# ON THE $p$-INDEPENDENCE BOUNDEDNESS PROPERTY OF CALDERÓN-ZYGMUND THEORY

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ABSTRACT. For $0 \leq \alpha < 1$ we construct examples of even integrable functions $\Omega$ on the unit sphere $\mathbf{S}^{d-1}$ with mean value zero satisfying

$$\underset{\xi \in \mathbf{S}^{d-1}}{\text{es sup}} \int_{\mathbf{S}^{d-1}} |\Omega(\theta)| \log^{1+\alpha} \frac{1}{|\theta \cdot \xi|}\, d\theta < +\infty\,,$$

such that the $L^2$-bounded singular integral operator $T_\Omega$ given by convolution with the distribution p.v. $\Omega(x/|x|)|x|^{-d}$ is not bounded on $L^p(\mathbf{R}^d)$ when $\left|\frac{1}{2} - \frac{1}{p}\right| > \frac{\alpha}{1+\alpha}$. In particular, we construct operators $T_\Omega$ that are bounded on $L^p$ exactly when $p = 2$.

## 1. INTRODUCTION

Let $\Omega$ be an even complex-valued integrable function on the sphere $\mathbf{S}^{d-1}$, with mean value zero with respect to the surface measure. We discuss the $L^p$ boundedness properties of the Calderón-Zygmund singular integral operator

$$(1) \qquad T_\Omega(f)(x) = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{\Omega(y/|y|)}{|y|^d} f(x-y)\, dy = \text{p.v.} \int_{\mathbf{R}^d} \frac{\Omega(y/|y|)}{|y|^d} f(x-y)\, dy\,,$$

initially defined for functions $f$ in the Schwartz class $\mathcal{S}(\mathbf{R}^d)$. The singular integral operator $T_\Omega$ is given by convolution with the distribution p.v. $\Omega(x/|x|)|x|^{-d}$ whose Fourier transform is the homogeneous of degree zero function

$$(2) \qquad m(\Omega)(\xi) := (\text{p.v.}\, \Omega(x/|x|)|x|^{-d})^\wedge(\xi) = \int_{\mathbf{S}^{d-1}} \Omega(\theta) \log \frac{1}{|\xi \cdot \theta|}\, d\theta\,.$$

The $L^2$ boundedness of $T_\Omega$ is equivalent to the condition that

$$(3) \qquad\qquad\qquad m(\Omega) \in L^\infty(\mathbf{R}^d)\,,$$

i.e. $m(\Omega)$ is an essentially bounded function. Calderón and Zygmund [1], [2] have developed the theory of such singular integrals and have established their $L^p$ boundedness in the range $1 < p < \infty$ for $\Omega$ in $L \log L(\mathbf{S}^{d-1})$. The more difficult issue of the weak type $(1,1)$ boundedness of such singular integrals with $\Omega$ in $L \log L(\mathbf{S}^{d-1})$ was settled by Christ and Rubio de Francia [5] and Seeger [14].

The $\Omega \in L \log L$ condition is the sharpest possible, in some sense, that implies $L^p$ boundedness for $T_\Omega$ in the whole range of $p \in (1, \infty)$, as indicated by Weiss and Zygmund [16]. A fundamental question in the subject is whether there exist other conditions on $\Omega$ that are $L^p$ "sensitive", i.e. they imply that $T_\Omega$ is bounded on $L^p$ for some $p$ but not on $L^q$ for some other index $q$. This question is motivated by the well-known $p$-independence boundedness property of Calderón-Zygmund operators with sufficiently smooth kernels, i.e. the fact that boundedness on one $L^{p_0}$ implies boundedness on all $L^p$ with $1 < p < \infty$. It has been an open question whether $p$-independence is still valid for all homogeneous singular integrals and if it fails, what condition is sensitive enough to differentiate boundedness between different $L^p$ spaces.

A starting point for investigating this question is to ask whether (3) or even the slightly stronger condition

$$(4) \qquad \operatorname*{essup}_{\xi \in \mathbf{S}^{d-1}} \int_{\mathbf{S}^{d-1}} |\Omega(\theta)| \log \frac{1}{|\xi \cdot \theta|} \, d\theta < \infty \,,$$

suffices to imply that $T_\Omega$ is bounded on $L^p(\mathbf{R}^d)$ for some $p \neq 2$. In this work we prove that this is not the case. In fact, we construct examples of functions $\Omega$ that satisfy (4) such that $T_\Omega$ is bounded on $L^p(\mathbf{R}^d)$ exactly when $p = 2$. This answers the above question in the negative: *The p-independence boundedness property of Calderón-Zygmund theory fails for insufficiently smooth homogeneous kernels.*

Next, for $\alpha \geq 0$ we denote by

$$m_\alpha(\Omega)(\xi) = \int_{\mathbf{S}^{d-1}} |\Omega(\theta)| \log^{1+\alpha} \frac{1}{|\theta \cdot \xi|} \, d\theta$$

a function defined for $\xi \in \mathbf{S}^{d-1}$ that measures the integrability of $\Omega$ against a power of the logarithm that appears in (4) and we introduce the following conditions on $\Omega$:

$$(5) \qquad \operatorname*{essup}_{\xi \in \mathbf{S}^{d-1}} m_\alpha(\Omega)(\xi) < +\infty \,.$$

These conditions become stronger as $\alpha$ increases and are known to imply boundedness for $T_\Omega$ when $1/p$ lies in some nontrivial open interval centered at $1/2$, see [10], [8].

We sharpen the aforementioned result by showing the existence of functions $\Omega$ that satisfy (5) for some $\alpha > 0$ such that the corresponding operators $T_\Omega$ are unbounded on $L^p(\mathbf{R}^d)$ for $p$ away from 2. Precisely, we have the following:

**Theorem 1.** *For every $\alpha$ satisfying $0 \leq \alpha < 1$ there is an even integrable function $\Omega$ on $\mathbf{S}^{d-1}$ with mean value zero that satisfies* (5) *such that the singular integral operator $T_\Omega$ is* unbounded *on $L^p(\mathbf{R}^d)$ whenever*

$$(6) \qquad \left| \frac{1}{2} - \frac{1}{p} \right| > \frac{\alpha}{1+\alpha} \,.$$

*In particular, there is a function $\Omega$ such that $T_\Omega$ is $L^p$ bounded exactly when $p = 2$.*

## 2. The counterexample of Theorem 1

We only discuss the proof when $d = 2$ as this example can be embedded in higher dimensional spaces. We fix an $0 \leq \alpha < 1$. We construct a sequence of even functions $\Omega_n$ with mean value zero such that $m(\Omega_n)$ fails to be an $L^p$ multiplier for any $p$ satisfying (6) while $\Omega_n$ satisfies condition (5) uniformly in $n$.

To prove that the norm of a Fourier multiplier on $\mathbf{R}^2$ is large, we are going to use a deLeeuw type transference argument (see [7]) in a way similar to that used in the article of Lebedev and Olevskii [11].

**Lemma 1.** *Let $\{x_k\}_{k=1}^l$ be an arithmetic progression in $\mathbf{R}^2$. (This means that there is a vector $v$ such that $x_k + v = x_{k+1}$ for $k = 1, \ldots, l-1$.) Let $m$ be a function on $\mathbf{R}^2$ which is continuous at the points $x_k$. We define a Fourier multiplier sequence $b$ on $\mathbf{Z}$ by the formula $b(k) = m(x_k)$ for $1 \leq k \leq l$ and $b(k) = 0$ otherwise. Then for some $c_p > 0$, dependent only on $1 < p < \infty$, the $L^p(\mathbf{R}^2)$ multiplier norm $\|m\|_{\mathcal{M}_p(\mathbf{R}^2)}$ of $m$ is at least $c_p$ times the $L^p(\mathbf{T})$ multiplier norm $\|\{b(k)\}_{k \in \mathbf{Z}}\|_{\mathcal{M}_p(\mathbf{Z})}$ of the sequence $b$.*

*Proof.* To prove the lemma, by applying a translation, a dilation and a rotation, we may assume that the points $\{x_k\}_{k=1}^l$ are the points $\{(k,0)\}_{k=1}^l$. Let $R$ be the rectangle $[-\frac{1}{4}, l + \frac{1}{4}] \times [-\frac{1}{4}, \frac{1}{4}]$. The continuity of $m$ at the points $\{(k,0)\}_{k=1}^l$ allows us to use a classical transference theorem (see for instance [9] Theorem 3.6.7) to deduce that

$$c_p^{-1}\|m\|_{\mathcal{M}_p(\mathbf{R}^2)} \geq \|m\chi_R\|_{\mathcal{M}_p(\mathbf{R}^2)} \geq \|\{b(k)\}_{k \in \mathbf{Z}}\|_{\mathcal{M}_p(\mathbf{Z})} .$$

$\square$

If $p \neq 2$, the Riesz basis $\{e^{2\pi ikx}\}_{k=-\infty}^\infty$ of $L^p(\mathbf{T})$ is not unconditional. That means that for every $n = 1, 2, \ldots$ there are two sequences $a_k^n$ and $|\varepsilon_k^n| \leq 1$ supported on the set $\{1, \ldots, n\}$ such that

$$(7) \qquad \left\| \sum_{k=1}^n \varepsilon_k^n a_k^n e^{2\pi ikx} \right\|_{L^p[0,1]} \geq K(n) \left\| \sum_{k=1}^n a_k^n e^{2\pi ikx} \right\|_{L^p[0,1]},$$

where $K(n) \to \infty$ as $n \to \infty$. Using properties of the Rademacher functions we see that we can take $K(n) = c_p' n^{|1/2 - 1/p|}$, where $c_p'$ depends only on $p$.

Moreover, we can choose $(\varepsilon_k^n)_{k=1}^n$ such that

$$\|(\ldots, 0, \ldots, 0, \varepsilon_1^n, \varepsilon_2^n, \ldots, \varepsilon_n^n, 0, \ldots)\|_{M^p(\mathbf{Z})} =$$

$$\sup\{\|(\ldots, 0, \ldots, 0, \delta_1^n, \delta_2^n, \ldots, \delta_n^n, 0, \ldots)\|_{M^p(\mathbf{Z})} : |\delta_k^n| \leq 1, \ k = 1, \ldots, n \}.$$

We fix $n \geq 1000$. Denote by $I$ the angular sector from $\pi/2$ to $3\pi/4$. Fix $s_0$ large enough. For $k \in \{1, 2, \ldots, 2n\}$, we introduce points $x_k = (t_k, s_0) \in \mathbf{Z}^2 \cap I$ and disjoint open cones $I^k$ centered at the origin whose bisector passes through the points $x_k$ so that (a) the arc $I^k \cap \mathbf{S}^1$ has length roughly $(20n)^{-1}$ and (b) the distance between the points $x_k/|x_k|$ and $x_{k+1}/|x_{k+1}|$ is about $n^{-1}$.

For $x \in \mathbf{R}^2$ let

$$x^\perp = \{y \in \mathbf{R}^2 : \ x \cdot y = 0\}.$$

Now let $\tilde{x}_k$ be the single element of $x_k^\perp \cap \mathbf{S}^1$ which lies in the first quadrant (its argument lies in $(0, \pi/4)$) and let $A_k^{\epsilon_n}$, $k = 1, \ldots, 2n$, be pairwise disjoint arcs of small length $\epsilon_n$ (to be chosen later) contained in $\mathbf{S}^1$ and centered at $\tilde{x}_k$.

Define

$$(8) \qquad \omega_k^{\epsilon_n} = C(n, \epsilon_n) \sum_{j=0}^{3} (-1)^j \chi_{A_k^{\epsilon_n} + \frac{j\pi}{2}},$$

where $A_k^{\epsilon_n} + \frac{j\pi}{2}$ are the translations of the arcs $A_k^{\epsilon_n}$ along $\mathbf{S}^1$ by the amounts $\frac{j\pi}{2}$. Finally we introduce the function

$$\Omega_n = \sum_{k=1}^{2n} (-1)^k \varepsilon_{[(k+1)/2]}^n \omega_k^{\epsilon_n},$$

where $[\,]$ denotes the integer part. The normalization constant $C(n, \epsilon_n)$ is chosen so that

$$m_\alpha(\omega_k^{\epsilon_n})(x_k/|x_k|) = 1/2.$$

Set

$$D(n, \epsilon_n) = m(\omega_k^{\epsilon_n})(x_k) = m(\omega_k^{\epsilon_n})(x_k/|x_k|)$$

and note that in view of rotational invariance the constants $C(n, \epsilon_n)$ and $D(n, \epsilon_n)$ do not depend on $k$. Also notice that

$$\sup_x m_\alpha(\omega_k^{\epsilon_n})(x) = m_\alpha(\omega_k^{\epsilon_n})(x_k/|x_k|) \text{ and}$$

$$\sup_x |m(\omega_k^{\epsilon_n})(x)| = |m(\omega_k^{\epsilon_n})(x_k)| = |m(\omega_k^{\epsilon_n})(x_k/|x_k|)|.$$

We will make use of the following auxiliary estimates.

**Lemma 2.** *We have*

$$C(n, \epsilon_n) \approx \epsilon_n^{-1} |\log \epsilon_n|^{-1-\alpha}$$

*and*

$$D(n, \epsilon_n) \approx |\log \epsilon_n|^{-\alpha},$$

*for all* $x \notin \cup_{j=0}^{3} (I^k + \frac{j\pi}{2}) \cap \mathbf{S}^1$

$$(9) \qquad m_\alpha(\omega_k^{\epsilon_n})(x) \lesssim (\log n)^{1+\alpha} |\log \epsilon_n|^{-1-\alpha}$$

*and for* $x \notin (\cup_{j=0}^{3}(I^{2k} + \frac{j\pi}{2})) \cup (\cup_{j=0}^{3}(I^{2k-1} + \frac{j\pi}{2})) \cap \mathbf{S}^1$ *and* $1 \leq k \leq n$

$$(10) \qquad |\varepsilon_k^n m(\omega_{2k}^{\epsilon_n})(x) - \varepsilon_k^n m(\omega_{2k-1}^{\epsilon_n})(x)| \lesssim \frac{|\log \epsilon_n|^{-1-\alpha}}{n|x - x_{2k}/|x_{2k}||}.$$

*Proof.* These estimates are straightforward. To prove (10) we use the mean value theorem for integrals and the fact that for $\theta_1, \theta_2$ in two successive arcs $A_{2k-1}^{\epsilon_n}$, $A_{2k}^{\epsilon_n}$ we can control $|\log |x \cdot \theta_1|^{-1} - \log |x \cdot \theta_2|^{-1}|$ by a multiple of $n^{-1}|x \cdot \theta_2|^{-1}$ which is at most a multiple of $n^{-1}|x - x_{2k}/|x_{2k}||$. $\qquad \square$

Recall the fixed constants $\varepsilon_k^n$ in (7). We examine properties of the function $\Omega_n$. Observe that

$$(11) \qquad \|\Omega_n\|_{L^1(\mathbf{S}^1)} \lesssim n \, |\log \epsilon_n|^{-1-\alpha}$$

and note that in view of (9) we have

$$(12) \qquad \|m_\alpha(\Omega_n)\|_{L^\infty(\mathbf{S}^1)} \lesssim \max \left(1, n \, (\log n)^{1+\alpha} \, |\log \epsilon_n|^{-1-\alpha}\right).$$

On the other hand,

$$m(\Omega_n)(x_k) = D(n, \epsilon_n)\varepsilon_k^n + \sum_{1 \le i \ne k \le 2n} (-1)^i \varepsilon_{[(i+1)/2]}^n m(\omega_i^{\epsilon_n})(x_k) = D(n, \epsilon_n)\varepsilon_k^n + o_k^n,$$

and (10) implies $|o_k^n| \le D(n, \epsilon_n)/4$ as long as

$$\text{constant } (\log n) \, |\log \epsilon_n|^{-1-\alpha} \le \frac{1}{4}|\log \epsilon_n|^{-\alpha}$$

which is equivalent to

$$(13) \qquad \qquad n^{4\text{constant}} \lesssim \epsilon_n^{-1}.$$

The function $m(\Omega_n)$ is continuous at the points $x_k$, since $m(\omega_k^{\epsilon_n})$ is continuous at $x_k/|x_k|$ as a circular convolution of $L^1(\mathbf{S}^1)$ and $L^\infty(\mathbf{S}^1)$ functions. By Lemma 1 applied to points $x_k$ we get that the $L^p$ multiplier norm of $m(\Omega_n)$ is comparable to $C_p D(n, \epsilon_n) n^{|\frac{1}{2} - \frac{1}{p}|}$. Indeed,

$$
\begin{aligned}
\|m(\Omega_n)\|_{M^p(\mathbf{R}^2)} \;\ge\;& c_p \|(..., 0, m(\Omega_n)(x_1), m(\Omega_n)(x_2), ..., m(\Omega_n)(x_n), 0, ...)\|_{M^p(\mathbf{Z})} \\
\ge\;& c_p \, D(n, \epsilon_n) \left( \|(..., 0, ..., 0, \varepsilon_1^n, \varepsilon_2^n, ..., \varepsilon_n^n, 0, ...)\|_{M^p(\mathbf{Z})} - \right. \\
& \left. \|(..., 0, o_1^n, o_2^n, ..., o_n^n, 0, ...)/D(n, \epsilon_n)\|_{M^p(\mathbf{Z})} \right) \\
\ge\;& \frac{1}{2} c_p \, D(n, \epsilon_n) n^{|\frac{1}{2} - \frac{1}{p}|},
\end{aligned}
$$

since the inequality

$$\left\| \frac{(..., 0, o_1^n, o_2^n, ..., o_n^n, 0, ...)}{D(n, \epsilon_n)} \right\|_{M^p(\mathbf{Z})} > \frac{1}{2} \left\| (..., 0, \varepsilon_1^n, \varepsilon_2^n, ..., \varepsilon_n^n, 0, ...) \right\|_{M^p(\mathbf{Z})}$$

would contradict the choice of $(\varepsilon_k^n)_{k=1}^n$. This shows that the $L^p$ operator norm of $T_{\Omega_n}$ is at least a constant multiple of

$$(14) \qquad \qquad D(n, \epsilon_n) n^{|\frac{1}{2} - \frac{1}{p}|} \approx |\log \epsilon_n|^{-\alpha} n^{|\frac{1}{2} - \frac{1}{p}|}.$$

We select $\epsilon_n$ satisfying

$$n(\log n)^{(1+\alpha)} = |\log \epsilon_n|^{1+\alpha},$$

and we note that (13) holds for this choice of $\epsilon_n$. Also observe that (12) gives $\|m_\alpha(\Omega_n)\|_{L^\infty(\mathbf{S}^1)} \lesssim 1$, (11) yields $\|\Omega_n\|_{L^1(\mathbf{S}^1)} \lesssim 1$, while (14) gives

$$\|T_{\Omega_n}\|_{L^p \to L^p} \ge n^{|\frac{1}{2} - \frac{1}{p}| - \frac{\alpha}{1+\alpha}} (\log n)^{-\alpha}.$$

We conclude that $\|T_{\Omega_n}\|_{L^p \to L^p}$ goes to infinity with $n$ as long as $|\frac{1}{2} - \frac{1}{p}| > \frac{\alpha}{1+\alpha}$.

The existence of the function $\Omega$ claimed in Theorem 1 is a consequence of the uniform boundedness principle. Denote by $\mathcal{B}_\alpha$ the Banach space of all even integrable functions $\Omega$ on $\mathbf{S}^1$ with mean value zero with norm

$$\|\Omega\|_{\mathcal{B}_\alpha} \equiv \|\Omega\|_{L^1(\mathbf{S}^1)} + \|m_\alpha(\Omega)\|_{L^\infty(\mathbf{S}^1)} < \infty.$$

Consider the family of linear maps from $\mathcal{B}_\alpha \to L^p(\mathbf{R}^n)$

$$\Omega \to T_\Omega(f)$$

indexed by functions in the set $U = \{f \in L^p(\mathbf{R}^n) : \|f\|_{L^p} = 1\}$. If no $\Omega$ as in Theorem 1 existed, then for all $\Omega \in \mathcal{B}_\alpha$ we would have

$$\sup_{f \in U} \|T_\Omega(f)\|_{L^p} \le C(\Omega) < \infty.$$

The uniform boundedness principle implies the existence of a constant $K < \infty$ such that

$$\|T_\Omega\|_{L^p \to L^p} = \sup_{f \in U} \|T_\Omega(f)\|_{L^p} \le K \|\Omega\|_{\mathcal{B}_\alpha}$$

for all $\Omega \in \mathcal{B}_\alpha$. But this clearly contradicts the construction of the $\Omega_n$'s whenever $\left|\frac{1}{2} - \frac{1}{p}\right| > \frac{\alpha}{1+\alpha}$. This concludes the proof of Theorem 1.

## 3. Final Remarks

It is natural to ask whether boundedness holds for $T_\Omega$ outside the region ruled out by Theorem 1. This question was previously addressed and partially answered by Grafakos and Stefanov [10] who showed that condition (5) implies the boundedness of $T_\Omega$ on $L^p(\mathbf{R}^d)$ for $p$ satisfying $\left|\frac{1}{2} - \frac{1}{p}\right| < \frac{\alpha}{2(2+\alpha)}$. A sharper version of this theorem where $\frac{\alpha}{2(2+\alpha)}$ is replaced by $\frac{\alpha}{2(1+\alpha)}$ was obtained by Fan, Guo, and Pan [8].

The issue of the sufficiency of condition (5) for the $L^p$ boundedness of $T_\Omega$ remains unanswered for $p$'s satisfying $\frac{\alpha}{2(1+\alpha)} \le \left|\frac{1}{2} - \frac{1}{p}\right| \le \frac{\alpha}{1+\alpha}$ whenever $0 < \alpha < 1$. It is possible that for $\alpha > 1$, $T_\Omega$ is bounded on $L^p$ for all $1 < p < \infty$ whenever (5) is satisfied, but this is also unknown at present. We note that for zonal functions $\Omega$, condition (4) suffices for the boundedness of $T_\Omega$ on all $L^p$ spaces ($1 < p < \infty$) as proved by Ryabogin and Rubin [13].

It should be noted that the counterexamples discussed in this paper are related to those that indicate the sharpness in the Coifman-Rubio de Francia-Semmes condition [6] in terms of the $s$-variation of the multipliers. They are also related in spirit to the work of Carbery, Christ, Vance, Wainger, and Watson [3], Christ [4], Seeger, Wainger, Wright, and Ziesler [15], as well as the work of Olevskii [12].

The authors would like to thank to N. Kalton for a hint on the best value of the constant in (7).

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