MULTILINEAR PARAPRODUCTS REVISITED

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ABSTRACT. We prove that mutlilinear paraproducts are bounded from products of Lebesgue spaces $L^{p_1} \times \cdots \times L^{p_{m+1}}$ to $L^{p,\infty}$, when $1 \le p_1, \dots, p_{m+1} < \infty$, $1/p_1 + \cdots + 1/p_{m+1} = 1/p$. We focus on the endpoint case when some indices p_j are equal to 1, in particular we obtain a new proof of the estimate $L^1 \times \cdots \times L^1 \to L^{1/(m+1),\infty}$.

In memory of Nigel Kalton

1. INTRODUCTION

Paraproducts have become tools of great use in analysis and PDEs. They are traditionally built by Littlewood-Paley square functions and may appear in different forms. Paraproducts first emerged in Bony's theory of paradifferential operators [5] which has taken a step further the pseudodifferential operator theory of Coifman and Meyer [6]. They provide important examples of operators with specific properties and have been used in significant applications, such as the proof of the $T1$ theorem by David and Journé [7]. The relationship of paraproducts with Carleson measures and BMO is so intimate that the former have been on the forefront of research in harmonic analysis through almost a quarter century. The boundedness of paraproducts on L^p spaces for $p > 1$ is easily achieved via duality, but the extension to indices $p \le 1$ is more delicate and was proved independently by Grafakos and Kalton [9] and by Auscher, Hofmann, Muscalu, Thiele, and Tao [1]; a different proof was given by Bényi, Maldonado, Nahmod, and Torres [2]. Hundreds of references exist on paraproducts today; of these the articles [4], [9], [13] and [14] focus on delicate boundedness properties of them. The expository article of Bényi, Maldonado, Naibo [3] presents a well-motivated introduction to paraproducts.

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Mutilinear paraproducts may have first appeared explicitly in the work of Yabuta [16] and later resurfaced in the work of Sato and Yabuta [15] who obtained their L^p boundedness for $p \geq 1$. Although paraproducts fit into the class of multilinear Calderón-Zygmund theory, one may wonder if there are insightful direct proofs of their L^p (reps. weak L^p) boundedness, especially in the difficult case $p < 1$. Such proofs would take into account the specific form of paraproducts and would reflect the interplay of their intrinsic orthogonality with the orthogonality of L^p (reps. weak L^p). In this work we undertake this task and we include the endpoint cases when at least one index is 1. Our work is based on a weak type square function inequality (Lemma 1.2) recently obtained in [11], which is valid for all $0 < p < \infty$. Another type of m-linear paraproducts built by sums of wave packets associated with dyadic intervals on the line has been studied by Lacey and Metcalfe [12] who obtained similar endpoint estimates to the ones in this article for the paraproducts built by the *Littlewood-Paley operators.*

We will be working on $\mathbb{R}^d$ for some natural number d. For a Schwartz function Φ we denote by Δ_j^Φ the Littlewood-Paley operator given by convolution with the function $\Phi_{2^{-j}}(x) = 2^{jd}\Phi(2^j x)$. We denote by $S_j^\Phi = \sum_{k\leq j} \Delta_k^\Phi$ the partial sum operator of the Δ_k^Φ's. For fixed smooth bumps Φ and Θ whose Fourier transforms have compact supports that do not contain the origin, we define the paraproduct operator

$$P_2(f,g) = \sum_{j\in\mathbf{Z}} \sum_{k\leq j} \Delta_j^\Theta(f)\, \Delta_k^\Phi(g) = \sum_{j\in\mathbf{Z}} \Delta_j^\Theta(f) S_j^\Phi(g)\,,$$

for Schwartz functions f, g. This operator and its $(m+1)$-linear version is the main object of study of this paper. This is defined by

$$P_{m+1}(f_0, f_1, \dots, f_m) = \sum_{j\in\mathbf{Z}} \Delta_j^\Theta(f_0) S_j^{\Theta_1}(f_1) \cdots S_j^{\Theta_m}(f_m)\,,$$

for Schwarz functions $f_0, f_1, \dots, f_m$ and smooth bumps $\Theta, \Theta_1, \dots, \Theta_m$.

For $0 < p < \infty$, we denote by L^p the space of all measurable functions on $\mathbb{R}^d$ whose pth power is integrable over $\mathbb{R}^d$ and by $L^{p,\infty}$ the space of all measurable functions h that satisfy

$$\|h\|_{L^{p,\infty}} = \sup_{\lambda>0} \lambda \Big|\{x \in \mathbb{R}^d : \ |h(x)| > \lambda\}\Big|^{\frac{1}{p}} < \infty\,.$$

Given a bump Ψ, we define the square function associated with Ψ by

$$\mathbf{S}^\Psi(f) = \Big(\sum_{\ell\in\mathbf{Z}} |\Delta_\ell^\Psi(f)|^2\Big)^{\frac{1}{2}}\,.$$

We will also work with the "lacunary" square function

$$\mathbf{S}_q^{\Psi}(f) = \Big(\sum_{\ell \in \mathbf{Z}} |\Delta_{q\ell}^{\Psi}(f)|^2 \Big)^{\frac{1}{2}} .$$

defined for a positive integer q. (Notice that $\mathbf{S}_1^{\Psi} = \mathbf{S}^{\Psi}$.) Under very mild assumptions on Ψ (such as $|\Psi(x)| + |\nabla\Psi(x)| \le A(1+|x|)^{-d-\varepsilon}$ and $\int_{\mathbb{R}^d} \Psi(x)\,dx = 0$), it is known that $\mathbf{S}^{\Psi}$ (also $\mathbf{S}_q^{\Psi}$) maps $L^r(\mathbb{R}^d)$ to $L^{r,\infty}(\mathbb{R}^d)$ for all $1 \le r < \infty$ (see [8]). Finally, we denote by $\mathbf{M}$ the Hardy-Littlewood maximal operator. We recall that

$$\sup_{j\in\mathbf{Z}} |\Delta_j^{\Theta}(f)| + \sup_{j\in\mathbf{Z}} |S_j^{\Theta}(f)| \le C_{\Theta}\, \mathbf{M}(f) \,,$$

for all Schwartz functions f, for some constant C_{Θ}.

The main goal of this paper is to indicate how to obtain boundedness for P_{m+1} from the product of Lebesgue spaces $L^{p_0} \times L^{p_1} \times \cdots \times L^{p_m}$ to $L^{p,\infty}$ whenever $1 \le p_0, p_1, \dots, p_m < \infty$ and $p = (p_0^{-1} + p_1^{-1} + \cdots + p_m^{-1})^{-1}$. The case $p \ge 1$ is quite easy to deal with via duality and Hölder's inequality, but the case $p < 1$ is more delicate and we will focus on it. In particular, we show paraproducts map $L^1 \times \cdots \times L^1 \to L^{1/(m+1),\infty}$ which is the strongest endpoint estimate concerning them.

When $m = 1$ this result is known, see for instance [9], [1], [12], but the contribution of this paper is to provide a simple proof of it that does not rely on deep technical machinery (tiles, Carleson measures) and which also works for all $m \ge 1$. The following is our main result.

Theorem 1.1. *Fix an integer $m \ge 1$ and smooth bumps $\Theta, \Theta_1, \dots, \Theta_m$ whose Fourier transforms are compactly supported in $\mathbb{R}^d \setminus \{0\}$. For each $0 \le k \le m-1$ and functions f_j in the Schwartz class of $\mathbb{R}^d$ define the $(m+1)$-linear paraproduct*

$$(1) \quad P_{m+1}^{(k)}(f_0, f_1, \dots, f_m) = \sum_{j\in\mathbf{Z}} \Big[\Delta_j^{\Theta}(f_0) \prod_{s=1}^{k} \Delta_j^{\Theta_s}(f_s) \prod_{s=k+1}^{m} S_j^{\Theta_s}(f_s) \Big] ,$$

with the understanding that when $k = 0$, the first product is missing. Let p be defined by $p^{-1} = p_0^{-1} + p_1^{-1} + \cdots + p_m^{-1}$. Then $P_{m+1}^{(k)}$ is is bounded from $L^{p_0}(\mathbb{R}^d) \times L^{p_1}(\mathbb{R}^d) \times \cdots \times L^{p_m}(\mathbb{R}^d)$ to $L^{p,\infty}(\mathbb{R}^d)$ when $1 \le p_j < \infty$ and into $L^p(\mathbb{R}^d)$ when $1 < p_j < \infty$ for all j.

We will need the following lemma which is Corollary 4 in [11].

Lemma 1.2. *Let Ψ be a smooth bump whose Fourier transform is supported in an annulus that does not contain the origin and satisfies* *for some positive integer* q*:*

$$\sum_{j\in\mathbf{Z}} \widehat{\Psi}(2^{-jq}\xi) = 1, \qquad \xi \in \mathbb{R}^d \setminus \{0\}.$$

Then for any $0 < p < \infty$ *there is a constant* $C_{p,d}$ *(that also depends on* Ψ*) such that for all functions* g *in* L^2 *we have*

$$\|g\|_{L^{p,\infty}} \le C_{p,d} \big\|\mathbf{S}_q^{\Psi}(g)\big\|_{L^{p,\infty}} .$$

2. The proof of the Theorem 1.1

When all $p_j > 1$, the fact $P^{(k)}_{m+1} : \; L^{p_0} \times L^{p_1} \times \cdots \times L^{p_m} \to L^p$ is a consequence of the corresponding weak type estimate via multilinear interpolation, see [10]. It will therefore suffice to prove that $P^{(k)}_{m+1}$ maps $L^{p_0} \times L^{p_1} \times \cdots \times L^{p_m}$ to $L^{p,\infty}$ when $1/(m+1) \le p < \infty$.

We suppose that the Fourier transform of Θ is supported in the annulus $a_0 < |\xi| < b_0$ for some $0 < a_0 < b_0 < \infty$, of Θ_j is supported in the annulus $a_j < |\xi| < b_j$ for some $0 < a_j < b_j < \infty$, $1 \le j \le m$.

Case 1: $m \ge 1$ **and** $k = m - 1$**.**
Subcase 1.a: $m \ge 2$**.**

When $k = m - 1$ only one partial sum operator S_j appears in the product in (1). Then, for $m \ge 2$, $P^{(m-1)}_m(f, f_1, \ldots, f_m)$ is pointwise bounded by

$$\Big(\sum_{j\in\mathbf{Z}} |\Delta_j^{\Theta_1}(f_1)\cdots\Delta_j^{\Theta_{m-1}}(f_{m-1})|^2\Big)^{\frac12}\Big(\sum_{j\in\mathbf{Z}} |\Delta_j^{\Theta}(f_0)S_j^{\Theta_m}(f_m)|^2\Big)^{\frac12}.$$

This is in turn controlled by

$$\mathbf{S}^{\Theta_1}(f_1)\big[\mathbf{M}(f_2)\cdots\mathbf{M}(f_{m-1})\big]\,\big[\mathbf{S}^{\Theta}(f_0)\mathbf{M}(f_m)\big] \tag{2}$$

(with the understanding that the middle factor does not appear when $m = 2$) which is easily shown to satisfy the claimed conclusion, by applying Hölder's inequality on weak L^p spaces (i.e. $\|g_0g_1\cdots g_m\|_{L^{p,\infty}} \le \|g_0\|_{L^{p_0,\infty}}\|g_1\|_{L^{p_1,\infty}}\cdots\|g_m\|_{L^{p_m,\infty}}$) and using the boundedness of the maximal and square functions from L^r to $L^{r,\infty}$ for $1 \le r < \infty$.

Subcase 1.b: $m = 1$**.**

In this case we write

$$S_j^{\Theta_1} = S_{j+r_0}^{\Theta_1} + \sum_{i=j+r_0+1}^{j} \Delta_i^{\Theta_1} .$$

for some $r_0 < 0$ chosen so that the spectra of $S^{\Theta_1}_{j+r_0}$ and Δ^{Θ}_j are disjoint; picking r_0 so that $b_1 2^{r_0+j} < a_0 2^j$ suffices. Then the function

$\Delta_j^{\Theta}(f)S_{j+r_0}^{\Theta_1}(f_1)$ is supported in the annulus

$$(a_0 - b_1 2^{r_0})2^j < |\xi| < (b_0 + b_1 2^{r_0})2^j\,.$$

We pick integers $n_0 < m_0$ such that

$$2^{n_0} < a_0 - b_1 2^{r_0} < b_0 + b_1 2^{r_0} < 2^{m_0}$$

and we choose a function Ω whose Fourier transform equals 1 on the annulus $2^{n_0} < |\xi| < 2^{m_0}$, vanishes off the annulus $2^{n_0-1} < |\xi| < 2^{m_0+1}$, and satisfies

$$\sum_{\ell\in\mathbf{Z}} \widehat{\Omega}(2^{(m_0-n_0+1)\ell}\xi) = 1, \qquad \xi \in \mathbb{R}^d \setminus \{0\}. \tag{3}$$

It follows from (3) that

$$\sum_{\ell\in\mathbf{Z}} \widehat{\Omega}(2^{\ell}\xi) = m_0 - n_0 + 1, \qquad \xi \in \mathbb{R}^d \setminus \{0\}. \tag{4}$$

Then we write

$$P_2^{(0)}(f_0, f_1) = \sum_{j\in\mathbf{Z}} \Delta_j^{\Omega}\big(\Delta_j^{\Theta}(f_0)S_{j+r_0}^{\Theta_1}(f_1)\big) + E, \tag{5}$$

where E is a finite sum of terms of the form $\sum_j \Delta_j^{\Theta}(f_0)\Delta_{j+c}^{\Theta_1}(f_1)$. Since E is pointwise bounded by a constant multiple of $\mathbf{S}^{\Theta}(f_0)\mathbf{S}^{\Theta_1}(f_1)$, the required conclusion follows for E via an application of Hölder's inequality for weak type spaces.

We need to argue a bit more to handle the first term on the right in (5). We pick a function Ψ whose Fourier transform is equal to 1 on the annulus $2^{n_0-2} < |\xi| < 2^{m_0+2}$ and vanishes off the annulus $2^{n_0-3} < |\xi| < 2^{m_0+3}$. Set $q = m_0 - n_0 + 5$. We split $\mathbb{Z}$ as a disjoint union of sets $I_s = \{\ell q + s,\ \ell \in \mathbb{Z}\}$, $0 \le s \le q-1$. Next we split the sum in (5) as a finite sum over $s \in \{0, 1, \dots, q-1\}$ of the sums

$$\Sigma_s = \sum_{j\in I_s} \Delta_j^{\Omega}[\Delta_j^{\Theta}(f_0)S_{j+r_0}^{\Theta_1}(f_1)]. \tag{6}$$

We also define a function Ψ_s by setting $\widehat{\Psi_s}(\xi) = \widehat{\Psi}(2^{-s}\xi)$ and we note that $\sum_\ell \widehat{\Psi_s}(2^{-\ell q}\xi) = 1$ for ξ in $\mathbb{R}^d \setminus \{0\}$.

We make the following crucial observation: for $j \in I_s$ and $\ell \in \mathbb{Z}$ the supports of the functions $\xi \to \widehat{\Psi_s}(2^{-\ell q}\xi)$ and $\xi \to \widehat{\Omega}(2^{-j}\xi)$ intersect exactly when $j = \ell q + s$ and this case $\Delta_j^{\Omega}\Delta_{\ell q}^{\Psi_s} = \Delta_j^{\Omega}$ as the first function equals 1 on the support of the second. We deduce that for $j \in I_s$ and $\ell \in \mathbb{Z}$ we have

$$\Delta_{\ell q}^{\Psi_s}\Big[\sum_{j\in I_s} \Delta_j^{\Omega}[\Delta_j^{\Theta}(f_0)S_{j+r}^{\Theta_1}(f_1)]\Big] = \Delta_{\ell q+s}^{\Omega}[\Delta_{\ell q+s}^{\Theta}(f_0)S_{\ell q+s+r_0}^{\Theta_1}(f_1)]\Big]$$

and this exactly equals $\Delta^{\Theta}_{\ell q+s}(f)S^{\Theta_1}_{\ell q+s+r_0}(f_1)$. It follows that

$$\mathbf{S}^{\mathbf{\Psi_s}}_{\mathbf{q}}(\Sigma_s) = \Big(\sum_{\ell\in\mathbf{Z}}\big|\Delta^{\Psi_s}_{\ell q}(\Sigma_s)\big|^2\Big)^{\frac{1}{2}} = \Big(\sum_{\ell\in\mathbf{Z}}\big|\Delta^{\Theta}_{\ell q+s}(f_0)S^{\Theta_1}_{\ell q+s+r_0}(f_1)\big|^2\Big)^{\frac{1}{2}}$$

which is pointwise controlled by a constant multiple of $\mathbf{S}^{\Theta}_{q}(f_0)\mathbf{M}(f_1)$. To apply Lemma 1.2 we need to show that Σ_s defined in (6) lies in L^2. By the orthogonality of L^2-norms, we have

$$\begin{aligned}\Big\|\sum_{j\in I_s}\Delta^{\Omega}_{j}[\Delta^{\Theta}_{j}(f_0)S^{\Theta_1}_{j+r_0}(f_1)]\Big\|^2_{L^2} &= \sum_{j\in I_s}\int_{\mathbf{R}^n}\big|\Delta^{\Omega}_{j}[\Delta^{\Theta}_{j}(f_0)S^{\Theta_1}_{j+r_0}(f_1)](x)\big|^2dx\\ &\le C\|M(f_1)\|_{L^\infty}\sum_{j\in I_s}\int_{\mathbf{R}^n}\big|\Delta^{\Theta}_{j}(f_0)(x)\big|^2dx\\ &\le C\|f_1\|_{L^\infty}\|f_0\|^2_{L^2} < \infty\,.\end{aligned}$$

Using Lemma 1.2, for each $s\in\{0,1,\dots,q-1\}$ we obtain that

$$\|\Sigma_s\|_{L^{p,\infty}} \le C_p\|\mathbf{S}^{\mathbf{\Psi_s}}_{\mathbf{q}}(\Sigma_s)\|_{L^{p,\infty}}$$

and by the previous discussion this expression at most a constant multiple of $\|\mathbf{S}^{\Theta}_{q}(f_0)\mathbf{M}(f_1)\|_{L^{p,\infty}}$. The required conclusion is an easy consequence of Hölder's inequality and of the boundedness of the maximal and square functions from L^r to $L^{r,\infty}$ for $1\le r<\infty$.

Case 2: $m\ge 2$ **and** $k<m-1$**.**

Having established the case $k=m-1$, we continue the proof by reverse induction on k. Fix a $k\in\{0,1,\dots,m-2\}$ and assume that the conclusion is valid for all $k'>k$ (and $k'\le m-1$.) We need to prove the same conclusion for k.

We begin by writing for all $s\in\{k+1,\dots,m\}$

$$S^{\Theta_s}_{j} = S^{\Theta_s}_{j+r_s} + \sum_{i=j+r_s+1}^{j}\Delta^{\Theta_s}_{i}$$

for some $r_s<0$ that satisfy

$$b_{k+1}2^{r_{k+1}}+\cdots+b_m2^{r_m} < a_0 \tag{7}$$

so that the spectra of $S^{\Theta_{k+1}}_{j+r_{k+1}}(f_{k+1})\cdots S^{\Theta_m}_{j+r_m}(f_m)$ and $\Delta^{\Theta}_{j}(f_0)$ are disjoint.

Then we express $P^{(k)}_{m+1}$ as a finite sum of operators of the form $P^{(k+1)}_{m+1}$, $P^{(k+2)}_{m+1}$, …, $P^{(m-1)}_{m+1}$ plus

$$\sum_{j\in\mathbf{Z}}\Big[\Delta^{\Theta}_{j}(f_0)\prod_{s=k+1}^{m}S^{\Theta_s}_{j+r_s}(f_s)\Big]\Big[\prod_{s=1}^{k}\Delta^{\Theta_s}_{j}(f_s)\Big], \tag{8}$$

with the understanding that if $k=0$, the last product does not appear. The induction hypothesis on k yields the boundedness of $P^{(k+1)}_{m+1}$, $P^{(k+2)}_{m+1}$, $\dots$, $P^{(m-1)}_{m+1}$, while the boundedness of (8) is discussed below considering two subcases.

Subcase 2.a: $k \geq 1$**.**

In this subcase things are straightforward. We apply the Cauchy-Schwarz inequality to control (8) by the product of the ℓ^2 norms of the expressions inside the square brackets and therefore by the product

$$\mathbf{S}^{\Theta}(f_0)\mathbf{S}^{\Theta_1}(f_1)\Big[\prod_{s=2}^{m}\mathbf{M}(f_s)\Big].$$

Obviously, this expression is bounded from $L^{p_0}\times\cdots\times L^{p_m}$ to $L^{p,\infty}$.

Subcase 2.b: $k=0$**.**

Condition (7) implies that the function $\Delta^{\Theta}_j(f_0)S^{\Theta_1}_{j+r_1}(f_1)\cdots S^{\Theta_m}_{j+r_m}(f_m)$ is supported in the annulus $2^{n_0}2^j < |\xi| < 2^{m_0}2^j$ where $n_0 < m_0$ are integers chosen so that

$$2^{n_0} < (a_0-(b_1 2^{r_1}+\cdots+b_m 2^{r_m})) < (b_0+b_1 2^{r_1}+\cdots+b_m 2^{r_m}) < 2^{m_0}.$$

We choose a smooth function Ω which is equal to 1 on the annulus $2^{n_0} < |\xi| < 2^{m_0}$ and vanishes off the annulus $2^{n_0-1} < |\xi| < 2^{m_0+1}$. Then we write the expression in (8) as follows:

$$\sum_{j\in\mathbf{Z}}\Delta^{\Omega}_j\Big[\Delta^{\Theta}(f_0)\prod_{s=1}^{m}S^{\Theta_s}_{j+r_s}(f_s)\Big]. \tag{9}$$

We now pick a function Ψ whose Fourier transform is equal to 1 on the annulus $2^{n_0-2} < |\xi| < 2^{m_0+2}$ and vanishes outside the annulus $2^{n_0-3} < |\xi| < 2^{m_0+3}$. Set $q = m_0-n_0+5$. We split $\mathbb{Z}$ as a disjoint union of sets $I_s=\{\ell q+s,\ \ell\in\mathbb{Z}\}$, $0\leq s\leq q-1$. Next we split the sum in (9) as a finite sum over $s\in\{0,1,\dots,q-1\}$ of the sums Σ_s where the indices j in (9) run over the set I_s. We also define a function Ψ_s by setting $\widehat{\Psi_s}(\xi)=\widehat{\Psi}(2^{-s}\xi)$ and we note that $\sum_\ell\widehat{\Psi_s}(2^{-\ell q}\xi)=1$ for ξ in $\mathbb{R}^d\setminus\{0\}$.

We observe that for $j\in I_s$ and $\ell\in\mathbb{Z}$ the supports of the functions $\xi\to\widehat{\Psi_s}(2^{-\ell q}\xi)$ and $\xi\to\widehat{\Omega}(2^{-j}\xi)$ intersect nontrivially exactly when $j=\ell q+s$ and this case $\Delta^{\Omega}_j\Delta^{\Psi_s}_{\ell q}=\Delta^{\Omega}_j$. We are therefore in a position to use Lemma 1.2, since again we can control the L^2-norm of $\sum_{j\in I_s}\Delta^{\Omega}_j[\Delta^{\Theta}(f_0)\prod_{s=1}^m S^{\Theta_s}_{j+r_s}(f_s)]$ by $C\prod_{s=1}^m\|f_s\|_{L^\infty}\|f_0\|_{L^2}<\infty$, and argue as in Subcase 2.2 to complete the proof.

Remark 2.1. *The exponent p_j can be taken to be equal to infinity whenever the maximal function $\mathbf{M}(f_j)$ appears in the estimate controlling $P^{(k)}_{m+1}$ (pointwise or in norm). For instance, when $m \geq 2$ and $k = m-1$, we may take $p_2 = \cdots = p_m = \infty$; see* (2).

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