ARE L^2-BOUNDED HOMOGENEOUS SINGULAR INTEGRALS NECESSARILY L^p-BOUNDED?

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ABSTRACT. We present a dyadic one-dimensional version of the construction of even integrable functions Ω on the unit sphere $\mathbf{S}^{d-1}$ with mean value zero satisfying

$$\operatorname{es\,sup}_{\xi\in\mathbf{S}^{d-1}} \int_{\mathbf{S}^{d-1}} |\Omega(\theta)| \log \frac{1}{|\theta\cdot\xi|}\, d\theta < +\infty\,,$$

such that the singular integral operator T_Ω given by convolution with the distribution p.v. $\Omega(x/|x|)|x|^{-d}$ is bounded on $L^p(\mathbf{R}^d)$ if and only if $p=2$.

1. Introduction and statements of results

Let Ω be an even complex-valued integrable function on the sphere $\mathbf{S}^{d-1}$, with mean value zero with respect to the surface measure. The classical theory of singular integral operators says that the Calderón and Zygmund principal-value singular integral initially defined for functions f in the Schwartz class $\mathcal{S}(\mathbf{R}^d)$

$$T_\Omega(f)(x) = \lim_{\varepsilon\to 0} \int\limits_{|y|>\varepsilon} \frac{\Omega(y/|y|)}{|y|^d} f(x-y)\, dy, \tag{1}$$

is given by a convolution with the distribution p.v. $\Omega(x/|x|)|x|^{-d}$, whose Fourier transform is the homogeneous of degree zero function

$$m(\Omega)(\xi) := (\text{p.v.}\,\Omega(x/|x|)|x|^{-d})\widehat{\ }(\xi) = \int_{\mathbf{S}^{d-1}} \Omega(\theta)\log\frac{1}{|\xi\cdot\theta|}\, d\theta\,. \tag{2}$$

Thus, the L^2 boundedness of T_Ω is equivalent to the condition that $m(\Omega)$ is an essentially bounded function, i.e. $m(\Omega)\in L^\infty(\mathbf{R}^d)$. The theory of singular integrals of the form (1) was developed by Calderón and Zygmund [1], [2] who established their L^p boundedness in the range $1<p<\infty$ for Ω in $L\log L(\mathbf{S}^{d-1})$. It was proved by Weiss and Zygmund [8] that T_Ω may be unbounded even on L^2 for Ω in $L(\log L)^{1-\varepsilon}(\mathbf{S}^{n-1})$ when $\varepsilon>0$. Thus the $L\log L$ condition on Ω is the sharpest possible, in this sense, that implies the L^p boundedness for in the whole range of $p\in(1,\infty)$. The weak type $(1,1)$ boundedness of such singular integrals with Ω in $L\log L(\mathbf{S}^{d-1})$ was studied much later by Christ and Rubio de Francia [3] and Seeger [7].

In [5] the following result was established:

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Theorem 1. *There is an integrable function Ω with mean value zero on the unit sphere $\mathbf{S}^{n-1}$, satisfying*

$$\operatorname*{es\,sup}_{\xi\in\mathbf{S}^{d-1}} \int_{\mathbf{S}^{d-1}} |\Omega(\theta)| \log \frac{1}{|\xi\cdot\theta|}\, d\theta < \infty\,, \tag{3}$$

but such that T_Ω is L^p bounded exactly when $p = 2$.

In this note we consider the one-dimensional dyadic model D_Ω of T_Ω,

$$\widehat{D_\Omega f}(x) = m(\Omega)(x)\widehat{f}(x), \qquad m(\Omega)(x) = \chi_{[0,1]}(x)\sum_{I\ni x}\int_I \Omega(y)dy, \qquad x\in\mathbf{R}. \tag{4}$$

Here the sum is extended over all dyadic subintervals I of $[0,1]$, and Ω is a nonnegative function in $L^1([0,1])$. We observe that

$$\sum_{I\ni x}\int_I \Omega(y)dy = \int_0^1 \sum_{I\ni x,y}\chi_I(y)\Omega(y)dy \le \int_0^1 \log\frac{1}{|x-y|}\Omega(y)dy,$$

provided x does not belong to a set of ends of dyadic intervals. We prove the following

Theorem 2. *There exists a nonnegative function $\Omega \in L^1([0,1])$ such that $m(\Omega)$ is bounded and is not a L^p Fourier multiplier for any $p \neq 2$.*

To show that the multiplier norm $\|m(\Omega)\|_{M_p(\mathbf{R})}$ is infinite for $p \neq 2$, we use deLeeuw [4] type result which comes from the work of Lebedev and Olevski [6]:

Theorem 3. *Let b be a function on the real line and let y_j be a sequence of real numbers such that $y_{j+1} - y_j$ is a constant for all j. Assume that the function b is regulated at the points y_j, i.e. the average of left and right limits of b at each y_j coincides with $b(y_j)$. Then we have*

$$\|b\|_{M_p(\mathbf{R})} \ge \|\{b(y_j)\}_j\|_{M_p(\mathbf{Z})}.$$

Here $\|\{b(y_j)\}_j\|_{M_p(\mathbf{Z})}$ is the norm of the operator $f \to \sum_j b(y_j)\widehat{f}(j)e^{2\pi ijx}$ acting on functions f on the circle $[0,1]$. For compactly supported sequences this norm is at most the size of the support of the sequence times its L^∞ norm.

Given a compactly supported sequence $\{\epsilon_j\}_j$ with a large norm $\|\{\epsilon_j\}_j\|_{M_p(\mathbf{Z})}$ we will construct an integrable function Ω and take an arithmetic progression $\{x_j\}_j$ such that $\|\{m(\Omega)(x_j)\}_j\|_{M_p(\mathbf{Z})} \ge c\|\{\epsilon_j\}_j\|_{M_p(\mathbf{Z})}$.

2. Proof of Theorem 2

To pick up a sequence $\{\epsilon_j\}_j$ with a large multiplier norm, we use the fact that the Riesz basis of $L^p(\mathbf{T})$, $\{e^{2\pi ijx}\}_{j=-\infty}^{+\infty}$ is not unconditional for $p \neq 2$. That means that for any $K > 0$ we can find a compactly supported sequence a_j and a sequence ε_j of 0's and 1's such that

$$\|\sum_j \varepsilon_j a_j e^{2\pi ijx}\|_p \ge K\|\sum_j a_j e^{2\pi ijx}\|_p. \tag{5}$$

Consider a decreasing sequence $p_1 > p_2 > p_3 > \dots$ which converges to 2 and let a_j^k be a sequence supported in $\{1, \dots, l_k\}$ and ε_j^k be a sequence of zeros and ones such that (5) holds with $p = p_k$ and $K = k$, i.e.

$$\|\sum_{j=1}^{l_k} \varepsilon_j^k a_j^k e^{2\pi ijx}\|_{p_k} \geq k \|\sum_{j=1}^{l_k} a_j^k e^{2\pi ijx}\|_{p_k}. \tag{6}$$

To construct Ω (depending on ε_j^k), we look at $m(\Omega)$ where $\Omega = \chi_{I_0}$ is the characteristic function of any dyadic interval $I_0 \subset [0,1]$ of length 2^{-i_0}. We observe that $m(\chi_{I_0})(x) = (i_0+1)2^{-i_0}$ for $x \in I_0$, and $m(\chi_{I_0})(y) \leq n_0 2^{-i_0}$, for y outside I_0. Here n_0 is the number of dyadic subintervals of $[0,1]$ that contain both I_0 and y. This means that given any dyadic interval I and any $\delta > 0$, one can find a centrally located (within I) dyadic subinterval J of I of length 2^{-j} and a function $\Omega_{\delta,I} = 2^j \chi_J/(j+1)$ such that $m(\Omega_{\delta,I})(x) = 1$ when $x \in J$ and $m(\Omega_{\delta,I})(x) \leq \delta$ when x is not in I. Note that the L^1 norm of $\Omega_{\delta,I}$ is $1/(j+1)$, $j = -\log|J|$, and it can be made small.

We set

$$\Omega = \sum_{k=0}^{\infty} \Omega_{I_k}, \qquad \Omega_{I_k} = \sum_{j=1}^{l_k} \varepsilon_j^k \, \Omega_{\delta_{k,j}, I_{k,j}}.$$

where Ω_{I_k} are supported in I_k, the dyadic subintervals of $[0,1]$,

$$I_1 = [0, 1/2], \quad I_2 = [1/2, 3/4], \quad I_3 = [3/4, 7/8], \quad I_4 = [7/8, 15/16], \quad ....,$$

and ε_j^k are as in (6). To define $\Omega_{\delta_{k,j}, I_{k,j}}$ we pick irrational points

$$x_{k,1} < x_{k,2} < \dots < x_{k,l_k}$$

inside I_k so that the intervals spanned by two consecutive such points have the same length. We choose small disjoint subintervals $I_{k,j}$ of I_k centered at the points $x_{k,j}$ for all $j \in \{1, 2, \dots, l_k\}$. Next, we select an interval $J_{k,j} \subset I_{k,j}$ such that the function

$$\Omega_{\delta_{k,j}, I_{k,j}} = \frac{\chi_{J_{k,j}}}{|J_{k,j}|(\log(1/|J_{k,j}|) + 1)}$$

satisfies

$$m(\Omega_{\delta_{k,j}, I_{k,j}})(x) = 1 \qquad \text{when} \qquad x \in J_{k,j}, \tag{7}$$

and

$$m(\Omega_{\delta_{k,j}, I_{k,j}})(x) \leq \delta_{k,j} = 2^{-2-j-k}/l_k^2 \qquad \text{when} \qquad x \notin I_{k,j}. \tag{8}$$

We can also assume that $J_{k,j}$ satisfies

$$\log \frac{1}{|J_{k,j}|} \geq k^2 \, l_k. \tag{9}$$

Observe that (9) implies

$$\|\Omega\|_1 \leq \sum_{k=1}^{\infty} \sum_{j=1}^{l_k} \frac{1}{\log(1/|J_{k,j}|)} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

Observe also that $m(\Omega)$ is a bounded function. Indeed, let $x \in [0,1]$. Then $x \in I_n$ for some $n \geq 1$, and

$$m(\Omega)(x) \leq \sum_{k=1, k\neq n}^{\infty} \sum_{j=1}^{l_k} m(\Omega_{\delta_{k,j}, I_{k,j}})(x) + \sum_{j=1}^{l_n} m(\Omega_{\delta_{n,j}, I_{n,j}})(x). \tag{10}$$

The first term in the right hand side of (10) is bounded due to the choice of $I_{k,j}$ and $\delta_{k,j}$, see (8). To estimate the second one, we consider two cases, a) $x \in I_{n,s} \setminus J_{n,s}$ for some fixed $s = 1, 2, \dots, l_n$, and b) $x \in J_{n,s}$, or $x \in I_n \setminus I_{n,s}$. We write

$$\sum_{j=1}^{l_n} m(\Omega_{\delta_{n,j}, I_{n,j}})(x) \leq \sum_{j=1, j\neq s}^{l_n} m(\Omega_{\delta_{n,j}, I_{n,j}})(x) + m(\Omega_{\delta_{n,s}, I_{n,s}})(x). \tag{11}$$

In the case a) we have

$$m(\Omega_{\delta_{n,s}, I_{n,s}})(x) \leq \sum_{I \ni x, J_{n,s}} \int_I \Omega_{\delta_{n,s}, I_{n,s}}(y) dy \leq \frac{|J_{n,s}| \sum_{I \ni x, J_{n,s}} 1}{|J_{n,s}|(\log(1/|J_{n,s}|) + 1)} < \infty,$$

and the boundedness of the right-hand side in (11) follows from (8). In the case b) we use (7) and (8). Thus, the second term in (10) is bounded and $m(\Omega)$ is bounded.

It remains to show that $m(\Omega)$ is not an L^p Fourier multiplier for any $p \neq 2$. We fix a $p > 2$ and pick a k_0 so that $2 < p_{k_0} < p$. Then $\|m(\Omega)\|_{M_p(\mathbf{R})} \geq \|m(\Omega)\|_{M_{p_{k_0}}(\mathbf{R})}$ and it suffices to show that the latter can become arbitrarily large.

Observe that the function $m(\Omega)$ is regulated at the points $\{x_{k_0,j}\}_{j=1}^{l_{k_0}}$, (this can be easily seen by splitting $m(\Omega)(x_{k_0,j})$ into the sums similar to (10), (11)), and by Theorem 3 we have

$$\|m(\Omega)\|_{M_{p_{k_0}}(\mathbf{R})} \geq \|\{m(\Omega)(x_{k_0,j})\}_{j=1}^{l_{k_0}}\|_{M_{p_{k_0}}(\mathbf{Z})}.$$

But the last expression is at least as big as

$$\|\{m(\Omega_{I_{k_0}})(x_{k_0,j})\}_{j=1}^{l_{k_0}}\|_{M_{p_{k_0}}(\mathbf{Z})} - \|\{\sum_{k\neq k_0} m(\Omega_{I_{k_0}})(x_{k_0,j})\}_{j=1}^{l_{k_0}}\|_{M_{p_{k_0}}(\mathbf{Z})}.$$

Note that the functions $\sum_{k\neq k_0} m(\Omega_{I_k})$ are constant on the interval I_{k_0} and therefore the sequence $\{\sum_{k\neq k_0} m(\Omega_{I_{k_0}})(x_{k_0,j})\}_{j=1}^{l_{k_0}}$ is constant of length l_{k_0}. The multiplier norm of this sequence is a constant $c(p_{k_0})$ which is bounded above by a constant $c(p) = \cot(\pi/2p)$ independent of k_0. Now

$$m(\Omega_{I_{k_0}})(x_{k_0,j}) = \epsilon_j^{k_0} + E_j^{k_0},$$

where

$$E_j^{k_0} = \sum_{1\leq j' \neq j \leq l_{k_0}} \epsilon_{j'}^{k_0} m(\Omega_{\delta_{k_0,j'}, I_{k_0,j'}})(x_{k_0,j}),$$

and (8) implies $|E_j^{k_0}| \leq 2^{-2-j-k}/l_k$, $\|\{E_j^{k_0}\}_{j=1}^{l_{k_0}}\|_{M_{p_{k_0}}(\mathbf{Z})} \leq 2^{-2-j-k}$, due to the compactness of the support of $\{E_j^{k_0}\}_j$. We conclude that $\|m(\Omega)\|_{M_p(\mathbf{R})} \geq k_0 - 1 - c(p)$ and this can be made arbitrarily large. Hence $\|m(\Omega)\|_{M_p(\mathbf{R})} = \infty$.

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