ON MULTILINEAR FOURIER MULTIPLIERS OF LIMITED SMOOTHNESS

LOUKAS GRAFAKOS, AKIHIKO MIYACHI, AND NAOHITO TOMITA

Abstract. In this paper, we prove certain L^2-estimate for multilinear Fourier multiplier operators with multipliers of limited smoothness. As a result, we extend the result of Calderón and Torchinsky in the linear theory to the multilinear case. The sharpness of our results and some related estimates in Hardy spaces are also discussed.

1. Introduction

The area of multilinear harmonic analysis originated in the fundamental work of Coifman and Meyer [4, 5, 6]. This area remained unexplored until about the late nineties when certain important advances were made. All these advances are too numerous to be included in this introduction; here we only mention the articles of Bényi and Torres [1], Grafakos and Torres [12], Kenig and Stein [16], and Lerner, Ombrosi, Pérez, Torres and Trujillo-González [17]. The results contained in these and in other known articles in the area concern multilinear operators whose kernels have an explicit form or satisfy some pointwise estimates (and their derivatives also satisfy analogous pointwise estimates). In the present paper, we shall consider multilinear Fourier multiplier operators whose multipliers have limited smoothness described in terms of a function space and not in a pointwise form.

We use the following notations. For Schwartz functions f on $\mathbb{R}^d$, we define the Fourier transform by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx$$

and the inverse Fourier transform by

$$\mathcal{F}^{-1} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi)\, d\xi\,.$$

For functions m on $\mathbb{R}^d$ and $j \in \mathbb{Z}$, we define

$$m_j(\xi) = m(2^j\xi)\Psi(\xi), \tag{1.1}$$

where we fix a $\Psi \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\operatorname{supp}\Psi \subset \{\xi \in \mathbb{R}^d : 1/2 \le |\xi| \le 2\}, \qquad \sum_{k\in\mathbb{Z}} \Psi(\xi/2^k) = 1 \text{ for all } \xi \in \mathbb{R}^d \setminus \{0\}. \tag{1.2}$$

We first recall the linear Fourier multiplier operators. For $m \in L^\infty(\mathbb{R}^n)$, the linear Fourier multiplier operator T_m is defined by

$$T_m f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} m(\xi) \widehat{f}(\xi)\, d\xi$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. The Mihlin multiplier theorem says that if m satisfies the differential estimates up to the order "[half of dimension]+1",

$$|\partial_\xi^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|} \qquad \text{for all } |\alpha| \le [n/2]+1, \tag{1.3}$$

then T_m is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. The Hörmander multiplier theorem [13] states that if $s > n/2$ and $m \in L^\infty(\mathbb{R}^n)$ satisfies

$$\sup_{j\in\mathbb{Z}} \|m_j\|_{W^s(\mathbb{R}^n)} < \infty, \tag{1.4}$$

2000 *Mathematics Subject Classification.* 42B15, 42B20.
Key words and phrases. Multilinear Fourier multipliers, Hörmander multiplier theorem, Hardy spaces.
The first author acknowledges support from the NSF (DMS 0900946).

then T_m is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$, where $W^s(\mathbb{R}^n)$ is the Sobolev space (see Section 2 for the definition). The Hörmander multiplier theorem improves the Mihlin multiplier theorem since the condition (1.4) with $n/2 < s < [n/2] + 1$ is certainly weaker than (1.3). Calderón and Torchinsky [2] extended the Hörmander multiplier theorem to the case $p \le 1$; they proved that if $0 < p \le 1$ and if $m \in L^\infty(\mathbb{R}^n)$ satisfies (1.4) with $s > n(1/p - 1/2)$, then T_m is bounded on the Hardy space $H^p(\mathbb{R}^n)$. The case $p = 1$ is due to Fefferman and Stein [8].

Now we shall consider the multilinear case. Let N be an integer strictly bigger than one. For $m \in L^\infty(\mathbb{R}^{Nn})$, the N-linear Fourier multiplier operator T_m is defined by

$$T_m(f_1, \dots, f_N)(x) = \frac{1}{(2\pi)^{Nn}} \int_{\mathbb{R}^{Nn}} e^{ix\cdot(\xi_1+\cdots+\xi_N)} m(\xi) \widehat{f_1}(\xi_1) \dots \widehat{f_N}(\xi_N)\, d\xi$$

for $f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n)$, where $x \in \mathbb{R}^n$, $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ and $d\xi = d\xi_1 \dots d\xi_N$. If $\mathcal{F}^{-1}m$, the inverse Fourier transform of m on $\mathbb{R}^{Nn}$, is an integrable function, then this can also be written as

$$T_m(f_1, \dots, f_N)(x) = \int_{\mathbb{R}^{Nn}} \mathcal{F}^{-1}m(x - y_1, \dots, x - y_N) f_1(y_1) \dots f_N(y_N)\, dy. \tag{1.5}$$

This representation of T_m is often valid even when $\mathcal{F}^{-1}m$ is not an integrable function via a principal value integral interpretation.

Coifman and Meyer [5] proved that if $m \in C^L(\mathbb{R}^{Nn} \setminus \{0\})$ satisfies

$$|\partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_N}^{\alpha_N} m(\xi_1, \dots, \xi_N)| \le C_{\alpha_1,\dots,\alpha_N} (|\xi_1| + \cdots + |\xi_N|)^{-(|\alpha_1|+\cdots+|\alpha_N|)} \tag{1.6}$$

for $|\alpha_1| + \cdots + |\alpha_N| \le L$ with L sufficiently large, then T_m is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1, \dots, p_N \le \infty$ and $1 < p < \infty$ satisfying $1/p_1 + \cdots + 1/p_N = 1/p$. Kenig and Stein [16] and Grafakos and Torres [12] extended the result to the range $p \le 1$. Finding the best possible L in these results is an important question that arises in applications.

Example: It is well known that the Kato-Ponce inequality (see [15]) can be studied via multilinear analysis. Let $\widehat{D^s(f)}(\xi) = \widehat{f}(\xi)|\xi|^s$ with $s > 0$ and $1/p = 1/p_1 + \cdots + 1/p_N$. In the study of the N-linear Kato-Ponce type inequality

$$\|D^s(f_1 \cdots f_N)\|_{L^p} \le C \sum_{i=1}^{N} \|D^s(f_i)\|_{L^{p_i}} \prod_{1\le j\ne i\le N} \|f_j\|_{L^{p_j}} \tag{1.7}$$

via Littlewood-Paley theory, the following N-linear multiplier arises:

$$m(\xi) = \sum_{j\ge 0} 2^{-sj} \sum_{k\in\mathbb{Z}} \Theta(2^{-k}(\xi_1 + \cdots + \xi_N)) \Psi_1(2^{-(j+k)}\xi_1) \cdots \Psi_N(2^{-(j+k)}\xi_N)$$

where $\Psi_1, \dots, \Psi_N$, and Θ are smooth functions supported in some annulus. It is straightforward to verify that m satisfies condition (1.6) for $|\alpha_1| + \cdots + |\alpha_N| < s$ but not for larger $|\alpha_1| + \cdots + |\alpha_N|$ and thus the smoothness of m is "limited". This example plays a motivating role in the theory developed hereby and its connection with (1.7) is discussed in Appendix B.

The L given in [5] is strictly greater than $2Nn$ and this seems to be too large compared with the case of linear operators. It will be natural to expect that we can take $L =$ "[half of dimension]+1"$=$ $[Nn/2] + 1$. In fact, Tomita [21] recently proved that if $m \in L^\infty(\mathbb{R}^{Nn})$ satisfies

$$\sup_{j\in\mathbb{Z}} \|m_j\|_{W^s(\mathbb{R}^{Nn})} < \infty \tag{1.8}$$

with $s > Nn/2$, then T_m is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1, \dots, p_N, p < \infty$ satisfying $1/p_1 + \cdots + 1/p_N = 1/p$. Grafakos and Si [11] extended the result to the case $p \le 1$ by using the L^r-based Sobolev space, $1 < r \le 2$.

In the present paper, we shall consider multipliers which satisfy (1.8) with the product type Sobolev space $W^{(s_1,\dots,s_N)}(\mathbb{R}^{Nn})$ (for the definition, see Section 2) in place of $W^s(\mathbb{R}^{Nn})$. We shall prove a basic L^2-estimate and give an extension of the Calderón-Torchinsky multiplier theorem to the multilinear case. We also give extensions and improvements of the results of [21] and [11].

The following is the first main result, which gives the basic L^2-estimate.

Theorem 1.1. *Assume that* $m \in L^\infty(\mathbb{R}^{Nn})$ *satisfies*

$$\sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,\dots,s_N)}(\mathbb{R}^{Nn})} < \infty \quad \text{with } s_1,\dots,s_N > n/2.$$

Then T_m *is bounded from* $L^2(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \times \cdots \times L^\infty(\mathbb{R}^n)$ *to* $L^2(\mathbb{R}^n)$.

In the case $N = 2$, Theorem 1.1 with $W^{(s_1,s_2)}(\mathbb{R}^{2n})$ replaced by $W^s(\mathbb{R}^{2n})$ with $s > n$ follows from the result of Grafakos and Si [11]. But there is a difference between $N = 2$ and $N \geq 3$ and the argument of [11] cannot be applied to the case $N \geq 3$ (even if the product type Sobolev norm $\|\cdot\|_{W^{(s_1,\dots,s_N)}(\mathbb{R}^{Nn})}$ is replaced by the usual Sobolev norm $\|\cdot\|_{W^{s_1+\cdots+s_N}(\mathbb{R}^{Nn})}$). In the case $N = 2$, it follows from duality that the boundedness of T_m from $L^2 \times L^\infty$ to L^2 is equivalent to that of $T_{m^{*2}}$ from $L^2 \times L^2$ to L^1, where m^{*2} is the multiplier of the dual operator with respect to the second variable (see Section 8), and, by using m^{*2} instead of m, we do not need to treat L^∞. However, in the case $N = 3$, the boundedness of T_m from $L^2 \times L^\infty \times L^\infty$ to L^2 is equivalent to that of $T_{m^{*2}}$ (resp. $T_{m^{*3}}$) from $L^2 \times L^2 \times L^\infty$ to L^1 (resp. from $L^2 \times L^\infty \times L^2$ to L^1), and we cannot remove L^∞. Our proof of Theorem 1.1 do not use this duality and can be applied to all $N \geq 2$. Notice also that in the framework of Sobolev spaces of product type, we cannot use the duality argument (see Section 8).

Using Theorem 1.1, we extend the multiplier theorem of Calderón and Torchinsky [2] to the multilinear case. The following is the second main result.

Theorem 1.2. *Let* $0 < p \leq 1$. *If* $m \in L^\infty(\mathbb{R}^{Nn})$ *satisfies*

$$\sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,\dots,s_N)}} < \infty \quad \text{with } s_1 > n(1/p - 1/2),\ s_2,\dots,s_N > n/2, \tag{1.9}$$

then T_m *is bounded from* $H^p(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \times \cdots \times L^\infty(\mathbb{R}^n)$ *to* $L^p(\mathbb{R}^n)$.

We shall also prove that the numbers $n/2$ and $n(1/p - 1/2)$ in Theorems 1.1 and 1.2 are sharp; see Propositions 7.1 and 7.2. In a recent paper [18], Theorems 1.1 and 1.2 are used as key tools to determine the minimal smoothness conditions on bilinear Fourier multipliers to assure the boundedness of the corresponding operators from $H^{p_1} \times H^{p_2}$ to L^p, $0 < p_1, p_2 \leq \infty$, $1/p = 1/p_1 + 1/p_2$.

From Theorems 1.1 and 1.2, by interpolation, we also obtain the boundedness of multilinear Fourier multiplier operators in

$$H^{p_1} \times H^{p_2} \times \cdots \times H^{p_N} \to L^p, \quad 0 < p_j \leq \infty,\ 0 < p \leq 2,\ \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_N} = \frac{1}{p}.$$

The results include some extensions and improvements of the results of [21] and [11]. For details, see Theorem 6.1 and Section 8.

It should be mentioned that Grafakos and Kalton [10] considered multilinear Calderón-Zygmund operators, and proved the boundedness of the operators on Hardy spaces. The definition of Calderón-Zygmund operators in [10], however, contains pointwise estimate of kernels, whereas the kernels of the multipliers of our theorems do not have pointwise estimate in general. Hence our results do not follow from the general results in [10].

We explain some ideas of the proofs of the main theorems. In the proof of Theorem 1.1, using a partition of unity with respect to $\xi/|\xi|$ and using the usual dyadic decomposition with respect to $|\xi|$, we reduce the problem to the case that m has appropriate compact support. One of our main tools is a pointwise estimate of $T_m(f_1, f_2, \dots, f_N)(x)$ for compactly supported m, which will be given in Lemma 3.3. Another main tool is a modified version of the Carleson measure estimate related to BMO functions, which will be given in Lemmas 3.1 and 3.2. After Theorem 1.1 is established, Theorem 1.2 can be proved by a rather straightforward generalization of the method used in the case of linear Fourier multiplier operators.

The paper is organized as follows. Sections 2 and 3 contain definitions and preliminary lemmas. In Sections 4 and 5, we prove Theorems 1.1 and 1.2 respectively. In Section 6, we use interpolation to give results on the boundedness from $H^{p_1} \times H^{p_2} \times \cdots \times H^{p_N}$ to L^p with $p \leq 1$. In Section 7, sharpness of the conditions of Theorems 1.1 and 1.2 is discussed. In Section 8, we comment on some results for the case $p > 1$.

2. Preliminaries

Throughout this paper, the letter C will denote a constant which may be different in each occasion but is independent of the essential variables. The set of all non-negative integers is denoted by $\mathbb{N}_0$. For $1 \le p \le \infty$, p' is the conjugate exponent of p, that is, $1/p + 1/p' = 1$. The symbols $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ denote the Schwartz space of rapidly decreasing smooth functions and the space of tempered distributions, respectively. As usual, for a function ψ on $\mathbb{R}^n$ and $t > 0$, we write $\psi_t(x) = t^{-n}\psi(x/t)$.

The Sobolev space $W^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{W^s} = \left(\int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\widehat{f}(\xi)|^2 \, d\xi \right)^{1/2} < \infty,$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. We also use the Sobolev space of product type $W^{(s_1,\dots,s_N)}(\mathbb{R}^{Nn})$, $(s_1, \dots, s_N) \in \mathbb{R} \times \cdots \times \mathbb{R}$, which is defined by the norm

$$\|F\|_{W^{(s_1,\dots,s_N)}} = \left(\int_{\mathbb{R}^{Nn}} \langle \xi_1 \rangle^{2s_1} \dots \langle \xi_N \rangle^{2s_N} |\widehat{F}(\xi)|^2 \, d\xi \right)^{1/2},$$

where $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ and $d\xi = d\xi_1 \dots d\xi_N$. For $s \in \mathbb{R}$, we set

$$w_s(x) = \langle x_1 \rangle^s \dots \langle x_N \rangle^s, \quad x = (x_1, \dots, x_N) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n.$$

The weighted Lebesgue space $L^q(w_s)$ consists of all measurable functions F on $\mathbb{R}^{Nn}$ such that

$$\|F\|_{L^q(w_s)} = \left(\int_{\mathbb{R}^{Nn}} |F(x)|^q \langle x_1 \rangle^s \dots \langle x_N \rangle^s \, dx \right)^{1/q} < \infty.$$

We recall the definition and some properties of Hardy spaces on $\mathbb{R}^n$ (see [19, Chapter 3]). Let $0 < p \le \infty$, and let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \Phi(x)\, dx \ne 0$. Then the Hardy space $H^p(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{H^p} = \| \sup_{0<t<\infty} |\Phi_t * f| \|_{L^p} < \infty.$$

It is known that $H^p(\mathbb{R}^n)$ does not depend on the choice of the function Φ ([19, Chapter 3, Theorem 1]). If $1 < p \le \infty$, then $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ ([19, Chapter 3, Section 1.2]). For $0 < p \le 1$, a function a on $\mathbb{R}^n$ is called an H^p-atom if there exists a cube $Q = Q_a$ such that

$$\operatorname{supp} a \subset Q, \quad \|a\|_{L^\infty} \le |Q|^{-1/p}, \quad \int_{\mathbb{R}^n} x^\alpha \, a(x)\, dx = 0 \ \text{ for } \ |\alpha| \le [n(1/p - 1)],$$

where $|Q|$ is the Lebesgue measure of Q and $[n(1/p-1)]$ is the integer part of $n(1/p-1)$. It is known that every $f \in H^p(\mathbb{R}^n)$ can be written as an infinite sum $f = \sum_{i=1}^\infty \lambda_i a_i$ convergent in $\mathcal{S}'(\mathbb{R}^n)$, where $\{a_i\}$ is a collection of H^p-atoms and $\{\lambda_i\}$ is a sequence of complex numbers with $\sum_{i=1}^\infty |\lambda_i|^p < \infty$. Moreover,

$$C^{-1} \inf \left(\sum_{i=1}^\infty |\lambda_i|^p \right)^{1/p} \le \|f\|_{H^p} \le C \inf \left(\sum_{i=1}^\infty |\lambda_i|^p \right)^{1/p},$$

where the infimum is taken over all representations of f ([19, Chapter 3, Theorem 2]).

We denote by $BMO(\mathbb{R}^n)$ the space of all locally integrable functions f on $\mathbb{R}^n$ that satisfy

$$\|f\|_{BMO} = \sup \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx < \infty,$$

where f_Q is the average of f over Q and the supremum is taken over all cubes Q in $\mathbb{R}^n$. A positive measure ν on $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty)$ is said to be a Carleson measure if there exists a constant $A > 0$ such that

$$\nu(Q \times (0, \ell(Q))) \le A|Q| \quad \text{for all cubes } Q \text{ in } \mathbb{R}^n,$$

where $\ell(Q)$ is the side length of Q. The infimum of the possible values of the constant A is called the Carleson constant of ν and is denoted by $\|\nu\|$.

We end this section by quoting the following facts which will be used in the sequel.

Lemma 2.1. *Let $2 \le q < \infty$, $r > 0$ and $s \ge 0$. Then there exists a constant $C > 0$ such that*

$$\|\widehat{F}\|_{L^q(w_{sq})} \le C\|\widehat{F}\|_{L^2(w_{2s})} = C\|F\|_{W^{(s,\ldots,s)}}$$

for all $F \in W^{(s,\ldots,s)}(\mathbb{R}^{Nn})$ with $\operatorname{supp} F \subset \{|x| \le r\}$.

Lemma 2.1 is a simple case of [9, Lemma A.1], but we shall give a proof for the reader's convenience in Appendix A.

Proposition 2.2. *If $s_j > n/2$ for $1 \le j \le N$, then $W^{(s_1,\ldots,s_N)}(\mathbb{R}^{Nn})$ is an algebra under pointwise multiplication.*

The proof of Proposition 2.2 is also given in Appendix A for the reader's convenience.

3. Lemmas

In this section, we prepare the lemmas that will be used in the proof of Theorem 1.1.

Lemma 3.1. *Let $b \in BMO(\mathbb{R}^n)$, $\zeta(x) = (1+|x|)^{-(n+\epsilon)}$ with $\epsilon > 0$, and let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \psi(x)\,dx = 0$. Then the measure ν defined by*

$$d\nu = (\zeta_t * |\psi_t * b|^2)(x)\,\frac{dxdt}{t}$$

is a Carleson measure with Carleson constant $\|\nu\| \le C\|b\|_{BMO}^2$.

Proof. Since $\|b(\cdot + x_0)\|_{BMO} = \|b(r\cdot)\|_{BMO} = \|b\|_{BMO}$, and since

$$\int_{x_0+Q}\int_0^{\ell(x_0+Q)} (\zeta_t * |\psi_t * b|^2)(x)\,\frac{dtdx}{t} = \int_Q\int_0^{\ell(Q)} (\zeta_t * |\psi_t * [b(\cdot + x_0)]|^2)(x)\,\frac{dtdx}{t}$$

and

$$\begin{aligned}\int_{[-r,r]^n}\int_0^{2r} (\zeta_t * |\psi_t * b|^2)(x)\,\frac{dtdx}{t} &= r^n \int_{[-1,1]^n}\int_0^{2r} (\zeta_{t/r} * |\psi_{t/r} * [b(r\cdot)]|^2)(x)\,\frac{dtdx}{t}\\ &= 2^{-n}|[-r,r]^n| \int_{[-1,1]^n}\int_0^{2} (\zeta_t * |\psi_t * [b(r\cdot)]|^2)(x)\,\frac{dtdx}{t},\end{aligned}$$

it is enough to prove

$$\int_{[-1,1]^n}\int_0^{2} (\zeta_t * |\psi_t * b|^2)(x)\,\frac{dtdx}{t} \le C\|b\|_{BMO}^2.$$

We recall the fact that if μ is a Carleson measure then

$$\int_{\mathbb{R}^{n+1}_+} |F(x,t)|\,d\mu(x,t) \le C\|\mu\| \int_{\mathbb{R}^n} F^*(x)\,dx, \tag{3.1}$$

where F^* is the nontangential maximal function of F, which is defined by

$$F^*(x) = \sup_{|x-y|<t} |F(y,t)|$$

(see, e.g., [19, Theorem 2, p.59] or [7, Proof of Theorem 9.5]). On the other hand, it is well known that $|\psi_t * b(x)|^2 \frac{dxdt}{t}$ is a Carleson measure with Carleson constant dominated by $C\|b\|_{BMO}^2$ (see, e.g., [7, Theorem 9.6]). Hence,

$$\begin{aligned}&\int_{[-1,1]^n}\int_0^{2} (\zeta_t * |\psi_t * b|^2)(x)\,\frac{dtdx}{t}\\ &= \int_{\mathbb{R}^{n+1}_+} \chi_{[-1,1]^n\times[0,2]}(x,t)\left(\int_{\mathbb{R}^n} \zeta_t(x-y)|\psi_t * b(y)|^2\,dy\right)\frac{dtdx}{t}\\ &= \int_{\mathbb{R}^{n+1}_+} \left(\int_{\mathbb{R}^n} \zeta_t(x-y)\chi_{[-1,1]^n\times[0,2]}(x,t)\,dx\right)|\psi_t * b(y)|^2\,\frac{dtdy}{t}\\ &\le C\|b\|_{BMO}^2 \int_{\mathbb{R}^n} G^*(x)\,dx,\end{aligned}$$

where $\chi_{[-1,1]^n\times[0,2]}$ is the characteristic function of $[-1,1]^n\times[0,2]$ and $G^*(x)$ is the nontangential maximal function of

$$G(y,t)=\int_{\mathbb{R}^n}\zeta_t(z-y)\chi_{[-1,1]^n\times[0,2]}(z,t)\,dz.$$

Thus it is sufficient to show that G^* is integrable on $\mathbb{R}^n$. Note that $G(y,t)=0$ for $t>2$, and $G(y,t)\le\|\zeta\|_{L^1}$ for $y\in\mathbb{R}^n$ and $t\le 2$. Moreover, since $|y-z|\ge|y|/2$ for $|y|\ge 2\sqrt{n}$ and $z\in[-1,1]^n$, we have $G(y,t)\le 2^nt^{-n}(|y|/2t)^{-(n+\epsilon)}\le C|y|^{-(n+\epsilon)}$ for $|y|\ge 2\sqrt{n}$ and $t\le 2$. Therefore $G^*(x)\le C(1+|x|)^{-(n+\epsilon)}$, and consequently G^* is integrable on $\mathbb{R}^n$. The proof is complete. □

Lemma 3.2. *Let $1<q<2$, $\zeta(x)=(1+|x|)^{-(n+\epsilon)}$ with $\epsilon>0$, and let $\psi\in\mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n}\psi(x)\,dx=0$. Then:*

$$\int_{\mathbb{R}^{n+1}_+}|\psi_t*f(x)|^2\,\frac{dxdt}{t}\le C\|f\|_{L^2}^2, \tag{3.2}$$

$$\int_{\mathbb{R}^{n+1}_+}(\zeta_t*|\psi_t*f|^q)(x)^{2/q}\,\frac{dxdt}{t}\le C\|f\|_{L^2}^2, \tag{3.3}$$

$$\int_{\mathbb{R}^{n+1}_+}(\zeta_t*|f|^q)(x)^{2/q}(\zeta_t*|\psi_t*g|^q)(x)^{2/q}\,\frac{dxdt}{t}\le C\|f\|_{L^2}^2\|g\|_{BMO}^2, \tag{3.4}$$

$$\int_{\mathbb{R}^{n+1}_+}(\zeta_t*|f|)(x)^2(\zeta_t*|\psi_t*g|^q)(x)^{2/q}\,\frac{dxdt}{t}\le C\|f\|_{L^2}^2\|g\|_{BMO}^2. \tag{3.5}$$

Proof. The inequality (3.2) is well known and is easily proved by an application of Plancherel's theorem; see, e.g., [5, p.148].

To prove (3.3), observe that Young's inequality gives

$$\int_{\mathbb{R}^n}(\zeta_t*|\psi_t*f|^q)(x)^{2/q}dx\le(\|\zeta_t\|_{L^1}\||\psi_t*f|^q\|_{L^{2/q}})^{2/q}=C\int_{\mathbb{R}^n}|\psi_t*f(x)|^2\,dx.$$

Integrating over $0<t<\infty$ and using (3.2), we obtain (3.3).

To prove (3.4), observe that Hölder's inequality gives

$$(\zeta_t*|\psi_t*g|^q)(x)^{2/q}\le\left(\|\zeta_t\|_{L^1}^{1-q/2}(\zeta_t*|\psi_t*g|^2)(x)^{q/2}\right)^{2/q}=C(\zeta_t*|\psi_t*g|^2)(x).$$

Thus, by Lemma 3.1, the measure

$$d\nu=(\zeta_t*|\psi_t*g|^q)(x)^{2/q}\frac{dxdt}{t}$$

is a Carleson measure with Carleson constant $\le C\|g\|_{BMO}^2$. Hence, by (3.1), the left hand side of (3.4) is majorized by

$$C\|g\|_{BMO}^2\int_{\mathbb{R}^n}\sup_{|z-x|<t}((\zeta_t*|f|^q)(x))^{2/q}\,dz=U.$$

But the integrand of the above integral is majorized by $CM(|f|^q)(z)^{2/q}$ with M denoting the Hardy-Littlewood maximal operator (see, e.g., [19, Chapter 1, Section 2.1] or [7, Chapter 2, Section 8.7]). Hence, since $q<2$, the maximal theorem gives

$$U\le C\|g\|_{BMO}^2\||f|^q\|_{L^{2/q}}^{2/q}=C\|g\|_{BMO}^2\|f\|_{L^2}^2.$$

Finally, (3.5) follows from (3.4) since Hölder's inequality gives

$$(\zeta_t*|f|)(x)^2\le\left(\|\zeta_t\|_{L^1}^{1-1/q}(\zeta_t*|f|^q)(x)^{1/q}\right)^2=C(\zeta_t*|f|^q)(x)^{2/q}.$$

The proof of Lemma 3.2 is complete. □

Using the idea given in [21], we prove the following.

Lemma 3.3. *Let $s>n/2$, $\max\{1,n/s\}<q<2$ and $\zeta(x)=(1+|x|)^{-sq}$. Suppose $m\in W^{(s,\dots,s)}(\mathbb{R}^{Nn})$, $t>0$ and $\operatorname{supp} m\subset\{|\xi|\le 2/t\}$. Then there exists a constant $C>0$ depending only on N, n, s and q such that*

$$|T_m(f_1,\dots,f_N)(x)|\le C\|m(\cdot/t)\|_{W^{(s,\dots,s)}}(\zeta_t*|f_1|^q)(x)^{1/q}\dots(\zeta_t*|f_N|^q)(x)^{1/q}$$

for all $x\in\mathbb{R}^n$.

Proof. We write $m(\xi/t) = \widetilde{m}(\xi)$. By (1.5) and by Hölder's inequality,

$$\begin{aligned}
&|T_m(f_1,\dots,f_N)(x)| \\
&\le \int_{\mathbb{R}^{Nn}} t^{-Nn} |\mathcal{F}^{-1}\widetilde{m}((x-y_1)/t,\dots,(x-y_N)/t) f_1(y_1)\dots f_N(y_N)|\,dy \\
&= t^{-Nn} \int_{\mathbb{R}^{Nn}} (1+|x-y_1|/t)^s \dots (1+|x-y_N|/t)^s |\mathcal{F}^{-1}\widetilde{m}((x-y_1)/t,\dots,(x-y_N)/t)| \\
&\qquad \times (1+|x-y_1|/t)^{-s} \dots (1+|x-y_N|/t)^{-s} |f_1(y_1)\dots f_N(y_N)|\,dy \\
&\le \left(\int_{\mathbb{R}^{Nn}} (1+|z_1|)^{sq'} \dots (1+|z_N|)^{sq'} |\mathcal{F}^{-1}\widetilde{m}(z_1,\dots,z_N)|^{q'}\,dz \right)^{1/q'} \\
&\qquad \times \left(\frac{1}{t^{Nn}} \int_{\mathbb{R}^{Nn}} \frac{|f_1(y_1)\dots f_N(y_N)|^q}{(1+|x-y_1|/t)^{sq}\dots(1+|x-y_N|/t)^{sq}}\,dy \right)^{1/q} \\
&= \|\mathcal{F}^{-1}\widetilde{m}\|_{L^{q'}(w_{sq'})} (\zeta_t * |f_1|^q)(x)^{1/q} \dots (\zeta_t * |f_N|^q)(x)^{1/q}.
\end{aligned}$$

Since $\operatorname{supp}\widetilde{m} \subset \{|\xi| \le 2\}$ and $2 < q' < \infty$, Lemma 2.1 gives $\|\mathcal{F}^{-1}\widetilde{m}\|_{L^{q'}(w_{sq'})} \le C\|\widetilde{m}\|_{W^{(s,\dots,s)}}$, which, combined with the above inequality, implies the desired estimate. The proof is complete. □

Lemma 3.4. *Let $s_1,\dots,s_N > n/2$, and let $\widetilde{\Psi} \in \mathcal{S}(\mathbb{R}^{Nn})$ be such that $\operatorname{supp}\widetilde{\Psi}$ is compact and does not contain the origin. Assume that $\Phi \in C^\infty(\mathbb{R}^{Nn}\setminus\{0\})$ satisfies*

$$|\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_N}^{\alpha_N}\Phi(\xi_1,\dots,\xi_N)| \le C_{\alpha_1,\dots,\alpha_N}(|\xi_1|+\dots+|\xi_N|)^{-(|\alpha_1|+\dots+|\alpha_N|)}$$

for all $\alpha_1,\dots,\alpha_N \in \mathbb{N}_0^n$. Then there exists a constant $C>0$ such that

$$\sup_{t>0}\|m(t\cdot)\Phi(t\cdot)\widetilde{\Psi}\|_{W^{(s_1,\dots,s_N)}} \le C\sup_{j\in\mathbb{Z}}\|m_j\|_{W^{(s_1,\dots,s_N)}}$$

for all $m \in L^\infty(\mathbb{R}^{Nn})$ satisfying $\sup_{j\in\mathbb{Z}}\|m_j\|_{W^{(s_1,\dots,s_N)}} < \infty$, where m_j is defined by (1.1).

Proof. We may assume that $\operatorname{supp}\widetilde{\Psi} \subset \{1/2^{j_0} \le |\xi| \le 2^{j_0}\}$ for some $j_0 \in \mathbb{N}$. Given $t>0$, take $j \in \mathbb{Z}$ satisfying $2^{j-1} \le t < 2^j$. Then, since $1 < 2^j/t \le 2$, by a change of variables,

$$\|m(t\cdot)\Phi(t\cdot)\widetilde{\Psi}\|_{W^{(s_1,\dots,s_N)}} \le C\|m(2^j\cdot)\Phi(2^j\cdot)\widetilde{\Psi}(2^jt^{-1}\cdot)\|_{W^{(s_1,\dots,s_N)}}.$$

Let $\Psi \in \mathcal{S}(\mathbb{R}^{Nn})$ be as in (1.2) with $d = Nn$, and note that $\operatorname{supp}\Psi(\cdot/2^k) \subset \{2^{k-1} \le |\xi| \le 2^{k+1}\}$. Using $\operatorname{supp}\widetilde{\Psi}(2^jt^{-1}\cdot) \subset \{1/2^{j_0+1} \le |\xi| \le 2^{j_0}\}$, we have by Proposition 2.2

$$\begin{aligned}
&\|m(2^j\cdot)\Phi(2^j\cdot)\widetilde{\Psi}(2^jt^{-1}\cdot)\|_{W^{(s_1,\dots,s_N)}} \le \sum_{k=-(j_0+1)}^{j_0} \|m(2^j\cdot)\Phi(2^j\cdot)\widetilde{\Psi}(2^jt^{-1}\cdot)\Psi(\cdot/2^k)\|_{W^{(s_1,\dots,s_N)}} \\
&\le C\sum_{k=-(j_0+1)}^{j_0} \|m(2^j\cdot)\Psi(\cdot/2^k)\|_{W^{(s_1,\dots,s_N)}} \|\Phi(2^j\cdot)\widetilde{\Psi}(2^jt^{-1}\cdot)\|_{W^{(s_1,\dots,s_N)}} \\
&\le C\sum_{k=-(j_0+1)}^{j_0} \|m(2^{j+k}\cdot)\Psi\|_{W^{(s_1,\dots,s_N)}} \|\Phi(t\cdot)\widetilde{\Psi}\|_{W^{(s_1,\dots,s_N)}} \\
&\le C\left(\sup_{j\in\mathbb{Z}}\|m_j\|_{W^{(s_1,\dots,s_N)}}\right)\left(\sup_{t>0}\|\Phi(t\cdot)\widetilde{\Psi}\|_{W^{(s_1,\dots,s_N)}}\right).
\end{aligned}$$

Since $|\partial_\xi^\alpha\Phi(t\xi)| \le C_\alpha|\xi|^{-|\alpha|}$ and $\operatorname{supp}\widetilde{\Psi}$ does not contain the origin,

$$\sup_{t>0}\|\Phi(t\cdot)\widetilde{\Psi}\|_{W^{(s_1,\dots,s_N)}} \le C\sup_{t>0}\left(\sum_{|\alpha_1|\le[s_1]+1}\dots\sum_{|\alpha_N|\le[s_N]+1}\left\|\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_N}^{\alpha_N}\left(\Phi(t\cdot)\widetilde{\Psi}\right)\right\|_{L^2}\right) < \infty.$$

The proof is complete. □

4. The boundedness from $L^2 \times L^\infty \times \cdots \times L^\infty$ to L^2

In this section, we prove Theorem 1.1.

Proof of Theorem 1.1. We shall give a proof in which the case $N \geq 3$ and the case $N = 2$ are treated in a parallel way. (Cf. the comments given in the paragraph just below Theorem 1.1.) To provide clarity in the exposition, we give the proof only in the typical case $N = 3$. It will be obvious that our argument can be generalized to every $N \geq 2$ with trivial modifications.

If we set $s = \min\{s_1, s_2, s_3\}$, then $W^{(s_1,s_2,s_3)}(\mathbb{R}^{3n}) \hookrightarrow W^{(s,s,s)}(\mathbb{R}^{3n})$. Hence, in the proof of Theorem 1.1, it is sufficient to consider the case $s_1 = s_2 = s_3 > n/2$. Thus we assume $s > n/2$ and consider m that satisfies $\sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s,s,s)}} < \infty$. We use the following notations: $\mathcal{A}_0$ denotes the set of even functions $\varphi \in \mathcal{S}(\mathbb{R}^n)$ for which $\operatorname{supp}\widehat{\varphi}$ is compact; $\mathcal{A}_1$ denotes the set of even functions $\psi \in \mathcal{S}(\mathbb{R}^n)$ for which $\operatorname{supp}\widehat{\psi}$ is a compact subset of $\mathbb{R}^n \setminus \{0\}$. Notice that the boundedness of T_m from $L^2 \times L^\infty \times L^\infty$ to L^2 is equivalent to the estimate

$$\left| \int_{\mathbb{R}^n} T_m(f_1, f_2, f_3)(x) g(x) dx \right| \leq C \|f_1\|_{L^2} \|f_2\|_{L^\infty} \|f_3\|_{L^\infty} \|g\|_{L^2}$$

for all f_i, $g \in \mathcal{S}$. We shall use the following identity:

$$\theta * T_m(f_1, f_2, f_3)(x) = \frac{1}{(2\pi)^{3n}} \int_{\mathbb{R}^{3n}} e^{ix\cdot(\xi_1+\xi_2+\xi_3)} \widehat{\theta}(\xi_1 + \xi_2 + \xi_3) m(\xi) \widehat{f_1}(\xi_1) \widehat{f_2}(\xi_2) \widehat{f_3}(\xi_3)\, d\xi. \tag{4.1}$$

We first decompose m into a finite number of multipliers each of which is supported on a cone in $\mathbb{R}^{3n}$. To do this, consider the unit sphere $\Sigma = \{\eta \in \mathbb{R}^{3n} \mid |\eta| = 1\}$ and notice the following simple fact: for each $\eta \in \Sigma$, at least two of the four $\mathbb{R}^n$-vectors η_1, η_2, η_3, and $\eta_1 + \eta_2 + \eta_3$ are not equal to 0. Hence there exists a constant $c_0 >$ such that the compact set Σ is covered by the following six open subsets:

$$\begin{aligned}
V_1 &= \{\eta \in \Sigma \mid |\eta_1| > c_0,\ |\eta_1 + \eta_2 + \eta_3| > c_0\}, \\
V_2 &= \{\eta \in \Sigma \mid |\eta_2| > c_0,\ |\eta_1 + \eta_2 + \eta_3| > c_0\}, \\
V_3 &= \{\eta \in \Sigma \mid |\eta_3| > c_0,\ |\eta_1 + \eta_2 + \eta_3| > c_0\}, \\
V_4 &= \{\eta \in \Sigma \mid |\eta_1| > c_0,\ |\eta_2| > c_0\}, \\
V_5 &= \{\eta \in \Sigma \mid |\eta_1| > c_0,\ |\eta_3| > c_0\}, \\
V_6 &= \{\eta \in \Sigma \mid |\eta_2| > c_0,\ |\eta_3| > c_0\}.
\end{aligned}$$

We write

$$\Gamma(V_i) = \{\xi \in \mathbb{R}^{3n} \setminus \{0\} \mid \xi/|\xi| \in V_i\}.$$

We take functions Φ_i, $i = 1, 2, \ldots, 6$, on $\mathbb{R}^{3n}$ such that each Φ_i is homogeneous of degree 0, smooth away from the origin, $\operatorname{supp}\Phi_i \subset \Gamma(V_i)$, and $\sum_{i=1}^6 \Phi_i(\xi) = 1$ for all $\xi \neq 0$. We decompose m as

$$m(\xi) = \sum_{i=1}^6 m(\xi)\Phi_i(\xi).$$

It is sufficient to prove the boundedness of each $T_{m\Phi_i}$. By Lemma 3.4, we see that each $m\Phi_i$ satisfies

$$\sup_{j\in\mathbb{Z}} \|(m\Phi_i)_j\|_{W^{(s,s,s)}} \leq C \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s,s,s)}}.$$

Thus, in the rest of the proof, writing simply m instead of $m\Phi_i$, we shall assume that the support of our multiplier m is included in one of $\Gamma(V_i)$. By symmetry of the situation, the cases $i = 2$ and $i = 3$ are treated in the same way, and $i = 4$ and $i = 5$ are also treated in the same way. Therefore, we shall only consider the four cases $i = 1, 2, 4, 6$.

We make another decomposition of m. We take a function $\Theta \in C^\infty(\mathbb{R}^{3n})$ such that $\operatorname{supp}\Theta \subset \{1/2 \leq |\xi| \leq 2\}$ and

$$\int_0^\infty \Theta(t\xi)\, \frac{dt}{t} = 1 \quad \text{for all} \quad \xi \in \mathbb{R}^{3n} \setminus \{0\},$$

and write m as

$$m(\xi) = \int_0^\infty m(\xi)\Theta(t\xi)\, \frac{dt}{t} = \int_0^\infty m_t(\xi)\, \frac{dt}{t},$$

where

$$m_t(\xi) = m(\xi)\Theta(t\xi).$$

Thus

$$\int_{\mathbb{R}^n} T_m(f_1, f_2, f_3)(x)g(x)\,dx = \int_0^\infty \int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x)g(x)\,\frac{dxdt}{t}.$$

For the operator T_{m_t}, $t > 0$, we use Lemmas 3.3 and 3.4 to obtain the following pointwise estimate:

$$|T_{m_t}(f_1, f_2, f_3)(x)| \le C(\zeta_t * |f_1|^q)(x)^{1/q}(\zeta_t * |f_2|^q)(x)^{1/q}(\zeta_t * |f_3|^q)(x)^{1/q}, \tag{4.2}$$

where q is a number satisfying $\max\{1, n/s\} < q < 2$ and $\zeta(x) = (1 + |x|)^{-sq}$. In the rest of the proof, we shall consider the four cases separately.

The case $\operatorname{supp} m \subset \Gamma(V_1)$. In this case, for $(\xi_1, \xi_2, \xi_3) \in \operatorname{supp} m_t$, we have $|\xi_1 + \xi_2 + \xi_3| \approx |\xi_1| \approx |\xi| \approx 1/t$. Hence we can find a function $\psi \in \mathcal{A}_1$ such that $\widehat{\psi}(t(\xi_1 + \xi_2 + \xi_3))\widehat{\psi}(t\xi_1) = 1$ on the support of m_t. Thus, using (4.1), we can write

$$\begin{aligned}
&T_{m_t}(f_1, f_2, f_3)(x)\\
&= \frac{1}{(2\pi)^{3n}} \int_{\mathbb{R}^{3n}} e^{ix\cdot(\xi_1+\xi_2+\xi_3)}\widehat{\psi}(t(\xi_1 + \xi_2 + \xi_3))m_t(\xi)\widehat{\psi}(t\xi_1)\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\widehat{f_3}(\xi_3)\,d\xi\\
&= \psi_t * T_{m_t}(\psi_t * f_1, f_2, f_3)(x)
\end{aligned}$$

and hence

$$\int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x)g(x)\,dx = \int_{\mathbb{R}^n} T_{m_t}(\psi_t * f_1, f_2, f_3)(x)\psi_t * g(x)\,dx.$$

From this expression and from (4.2), we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x)g(x)\,dx\right|\\
&\le \int_{\mathbb{R}^n} |T_{m_t}(\psi_t * f_1, f_2, f_3)(x)||\psi_t * g(x)|\,dx\\
&\le C\int_{\mathbb{R}^n} (\zeta_t * |\psi_t * f_1|^q)(x)^{1/q}(\zeta_t * |f_2|^q)(x)^{1/q}(\zeta_t * |f_3|^q)(x)^{1/q}|\psi_t * g(x)|\,dx\\
&\le C\|f_2\|_{L^\infty}\|f_3\|_{L^\infty}\int_{\mathbb{R}^n} (\zeta_t * |\psi_t * f_1|^q)(x)^{1/q}|\psi_t * g(x)|\,dx.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} T_m(f_1, f_2, f_3)(x)g(x)\,dx\right| = \left|\int_0^\infty \int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x)g(x)\,\frac{dxdt}{t}\right|\\
&\le C\|f_2\|_{L^\infty}\|f_3\|_{L^\infty}\int_0^\infty \int_{\mathbb{R}^n} (\zeta_t * |\psi_t * f_1|^q)(x)^{1/q}|\psi_t * g(x)|\,\frac{dxdt}{t}\\
&\le C\|f_2\|_{L^\infty}\|f_3\|_{L^\infty}\left(\int_0^\infty \int_{\mathbb{R}^n} (\zeta_t * |\psi_t * f_1|^q)(x)^{2/q}\,\frac{dxdt}{t}\right)^{1/2}\\
&\quad\times\left(\int_0^\infty \int_{\mathbb{R}^n} |\psi_t * g(x)|^2\,\frac{dxdt}{t}\right)^{1/2}
\end{aligned}$$

by Schwarz's inequality. By Lemma 3.2 (3.3) and (3.2), the last quantity is majorized by

$$C\|f_2\|_{L^\infty}\|f_3\|_{L^\infty}\|f_1\|_{L^2}\|g\|_{L^2}$$

as desired.

The case $\operatorname{supp} m \subset \Gamma(V_2)$. In this case, for $(\xi_1, \xi_2, \xi_3) \in \operatorname{supp} m_t$, we have $|\xi_1 + \xi_2 + \xi_3| \approx |\xi_2| \approx |\xi| \approx 1/t$. Hence we can find a $\psi \in \mathcal{A}_1$ such that $\widehat{\psi}(t(\xi_1 + \xi_2 + \xi_3))\widehat{\psi}(t\xi_2) = 1$ on the support of m_t. Thus, in the same way as in the first case, we can write

$$\int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x)g(x)\,dx = \int_{\mathbb{R}^n} T_{m_t}(f_1, \psi_t * f_2, f_3)(x)\psi_t * g(x)\,dx.$$

From this expression and (4.2), we obtain

$$\left|\int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x)g(x)\,dx\right|$$

$$\leq \int_{\mathbb{R}^n} |T_{m_t}(f_1, \psi_t * f_2, f_3)(x)||\psi_t * g(x)| \, dx$$
$$\leq C \int_{\mathbb{R}^n} (\zeta_t * |f_1|^q)(x)^{1/q} (\zeta_t * |\psi_t * f_2|^q)(x)^{1/q} (\zeta_t * |f_3|^q)(x)^{1/q} |\psi_t * g(x)| \, dx$$
$$\leq C \|f_3\|_{L^\infty} \int_{\mathbb{R}^n} (\zeta_t * |f_1|^q)(x)^{1/q} (\zeta_t * |\psi_t * f_2|^q)(x)^{1/q} |\psi_t * g(x)| \, dx.$$

Thus

$$\left| \int_{\mathbb{R}^n} T_m(f_1, f_2, f_3)(x) g(x) \, dx \right| = \left| \int_0^\infty \int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x) g(x) \, \frac{dxdt}{t} \right|$$
$$\leq C \|f_3\|_{L^\infty} \int_0^\infty \int_{\mathbb{R}^n} (\zeta_t * |f_1|^q)(x)^{1/q} (\zeta_t * |\psi_t * f_2|^q)(x)^{1/q} |\psi_t * g(x)| \, \frac{dxdt}{t}$$
$$\leq C \|f_3\|_{L^\infty} \left(\int_0^\infty \int_{\mathbb{R}^n} (\zeta_t * |f_1|^q)(x)^{2/q} (\zeta_t * |\psi_t * f_2|^q)(x)^{2/q} \, \frac{dxdt}{t} \right)^{1/2}$$
$$\times \left(\int_0^\infty \int_{\mathbb{R}^n} |\psi_t * g(x)|^2 \, \frac{dxdt}{t} \right)^{1/2}.$$

By Lemma 3.2 (3.4) and (3.2), the last quantity is majorized by

$$C \|f_3\|_{L^\infty} \|f_2\|_{BMO} \|f_1\|_{L^2} \|g\|_{L^2} \leq C \|f_3\|_{L^\infty} \|f_2\|_{L^\infty} \|f_1\|_{L^2} \|g\|_{L^2}.$$

The case $\operatorname{supp} m \subset \Gamma(V_4)$. In this case, for $(\xi_1, \xi_2, \xi_3) \in \operatorname{supp} m_t$, we have $|\xi_1| \approx |\xi_2| \approx |\xi| \approx 1/t$. Hence we can find functions $\psi \in \mathcal{A}_1$ and $\varphi \in \mathcal{A}_0$ such that $\widehat{\varphi}(t(\xi_1 + \xi_2 + \xi_3)) \widehat{\psi}(t\xi_1) \widehat{\psi}(t\xi_2) = 1$ on the support of m_t. Thus we can write

$$T_{m_t}(f_1, f_2, f_3)(x)$$
$$= \frac{1}{(2\pi)^{3n}} \int_{\mathbb{R}^{3n}} e^{ix \cdot (\xi_1 + \xi_2 + \xi_3)} \widehat{\varphi}(t(\xi_1 + \xi_2 + \xi_3)) m_t(\xi) \widehat{\psi}(t\xi_1) \widehat{f_1}(\xi_1) \widehat{\psi}(t\xi_2) \widehat{f_2}(\xi_2) \widehat{f_3}(\xi_3) \, d\xi$$
$$= \varphi_t * T_{m_t}(\psi_t * f_1, \psi_t * f_2, f_3)(x)$$

and hence

$$\int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x) g(x) \, dx = \int_{\mathbb{R}^n} T_{m_t}(\psi_t * f_1, \psi_t * f_2, f_3)(x) \varphi_t * g(x) \, dx.$$

From this expression and (4.2), we obtain

$$\left| \int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x) g(x) \, dx \right|$$
$$\leq \int_{\mathbb{R}^n} |T_{m_t}(\psi_t * f_1, \psi_t * f_2, f_3)(x)||\varphi_t * g(x)| \, dx$$
$$\leq C \int_{\mathbb{R}^n} (\zeta_t * |\psi_t * f_1|^q)(x)^{1/q} (\zeta_t * |\psi_t * f_2|^q)(x)^{1/q} (\zeta_t * |f_3|^q)(x)^{1/q} |\varphi_t * g(x)| \, dx$$
$$\leq C \|f_3\|_{L^\infty} \int_0^\infty \int_{\mathbb{R}^n} (\zeta_t * |\psi_t * f_1|^q)(x)^{1/q} (\zeta_t * |\psi_t * f_2|^q)(x)^{1/q} |\varphi_t * g(x)| \, dx.$$

Thus

$$\left| \int_{\mathbb{R}^n} T_m(f_1, f_2, f_3)(x) g(x) \, dx \right| = \left| \int_0^\infty \int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x) g(x) \, \frac{dxdt}{t} \right|$$
$$\leq C \|f_3\|_{L^\infty} \int_0^\infty \int_{\mathbb{R}^n} (\zeta_t * |\psi_t * f_1|^q)(x)^{1/q} (\zeta_t * |\psi_t * f_2|^q)(x)^{1/q} |\varphi_t * g(x)| \, \frac{dxdt}{t}$$
$$\leq C \|f_3\|_{L^\infty} \left(\int_0^\infty \int_{\mathbb{R}^n} (\zeta_t * |\psi_t * f_1|^q)(x)^{2/q} \, \frac{dxdt}{t} \right)^{1/2}$$
$$\times \left(\int_0^\infty \int_{\mathbb{R}^n} (\zeta_t * |\psi_t * f_2|^q)(x)^{2/q} |\varphi_t * g(x)|^2 \, \frac{dxdt}{t} \right)^{1/2}.$$

By Lemma 3.2 (3.3) and (3.5), the last quantity is majorized by

$$C \|f_3\|_{L^\infty} \|f_1\|_{L^2} \|f_2\|_{BMO} \|g\|_{L^2} \leq C \|f_3\|_{L^\infty} \|f_1\|_{L^2} \|f_2\|_{L^\infty} \|g\|_{L^2}.$$

The case $\operatorname{supp} m \subset \Gamma(V_6)$. In this case, for $(\xi_1,\xi_2,\xi_3) \in \operatorname{supp} m_t$, we have $|\xi_2| \approx |\xi_3| \approx |\xi| \approx 1/t$. Hence we can find functions $\psi \in \mathcal{A}_1$ and $\varphi \in \mathcal{A}_0$ such that $\widehat{\varphi}(t(\xi_1+\xi_2+\xi_3))\widehat{\psi}(t\xi_2)\widehat{\psi}(t\xi_3) = 1$ on the support of m_t. Thus, in the same way as in the third case, we can write

$$\int_{\mathbb{R}^n} T_{m_t}(f_1,f_2,f_3)(x)g(x)\,dx = \int_{\mathbb{R}^n} T_{m_t}(f_1,\psi_t * f_2,\psi_t * f_3)(x)\varphi_t * g(x)\,dx.$$

From this expression and (4.2), we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} T_{m_t}(f_1,f_2,f_3)(x)g(x)\,dx\right| \\
&\le \int_{\mathbb{R}^n} |T_{m_t}(f_1,\psi_t * f_2,\psi_t * f_3)(x)||\varphi_t * g(x)|\,dx \\
&\le C\int_{\mathbb{R}^n} (\zeta_t * |f_1|^q)(x)^{1/q}(\zeta_t * |\psi_t * f_2|^q)(x)^{1/q}(\zeta_t * |\psi_t * f_3|^q)(x)^{1/q}|\varphi_t * g(x)|\,dx.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} T_m(f_1,f_2,f_3)(x)g(x)\,dx\right| = \left|\int_0^\infty\int_{\mathbb{R}^n} T_{m_t}(f_1,f_2,f_3)(x)g(x)\,\frac{dxdt}{t}\right| \\
&\le C\int_0^\infty\int_{\mathbb{R}^n} (\zeta_t * |f_1|^q)(x)^{1/q}(\zeta_t * |\psi_t * f_2|^q)(x)^{1/q}(\zeta_t * |\psi_t * f_3|^q)(x)^{1/q}|\varphi_t * g(x)|\,\frac{dxdt}{t} \\
&\le C\left(\int_0^\infty\int_{\mathbb{R}^n} (\zeta_t * |f_1|^q)(x)^{2/q}(\zeta_t * |\psi_t * f_2|^q)(x)^{2/q}\,\frac{dxdt}{t}\right)^{1/2} \\
&\quad\times\left(\int_0^\infty\int_{\mathbb{R}^n} (\zeta_t * |\psi_t * f_3|^q)(x)^{2/q}|\varphi_t * g(x)|^2\,\frac{dxdt}{t}\right)^{1/2}.
\end{aligned}$$

By Lemma 3.2 (3.4) and (3.5), the last quantity is majorized by

$$C\|f_1\|_{L^2}\|f_2\|_{BMO}\|f_3\|_{BMO}\|g\|_{L^2} \le C\|f_1\|_{L^2}\|f_2\|_{L^\infty}\|f_3\|_{L^\infty}\|g\|_{L^2}.$$

This completes the proof of Theorem 1.1 □

5. The boundedness from $H^p \times L^\infty \times \cdots \times L^\infty$ to L^p with $p \le 1$

In this section, we prove Theorem 1.2.

Proof of Theorem 1.2. We shall give the proof for the case $N = 3$. The general case $N \ge 2$ can be proved in a similar way.

Let p, s_1, s_2, s_3, and m satisfy the assumptions of Theorem 1.2 with $N = 3$. Without loss of generality, we may assume $\sup_{j\in\mathbb{Z}}\|m_j\|_{W^{(s_1,s_2,s_3)}} = 1$. We write $L = [n(1/p-1)]$. It is sufficient to consider the case

$$n(1/p-1/2) < s_1 < L + n/2 + 1 \tag{5.1}$$

We first observe that the desired boundedness of T_m follows if we prove the estimate

$$\|T_m(f_1,f_2,f_3)\|_{L^p} \le C\|f_2\|_{L^\infty}\|f_3\|_{L^\infty} \tag{5.2}$$

for all f_1 such that

$$\operatorname{supp} f_1 \subset \{x\in\mathbb{R}^n : |x| \le r\}, \quad \|f_1\|_{L^\infty} \le r^{-n/p}, \quad \int_{\mathbb{R}^n} x^\alpha f_1(x)\,dx = 0 \quad (|\alpha| \le L) \tag{5.3}$$

with some $r > 0$ (where r depends on f_1). Indeed, since the norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{L^\infty}$ are translation invariant and since the operator T_m is also translation invariant in the sense that

$$T_m(f_1,f_2,f_3)(x+x_0) = T_m(f_1(\cdot+x_0),f_2(\cdot+x_0),f_3(\cdot+x_0))(x),$$

if (5.2) holds for all f_1 satisfying (5.3), then it follows that (5.2) holds for all H^p-atoms f_1. Hence, by considering the linear operator $f_1 \mapsto T_m(f_1,f_2,f_3)$ and applying the usual argument of using the atomic decomposition, we obtain the desired estimate

$$\|T_m(f_1,f_2,f_3)\|_{L^p} \le C\|f_1\|_{H^p}\|f_2\|_{L^\infty}\|f_3\|_{L^\infty}.$$

(Notice that here we benefited from the particular combination of the exponents (p,∞,∞); the atomic decomposition could not be directly used to prove the $H^{p_1} \to L^p$ estimate if $1 \ge p_1 > p$.)

In the rest of the proof, we assume that f_1 is a function satisfying (5.3).

We first prove

$$\|T_m(f_1,f_2,f_3)(x)\|_{L^p(|x|\le 2r)} \le C\|f_2\|_{L^\infty}\|f_3\|_{L^\infty}.$$

Since $s_1 > n(1/p-1/2) \ge n/2$ and $s_2, s_3 > n/2$, it follows from Theorem 1.1 that T_m is bounded from $L^2(\mathbb{R}^n)\times L^\infty(\mathbb{R}^n)\times L^\infty(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Note that $\|f_1\|_{L^2} \le Cr^{-n(1/p-1/2)}$. Thus, by Hölder's inequality, we obtain

$$\begin{aligned}\|T_m(f_1,f_2,f_3)(x)\|_{L^p(|x|\le 2r)} &\le C(2r)^{n(1/p-1/2)}\|T_m(f_1,f_2,f_3)\|_{L^2}\\ &\le Cr^{n(1/p-1/2)}\|f_1\|_{L^2}\|f_2\|_{L^\infty}\|f_3\|_{L^\infty}\\ &\le C\|f_2\|_{L^\infty}\|f_3\|_{L^\infty}.\end{aligned}$$

Thus, what is left is to prove the estimate

$$\|T_m(f_1,f_2,f_3)(x)\|_{L^p(|x|>2r)} \le C\|f_2\|_{L^\infty}\|f_3\|_{L^\infty}. \tag{5.4}$$

To prove this, we take a function Ψ satisfying (1.2) with $d = 3n$ and decompose m as

$$m(\xi) = \sum_{j=-\infty}^{\infty} m(\xi)\Psi(\xi/2^j).$$

We define

$$K_j = \mathcal{F}^{-1}[m(\cdot)\Psi(\cdot/2^j)] = \mathcal{F}^{-1}[m_j(\cdot/2^j)].$$

If we write $\widetilde{K}_j = \mathcal{F}^{-1}[m_j]$, then $K_j(x) = 2^{3jn}\widetilde{K}_j(2^jx)$. The function $T_m(f_1,f_2,f_3)$ can be written as

$$\begin{aligned}&T_m(f_1,f_2,f_3)(x)\\ &=\sum_{j\in\mathbb{Z}} T_{m(\cdot)\Psi(\cdot/2^j)}(f_1,f_2,f_3)(x)\\ &=\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^{3n}} K_j(x-y_1,x-y_2,x-y_3)f_1(y_1)f_2(y_2)f_3(y_3)\,dy\\ &=\sum_{j\in\mathbb{Z}} F_j(x),\quad \text{say.}\end{aligned}$$

By the subadditivity of the p-th power of the L^p-norm, $p\le 1$, and by Hölder's inequality, we have

$$\begin{aligned}&\|T_m(f_1,f_2,f_3)(x)\|^p_{L^p(|x|>2r)}\\ &\le \sum_{j\in\mathbb{Z}}\|F_j(x)\|^p_{L^p(|x|>2r)}\\ &\le \sum_{j\in\mathbb{Z}}\left(\int_{|x|>2r}|x|^{-s_1/(1/p-1/2)}dx\right)^{1-p/2}\left(\int_{|x|>2r}|x|^{2s_1}|F_j(x)|^2dx\right)^{p/2}\\ &= C\sum_{j\in\mathbb{Z}}\left\{r^{-s_1+n/p-n/2}\||x|^{s_1}F_j(x)\|_{L^2(|x|>2r)}\right\}^p,\end{aligned} \tag{5.5}$$

where we have used the assumption $s_1 > n(1/p-1/2)$ to obtain the last equality.

We shall estimate the function $F_j(x)$. Writing $\partial_1^\alpha K_j(y_1,y_2,y_3) = \partial_{y_1}^\alpha K_j(y_1,y_2,y_3)$ and using the moment condition on f_1, we have

$$\begin{aligned}&F_j(x)\\ &=\int_{\mathbb{R}^{3n}}\Big\{K_j(x-y_1,x-y_2,x-y_3)\\ &\qquad -\sum_{|\alpha|\le L}\frac{(-y_1)^\alpha}{\alpha!}\partial_1^\alpha K_j(x,x-y_2,x-y_3)\Big\}f_1(y_1)f_2(y_2)f_3(y_3)\,dy\\ &=(L+1)\sum_{|\alpha|=L+1}\int_{y\in\mathbb{R}^{3n}}\int_{0<t<1}\frac{(-y_1)^\alpha}{\alpha!}(1-t)^L\partial_1^\alpha K_j(x-ty_1,x-y_2,x-y_3)\\ &\qquad\times f_1(y_1)f_2(y_2)f_3(y_3)\,dtdy\end{aligned}$$

$$= (L+1) \sum_{|\alpha|=L+1} \int_{y\in\mathbb{R}^{3n}} \int_{0<t<1} \frac{(-y_1)^\alpha}{\alpha!} (1-t)^L \partial_1^\alpha K_j(x-ty_1, y_2, y_3) \times f_1(y_1) f_2(x-y_2) f_3(x-y_3)\, dtdy.$$

Thus using the support condition and the L^∞-norm condition on f_1, we have

$$|F_j(x)| \le C r^{L+1-n/p} \|f_2\|_{L^\infty} \|f_3\|_{L^\infty} \times \sum_{|\alpha|=L+1} \int_{\substack{|y_1|\le r\\ y_2,y_3\in\mathbb{R}^n}} \int_{0<t<1} |\partial_1^\alpha K_j(x-ty_1,y_2,y_3)|\, dtdy. \tag{5.6}$$

In the same way, without using the moment condition of f_1, we obtain

$$|F_j(x)| \le C r^{-n/p} \|f_2\|_{L^\infty} \|f_3\|_{L^\infty} \int_{\substack{|y_1|\le r\\ y_2,y_3\in\mathbb{R}^n}} |K_j(x-y_1,y_2,y_3)|\, dy. \tag{5.7}$$

We shall estimate the weighted L^2-norm $\||x|^{s_1} F_j(x)\|_{L^2(|x|>2r)}$ using (5.6) and (5.7). First consider the integral appearing in (5.6). Notice that $|x|/2 \le |x-ty_1| \le 3|x|/2$ for $|x|>2r$, $|y_1|\le r$, and $0\le t\le 1$. Hence, using Minkowski's inequality for integrals, we have

$$\begin{aligned}
&\Big\| |x|^{s_1} \int_{\substack{|y_1|\le r\\ y_2,y_3\in\mathbb{R}^n}} \int_{0<t<1} |\partial_1^\alpha K_j(x-ty_1,y_2,y_3)|\, dtdy \Big\|_{L^2(|x|>2r)} \\
&\le C \int_{\substack{|y_1|\le r\\ y_2,y_3\in\mathbb{R}^n}} \int_{0<t<1} \Big\| |x-ty_1|^{s_1} \partial_1^\alpha K_j(x-ty_1,y_2,y_3) \Big\|_{L^2(|x|>2r)} dtdy \\
&\le C r^n \int_{y_2,y_3\in\mathbb{R}^n} \Big\| |x|^{s_1} \partial_1^\alpha K_j(x,y_2,y_3) \Big\|_{L^2(\mathbb{R}^n_x)} dy_2 dy_3 \\
&= (*).
\end{aligned}$$

In terms of $\widetilde{K}_j$ and $\partial_1^\alpha \widetilde{K}_j(z_1,z_2,z_3) = \partial_{z_1}^\alpha \widetilde{K}_j(z_1,z_2,z_3)$, the last term can be written as

$$\begin{aligned}
(*) &= C r^n \int_{y_2,y_3\in\mathbb{R}^n} 2^{j(-s_1+3n+|\alpha|)} \Big\| |2^j x|^{s_1} \partial_1^\alpha \widetilde{K}_j(2^j x, 2^j y_2, 2^j y_3) \Big\|_{L^2(\mathbb{R}^n_x)} dy_2 dy_3 \\
&= C r^n 2^{j(-s_1+n/2+|\alpha|)} \int_{z_2,z_3\in\mathbb{R}^n} \Big\| |z_1|^{s_1} \partial_1^\alpha \widetilde{K}_j(z_1,z_2,z_3) \Big\|_{L^2(\mathbb{R}^n_{z_1})} dz_2 dz_3.
\end{aligned}$$

Since $s_2, s_3 > n/2$, Schwarz's inequality gives

$$\begin{aligned}
(*) &\le C r^n 2^{j(-s_1+n/2+|\alpha|)} \Big(\int_{\mathbb{R}^{2n}} \Big\| |z_1|^{s_1} \langle z_2\rangle^{s_2} \langle z_3\rangle^{s_3} \partial_1^\alpha \widetilde{K}_j(z_1,z_2,z_3) \Big\|^2_{L^2(\mathbb{R}^n_{z_1})} dz_2 dz_3 \Big)^{1/2} \\
&\le C r^n 2^{j(-s_1+n/2+|\alpha|)} \Big\| \langle z_1\rangle^{s_1} \langle z_2\rangle^{s_2} \langle z_3\rangle^{s_3} \partial_1^\alpha \widetilde{K}_j(z_1,z_2,z_3) \Big\|_{L^2(\mathbb{R}^{3n}_z)} \\
&= C r^n 2^{j(-s_1+n/2+|\alpha|)} \|\xi_1^\alpha m_j(\xi_1,\xi_2,\xi_2)\|_{W^{(s_1,s_2,s_3)}}.
\end{aligned}$$

We shall see that the last $\|\dots\|_{W^{(s_1,s_2,s_3)}}$ is majorized by C. In fact, if we take a function $\widetilde{\Psi} \in \mathcal{S}(\mathbb{R}^{3n})$ such that $\widetilde{\Psi} = 1$ on $\{1/2 \le |\xi| \le 2\}$, then, by Proposition 2.2,

$$\begin{aligned}
\|\xi_1^\alpha m_j\|_{W^{(s_1,s_2,s_3)}} &= \|\xi_1^\alpha m(2^j\xi)\Psi(\xi)\|_{W^{(s_1,s_2,s_3)}} = \|\xi_1^\alpha m(2^j\xi)\Psi(\xi)\widetilde{\Psi}(\xi)\|_{W^{(s_1,s_2,s_3)}} \\
&\le C\|m(2^j\xi)\Psi(\xi)\|_{W^{(s_1,s_2,s_3)}} \|\xi_1^\alpha \widetilde{\Psi}(\xi)\|_{W^{(s_1,s_2,s_3)}} = C\|m_j\|_{W^{(s_1,s_2,s_3)}} \le C.
\end{aligned}$$

Thus we have

$$\Big\| |x|^{s_1} \int_{\substack{|y_1|\le r\\ y_2,y_3\in\mathbb{R}^n}} \int_{0<t<1} |\partial_1^\alpha K_j(x-ty_1,y_2,y_3)|\, dtdy \Big\|_{L^2(|x|>2r)} \le C r^n 2^{j(-s_1+n/2+|\alpha|)}.$$

Combining this with (5.6), we obtain

$$r^{-s_1+n/p-n/2} \||x|^{s_1} F_j(x)\|_{L^2(|x|>2r)} \le C(2^j r)^{-s_1+n/2+L+1} \|f_2\|_{L^\infty} \|f_3\|_{L^\infty}. \tag{5.8}$$

In the same way, using (5.7) in place of (5.6), we obtain

$$r^{-s_1+n/p-n/2} \||x|^{s_1} F_j(x)\|_{L^2(|x|>2r)} \le C(2^j r)^{-s_1+n/2} \|f_2\|_{L^\infty} \|f_3\|_{L^\infty}. \tag{5.9}$$

Now using (5.5), (5.8) and (5.9), we obtain

$$\|T_m(f_1,f_2,f_3)\|^p_{L^p(|x|\ge 2r)}$$

$$\le C(\|f_2\|_{L^\infty}\|f_3\|_{L^\infty})^p\left(\sum_{2^jr\le 1}(2^jr)^{p(-s_1+n/2+L+1)}+\sum_{2^jr>1}(2^jr)^{p(-s_1+n/2)}\right)$$
$$\le C(\|f_2\|_{L^\infty}\|f_3\|_{L^\infty})^p,$$

where we have used (5.1). Thus we proved (5.4). The proof of Theorem 1.2 is complete. □

6. The boundedness from $H^{p_1}\times H^{p_2}\times\cdots\times H^{p_N}$ to L^p with $p\le 1$

In this section, using interpolation, we prove the following.

Theorem 6.1. *Let $0<p_1,\dots,p_N\le\infty$, $0<p\le 1$ and $1/p_1+\cdots+1/p_N=1/p$. If $m\in L^\infty(\mathbb{R}^{Nn})$ satisfies*

$$\sup_{j\in\mathbb{Z}}\|m_j\|_{W^{(s_1,\dots,s_N)}(\mathbb{R}^{Nn})}<\infty \qquad \text{with } s_j>\frac{n}{p_j}(1-p)+\frac{n}{2},\ j=1,\dots,N,$$

then T_m is bounded from $H^{p_1}(\mathbb{R}^n)\times\cdots\times H^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

Proof. We give the proof for the case $N=3$. The argument can be easily extended to the case $N\ge 2$. We shall divide the proof into two steps.

Step 1. Let $0<\theta<1$, $0<p_j,p_{j,k}\le\infty$ and $s_{j,k}>n/2$ ($j\in\{0,1\}$, $k\in\{1,2,3\}$). Set $1/p=(1-\theta)/p_0+\theta/p_1$, $1/p_k=(1-\theta)/p_{0,k}+\theta/p_{1,k}$ and $s_k=(1-\theta)s_{0,k}+\theta s_{1,k}$. In this step, we prove that if

$$\|T_m\|_{H^{p_{0,1}}\times H^{p_{0,2}}\times H^{p_{0,3}}\to L^{p_0}}\le C\sup_{k\in\mathbb{Z}}\|m_k\|_{W^{(s_{0,1},s_{0,2},s_{0,3})}}$$

and

$$\|T_m\|_{H^{p_{1,1}}\times H^{p_{1,2}}\times H^{p_{1,3}}\to L^{p_1}}\le C\sup_{k\in\mathbb{Z}}\|m_k\|_{W^{(s_{1,1},s_{1,2},s_{1,3})}},$$

then

$$\|T_m\|_{H^{p_1}\times H^{p_2}\times H^{p_3}\to L^p}\le C\sup_{k\in\mathbb{Z}}\|m_k\|_{W^{(s_1,s_2,s_3)}}. \tag{6.1}$$

To do this, we construct a family of multilinear Fourier multipliers m_z as follows:

$$m_z(\xi)=\sum_{j\in\mathbb{Z}}m_{z,j}(\xi/2^j)\Phi(\xi/2^j),\qquad z\in\{z\in\mathbb{C}:0\le\operatorname{Re}z\le 1\},$$

where $\Phi\in\mathcal{S}(\mathbb{R}^{3n})$ satisfies $\Phi=1$ on $\{1/2\le|\xi|\le 2\}$ and $\operatorname{supp}\Phi\subset\{1/4\le|\xi|\le 4\}$, and

$$\begin{aligned}m_{z,j}(\xi)&=\langle D_1\rangle^{(s_{0,1}-s_{1,1})(z-\theta)}\langle D_2\rangle^{(s_{0,2}-s_{1,2})(z-\theta)}\langle D_3\rangle^{(s_{0,3}-s_{1,3})(z-\theta)}m_j(\xi)\\&=\frac{1}{(2\pi)^{3n}}\int_{\mathbb{R}^{3n}}e^{i(x_1\cdot\xi_1+x_2\cdot\xi_2+x_3\cdot\xi_3)}\\&\qquad\times\langle x_1\rangle^{(s_{0,1}-s_{1,1})(z-\theta)}\langle x_2\rangle^{(s_{0,2}-s_{1,2})(z-\theta)}\langle x_3\rangle^{(s_{0,3}-s_{1,3})(z-\theta)}\widehat{m_j}(x)\,dx.\end{aligned}$$

Since $m_{\theta,j}(\xi)=m_j(\xi)$ and $\Phi=1$ on $\operatorname{supp}\Psi$,

$$m_\theta(\xi)=\sum_{j\in\mathbb{Z}}m(\xi)\Psi(\xi/2^j)\Phi(\xi/2^j)=\sum_{j\in\mathbb{Z}}m(\xi)\Psi(\xi/2^j)=m(\xi).$$

Then, it follows from the interpolation theorem for analytic families of operators ([14, 20]) that

$$\begin{aligned}\|T_m\|_{H^{p_1}\times H^{p_2}\times H^{p_3}\to L^p}&\le\left(\sup_{t\in\mathbb{R}}\|T_{m_{it}}\|_{H^{p_{0,1}}\times H^{p_{0,2}}\times H^{p_{0,3}}\to L^{p_0}}\right)^{1-\theta}\\&\qquad\times\left(\sup_{t\in\mathbb{R}}\|T_{m_{1+it}}\|_{H^{p_{1,1}}\times H^{p_{1,2}}\times H^{p_{1,3}}\to L^{p_1}}\right)^{\theta}.\end{aligned} \tag{6.2}$$

Using $\operatorname{supp}\Psi\subset\{1/2\le|\xi|\le 2\}$ and $\operatorname{supp}\Phi(2^{k-j}\cdot)\subset\{2^{j-k-2}\le|\xi|\le 2^{j-k+2}\}$, we have

$$\begin{aligned}(m_{it})_k(\xi)=m_{it}(2^k\xi)\Psi(\xi)&=\left(\sum_{j\in\mathbb{Z}}m_{it,j}(2^{k-j}\xi)\Phi(2^{k-j}\xi)\right)\Psi(\xi)\\&=\sum_{j=k-2}^{k+2}m_{it,j}(2^{k-j}\xi)\Phi(2^{k-j}\xi)\Psi(\xi).\end{aligned}$$

Recall that $s_{0,1}, s_{0,2}, s_{0,3} > n/2$. Then, by Proposition 2.2 and a change of variables,

$$\begin{aligned}\|(m_{it})_k\|_{W^{(s_{0,1},s_{0,2},s_{0,3})}} &\le C \sum_{j=k-2}^{k+2} \|m_{it,j}(2^{k-j}\cdot)\|_{W^{(s_{0,1},s_{0,2},s_{0,3})}} \|\Phi(2^{k-j}\cdot)\Psi\|_{W^{(s_{0,1},s_{0,2},s_{0,3})}} \\ &\le C \sum_{j=k-2}^{k+2} \|m_{it,j}\|_{W^{(s_{0,1},s_{0,2},s_{0,3})}} \le C \sup_{j\in\mathbb{Z}} \|\langle D_1\rangle^{s_{0,1}}\langle D_2\rangle^{s_{0,2}}\langle D_3\rangle^{s_{0,3}} m_{it,j}\|_{L^2} \\ &= C \sup_{j\in\mathbb{Z}} \Big(\int_{\mathbb{R}^{3n}} |\langle x_1\rangle^{s_{0,1}+(s_{0,1}-s_{1,1})(it-\theta)}|^2 |\langle x_2\rangle^{s_{0,2}+(s_{0,2}-s_{1,2})(it-\theta)}|^2 \\ &\qquad\qquad \times |\langle x_3\rangle^{s_{0,3}+(s_{0,3}-s_{1,3})(it-\theta)}|^2 |\widehat{m_j}(x)|^2\, dx \Big)^{1/2} \\ &= C \sup_{j\in\mathbb{Z}} \Big(\int_{\mathbb{R}^{3n}} \langle x_1\rangle^{2s_1}\langle x_2\rangle^{2s_2}\langle x_3\rangle^{2s_3} |\widehat{m_j}(x)|^2\, dx \Big)^{1/2} = C \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,s_2,s_3)}}.\end{aligned}$$

Hence, our assumption implies

$$\|T_{m_{it}}\|_{H^{p_{0,1}}\times H^{p_{0,2}}\times H^{p_{0,3}}\to L^{p_0}} \le C \sup_{k\in\mathbb{Z}} \|(m_{it})_k\|_{W^{(s_{0,1},s_{0,2},s_{0,3})}} \le C \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,s_2,s_3)}}. \tag{6.3}$$

Similarly we have

$$\|T_{m_{1+it}}\|_{H^{p_{1,1}}\times H^{p_{1,2}}\times H^{p_{1,3}}\to L^{p_1}} \le C \sup_{k\in\mathbb{Z}} \|(m_{1+it})_k\|_{W^{(s_{1,1},s_{1,2},s_{1,3})}} \le C \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,s_2,s_3)}}. \tag{6.4}$$

The estimate (6.1) now follows from (6.2)-(6.4).

Step 2. Let $0 < p \le 1$ and $\epsilon > 0$. By interchanging the role of p_1 and p_2 or p_3, we have by Theorem 1.2

$$\|T_m\|_{H^p\times L^\infty\times L^\infty\to L^p} \le C \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(n(1/p-1/2)+\epsilon,\, n/2+\epsilon,\, n/2+\epsilon)}}, \tag{6.5}$$

$$\|T_m\|_{L^\infty\times H^p\times L^\infty\to L^p} \le C \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(n/2+\epsilon,\, n(1/p-1/2)+\epsilon,\, n/2+\epsilon)}}, \tag{6.6}$$

$$\|T_m\|_{L^\infty\times L^\infty\times H^p\to L^p} \le C \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(n/2+\epsilon,\, n/2+\epsilon,\, n(1/p-1/2)+\epsilon)}}. \tag{6.7}$$

Then, it follows from Step 1 that (6.5) and (6.6) give

$$\|T_m\|_{H^{p_1}\times H^{p_2}\times L^\infty\to L^p} \le C \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(n(1-p)/p_1+n/2+\epsilon,\, n(1-p)/p_2+n/2+\epsilon,\, n/2+\epsilon)}}, \tag{6.8}$$

where $p \le p_1, p_2 \le \infty$ and $1/p_1 + 1/p_2 = 1/p$. Furthermore, (6.7) and (6.8) give

$$\|T_m\|_{H^{p_1}\times H^{p_2}\times H^{p_3}\to L^p} \le C \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(n(1-p)/p_1+n/2+\epsilon,\, n(1-p)/p_2+n/2+\epsilon,\, n(1-p)/p_3+n/2+\epsilon)}},$$

where $p \le p_1, p_2, p_3 \le \infty$ and $1/p_1+1/p_2+1/p_3 = 1/p$. The proof of Theorem 6.1 is complete. □

7. Sharpness of the conditions of Theorems 1.1 and 1.2

In this section, we consider the sharpness of Theorems 1.1 and 1.2.

Proposition 7.1. *The estimate*

$$\begin{aligned}&\|T_m(f_1, f_2, \dots, f_N)\|_{L^2(\mathbb{R}^n)} \\ &\quad \le C \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,s_2,\dots,s_N)}(\mathbb{R}^{Nn})} \|f_1\|_{L^2(\mathbb{R}^n)} \|f_2\|_{L^\infty(\mathbb{R}^n)} \cdots \|f_N\|_{L^\infty(\mathbb{R}^n)}\end{aligned} \tag{7.1}$$

holds only if $s_1, s_2, \dots, s_N \ge n/2$.

Proof. We give the proof in the case $N = 2$. Generalization to $N \ge 3$ will be obvious. We take functions ψ and φ such that

$$\begin{aligned}&\psi \in \mathcal{S}(\mathbb{R}^n), \quad \psi \ne 0, \qquad \operatorname{supp}\widehat{\psi} \subset \{\xi \in \mathbb{R}^n \mid 9/10 \le |\xi| \le 11/10\}, \\ &\varphi \in \mathcal{S}(\mathbb{R}^n), \quad \widehat{\varphi}(0) \ne 0, \quad \operatorname{supp}\widehat{\varphi} \subset \{\xi \in \mathbb{R}^n \mid |\xi| \le 1\}.\end{aligned}$$

To prove the necessity of the condition $s_2 \ge n/2$, we set, for sufficiently small $\epsilon > 0$,

$$m(\xi_1, \xi_2) = \widehat{\psi}(\xi_1)\widehat{\varphi}(\xi_2/\epsilon). \tag{7.2}$$

For this m, we have

$$T_m(f_1,f_2)(x) = \mathcal{F}^{-1}[\widehat{\psi}\widehat{f_1}](x)\mathcal{F}^{-1}[\widehat{\varphi}(\cdot/\epsilon)\widehat{f_2}(\cdot)](x),$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform on $\mathbb{R}^n$. To estimate the norm $\|m_j\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})}$, we choose the function $\Psi \in \mathcal{S}(\mathbb{R}^{2n})$, which appeared in the definition of m_j, so that we have

$$\begin{aligned} &\operatorname{supp}\Psi \subset \{\xi \in \mathbb{R}^{2n} \mid 2^{-1/2-\alpha} \le |\xi| \le 2^{1/2+\alpha}\}, \\ &\sum_{k\in\mathbb{Z}} \Psi(2^{-k}\xi) = 1 \quad \text{for all } \xi \neq 0, \\ &\Psi(\xi) = 1 \quad \text{if} \quad 2^{-1/2+\alpha} \le |\xi| \le 2^{1/2-\alpha}, \end{aligned}$$

where $\alpha > 0$ is a sufficiently small number. Then, for sufficiently small $\epsilon > 0$, we have

$$\begin{aligned} \operatorname{supp} m &\subset \{(\xi_1,\xi_2) \in \mathbb{R}^{2n} \mid 9/10 \le |\xi_1| \le 11/10,\ |\xi_2| \le \epsilon\} \\ &\subset \{\xi \in \mathbb{R}^{2n} \mid 2^{-1/2+\alpha} \le |\xi| \le 2^{1/2-\alpha}\}. \end{aligned}$$

Hence

$$m_j(\xi) = m(2^j\xi)\Psi(\xi) = \begin{cases} m(\xi) & (j=0) \\ 0 & (j \neq 0) \end{cases} \tag{7.3}$$

and

$$\begin{aligned} \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})} = \|m\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})} &= \|\widehat{\psi}(\xi_1)\widehat{\varphi}(\xi_2/\epsilon)\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})} \\ &= \|\widehat{\psi}\|_{W^{s_1}(\mathbb{R}^n)}\|\widehat{\varphi}(\cdot/\epsilon)\|_{W^{s_2}(\mathbb{R}^n)}. \end{aligned}$$

Thus the inequality (7.1) for m of (7.2) is equivalent to

$$\|\mathcal{F}^{-1}[\widehat{\psi}\widehat{f_1}](x)\mathcal{F}^{-1}[\widehat{\varphi}(\cdot/\epsilon)\widehat{f_2}(\cdot)](x)\|_{L^2} \le C\|\widehat{\psi}\|_{W^{s_1}}\|\widehat{\varphi}(\cdot/\epsilon)\|_{W^{s_2}}\|f_1\|_{L^2}\|f_2\|_{L^\infty}. \tag{7.4}$$

We have

$$\|\widehat{\psi}\|_{W^{s_1}} = C$$

and

$$\begin{aligned} \|\widehat{\varphi}(\cdot/\epsilon)\|_{W^{s_2}} &= \|\epsilon^n\varphi(\epsilon x)\langle x\rangle^{s_2}\|_{L^2} \\ &\lesssim \epsilon^n\left(\int_{\mathbb{R}^n}(1+|x|)^{2s_2}(1+\epsilon|x|)^{-2N}\,dx\right)^{1/2} \qquad (N>0 \quad \text{large}) \\ &\approx \epsilon^n\left(\int_{|x|\le 1}dx + \int_{1<|x|\le 1/\epsilon}|x|^{2s_2}\,dx + \int_{1/\epsilon<|x|<\infty}|x|^{2s_2}(\epsilon|x|)^{-2N}\,dx\right)^{1/2} \\ &\approx \epsilon^{-s_2+n/2}. \end{aligned}$$

Hence (7.4) implies

$$\|\mathcal{F}^{-1}[\widehat{\psi}\widehat{f_1}](x)\mathcal{F}^{-1}[\widehat{\varphi}(\cdot/\epsilon)\widehat{f_2}(\cdot)](x)\|_{L^2} \le C\epsilon^{-s_2+n/2}\|f_1\|_{L^2}\|f_2\|_{L^\infty}. \tag{7.5}$$

We test (7.5) for

$$f_1(x) = \psi(x), \qquad f_2(x) = 1.$$

Then

$$\begin{aligned} &(\text{the left hand side of (7.5)}) = \|\mathcal{F}^{-1}[\widehat{\psi}^2](x)\widehat{\varphi}(0)\|_{L^2} = C, \\ &(\text{the right hand side of (7.5)}) = C\epsilon^{-s_2+n/2}\|\psi\|_{L^2} = C\epsilon^{-s_2+n/2}. \end{aligned}$$

Thus (7.5) holds only if $s_2 \ge n/2$.

To prove the necessity of the condition $s_1 \ge n/2$, we set, for sufficiently small $\epsilon > 0$,

$$m(\xi_1,\xi_2) = \widehat{\varphi}(\xi_1/\epsilon)\widehat{\psi}(\xi_2). \tag{7.6}$$

By the same reason as above, we have

$$\begin{aligned} \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})} &= \|m\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})} \\ &= \|\widehat{\varphi}(\xi_1/\epsilon)\widehat{\psi}(\xi_2)\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})} \\ &= \|\widehat{\varphi}(\cdot/\epsilon)\|_{W^{s_1}(\mathbb{R}^n)}\|\widehat{\psi}\|_{W^{s_2}(\mathbb{R}^n)} \end{aligned}$$

$$\le C\epsilon^{-s_1+n/2}$$

and the inequality (7.1) with m of (7.6) implies

$$\|\mathcal{F}^{-1}[\widehat{\varphi}(\cdot/\epsilon)\widehat{f_1}(\cdot)](x)\mathcal{F}^{-1}[\widehat{\psi}\widehat{f_2}](x)\|_{L^2} \le C\epsilon^{-s_1+n/2}\|f_1\|_{L^2}\|f_2\|_{L^\infty}. \tag{7.7}$$

We test (7.7) for

$$\widehat{f_1}(\xi_1) = \epsilon^{-n/2}\widehat{\varphi}(\xi_1/\epsilon), \qquad f_2(x) = e^{i\eta^\circ\cdot x},$$

where we choose η° so that $\widehat{\psi}(\eta^\circ) \neq 0$. Then

$$\begin{aligned}
(\text{the left hand side of (7.7)}) &= \|\mathcal{F}^{-1}[\epsilon^{-n/2}\widehat{\varphi}(\cdot/\epsilon)^2]\|_{L^2}|\widehat{\psi}(\eta^\circ)| = \|\widehat{\varphi}^2\|_{L^2}|\widehat{\psi}(\eta^\circ)| = C,\\
(\text{the right hand side of (7.7)}) &= C\epsilon^{-s_1+n/2}\|\mathcal{F}^{-1}[\epsilon^{-n/2}\widehat{\varphi}(\cdot/\epsilon)]\|_{L^2}\\
&= C\epsilon^{-s_1+n/2}\|\widehat{\varphi}\|_{L^2} = C\epsilon^{-s_1+n/2}.
\end{aligned}$$

Thus (7.7) holds only if $s_1 \ge n/2$. Proposition 7.1 is proved. □

Proposition 7.2. *Let $0 < p \le 1$. Then the estimate*

$$\begin{aligned}
&\|T_m(f_1, f_2, \dots, f_N)\|_{L^p(\mathbb{R}^n)}\\
&\quad \le C\sup_{j\in\mathbb{Z}}\|m_j\|_{W^{(s_1,s_2,\dots,s_N)}(\mathbb{R}^{Nn})}\|f_1\|_{H^p(\mathbb{R}^n)}\|f_2\|_{L^\infty(\mathbb{R}^n)}\cdots\|f_N\|_{L^\infty(\mathbb{R}^n)}
\end{aligned} \tag{7.8}$$

holds only if $s_1 \ge n(1/p - 1/2)$ and $s_2, \dots, s_N \ge n/2$.

Proof. We give the proof in the case $N = 2$. Generalization to $N \ge 3$ will be obvious. The necessity of the condition $s_2 \ge n/2$ can be proved in the same way as in the first part of the proof of Proposition 7.1.

To prove the necessity of the condition $s_1 \ge n(1/p - 1/2)$, we take the functions ψ and φ as in the proof of Proposition 7.1 and take a $\zeta^\circ \in \mathbb{R}^n$ such that $|\zeta^\circ| = 1/10$. We set

$$m(\xi_1, \xi_2) = \widehat{\varphi}((\xi_1 - \zeta^\circ)/\epsilon)\widehat{\psi}(\xi_2). \tag{7.9}$$

For sufficiently small ϵ,

$$\begin{aligned}
\operatorname{supp} m &\subset \{(\xi_1, \xi_2) \in \mathbb{R}^{2n} \mid |\xi_1 - \zeta^\circ| \le \epsilon,\ 9/10 \le |\xi_2| \le 11/10\}\\
&\subset \{\xi \in \mathbb{R}^{2n} \mid 2^{-1/2+\alpha} \le |\xi| \le 2^{1/2-\alpha}\}
\end{aligned}$$

and hence (7.3) holds again and we have

$$\begin{aligned}
\sup_{j\in\mathbb{Z}}\|m_j\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})} &= \|m\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})}\\
&= \|\widehat{\varphi}((\xi_1 - \zeta^\circ)/\epsilon)\widehat{\psi}(\xi_2)\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})}\\
&= \|\widehat{\varphi}((\cdot - \zeta^\circ)/\epsilon)\|_{W^{s_1}(\mathbb{R}^n)}\|\widehat{\psi}\|_{W^{s_2}(\mathbb{R}^n)}\\
&\le C\epsilon^{-s_1+n/2}.
\end{aligned}$$

Thus the inequality (7.8) for m of (7.9) implies

$$\|\mathcal{F}^{-1}[\widehat{\varphi}((\cdot - \zeta^\circ)/\epsilon)\widehat{f_1}(\cdot)](x)\mathcal{F}^{-1}[\widehat{\psi}\widehat{f_2}](x)\|_{L^p} \le C\epsilon^{-s_1+n/2}\|f_1\|_{H^p}\|f_2\|_{L^\infty}. \tag{7.10}$$

We test (7.10) for

$$f_1(x) = \psi'(x), \qquad f_2(x) = e^{i\eta^\circ\cdot x},$$

where ψ' and η° are chosen so that

$$\begin{aligned}
&\psi' \in \mathcal{S}(\mathbb{R}^n), \quad \operatorname{supp}\widehat{\psi'} \text{ is a compact subset of } \mathbb{R}^n \setminus \{0\},\\
&\qquad\qquad \widehat{\psi'}(\xi_1) = 1 \text{ in a neighborhood of } \zeta^\circ,\\
&\eta^\circ \in \mathbb{R}^n, \qquad \widehat{\psi}(\eta^\circ) \neq 0.
\end{aligned}$$

Then

$$\begin{aligned}
(\text{the left hand side of (7.10)}) &= \|\mathcal{F}^{-1}[\widehat{\varphi}((\cdot - \zeta^\circ)/\epsilon)](x)\|_{L^p}|\widehat{\psi}(\eta^\circ)|\\
&= \epsilon^{n-n/p}\|\varphi\|_{L^p}|\widehat{\psi}(\eta^\circ)| = C\epsilon^{n-n/p},\\
(\text{the right hand side of (7.10)}) &= C\epsilon^{-s_1+n/2}\|\psi'\|_{H^p} = C\epsilon^{-s_1+n/2}.
\end{aligned}$$

Thus (7.10) holds only if $s_1 \ge n/p - n/2$. Proposition 7.2 is proved. □

8. Related results and comments

As a corollary of Theorems 1.1 and 1.2, we can also prove the boundedness of T_m from $L^{p_1} \times L^{p_2} \times L^{p_3}$ to L^p for $1 < p \le 2$ and for m satisfying the product type estimate. (We have treated the case $0 < p \le 1$ in Theorem 6.1). In fact, by Theorems 1.1 and 1.2,

$$\|T_m\|_{L^2 \times L^\infty \times L^\infty \to L^2} \le C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(n/2+\epsilon, n/2+\epsilon, n/2+\epsilon)}},$$
$$\|T_m\|_{H^1 \times L^\infty \times L^\infty \to L^1} \le C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(n/2+\epsilon, n/2+\epsilon, n/2+\epsilon)}}$$

Then, it follows from Step 1 in Section 6 that

$$\|T_m\|_{L^p \times L^\infty \times L^\infty \to L^p} \le C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(n/2+\epsilon, n/2+\epsilon, n/2+\epsilon)}}, \tag{8.1}$$

where $1 < p \le 2$. By interchanging the role of p_1 and p_2 or p_3, we have

$$\|T_m\|_{L^\infty \times L^p \times L^\infty \to L^p} \le C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(n/2+\epsilon, n/2+\epsilon, n/2+\epsilon)}},$$
$$\|T_m\|_{L^\infty \times L^\infty \times L^p \to L^p} \le C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(n/2+\epsilon, n/2+\epsilon, n/2+\epsilon)}}.$$

Hence, by the same argument as in Section 6, Step 2, we obtain

$$\|T_m\|_{L^{p_1} \times L^{p_2} \times L^{p_3} \to L^p} \le C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(n/2+\epsilon, n/2+\epsilon, n/2+\epsilon)}}, \tag{8.2}$$

where $1/p_1 + 1/p_2 + 1/p_3 = 1/p$, $1 < p_j \le \infty$ and $1 < p \le 2$.

We remark that (8.2) also holds for all $2 < p_1, p_2, p_3, p < \infty$ satisfying $1/p_1 + 1/p_2 + 1/p_3 = 1/p$ (see [9, Theorem 6.2]). But at present, it is unclear to the authors whether one or more of indices p_1, p_2, p_3 can be equal to ∞ in the case $2 < p < \infty$.

We end this section by giving the remark on duality. For $m \in L^\infty(\mathbb{R}^{Nn})$ and $1 \le k \le N$, we set

$$m^{*k}(\xi) = m(\xi_1, \dots, \xi_{k-1}, -(\xi_1 + \dots + \xi_N), \xi_{k+1}, \dots, \xi_N),$$

where $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} T_m(f_1, \dots, f_N) g \, dx = \int_{\mathbb{R}^n} T_{m^{*k}}(f_1, \dots, f_{k-1}, g, f_{k+1}, \dots, f_N) f_k \, dx \tag{8.3}$$

for all $f_1, \dots, f_N, g \in \mathcal{S}(\mathbb{R}^n)$. The formula (8.3) says that the boundedness of T_m from $L^p \times L^\infty \times L^\infty$ to L^p is equivalent to that of $T_{m^{*1}}$ from $L^{p'} \times L^\infty \times L^\infty$ to $L^{p'}$. However, in the framework of Sobolev spaces of product type we cannot use the duality argument, because Sobolev spaces of product type are not invariant under the map $m \mapsto m^{*1}$. More precisely, the following inequality does not hold:

$$\sup_{j \in \mathbb{Z}} \|m(-2^j(\xi_1 + \xi_2 + \xi_3), 2^j \xi_2, 2^j \xi_3) \Psi(\xi)\|_{W^{(s_1, s_2, s_3)}} \le C \sup_{j \in \mathbb{Z}} \|m(2^j \xi) \Psi(\xi)\|_{W^{(s_1, s_2, s_3)}}, \tag{8.4}$$

where $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ and Ψ is as in (1.2) with $d = 3n$.

It should be pointed out that (8.4) holds if we replace $W^{(s_1, s_2, s_3)}$ by the (usual) Sobolev space W^s with $s > 3n/2$ (see [11, 21]). Then, since $W^{3(n/2+\epsilon)} \hookrightarrow W^{(n/2+\epsilon, n/2+\epsilon, n/2+\epsilon)}$, it follows from (8.1) and duality that

$$\begin{aligned} \|T_m\|_{L^{p'} \times L^\infty \times L^\infty \to L^{p'}} &= \|T_{m^{*1}}\|_{L^p \times L^\infty \times L^\infty \to L^p} \le C \sup_{j \in \mathbb{Z}} \|(m^{*1})_j\|_{W^{(n/2+\epsilon, n/2+\epsilon, n/2+\epsilon)}} \\ &\le C \sup_{j \in \mathbb{Z}} \|(m^{*1})_j\|_{W^{3(n/2+\epsilon)}} \le C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{3(n/2+\epsilon)}}, \end{aligned}$$

where $(m^{*1})_j(\xi) = m^{*1}(2^j \xi) \Psi(\xi)$ and $1 < p < 2$. Therefore, if $\sup_{j \in \mathbb{Z}} \|m_j\|_{W^s} < \infty$ with $s > 3n/2$, then T_m is bounded from $L^p \times L^\infty \times L^\infty$ to L^p for $2 < p < \infty$.

Appendix A

We shall give proofs of Lemma 2.1 and Proposition 2.2.

Proof of Lemma 2.1. We only consider the case $N = 3$; the argument can be immediately extended to the case $N \geq 2$. Suppose $F \in W^{(s,s,s)}(\mathbb{R}^{3n})$ and $\operatorname{supp} F \subset \{|x| \leq r\}$, where $x = (x_1, x_2, x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$. Take a $\Phi \in \mathcal{S}(\mathbb{R}^{3n})$ such that $\Phi = 1$ on $\{|x| \leq r\}$ and $\operatorname{supp}\Phi \subset \{|x| \leq 2r\}$. Then $F(x) = \Phi(x)F(x)$. Hence, by Schwarz's inequality and Young's inequality,

$$
\begin{aligned}
&\|\widehat{F}\|_{L^q(w_{sq})}^q = \frac{1}{(2\pi)^{3nq}} \int_{\mathbb{R}^{3n}} \langle\xi_1\rangle^{sq}\langle\xi_2\rangle^{sq}\langle\xi_3\rangle^{sq} |\widehat{\Phi} * \widehat{F}(\xi)|^q \, d\xi \\
&\leq C \int_{\mathbb{R}^{3n}} \left(\int_{\mathbb{R}^{3n}} \langle\xi_1-\eta_1\rangle^s\langle\xi_2-\eta_2\rangle^s\langle\xi_3-\eta_3\rangle^s |\widehat{\Phi}(\xi-\eta)| \langle\eta_1\rangle^s\langle\eta_2\rangle^s\langle\eta_3\rangle^s |\widehat{F}(\eta)| \, d\eta \right)^q d\xi \\
&\leq C \left(\sup_{\xi\in\mathbb{R}^{3n}} \int_{\mathbb{R}^{3n}} \langle\xi_1-\eta_1\rangle^s\langle\xi_2-\eta_2\rangle^s\langle\xi_3-\eta_3\rangle^s |\widehat{\Phi}(\xi-\eta)| \langle\eta_1\rangle^s\langle\eta_2\rangle^s\langle\eta_3\rangle^s |\widehat{F}(\eta)| \, d\eta \right)^{q-2} \\
&\times \left\{ \int_{\mathbb{R}^{3n}} \left(\int_{\mathbb{R}^{3n}} \langle\xi_1-\eta_1\rangle^s\langle\xi_2-\eta_2\rangle^s\langle\xi_3-\eta_3\rangle^s |\widehat{\Phi}(\xi-\eta)| \langle\eta_1\rangle^s\langle\eta_2\rangle^s\langle\eta_3\rangle^s |\widehat{F}(\eta)| \, d\eta \right)^2 d\xi \right\} \\
&\leq C \left(\|\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s \widehat{\Phi}\|_{L^2} \|\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s \widehat{F}\|_{L^2} \right)^{q-2} \\
&\qquad \times \left(\|\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s \widehat{\Phi}\|_{L^1} \|\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s \widehat{F}\|_{L^2} \right)^2 \\
&= C \|\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s \widehat{F}\|_{L^2}^q = C\|F\|_{W^{(s,s,s)}}^q.
\end{aligned}
$$

The proof is complete. □

Proof of Proposition 2.2. We only consider the case $N = 3$; the argument is easily extended to the case of a general N. Since

$$
\langle\xi_1\rangle^{s_1}\langle\xi_2\rangle^{s_2}\langle\xi_3\rangle^{s_3} \leq C\left(\langle\xi_1-\eta_1\rangle^{s_1} + \langle\eta_1\rangle^{s_1}\right)\left(\langle\xi_2-\eta_2\rangle^{s_2} + \langle\eta_2\rangle^{s_2}\right)\left(\langle\xi_3-\eta_3\rangle^{s_3} + \langle\eta_3\rangle^{s_3}\right),
$$

we see that

$$
\begin{aligned}
\|FG\|_{W^{(s_1,s_2,s_3)}} &= \frac{1}{(2\pi)^{3n}} \left(\int_{\mathbb{R}^{3n}} \langle\xi_1\rangle^{2s_1}\langle\xi_2\rangle^{2s_2}\langle\xi_3\rangle^{2s_3} \left| \int_{\mathbb{R}^{3n}} \widehat{F}(\xi-\eta)\widehat{G}(\eta)\, d\eta \right|^2 d\xi \right)^{1/2} \\
&\leq C \sum_{i_1,i_2,i_3=0}^{1} \|\widehat{F}_{(i_1,i_2,i_3)} * \widehat{G}_{(1-i_1,1-i_2,1-i_3)}\|_{L^2},
\end{aligned}
$$

where

$$
\begin{aligned}
\widehat{F}_{(i_1,i_2,i_3)}(\xi) &= \langle\xi_1\rangle^{i_1 s_1}\langle\xi_2\rangle^{i_2 s_2}\langle\xi_3\rangle^{i_3 s_3}|\widehat{F}(\xi)|, \\
\widehat{G}_{(1-i_1,1-i_2,1-i_3)}(\xi) &= \langle\xi_1\rangle^{(1-i_1)s_1}\langle\xi_2\rangle^{(1-i_2)s_2}\langle\xi_3\rangle^{(1-i_3)s_3}|\widehat{G}(\xi)|.
\end{aligned}
$$

It is not difficult to estimate $\widehat{F}_{(1,1,1)} * \widehat{G}_{(0,0,0)}$ and $\widehat{F}_{(0,0,0)} * \widehat{G}_{(1,1,1)}$. In fact, since $s_1, s_2, s_3 > n/2$, by Young's inequality and Schwarz's inequality, we have

$$
\begin{aligned}
\|\widehat{F}_{(1,1,1)} * \widehat{G}_{(0,0,0)}\|_{L^2} &\leq \|\widehat{F}_{(1,1,1)}\|_{L^2}\|\widehat{G}_{(0,0,0)}\|_{L^1} = \|F\|_{W^{(s_1,s_2,s_3)}}\|\widehat{G}\|_{L^1} \\
&\leq C\|F\|_{W^{(s_1,s_2,s_3)}}\|\langle\xi_1\rangle^{s_1}\langle\xi_2\rangle^{s_2}\langle\xi_3\rangle^{s_3}\widehat{G}\|_{L^2} = C\|F\|_{W^{(s_1,s_2,s_3)}}\|G\|_{W^{(s_1,s_2,s_3)}}
\end{aligned}
$$

and the similar estimate for $\widehat{F}_{(0,0,0)} * \widehat{G}_{(1,1,1)}$.

As an example of the remaining terms, let us consider $\widehat{F}_{(1,0,1)} * \widehat{G}_{(0,1,0)}$. By Minkowski's inequality for integrals and Young's inequality,

$$
\begin{aligned}
&\left\| \int_{\mathbb{R}^{3n}} \widehat{F}_{(1,0,1)}(\xi_1-\eta_1, \xi_2-\eta_2, \xi_3-\eta_3)\widehat{G}_{(0,1,0)}(\eta_1,\eta_2,\eta_3)\, d\eta_1 d\eta_2 d\eta_3 \right\|_{L^2_{\xi_1,\xi_2,\xi_3}} \\
&\leq \left\| \int_{\mathbb{R}^{2n}} \left\| \int_{\mathbb{R}^n} \widehat{F}_{(1,0,1)}(\xi_1-\eta_1, \xi_2-\eta_2, \xi_3-\eta_3)\widehat{G}_{(0,1,0)}(\eta_1,\eta_2,\eta_3)\, d\eta_1 \right\|_{L^2_{\xi_1}} d\eta_2 d\eta_3 \right\|_{L^2_{\xi_2,\xi_3}} \\
&\leq \left\| \int_{\mathbb{R}^{2n}} \left\| \widehat{F}_{(1,0,1)}(\xi_1, \xi_2-\eta_2, \xi_3-\eta_3) \right\|_{L^2_{\xi_1}} \left\| \widehat{G}_{(0,1,0)}(\xi_1,\eta_2,\eta_3) \right\|_{L^1_{\xi_1}} d\eta_2 d\eta_3 \right\|_{L^2_{\xi_2,\xi_3}}.
\end{aligned}
$$

Repeating this argument, we have

$$
\begin{aligned}
&\left\| \int_{\mathbb{R}^{2n}} \left\| \widehat{F}_{(1,0,1)}(\xi_1, \xi_2 - \eta_2, \xi_3 - \eta_3) \right\|_{L^2_{\xi_1}} \left\| \widehat{G}_{(0,1,0)}(\xi_1, \eta_2, \eta_3) \right\|_{L^1_{\xi_1}} d\eta_2 d\eta_3 \right\|_{L^2_{\xi_2,\xi_3}} \\
&\le \left\| \int_{\mathbb{R}^n} \left\| \int_{\mathbb{R}^n} \left\| \widehat{F}_{(1,0,1)}(\xi_1, \xi_2 - \eta_2, \xi_3 - \eta_3) \right\|_{L^2_{\xi_1}} \left\| \widehat{G}_{(0,1,0)}(\xi_1, \eta_2, \eta_3) \right\|_{L^1_{\xi_1}} d\eta_2 \right\|_{L^2_{\xi_2}} d\eta_3 \right\|_{L^2_{\xi_3}} \\
&\ \vdots \\
&\le \left\| \left\| \left\| \widehat{F}_{(1,0,1)}(\xi_1, \xi_2, \xi_3) \right\|_{L^2_{\xi_1}} \right\|_{L^1_{\xi_2}} \right\|_{L^2_{\xi_3}} \left\| \left\| \left\| \widehat{G}_{(0,1,0)}(\xi_1, \xi_2, \xi_3) \right\|_{L^1_{\xi_1}} \right\|_{L^2_{\xi_2}} \right\|_{L^1_{\xi_3}} .
\end{aligned}
$$

By Schwarz's inequality,

$$
\begin{aligned}
&\left[\int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \widehat{F}_{(1,0,1)}(\xi_1, \xi_2, \xi_3)^2 \, d\xi_1 \right)^{1/2} d\xi_2 \right\}^2 d\xi_3 \right]^{1/2} \\
&\le C \left(\int_{\mathbb{R}^{3n}} \langle \xi_2 \rangle^{2s_2} \widehat{F}_{(1,0,1)}(\xi_1, \xi_2, \xi_3)^2 \, d\xi_1 d\xi_2 d\xi_3 \right)^{1/2} = C \|F\|_{W^{(s_1,s_2,s_3)}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \widehat{G}_{(0,1,0)}(\xi_1, \xi_2, \xi_3) \, d\xi_1 \right)^2 d\xi_2 \right\}^{1/2} d\xi_3 \\
&\le C \left(\int_{\mathbb{R}^{3n}} \langle \xi_1 \rangle^{2s_1} \langle \xi_3 \rangle^{2s_3} \widehat{G}_{(0,1,0)}(\xi_1, \xi_2, \xi_3)^2 \, d\xi_1 d\xi_2 d\xi_3 \right)^{1/2} = C \|G\|_{W^{(s_1,s_2,s_3)}}.
\end{aligned}
$$

Thus $\|\widehat{F}_{(1,0,1)} * \widehat{G}_{(0,1,0)}\|_{L^2} \le C\|F\|_{W^{(s_1,s_2,s_3)}} \|G\|_{W^{(s_1,s_2,s_3)}}$. We can estimate the other terms in the same way. □

Appendix B

We explain the connection of the Kato-Ponce inequality (1.7) and multilinear multipliers of limited smoothness.

For matters of simplicity in the presentation we take $N = 2$ and we work with Schwartz functions f and g. Introduce a smooth bump Ψ which is supported in the annulus $6/7 < |\xi| < 2$ and is equal to one on $1 < |\xi| < 12/7$ and such that

$$\sum_j \Psi(2^{-j}\xi) = 1$$

for all $\xi \neq 0$. Let Δ_j be the associated Littlewood-Paley operator given by $\widehat{\Delta_j(f)}(\xi) = \Psi(2^{-j}\xi)\widehat{f}(\xi)$. Then we have

$$fg = \sum_{j,k} \Delta_j(f)\Delta_k(g)$$

and this identity holds for every $x \in \mathbb{R}^n$. We introduce the operator $S_k = \sum_{j \le k} \Delta_j$ and we note that it is given by multiplication on the Fourier transform by a function $\Phi(2^{-k}\xi)$ which is equal to one on the ball $|\xi| \le 2^k$.

We write the product fg as a sum of three terms:

$$
\begin{aligned}
\Pi_1(f,g) &= \sum_{j<k-1} \Delta_j(f)\Delta_k(g), \\
\Pi_2(f,g) &= \sum_{k<j-1} \Delta_j(f)\Delta_k(g), \\
\Pi_3(f,g) &= \sum_{|j-k|\le 1} \Delta_j(f)\Delta_k(g).
\end{aligned}
$$

Let $s > 0$. Then

$$D^s(\Pi_1(f,g))(x) = \frac{1}{(2\pi)^{2n}} \sum_k \int\int e^{ix\cdot(\xi+\eta)} \widehat{S_{k-2}(f)}(\xi)\widehat{\Delta_k(g)}(\eta)|\xi+\eta|^s d\xi d\eta,$$

which equals

$$D^s(\Pi_1(f,g))(x) = \frac{1}{(2\pi)^{2n}} \int\int \widehat{f}(\xi)|\eta|^s\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)} \left[\sum_k \Phi(2^{-k+2}\xi)\Psi(2^{-k}\eta)\frac{|\xi+\eta|^s}{|\eta|^s}\right] d\xi d\eta,$$

and the expression inside the square brackets is a bilinear Coifman-Meyer multiplier, hence boundedness holds. A similar argument applies for Π_2.

Now we look at Π_3. For simplicity let us only consider the term where $j = k$, i.e.,

$$\sum_j \Delta_j(f)\Delta_j(g).$$

Then we write

$$\begin{aligned}
D^s(\Pi_3(f,g)) &= \sum_k \Delta_k D^s\Big(\sum_j \Delta_j(f)\Delta_j(g)\Big)\\
&= \sum_k D^s\Delta_k\Big(\sum_{j\geq k-2} \Delta_j(f)\Delta_j(g)\Big)\\
&= \sum_k 2^{ks}\widetilde{\Delta}_k\Big(\sum_{j\geq k-2} \Delta_j(f)\Delta_j(g)\Big)\\
&= \sum_k 2^{ks}\widetilde{\Delta}_k\Big(\sum_{j'\geq -2} \Delta_{j'+k}(f)\Delta_{j'+k}(g)\Big)\\
&= \sum_{j'\geq -2} 2^{-sj'} \sum_k \widetilde{\Delta}_k\Big(2^{(j'+k)s}\Delta_{j'+k}(f)\Delta_{j'+k}(g)\Big)\\
&= \sum_{j\geq -2} 2^{-sj} \sum_k \widetilde{\Delta}_k\Big(\Delta'_{j+k}(D^s f)\Delta_{j+k}(g)\Big),
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{\Delta}_k f &= \mathcal{F}^{-1}\left(\widehat{f}(\xi)|2^{-k}\xi|^s\Psi(2^{-k}\xi)\right),\\
\Delta'_k f &= \mathcal{F}^{-1}\left(\widehat{f}(\xi)|2^{-k}\xi|^{-s}\Psi(2^{-k}\xi)\right).
\end{aligned}$$

The symbol of the preceding bilinear operator is

$$\sum_{j\geq -2} 2^{-sj} \sum_{k\in Z} \Theta(2^{-k}(\xi+\eta))\Psi_1(2^{-(j+k)}\xi)\Psi(2^{-(j+k)}\eta)$$

with $\Theta(\xi) = |\xi|^s\Psi(\xi)$ and $\Psi_1(\xi) = |\xi|^{-s}\Psi(\xi)$. This is a type of multiplier of limited smoothness, which is studied in this work. The study of the Kato-Ponce inequality via this approach is motivated by the work of Christ and Weinstein [3].

References

[1] Á. Bényi and R. Torres, Symbolic calculus and the transpose of bilinear pseudodifferential operators, Comm. PDE **28** (2003), 1161–1181.

[2] A. P. Calderón and A. Torchinsky, Parabolic maximal functions associated with a distribution II, Adv. in Math. **24** (1977), 101–171.

[3] F. M. Christ and M. I. Weinstein, Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation, J. Funct. Anal. **100** (1991), 87–109.

[4] R. R. Coifman and Y. Meyer, On commutators of singular integrals and bilinear singular integrals, Trans. Amer. Math. Soc. **212** (1975), 315–331.

[5] R. R. Coifman and Y. Meyer, Au delà des opérateurs pseudo-différentiels, Astérisque **57** (1978), 1–185.

[6] R. R. Coifman and Y. Meyer, Commutateurs d'intégrales singulières et opérateurs multilinéaires, Ann. Inst. Fourier (Grenoble) **28** (1978), 177–202.

[7] J. Duoandikoetxea, Fourier Analysis, Graduate Studies in Mathematics 29, Amer. Math. Soc., Providence, RI, 2001.

[8] C. Fefferman and E.M. Stein, H^p spaces of several variables, Acta Math. **129** (1972), 137–193.

[9] M. Fujita and N. Tomita, Weighted norm inequalities for multilinear Fourier multipliers, Trans. Amer. Math. Soc, to appear.

[10] L. Grafakos and N. Kalton, Multilinear Calderón-Zygmund operators on Hardy spaces, Collect. Math. **52** (2001), 169–179.

[11] L. Grafakos and Z. Si, The Hörmander multiplier theorem for multilinear operators, J. Reine Angew. Math., DOI: 10.1515/crelle.2011.137, to appear.

[12] L. Grafakos and R. Torres, Multilinear Calderón-Zygmund theory, Adv in Math. **165** (2002), 124–164.

[13] L. Hörmander, Estimates for translation invariant operators in L^p spaces, Acta Math. **104** (1960), 93–140.

[14] S. Janson and P. W. Jones, Interpolation between H^p spaces: The complex method, J. Funct. Anal. **48** (1982), 58–80.

[15] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, Comm. Pure Appl. Math. 41 (1988), 891–907.

[16] C. Kenig and E. M. Stein, Multilinear estimates and fractional integrals, Math. Res. Lett. **6** (1999), 1–15.

[17] A. Lerner, S. Ombrosi, C. Pérez, R. Torres and R. Trujillo-González, New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory, Adv. in Math. **220** (2009), 1222–1264.

[18] A. Miyachi and N. Tomita, Minimal smoothness conditions for bilinear Fourier multipliers, Rev. Mat. Iberoam., to appear.

[19] E. M. Stein, Harmonic Analysis, Real Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton University Press, Princeton, 1993.

[20] E. M. Stein and G. Weiss, On the interpolation of analytic families of operators acting on H^p-spaces, Tôhoku Math. J. **9** (1957), 318–339.

[21] N. Tomita, A Hörmander type multiplier theorem for multilinear operators, J. Funct. Anal. **259** (2010), 2028–2044.

Loukas Grafakos, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: grafakosl@missouri.edu

Akihiko Miyachi, Department of Mathematics, Tokyo Woman's Christian University, Zempukuji, Suginami-ku, Tokyo 167-8585, Japan
E-mail address: miyachi@lab.twcu.ac.jp

Naohito Tomita, Department of Mathematics, Osaka University, Toyonaka, Osaka 560-0043, Japan
E-mail address: tomita@math.sci.osaka-u.ac.jp