad (f, h) \in L^p \times L^1.$$

We obtain our theorem as a consequence of Theorem 3, but in order to apply the latter, we need the following two lemmas.

Lemma 5.1. *For each $m \geq 1$, the positive bilinear operator $\mathcal{B}_m : L^p(X) \times L^q(X) \rightarrow \mathcal{M}(X)$ is continuous-in-measure in each variable.*

Proof. Let $f_k \rightarrow f$ in $L^p(X)$ and $h_k \rightarrow h$ in $L^q(X)$. Since τ is measure preserving, composition with τ^m and τ^{2m} preserves L^p and L^q norms, hence

$$f_k \circ \tau^m \rightarrow f \circ \tau^m \quad \text{in } L^p(X), \quad h_k \circ \tau^{2m} \rightarrow h \circ \tau^{2m} \quad \text{in } L^q(X).$$

Set $r > 0$ by $1/r = 1/p + 1/q$. We express the difference as follows:

$$\mathcal{B}_m(f_k, h_k) - \mathcal{B}_m(f, h) = \frac{1}{|m| + 1} \left[(f_k - f) \circ \tau^m \cdot h_k \circ \tau^{2m} + f \circ \tau^m \cdot (h_k - h) \circ \tau^{2m} \right].$$

By Hölder's inequality,

$$\|(f_k - f) \circ \tau^m \cdot h_k \circ \tau^{2m}\|_{L^r} \leq \|f_k - f\|_{L^p} \|h_k\|_{L^q},$$

and

$$\|f \circ \tau^m \cdot (h_k - h) \circ \tau^{2m}\|_{L^r} \leq \|f\|_{L^p} \|h_k - h\|_{L^q}.$$

Since $f_k \rightarrow f$ in $L^p(X)$ and $h_k \rightarrow h$ in $L^q(X)$, the right-hand sides tend to zero as $k \rightarrow \infty$. Consequently,

$$\|\mathcal{B}_m(f_k, h_k) - \mathcal{B}_m(f, h)\|_{L^r} \rightarrow 0.$$

Finally, Chebyshev's inequality yields, for every $\varepsilon > 0$,

$$\mu\left(\left|\mathcal{B}_m(f_k, h_k) - \mathcal{B}_m(f, h)\right| > \varepsilon\right) \leq \varepsilon^{-r} \|\mathcal{B}_m(f_k, h_k) - \mathcal{B}_m(f, h)\|_{L^r}^r \rightarrow 0,$$

as $k \rightarrow \infty$, which is exactly convergence in measure.

It remains to consider the case $p = q = \infty$. In this case, $f_k \rightarrow f$ and $h_k \rightarrow h$ in $L^\infty(X)$, and hence $\{\|h_k\|_{L^\infty}\}$ is bounded. Since composition with a measure-preserving transformation preserves the L^∞ norm,

$$\begin{aligned} & \|\mathcal{B}_m(f_k, h_k) - \mathcal{B}_m(f, h)\|_{L^\infty} \\ & \leq \frac{1}{|m| + 1} \left(\|f_k - f\|_{L^\infty} \|h_k\|_{L^\infty} + \|f\|_{L^\infty} \|h_k - h\|_{L^\infty} \right) \rightarrow 0. \end{aligned}$$

Thus

$$\mathcal{B}_m(f_k, h_k) \rightarrow \mathcal{B}_m(f, h) \quad \text{in } L^\infty(X),$$

and therefore also in measure. $\square$

Lemma 5.2. *Let $\mathcal{T} = \{\tau^k : k \in \mathbb{Z}\}$. If τ is ergodic, then $\mathcal{T}$ is an ergodic family of measure-preserving transformations. Moreover, the bilinear family $\{\mathcal{B}_m\}_{m \geq 1}$ is distributive with respect to $\mathcal{T}$, in the sense that*

$$\mathcal{B}_m(f, h)(\tau^k x) = \mathcal{B}_m(f \circ \tau^k, h \circ \tau^k)(x)$$

for all integers m, k and all $x \in X$.

Proof. By definition, a family of measure-preserving transformations is ergodic if the only measurable sets invariant under every transformation in the family have measure 0 or 1.

Since invariance under all τ^k is equivalent to invariance under τ , the ergodicity of τ implies that $\mathcal{T}$ is ergodic.

The distributive identity follows from a direct computation:

$$\mathcal{B}_m(f, h)(\tau^k x) = \frac{f(\tau^{m+k} x) h(\tau^{2m+k} x)}{|m| + 1} = \mathcal{B}_m(f \circ \tau^k, h \circ \tau^k)(x).$$

Taking suprema in m yields

$$\mathcal{B}^*(f, h) \circ \tau^k = \mathcal{B}^*(f \circ \tau^k, h \circ \tau^k),$$

which completes the proof. $\square$

Proof of Theorem 5. In view of Lemmas 5.1 and 5.2 the hypotheses of Theorem 3 are satisfied. By Theorem 4, we know that for $1 \leq p, q \leq \infty$ we have

$$\mathcal{B}^*(f, h)(x) = \sup_{m \in \mathbb{N}_0} \left| \frac{f(\tau^m x) h(\tau^{2m} x)}{|m| + 1} \right|$$

maps $L^p(X) \times L^q(X)$ into $L^r(X)$ for every exponent

$$0 < r < \frac{1}{2}.$$

This implies that

$$\mathcal{B}^*(f, h) < \infty \quad \text{a.e.}$$

Applying Theorem 3, we obtain the weak-type estimate

$$(28) \quad \mu\left(\{x \in X : \mathcal{B}^*(f, h)(x) > \alpha\}\right) \leq \frac{C}{\alpha^r} \|f\|_{L^p(X)}^r \|h\|_{L^q(X)}^r,$$

with the interpretation $\|\mathcal{B}^*(f, h)\|_{L^\infty} \leq C \|f\|_{L^\infty} \|h\|_{L^\infty}$, when $r = \infty$. Thus (26) and (27) hold. Moreover, the strong type bound (25) also holds when p or q equals ∞ .

Finally we examine the region $1 < p, q < \infty$. Here the weak-type estimates (28) hold on an open convex region of exponents. Hence, by bilinear interpolation (see [11]), the weak type estimates can be upgraded to the strong-type estimate

$$\|\mathcal{B}^*(f, h)\|_{L^r} \leq C \|f\|_{L^p} \|h\|_{L^q}$$

in the same open convex region. Thus (25) is also valid. $\square$

7. AN APPLICATION CONCERNING MAXIMAL BILINEAR BOCHNER–RIESZ OPERATORS

The Bochner–Riesz means present natural ways to obtain the summability and pointwise convergence of Fourier series and Fourier integrals in higher dimensions. The study of bilinear Bochner–Riesz means has attracted attention in recent years, beginning with the works [3, 14], where $L^p \times L^q \rightarrow L^r$ boundedness is investigated.

In the bilinear setting, for $\alpha \geq 0$ and $\lambda > 0$, the bilinear Bochner–Riesz operator on $\mathbb{R}^n$ is defined by

$$\mathcal{B}_{\lambda, \mathbb{R}^n}^\alpha(f, g)(x) := \iint_{\mathbb{R}^n \times \mathbb{R}^n} (1 - \lambda^{-2}(|\xi|^2 + |\eta|^2))_+^\alpha \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta.$$

The associated Euclidean maximal bilinear Bochner–Riesz operator is

$$\mathcal{B}_{*, \mathbb{R}^n}^\alpha(f, g)(x) := \sup_{\lambda > 0} |\mathcal{B}_{\lambda, \mathbb{R}^n}^\alpha(f, g)(x)|.$$

Motivated by questions of pointwise convergence and almost everywhere control, maximal variants of bilinear Bochner–Riesz operators were subsequently introduced and studied in [6, 15, 16], where necessary and sufficient conditions for boundedness were obtained in various regimes of indices. In particular, He [6, Proposition 4.1] established necessary conditions for the boundedness of the maximal bilinear Bochner–Riesz operator, highlighting fundamental obstructions that do not appear in the linear theory.

We now turn to the periodic setting. Let f and g be trigonometric polynomials on $\mathbb{T}^n$ with Fourier coefficients $\widehat{f}(k)$ and $\widehat{g}(\ell)$. For $\alpha \geq 0$ and $m \in \mathbb{N}$, we define the bilinear Bochner–Riesz means by

$$\mathcal{B}_{m, \mathbb{T}^n}^\alpha(f, g)(x) = \sum_{k, \ell \in \mathbb{Z}^n} \left(1 - \frac{|k|^2 + |\ell|^2}{m^2}\right)_+^\alpha \widehat{f}(k) \widehat{g}(\ell) e^{2\pi i (k + \ell) \cdot x}.$$

The associated periodic maximal bilinear Bochner–Riesz operator is then given by

$$\mathcal{B}_{*, \mathbb{T}^n}^\alpha(f, g)(x) := \sup_{m \in \mathbb{N}} |\mathcal{B}_{m, \mathbb{T}^n}^\alpha(f, g)(x)|, \quad x \in \mathbb{T}^n.$$

Although $L^p \times L^q \rightarrow L^r$ boundedness for $\mathcal{B}_{*, \mathbb{T}^n}^\alpha$ is known for certain indices p, q , we are interested here in the range of indices where boundedness fails. A necessary condition for the boundedness of the maximal bilinear Bochner–Riesz operator is given below.

Proposition 6 (He [6]). *A necessary condition for the Euclidean maximal bilinear Bochner–Riesz operator $\mathcal{B}_{*, \mathbb{R}^n}^\alpha$ to be bounded from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to weak $L^r(\mathbb{R}^n)$ is that*

$$\alpha \geq \frac{2n-1}{2r} - \frac{2n-1}{2}.$$

Remark. When $r \geq 1$, Proposition 6 imposes no restriction on α , since

$$\frac{2n-1}{2} \left(\frac{1}{r} - 1 \right) \leq 0$$

and we already assume $\alpha \geq 0$. When $r < 1$, however, the weak-type estimate

$$\mathcal{B}_{*,\mathbb{R}^n}^\alpha : L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \longrightarrow L^{r,\infty}(\mathbb{R}^n)$$

can hold only if

$$\alpha \geq \frac{2n-1}{2} \left(\frac{1}{r} - 1 \right).$$

The same necessary condition holds for the periodic maximal bilinear Bochner–Riesz operator $\mathcal{B}_{*,\mathbb{T}^n}^\alpha$. Indeed, a standard rescaling and periodization argument shows that a weak-type estimate

$$\mathcal{B}_{*,\mathbb{T}^n}^\alpha : L^p(\mathbb{T}^n) \times L^q(\mathbb{T}^n) \longrightarrow L^{r,\infty}(\mathbb{T}^n)$$

would imply the corresponding weak-type estimate for $\mathcal{B}_{*,\mathbb{R}^n}^\alpha$. Consequently, Proposition 6 implies that

$$\alpha \geq \frac{2n-1}{2} \left(\frac{1}{r} - 1 \right).$$

Theorem 7. *Let $n > 1$, $1 \leq p, q < \infty$, $\alpha \geq 0$ and suppose $r = (1/p + 1/q)^{-1} < 1$. Define*

$$\mathcal{S}^\alpha := \left\{ (p, q, r) \in (1, 2] \times (1, 2] \times \left(\frac{1}{2}, 1\right) : \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \text{ and } \alpha < \frac{2n-1}{2} \left(\frac{1}{r} - 1 \right) \right\}.$$

Then if $(p, q, r) \in \mathcal{S}^\alpha$, then there exist functions $f \in L^p(\mathbb{T}^n)$ and $g \in L^q(\mathbb{T}^n)$ such that the limit

$$\lim_{m \rightarrow \infty} \mathcal{B}_{m,\mathbb{T}^n}^\alpha(f, g)(x)$$

does not exist on a set of positive measure in $\mathbb{T}^n$.

Proof. We apply Theorem 1 with $M = G = \mathbb{T}^n$. The group G acts on the space M by periodic translations. Let

$$(\tau_y f)(x) := f(x - y), \quad y \in \mathbb{T}^n.$$

We first verify the boundedness hypothesis of Theorem 1. For each fixed m , the defining sum for $\mathcal{B}_{m,\mathbb{T}^n}^\alpha$ contains only finitely many pairs $(k, \ell) \in \mathbb{Z}^n \times \mathbb{Z}^n$. Moreover,

$$|\widehat{f}(k)| \leq \|f\|_{L^1(\mathbb{T}^n)} \leq \|f\|_{L^p(\mathbb{T}^n)}$$

and similarly

$$|\widehat{g}(\ell)| \leq \|g\|_{L^1(\mathbb{T}^n)} \leq \|g\|_{L^q(\mathbb{T}^n)}.$$

Consequently, for each fixed m there exists $C_m < \infty$ such that

$$\|\mathcal{B}_{m,\mathbb{T}^n}^\alpha(f, g)\|_{L^\infty(\mathbb{T}^n)} \leq C_m \|f\|_{L^p(\mathbb{T}^n)} \|g\|_{L^q(\mathbb{T}^n)}.$$

Since $\mathbb{T}^n$ has normalized measure one, this also gives

$$\|\mathcal{B}_{m,\mathbb{T}^n}^\alpha(f, g)\|_{L^r(\mathbb{T}^n)} \leq C_m \|f\|_{L^p(\mathbb{T}^n)} \|g\|_{L^q(\mathbb{T}^n)}.$$

Thus each $\mathcal{B}_{m,\mathbb{T}^n}^\alpha$ is a bounded bilinear operator

$$L^p(\mathbb{T}^n) \times L^q(\mathbb{T}^n) \longrightarrow L^r(\mathbb{T}^n),$$

as required in Theorem 1. No uniformity of C_m in m is required.

Then the Fourier coefficients satisfy

$$\widehat{\tau_y f}(k) = e^{-2\pi i k \cdot y} \widehat{f}(k), \quad \widehat{\tau_y g}(\ell) = e^{-2\pi i \ell \cdot y} \widehat{g}(\ell).$$

A straightforward calculation yields: for $x \in \mathbb{T}^n$ we have

$$\begin{aligned}
 \mathcal{B}_{m, \mathbb{T}^n}^\alpha(\tau_y f, \tau_y g)(x) &= \sum_{k, \ell \in \mathbb{Z}^n} \left(1 - \frac{|k|^2 + |\ell|^2}{m^2}\right)_+^\alpha \widehat{\tau_y f}(k) \widehat{\tau_y g}(\ell) e^{2\pi i(k+\ell) \cdot x} \\
 &= \sum_{k, \ell \in \mathbb{Z}^n} \left(1 - \frac{|k|^2 + |\ell|^2}{m^2}\right)_+^\alpha e^{-2\pi i k \cdot y} \hat{f}(k) e^{-2\pi i \ell \cdot y} \hat{g}(\ell) e^{2\pi i(k+\ell) \cdot x} \\
 &= \sum_{k, \ell \in \mathbb{Z}^n} \left(1 - \frac{|k|^2 + |\ell|^2}{m^2}\right)_+^\alpha \hat{f}(k) \hat{g}(\ell) e^{2\pi i(k+\ell) \cdot (x-y)} \\
 &= \mathcal{B}_{m, \mathbb{T}^n}^\alpha(f, g)(x-y) \\
 &= \tau_y \mathcal{B}_{m, \mathbb{T}^n}^\alpha(f, g)(x).
 \end{aligned}$$

Hence $\mathcal{B}_{m, \mathbb{T}^n}^\alpha$ commutes with simultaneous translations, in the sense of Definition 6.

Suppose, contrary to the claim, that for every $f \in L^p(\mathbb{T}^n)$ and $g \in L^q(\mathbb{T}^n)$ the limit

$$\lim_{m \rightarrow \infty} \mathcal{B}_{m, \mathbb{T}^n}^\alpha(f, g)(x)$$

exists for almost every $x \in \mathbb{T}^n$. Under this assumption, Theorem 1 applies with $M = \mathbb{T}^n$ and yields the weak-type estimate

$$\mu(\{x \in \mathbb{T}^n : \mathcal{B}_{*, \mathbb{T}^n}^\alpha(f, g)(x) > \lambda\}) \leq \frac{C}{\lambda^r} \|f\|_{L^p(\mathbb{T}^n)}^r \|g\|_{L^q(\mathbb{T}^n)}^r, \quad \lambda > 0,$$

for all $f \in L^p(\mathbb{T}^n)$ and $g \in L^q(\mathbb{T}^n)$. Thus $\mathcal{B}_{*, \mathbb{T}^n}^\alpha$ is of weak type $(p, q; r)$ on $\mathbb{T}^n$. This contradicts the periodic necessary condition since

$$\alpha < \frac{2n-1}{2} \left(\frac{1}{r} - 1 \right).$$

Therefore, the assumption of almost everywhere convergence must be false, and there must exist $f \in L^p(\mathbb{T}^n)$ and $g \in L^q(\mathbb{T}^n)$ for which the sequence

$$\{\mathcal{B}_{m, \mathbb{T}^n}^\alpha(f, g)\}_{m \in \mathbb{N}}$$

fails to converge on a set of positive measure. □

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