SHARP BOUNDS FOR m-LINEAR HARDY AND HILBERT OPERATORS

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Abstract. The precise norms of m-linear Hardy operators and m-linear Hilbert operators on Lebesgue spaces with power weights are computed. Analogous results are also obtained for Morrey spaces and central Morrey spaces.

1. Introduction

Averaging operators are of fundamental importance in analysis and it is often desirable to obtain sharp norm estimates for them. The study of these operators may not be as delicate as that of maximal operators (cf. [7], [11]) but still requires the use of certain beautiful and elegant ideas. The most fundamental averaging operator is the Hardy functional

$$H(f)(x) = \frac{1}{x}\int_0^x f(t)\,dt$$

defined for locally integrable functions f on the line. A classical inequality, due to Hardy [9], states that

$$\int_0^\infty |H(f)(x)|^p\,dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |f(x)|^p\,dx.$$

for $1 < p < \infty$, and the constant $(\frac{p}{p-1})^p$ is best possible. Thus the classical Hardy inequality says that the norm of the Hardy operator on $L^p(\mathbb{R}^+)$ is equal to $\frac{p}{p-1}$.

In 1976 Faris [5] introduced the following n-dimensional Hardy operator

$$\mathcal{H}(f)(x) = \frac{1}{\Omega_n |x|^n}\int_{|y|<|x|} f(y)\,dy, \quad x \in \mathbb{R}^n\backslash\{0\}, \tag{1}$$

defined for nonnegative functions on $\mathbb{R}^n$, where $\Omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$ is the volume of the unit ball in $\mathbb{R}^n$. The norm of $\mathcal{H}$ on $L^p(\mathbb{R}^n)$ was evaluated in [3] and found to be equal to that of the 1-dimensional Hardy operators, i.e.,

$$\|\mathcal{H}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \frac{p}{p-1}\,.$$

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In this article we study m-linear averaging operators analogous to the operator in (1). The study of multilinear averaging operators is a byproduct of the recent interest in multilinear singular integral operator theory; this subject was founded in the 1970s by Coifman and Meyer [4] in their comprehensive study of many singular integral operators of multiparameter function input, such as the Calderón commutators, paraproducts, and pseudodifferential operators.

In the sequel we use the following notation: for $1 \le i \le m$, $y_i = (y_{1i}, y_{2i}, \dots, y_{ni})$ will denote elements of $\mathbb{R}^n$. The Euclidean norms of each y_i is $|y_i| = \sqrt{\sum_{j=1}^n |y_{ji}|^2}$ and of the m-tuple $(y_1, y_2, \dots, y_m)$ is $|(y_1, y_2, \dots, y_m)| = \sqrt{\sum_{i=1}^m |y_i|^2}$. We use this notation in the following definition of the m-linear Hardy operator.

Definition 1. *Let $m \in \mathbb{N}$, $f_1, f_2, \dots, f_m$ be nonnegative locally integrable functions on $\mathbb{R}^n$. The m-linear Hardy operator is defined by*

$$\mathcal{H}^m(f_1, \dots, f_m)(x) = \frac{1}{\Omega_{mn}} \frac{1}{|x|^{mn}} \int\limits_{|(y_1,\dots,y_m)|<|x|} f_1(y_1) \dots f_m(y_m)\, dy_1 dy_2 \dots dy_m, \tag{2}$$

where $x \in \mathbb{R}^n \setminus \{0\}$. The 2-linear operator will be referred to as bilinear.

Two other variants of m-linear Hardy operators (of one-dimensional nature) were introduced and studied by Bényi and Oh [2]. The n-dimensional m-linear Hardy-type operator $\mathcal{H}^m$ defined above does not seem to have previously appeared in the literature. Our approach is simpler than that in both [2] and [3] and easily adapts to the multilinear setting. It relies on the method of rotations and on the principle that many positive averaging operators attain their (weighted or unweighted) Lebesgue space operator norms on the subspace of radial functions.

We recall the definitions of the usual beta function $B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1} dt$, where z and w are complex numbers with positive real parts, and the gamma function $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$, where z is a complex number with positive real part. The following relationship between the gamma and beta functions is valid: $B(z, w)\Gamma(z + w) = \Gamma(z)\Gamma(w)$, when z and w have positive real parts.

2. Hardy operators on weighted Lebesgue spaces

The main result of this article is the following:

Theorem 1. *Let $m \in \mathbb{N}$, $f_i \in L^{p_i}(|x|^{\frac{\alpha_i p_i}{p}} dx)$, $1 < p_i < \infty$, $1 \le p < \infty$, $i = 1, 2, \dots, m$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m}$, $\alpha_i < pn(1 - 1/p_i)$, and $\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_m$. Then the m-linear Hardy operator $\mathcal{H}^m$ defined in (2) maps the product of weighted Lebesgue spaces $L^{p_1}(|x|^{\frac{\alpha_1 p_1}{p}} dx) \times L^{p_2}(|x|^{\frac{\alpha_2 p_2}{p}} dx) \times \cdots \times L^{p_m}(|x|^{\frac{\alpha_m p_m}{p}} dx)$ to $L^p(|x|^\alpha dx)$ with norm equal to the constant*

$$\frac{{\omega_n}^m}{\omega_{mn}} \frac{pmn}{pmn - n - \alpha} \frac{1}{2^{m-1}} \frac{\prod_{i=1}^m \Gamma(\frac{n}{2}(1 - \frac{1}{p_i} - \frac{\alpha_i}{pn}))}{\Gamma(\frac{n}{2}(m - \frac{1}{p} - \frac{\alpha}{pn}))}.$$

Remark 1. *The norm from $L^p(|x|^\alpha dx)$ to itself of the n-dimensional Hardy operator $\mathcal{H}$ defined in (1) is independent of n exactly when $\alpha = 0$.*

Proof. For clarity, we break up the proof in cases that represent several important special cases. The general case follows by combining the special cases.

1. Weighted case when $m = 1$.

We were informed that the result in this case was independently obtained by J. Soria, possibly by a different method. Set

$$g(x) = \frac{1}{\omega_n} \int_{|\xi|=1} f(|x|\xi)\, d\xi, \quad x \in \mathbb{R}^n,$$

where $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$. It is easy to see (cf. the next case in the proof of the same theorem) that the operator $\mathcal{H}$ and its restriction to radial functions have the same operator norm on $L^p(|x|^\alpha)$. We may therefore assume that f is a radial function.

Fix $\alpha < n(p-1)$. Let ω_n denote the area of the unite sphere $\mathbb{S}^{n-1}$ and $B(0,R)$ denote a ball of radius R centered at zero. By Minkowski's integral inequality, we have

$$\begin{aligned}
\|\mathcal{H}(f)\|_{L^p(|x|^\alpha dx)} &= \frac{1}{\Omega_n}\left(\int_{\mathbb{R}^n}\left|\frac{1}{|x|^n}\int_{B(0,|x|)} f(y)\,dy\right|^p |x|^\alpha\,dx\right)^{\frac{1}{p}} \\
&= \frac{1}{\Omega_n}\left(\int_{\mathbb{R}^n}\Big|\int_{B(0,1)} f(|x|y)\,dy\Big|^p |x|^\alpha\,dx\right)^{\frac{1}{p}} \\
&\le \frac{1}{\Omega_n}\int_{B(0,1)}\left(\int_{\mathbb{R}^n}|f(|y|x)|^p|x|^\alpha dx\right)^{\frac{1}{p}} dy \\
&\le \frac{1}{\Omega_n}\int_{B(0,1)}|y|^{-n/p-\alpha/p}dy\|f\|_{L^p(|x|^\alpha dx)} \\
&= \frac{\omega_n}{\Omega_n}\,\frac{1}{n(1-\frac{1}{p}-\frac{\alpha}{pn})}\|f\|_{L^p(|x|^\alpha dx)} \\
&= \frac{pn}{pn-n-\alpha}\,\|f\|_{L^p(|x|^\alpha dx)}.
\end{aligned}$$

Therefore, we have obtained the upper estimate

$$\|\mathcal{H}\|_{L^p(|x|^\alpha dx)\to L^p(|x|^\alpha dx)} \le \frac{pn}{pn-n-\alpha}\,.$$

On the other hand, for $0 < \varepsilon < \min\{1, \frac{n}{p'} - \frac{\alpha}{p}\}$, we take

$$f_\varepsilon(x) = \begin{cases} 0, & |x| \le 1, \\ |x|^{-\frac{n}{p}-\frac{\alpha}{p}-\varepsilon}, & |x| > 1. \end{cases}$$

By a simple calculation, we see that $\|f_\varepsilon\|^p_{L^p(|x|^\alpha dx)} = \frac{\omega_n}{p\varepsilon}$. It follows that

$$\mathcal{H}(f_\varepsilon)(x) = \begin{cases} 0, & |x| \le 1, \\ \frac{1}{\Omega_n}|x|^{-\frac{n}{p}-\frac{\alpha}{p}-\varepsilon}\int_{\frac{1}{|x|}<|y|<1}|y|^{-\frac{n}{p}-\frac{\alpha}{p}-\varepsilon}dy, & |x| > 1. \end{cases}$$

We have

$$
\begin{aligned}
\|\mathcal{H}(f_\varepsilon)\|_{L^p(|x|^\alpha dx)} &= \frac{1}{\Omega_n}\left(\int_{|x|>1}\left(|x|^{-\frac{n}{p}-\frac{\alpha}{p}-\varepsilon}\int_{\frac{1}{|x|}<|y|<1}|y|^{-\frac{n}{p}-\frac{\alpha}{p}-\varepsilon}dy\right)^p|x|^\alpha dx\right)^{\frac{1}{p}}\\
&\geq \frac{1}{\Omega_n}\left(\int_{|x|>1/\varepsilon}\left(|x|^{-\frac{n}{p}-\frac{\alpha}{p}-\varepsilon}\int_{\varepsilon<|y|<1}|y|^{-\frac{n}{p}-\frac{\alpha}{p}-\varepsilon}dy\right)^p|x|^\alpha dx\right)^{\frac{1}{p}}\\
&= \frac{\omega_n}{\Omega_n}\left(\int_{|x|>1/\varepsilon}|x|^{-n-p\varepsilon}dx\right)^{\frac{1}{p}}\int_\varepsilon^1 r^{n-\frac{n}{p}-\frac{\alpha}{p}-\varepsilon-1}dr\\
&= \frac{\omega_n}{\Omega_n}\frac{1-\varepsilon^{n-\frac{n}{p}-\frac{\alpha}{p}-\varepsilon}}{n-\frac{n}{p}-\frac{\alpha}{p}-\varepsilon}\,\varepsilon^\varepsilon\,\|f_\varepsilon\|_{L^p(|x|^\alpha x)}.
\end{aligned}
$$

Consequently,

$$
\|\mathcal{H}\|_{L^p(|x|^\alpha dx)\to L^p(|x|^\alpha dx)} \geq \frac{\omega_n}{\Omega_n}\frac{1-\varepsilon^{n-\frac{n}{p}-\frac{\alpha}{p}-\varepsilon}}{n-\frac{n}{p}-\frac{\alpha}{p}-\varepsilon}\,\varepsilon^\varepsilon,
$$

and letting $\varepsilon\to 0$ (using that $\varepsilon^\varepsilon\to 1$), we deduce

$$
\|\mathcal{H}\|_{L^p(|x|^\alpha dx)\to L^p(|x|^\alpha dx)} \geq \frac{pn}{pn-n-\alpha}\,.
$$

2. Unweighted case when $m = 2$.

Set as before $\omega_n = 2\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})$. For $i = 1, 2$ let

$$
g_i(y_i) = \frac{1}{\omega_n}\int_{|\xi_i|=1} f_i(|y_i|\xi_i)\,d\xi_i, \quad y_i\in\mathbb{R}^n\,.
$$

Obviously, $g_1(y_1)$ and $g_2(y_2)$ are radial functions and $\mathcal{H}^2(g_1, g_2)(x)$ is equal to

$$
\begin{aligned}
&\frac{1}{\Omega_{2n}|x|^{2n}}\int_{|(y_1,y_2)|<|x|} g_1(y_1)g_2(y_2)\,dy_1dy_2\\
=\;&\frac{1}{\Omega_{2n}|x|^{2n}}\int_{|(y_1,y_2)|<|x|}\prod_{i=1}^{2}\left(\frac{1}{\omega_n}\int_{|\xi_i|=1} f_i(|y_i|\xi_i)d\xi_i\right)dy_1dy_2\\
=\;&\frac{1}{\Omega_{2n}|x|^{2n}}\int_{|\xi_1|=1}\int_{|\xi_2|=1}\left(\frac{1}{{\omega_n}^2}\int_{|(y_1,y_2)|<|x|} f_1(|y_1|\xi_1)f_2(|y_2|\xi_2)dy_1dy_2\right)d\xi_1d\xi_2\\
=\;&\frac{1}{\Omega_{2n}|x|^{2n}}\int_{|(y_1,y_2)|<|x|} f_1(y_1)f_2(y_2)\,dy_1dy_2\\
=\;&\mathcal{H}^2(f_1,f_2)(x).
\end{aligned}
$$

Also by Minkowski's integral inequality, we have

$$\begin{aligned}
\|g_1\|_{L^{p_1}} &= \left(\int_{\mathbb{R}^n} |g_1(x_1)|^{p_1} dx_1\right)^{\frac{1}{p_1}} \\
&= \left[\omega_n \int_0^\infty \left|\frac{1}{\omega_n}\int_{|\xi_1|=1} f_1(r\xi_1)d\xi_1\right|^{p_1} r^{n-1}dr\right]^{\frac{1}{p_1}} \\
&\le \left[\omega_n \int_0^\infty \frac{1}{\omega_n}\int_{|\xi_1|=1} |f_1(r\xi_1)|^{p_1} d\xi_1 r^{n-1}dr\right]^{\frac{1}{p_1}} \\
&= \|f_1\|_{L^{p_1}}.
\end{aligned}$$

Similarly, for g_2, we obtain $\|g_2\|_{L^{p_2}} \le \|f_2\|_{L^{p_2}}$. Therefore one has that

$$\frac{\|\mathcal{H}^2(f_1,f_2)\|_{L^p}}{\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}} \le \frac{\|\mathcal{H}^2(g_1,g_2)\|_{L^p}}{\|g_1\|_{L^{p_1}}\|g_2\|_{L^{p_2}}}.$$

This implies that the operator $\mathcal{H}^2$ and its restriction to radial functions have the same operator norm in L^p. So, without loss of generality, we assume that f_i, $i = 1, 2$ are radial functions in the rest of the proof.

By Minkowski's integral inequality and Hölder's inequality, we have

$$\begin{aligned}
\|\mathcal{H}^2(f_1,f_2)\|_{L^p} &= \frac{1}{\Omega_{2n}}\left(\int_{\mathbb{R}^n}\left|\frac{1}{|x|^{2n}}\int_{|(y_1,y_2)|<|x|} f_1(y_1)f_2(y_2)\,dy_1dy_2\right|^p dx\right)^{\frac{1}{p}} \\
&= \frac{1}{\Omega_{2n}}\left(\int_{\mathbb{R}^n}\left|\int_{|(z_1,z_2)|<1} f_1(|x|z_1)f_2(|x|z_2)\,dz_1dz_2\right|^p dx\right)^{\frac{1}{p}} \\
&= \frac{1}{\Omega_{2n}}\left(\int_{\mathbb{R}^n}\left|\int_{|(z_1,z_2)|<1} f_1(|z_1|x)f_2(|z_2|x)\,dz_1dz_2\right|^p dx\right)^{\frac{1}{p}} \\
&\le \frac{1}{\Omega_{2n}}\int_{|(z_1,z_2)|<1}\left(\int_{\mathbb{R}^n}|f_1(|z_1|x)f_2(|z_2|x)|^p\,dx\right)^{\frac{1}{p}} dz_1dz_2 \\
&\le \frac{1}{\Omega_{2n}}\int_{|(z_1,z_2)|<1}\prod_{i=1}^{2}\left(\int_{\mathbb{R}^n}|f_i(x)|^{p_i}\,dx\right)^{\frac{1}{p_i}} |z_1|^{-\frac{n}{p_1}}|z_2|^{-\frac{n}{p_2}}\,dz_1dz_2 \\
&= \frac{1}{\Omega_{2n}}\int_{|(z_1,z_2)|<1}|z_1|^{-\frac{n}{p_1}}|z_2|^{-\frac{n}{p_2}}\,dz_1dz_2\,\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}} \\
&= \frac{1}{\Omega_{2n}}C_1\,\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}\,,
\end{aligned}$$

where C_1 is the following constant:

$$\begin{aligned}
C_1 &= \int_{|(z_1,z_2)|<1} |z_1|^{-\frac{n}{p_1}} |z_2|^{-\frac{n}{p_2}}\, dz_1 dz_2 \\
&= \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \int_{\rho_1^2+\rho_2^2\le 1, \rho_1\ge 0, \rho_2\ge 0} \rho_1^{-\frac{n}{p_1}} \rho_2^{-\frac{n}{p_2}} \rho_1^{n-1} \rho_2^{n-1}\, d\rho_1 d\rho_2\, dz_1' dz_2' \\
&= \omega_n^2 \int_0^{\frac{\pi}{2}} \int_0^1 r^{n-\frac{n}{p_1}-1} r^{n-\frac{n}{p_2}-1} (\cos\theta)^{n-\frac{n}{p_1}-1} (\sin\theta)^{n-\frac{n}{p_2}-1}\; r dr d\theta \\
&= \frac{\omega_n^2}{n(2-\frac{1}{p})} \int_0^{\frac{\pi}{2}} (\cos\theta)^{n-\frac{n}{p_1}-1} (\sin\theta)^{n-\frac{n}{p_2}-1}\, d\theta \\
&= \frac{\omega_n^2}{n(2-\frac{1}{p})} \int_0^1 (1-t^2)^{\frac{1}{2}(n-\frac{n}{p_1}-1)} t^{n-\frac{n}{p_2}-1} (1-t^2)^{-\frac{1}{2}} dt \quad (t=\sin\theta) \\
&= \frac{\omega_n^2}{n} \frac{p}{2p-1} \frac{1}{2} \int_0^1 (1-x)^{\frac{n}{2}-\frac{n}{2p_1}-1} x^{\frac{n}{2}-\frac{n}{2p_2}-1} dx \quad (x=t^2) \\
&= \frac{\omega_n^2}{2n} \frac{p}{2p-1}\, B\Big(\frac{n}{2}-\frac{n}{2p_1}, \frac{n}{2}-\frac{n}{2p_2}\Big).
\end{aligned}$$

(We wrote above $z_i = \rho_i z_i'$, where $z_i' \in \mathbb{S}^{n-1}$ and $\rho_i > 0$.) Therefore, it follows that

$$\|\mathcal{H}^2\|_{L^{p_1}\times L^{p_2}\to L^p} \le \frac{\omega_n^2}{\omega_{2n}} \frac{p}{2p-1}\, B\Big(\frac{n}{2}-\frac{n}{2p_1}, \frac{n}{2}-\frac{n}{2p_2}\Big).$$

Now for $0<\varepsilon<\min\{1, \frac{(p_1-1)n}{p_2}, \frac{n}{p_2'}\}$ we define

$$f_1^\varepsilon(x_1) = \begin{cases} 0, & |x_1| \le \frac{\sqrt{2}}{2}, \\ |x_1|^{-\frac{n}{p_1}-\frac{p_2\varepsilon}{p_1}}, & |x_1| > \frac{\sqrt{2}}{2}; \end{cases} \qquad f_2^\varepsilon(x_2) = \begin{cases} 0, & |x_2| \le \frac{\sqrt{2}}{2}, \\ |x_2|^{-\frac{n}{p_2}-\varepsilon}, & |x_2| > \frac{\sqrt{2}}{2}. \end{cases}$$

By an elementary calculation, we obtain that

$$\|f_1^\varepsilon\|_{L^{p_1}}^{p_1} = \|f_2^\varepsilon\|_{L^{p_2}}^{p_2} = \frac{\omega_n}{p_2\varepsilon} \Big(\frac{\sqrt{2}}{2}\Big)^{-p_2\varepsilon}.$$

Consequently, we have that $\mathcal{H}^2(f_1^\varepsilon, f_2^\varepsilon)(x) = 0$ when $|x|\le 1$ and that

$$\mathcal{H}^2(f_1^\varepsilon, f_2^\varepsilon)(x) = \frac{|x|^{-\frac{n}{p}-\frac{p_2\varepsilon}{p}}}{\Omega_{2n}} \int_{|(y_1,y_2)|<1;\, |y_1|>\frac{\sqrt{2}}{2|x|};\, |y_2|>\frac{\sqrt{2}}{2|x|}} |y_1|^{-\frac{n}{p_1}-\frac{p_2\varepsilon}{p_1}} |y_2|^{-\frac{n}{p_2}-\varepsilon} dy_1 dy_2\,,$$

when $|x|>1$. We write

$$
\begin{aligned}
&\|\mathcal{H}^2(f_1^\varepsilon,f_2^\varepsilon)\|_{L^p}\\
&=\frac{1}{\Omega_{2n}}\Bigg\{\int_{|x|>1}\Bigg[|x|^{-\frac{n}{p}-\frac{p_2\varepsilon}{p}}\int_{|(y_1,y_2)|<1;\,|y_1|>\frac{\sqrt{2}}{2|x|};\,|y_2|>\frac{\sqrt{2}}{2|x|}}|y_1|^{-\frac{n}{p_1}-\frac{p_2\varepsilon}{p_1}}|y_2|^{-\frac{n}{p_2}-\varepsilon}dy_1dy_2\Bigg]^p dx\Bigg\}^{\frac{1}{p}}\\
&\geq\frac{1}{\Omega_{2n}}\Bigg\{\int_{|x|>\frac{1}{\varepsilon}}\Bigg[|x|^{-\frac{n}{p}-\frac{p_2\varepsilon}{p}}\int_{|(y_1,y_2)|<1;\,|y_1|>\frac{\sqrt{2}}{2/\varepsilon};\,|y_2|>\frac{\sqrt{2}}{2/\varepsilon}}|y_1|^{-\frac{n}{p_1}-\frac{p_2\varepsilon}{p_1}}|y_2|^{-\frac{n}{p_2}-\varepsilon}dy_1dy_2\Bigg]^p dx\Bigg\}^{\frac{1}{p}}\\
&=\frac{1}{\Omega_{2n}}\Bigg(\int_{|x|>\frac{1}{\varepsilon}}|x|^{-n-p_2\varepsilon}dx\Bigg)^{\frac{1}{p}}\int_{|(y_1,y_2)|<1;\,|y_1|>\frac{\sqrt{2}}{2/\varepsilon};\,|y_2|>\frac{\sqrt{2}}{2/\varepsilon}}|y_1|^{-\frac{n}{p_1}-\frac{p_2\varepsilon}{p_1}}|y_2|^{-\frac{n}{p_2}-\varepsilon}dy_1dy_2\\
&=\frac{1}{\Omega_{2n}}\,C_2\,C_3,
\end{aligned}
$$

where C_2 and C_3 are the second and third factors in the last term, respectively. We now compute the values of the constants C_2 and C_3 . Writing $y_i=\rho_i z_i'$ we have

$$
\begin{aligned}
C_3 &= \int_{|(y_1,y_2)|<1;\,|y_1|>\frac{\sqrt{2}}{2/\varepsilon};\,|y_2|>\frac{\sqrt{2}}{2/\varepsilon}}|y_1|^{-\frac{n}{p_1}-\frac{p_2\varepsilon}{p_1}}|y_2|^{-\frac{n}{p_2}-\varepsilon}dy_1dy_2\\
&= \int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\int_{\rho_1^2+\rho_2^2<1;\,\rho_1>\frac{\sqrt{2}}{2/\varepsilon};\,\rho_2>\frac{\sqrt{2}}{2/\varepsilon}}\rho_1^{-\frac{n}{p_1}-\frac{p_2\varepsilon}{p_1}}\rho_2^{-\frac{n}{p_2}-\varepsilon}\rho_1^{n-1}\rho_2^{n-1}d\rho_1d\rho_2dz_1'dz_2'\\
&= \omega_n^2\int_{\rho_1=\frac{\sqrt{2}}{2/\varepsilon}}^{1}\int_{\rho_2=\frac{\sqrt{2}}{2/\varepsilon}}^{\sqrt{1-\rho_1^2}}\rho_1^{n-\frac{n}{p_1}-\frac{p_2\varepsilon}{p_1}-1}\rho_2^{n-\frac{n}{p_2}-\varepsilon-1}d\rho_2d\rho_1\\
&= \frac{\omega_n^2}{n-\frac{n}{p_2}-\varepsilon}\Bigg\{\frac{1}{2}\int_{\frac{1}{2\varepsilon^{-2}}}^{1}(1-t)^{\frac{1}{2}(n-\frac{n}{p_2}-\varepsilon)}t^{\frac{1}{2}(n-\frac{n}{p_1}-\frac{p_2\varepsilon}{p_1})-1}dt\\
&\quad-\Big(\frac{\sqrt{2}}{2/\varepsilon}\Big)^{n-\frac{n}{p_2}-\varepsilon}\frac{1}{n-\frac{n}{p_1}-\frac{p_2\varepsilon}{p_1}}\Big(1-\Big(\frac{\sqrt{2}}{2/\varepsilon}\Big)^{n-\frac{n}{p_1}-\frac{p_2\varepsilon}{p_1}}\Big)\Bigg\}\\
&= \frac{\omega_n^2}{n-\frac{n}{p_2}-\varepsilon}\Bigg\{\frac{1}{2}\Big(\int_0^1-\int_0^{\frac{\varepsilon^2}{2}}\Big)(1-t)^{\frac{1}{2}(n-\frac{n}{p_2}-\varepsilon)}t^{\frac{1}{2}(n-\frac{n}{p_1}-\frac{p_2\varepsilon}{p_1})-1}dt\\
&\quad-\Big(\frac{\sqrt{2}\,\varepsilon}{2}\Big)^{n-\frac{n}{p_2}}\frac{(\sqrt{2}\,)^{\varepsilon}}{\varepsilon^{\varepsilon}}\frac{1}{n-\frac{n}{p_1}-\frac{p_2\varepsilon}{p_1}}\Big(1-\Big(\frac{\sqrt{2}\,\varepsilon}{2}\Big)^{n-\frac{n}{p_1}}\Big(\frac{\sqrt{2}}{2}\Big)^{-\frac{p_2\varepsilon}{p_1}}(\varepsilon^{\varepsilon})^{-\frac{p_2}{p_1}}\Big)\Bigg\}\\
&= \frac{\omega_n^2}{n-\frac{n}{p_2}-\varepsilon}\Bigg\{\frac{1}{2}B\Big(\frac{n}{2}-\frac{n}{2p_1}-\frac{p_2\varepsilon}{2p_1},\frac{n-\varepsilon}{2}-\frac{n}{2p_2}+1\Big)\\
&\quad-\frac{1}{2}\int_0^{\frac{\varepsilon^2}{2}}(1-t)^{\frac{1}{2}(n-\frac{n}{p_2}-\varepsilon)}t^{\frac{1}{2}(n-\frac{n}{p_1}-\frac{p_2\varepsilon}{p_1})-1}dt\\
&\quad-\Big(\frac{\sqrt{2}\,\varepsilon}{2}\Big)^{n-\frac{n}{p_2}}\frac{(\sqrt{2}\,)^{\varepsilon}}{\varepsilon^{\varepsilon}}\frac{1}{n-\frac{n}{p_1}-\frac{p_2\varepsilon}{p_1}}\Big(1-\Big(\frac{\sqrt{2}\,\varepsilon}{2}\Big)^{n-\frac{n}{p_1}}\Big(\frac{\sqrt{2}}{2}\Big)^{-\frac{p_2\varepsilon}{p_1}}(\varepsilon^{\varepsilon})^{-\frac{p_2}{p_1}}\Big)\Bigg\}.
\end{aligned}
$$

For C_2 we have

$$\begin{aligned} C_2 &= \left(\int_{|x|>1/\varepsilon} |x|^{-n-p_2\varepsilon} dx\right)^{\frac{1}{p}} \\ &= \omega_n^{\frac{1}{p}} \left(\int_{1/\varepsilon}^{\infty} r^{-n-p_2\varepsilon} r^{n-1} dr\right)^{\frac{1}{p}} \\ &= \omega_n^{\frac{1}{p}} \left(\frac{\varepsilon^{p_2\varepsilon}}{p_2\varepsilon}\right)^{\frac{1}{p}} \left(\frac{\omega_n}{p_2\varepsilon}\right)^{-\frac{1}{p_1}} \left(\frac{\sqrt{2}}{2}\right)^{\frac{p_2\varepsilon}{p_1}} \left(\frac{\omega_n}{p_2\varepsilon}\right)^{-\frac{1}{p_2}} \left(\frac{\sqrt{2}}{2}\right)^{\varepsilon} \|f_1^\varepsilon\|_{L^{p_1}} \|f_2^\varepsilon\|_{L^{p_2}} \\ &= (\varepsilon^\varepsilon)^{\frac{p_2}{p}} \left(\frac{\sqrt{2}}{2}\right)^{\frac{p_2\varepsilon}{p}} \frac{p_2^{\frac{1}{p_1}} p_2^{\frac{1}{p_2}}}{p_2^{\frac{1}{p}}} \|f_1^\varepsilon\|_{L^{p_1}} \|f_2^\varepsilon\|_{L^{p_2}} \\ &= (\varepsilon^\varepsilon)^{\frac{p_2}{p}} \left(\frac{\sqrt{2}}{2}\right)^{\frac{p_2\varepsilon}{p}} \|f_1^\varepsilon\|_{L^{p_1}} \|f_2^\varepsilon\|_{L^{p_2}}. \end{aligned}$$

Let $\varepsilon \to 0$. By the beta-function property $B(p, q+1) = \frac{q}{p+q} B(p,q)$, $p, q > 0$, we obtain

$$\|\mathcal{H}^2\|_{L^{p_1} \times L^{p_2} \to L^p} \geq \frac{\omega_n^2}{\omega_{2n}} \frac{p}{2p-1} B\Big(\frac{n}{2} - \frac{n}{2p_1}, \frac{n}{2} - \frac{n}{2p_2}\Big).$$

3. Weighted case when $m = 2$.

The proof of the upper bound in this case is similar to that of the previous case. For the proof of the lower bound, for a sufficiently small $\varepsilon \in (0,1)$, we take

$$f_1^\varepsilon(x_1) = \begin{cases} 0, & |x_1| \leq \frac{\sqrt{2}}{2}, \\ |x_1|^{-\frac{n}{p_1} - \frac{\alpha_1}{p} - \frac{p_2\varepsilon}{p_1}}, & |x_1| > \frac{\sqrt{2}}{2}; \end{cases} \quad f_2^\varepsilon(x_2) = \begin{cases} 0, & |x_2| \leq \frac{\sqrt{2}}{2}, \\ |x_2|^{-\frac{n}{p_2} - \frac{\alpha_2}{p} - \varepsilon}, & |x_2| > \frac{\sqrt{2}}{2}. \end{cases}$$

These functions show that the claimed norm of the operator on weighted Lebesgue spaces is indeed obtained as $\varepsilon \to 0$.

4. The case $m \geq 3$.

Since the proof of the weighted case when $m \geq 3$ is similar to that of case 3, we only give the outline of the unweighted case. The most delicate part of the proof in the case $m \geq 3$ is the calculation of the best constant.

To obtain the upper bound estimate for the norm, we employ the idea of the proof of Theorem 1 to conclude that the claimed norm is bounded from above by the constant

$$C_4 = \frac{1}{\Omega_{mn}} \int_{|(z_1, z_2, \ldots, z_m)|<1} |z_1|^{-\frac{n}{p_1}} |z_2|^{-\frac{n}{p_2}} \ldots |z_m|^{-\frac{n}{p_m}} dz_1 dz_2 \ldots dz_m.$$

Expressing each $z_i = \rho_i z_i'$ in polar coordinates we can write

$$C_4 = \frac{\omega_n^m}{\Omega_{mn}} \int_{\sum_{i=1}^{m} \rho_i^2 \leq 1;\, \rho_i > 0,\, i=1,2,\ldots,m} \prod_{i=1}^{m} \rho_i^{-\frac{n}{p_i}} \rho_i^{n-1} d\rho_1 d\rho_2 \ldots d\rho_m. \tag{3}$$

Switching to spherical coordinates, we evaluate the integral in (3) as follows:

$$
\begin{aligned}
&\int_0^1 r^{\sum\limits_{i=1}^{m}(n-\frac{n}{p_i}-1)} r^{m-1} \int_0^{\pi/2}\int_0^{\pi/2}\cdots\int_0^{\pi/2} (\sin\theta_1)^{m-2}(\sin\theta_2)^{m-3}\dots(\sin\theta_{m-2})\\
&(\sin\theta_1)^{\sum\limits_{i=2}^{m}(n-\frac{n}{p_i}-1)}(\sin\theta_2)^{\sum\limits_{i=3}^{m}(n-\frac{n}{p_i}-1)}\dots(\sin\theta_{m-2})^{\sum\limits_{i=m-1}^{m}(n-\frac{n}{p_i}-1)}(\sin\theta_{m-1})^{n-\frac{n}{p_m}-1}\\
&(\cos\theta_{m-1})^{n-\frac{n}{p_{m-1}}-1}(\cos\theta_{m-2})^{n-\frac{n}{p_{m-2}}-1}\dots(\cos\theta_1)^{n-\frac{n}{p_1}-1}\,d\theta_1\dots d\theta_{m-1}dr\\
=\;&\frac{1}{nm-n\sum\limits_{i=1}^{m}\frac{1}{p_i}}\int_0^{\pi/2}\int_0^{\pi/2}\cdots\int_0^{\pi/2}\\
&\qquad(\sin\theta_1)^{m-2+\sum\limits_{i=2}^{m}(n-\frac{n}{p_i}-1)}(\sin\theta_2)^{m-3+\sum\limits_{i=3}^{m}(n-\frac{n}{p_i}-1)}\dots(\sin\theta_{m-2})^{2n-1-\frac{n}{p_{m-1}}-\frac{n}{p_m}}\\
&\qquad\quad(\cos\theta_{m-2})^{n-\frac{n}{p_{m-2}}-1}\dots(\cos\theta_1)^{n-\frac{n}{p_1}-1}\,d\theta_1\dots d\theta_{m-2}\\
&\int_0^1 t^{n-\frac{n}{p_m}-1}(1-t^2)^{\frac{1}{2}(n-\frac{n}{p_{m-1}}-1)-\frac{1}{2}}dt\\
=\;&\frac{1}{nm-\frac{n}{p}}\frac{1}{2^{m-1}}B\left(\frac{n}{2}\sum_{i=2}^{m}(1-\frac{1}{p_i}),\frac{n}{2}(1-\frac{1}{p_1})\right)\\
&B\left(\frac{n}{2}\sum_{i=3}^{m}\Big(1-\frac{1}{p_i}\Big),\frac{n}{2}\Big(1-\frac{1}{p_2}\Big)\right)\dots B\left(\frac{n}{2}\sum_{i=m-1}^{m}\Big(1-\frac{1}{p_i}\Big),\frac{n}{2}\Big(1-\frac{1}{p_{m-2}}\Big)\right)\\
&B\left(\frac{n}{2}\Big(1-\frac{1}{p_m}\Big),\frac{n}{2}\Big(1-\frac{1}{p_{m-1}}\Big)\right).
\end{aligned}
$$

Multiplying by $\frac{\omega_n^m}{\Omega_{mn}}$ we obtain the value of C_4 claimed in the statement of the theorem. To show that C_4 is the best possible constant, for a sufficiently small ε in $(0,1)$ we define

$$
f_i^\varepsilon(x_i)=\begin{cases}0, & |x_1|\le\frac{1}{\sqrt{m}},\\ |x_i|^{-\frac{n}{p_i}-\frac{p_m\varepsilon}{p_i}}, & |x_1|>\frac{1}{\sqrt{m}},\end{cases}
$$

where $i=1,2,\dots,m-1$ and

$$
f_m^\varepsilon(x_m)=\begin{cases}0, & |x_m|\le\frac{1}{\sqrt{m}},\\ |x_m|^{-\frac{n}{p_m}-\varepsilon}, & |x_m|>\frac{1}{\sqrt{m}}.\end{cases}
$$

We have that

$$
\|f_1^\varepsilon\|_{L^{p_1}}^{p_1}=\|f_2^\varepsilon\|_{L^{p_2}}^{p_2}=\dots\|f_m^\varepsilon\|_{L^{p_m}}^{p_m}=\frac{\omega_n}{p_m\varepsilon}\Big(\frac{1}{\sqrt{m}}\Big)^{-p_m\varepsilon}
$$

and that $\mathcal{H}^m(f_1^\varepsilon,\dots,f_m^\varepsilon)(x)$ is equal to zero when $|x|\le1$ and is equal to

$$
\frac{1}{\Omega_{mn}}|x|^{-\frac{n}{p}-\frac{p_m\varepsilon}{p}}\int\limits_{|(y_1,\dots,y_m)|<|x|;\,|y_i|>\frac{\sqrt{m}}{m/\varepsilon},\,i=1,\dots,m}\prod_{i=1}^{m-1}|y_i|^{-\frac{n}{p_i}-\frac{p_m\varepsilon}{p_i}}|y_m|^{-\frac{n}{p_m}-\varepsilon}dy_1\dots dy_m
$$

when $|x| > 1$. It follows from this expression that

$$\|\mathcal{H}^m\|_{L^{p_1}\times L^{p_2}\times \dots L^{p_m}\to L^p} \geq C_4$$

by letting $\varepsilon \to 0$. □

3. Hardy's inequality on Morrey spaces and central Morrey spaces

For purposes of this section, we introduce some notation and review some definitions. In what follows, $Q(x, R)$ denotes the cube centered at x with side length R and with sides parallel to the coordinate axes. Moreover, $|Q(x, R)|$ denotes the Lebesgue measure of $Q(x, R)$. Also, $B(0, R)$ denotes a ball of radius R centered at the origin. To study the local behavior of solutions to second order elliptic partial differential equations, Morrey [12] introduced the $L^{q,\lambda}(\mathbb{R}^n)$ spaces.

Definition 2. *Let $1 \leq q < \infty$ and $-1/q \leq \lambda$. The classical Morrey space $L^{q,\lambda}(\mathbb{R}^n)$ is defined by*

$$L^{q,\lambda}(\mathbb{R}^n) = \{f \in L^q_{loc}(\mathbb{R}^n) : \|f\|_{L^{q,\lambda}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{L^{q,\lambda}(\mathbb{R}^n)} = \sup_{a\in\mathbb{R}^n, R>0} \left(\frac{1}{|Q(a,R)|^{1+\lambda q}} \int_{Q(a,R)} |f(x)|^q dx\right)^{1/q}.$$

Obviously, $L^{q,-1/q}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$, $L^{q,0}(\mathbb{R}^n) = L^\infty$. When $\lambda > 0$, $L^{q,\lambda}(\mathbb{R}^n) = \{0\}$. For this reason, we only consider the case $-1/q < \lambda < 0$ below. Recently, Alvarez, Guzmán-Partida and Lakey [1] introduced the notion of central Morrey spaces.

Definition 3. *Let $1 \leq q < \infty$ and $-1/q \leq \lambda < 0$. The central homogeneous Morrey space $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ is defined by*

$$\dot{B}^{q,\lambda}(\mathbb{R}^n) = \{f \in L^q_{loc}(\mathbb{R}^n) : \|f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)} = \sup_{R>0} \left(\frac{1}{|B(0,R)|^{1+\lambda q}} \int_{B(0,R)} |f(x)|^q dx\right)^{1/q}.$$

The inhomogeneous central Morrey space $B^{q,\lambda}(\mathbb{R}^n)$ is defined analogously with the exception that in the previous norm the supremum over $R > 0$ is restricted to $R \geq 1$.

Obviously, $\dot{B}^{q,\lambda}(\mathbb{R}^n) \subset B^{q,\lambda}(\mathbb{R}^n)$ for $\lambda \geq -1/q$ and $1 < q < \infty$.

Remark 2. *$\dot{B}^{q,\lambda}(\mathbb{R}^n)$ and $B^{q,\lambda}(\mathbb{R}^n)$ reduce to $\{0\}$ when $\lambda < -1/q$, and it is true that $\dot{B}^{q,-1/q} = B^{q,-1/q} = L^q$.*

Remark 3. *When $\lambda_1 < \lambda_2$, we have $B^{q,\lambda_1} \subset B^{q,\lambda_2}$ for $1 < q < \infty$. If $1 < q_1 < q_2 < \infty$, then Hölder's inequality yields that $\dot{B}^{q_2,\lambda} \subset \dot{B}^{q_1,\lambda}$ and $B^{q_2,\lambda} \subset B^{q_1,\lambda}$ for all $\lambda \in \mathbb{R}$.*

The following is the main result of this section. It is new even when $m = 1$.

Theorem 2. *Let $m \in \mathbb{N}$, f_i be in $\dot{B}^{p_i,\lambda_i}(\mathbb{R}^n)$, $1 < p_i < \infty$, $1 < p < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m}$, $-1/p_i \leq \lambda_i < 0$, $i = 1, 2, \ldots, m$, and $\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_m$. Then $\mathcal{H}^m$ maps $\dot{B}^{p_1,\lambda_1} \times \dot{B}^{p_2,\lambda_2} \times \cdots \times \dot{B}^{p_m,\lambda_m}$ to $\dot{B}^{p,\lambda}$ with norm*

$$\|\mathcal{H}^m\|_{\dot{B}^{p_1,\lambda_1}\times\dot{B}^{p_2,\lambda_2}\times\cdots\times\dot{B}^{p_m,\lambda_m}\to\dot{B}^{p,\lambda}} = \frac{{\omega_n}^m}{\omega_{mn}} \frac{m}{\lambda+m} \frac{1}{2^{m-1}} \frac{\prod_{i=1}^m \Gamma(\frac{n}{2}(\lambda_i+1))}{\Gamma(\frac{n}{2}(m+\lambda))}.$$

Proof. We consider the cases $m = 1$, $m = 2$, and $m \geq 2$.

1. Case $m = 1$.

We observe that, as in the proof of Theorem 1, the operator $\mathcal{H}$ and its restriction to radial functions have the same operator norm on the spaces $\dot{B}^{q,\lambda}$. Let

$$\mathcal{H}(f)(x) = \frac{1}{\Omega_n} \int_{B(0,1)} f(t|x|)\,dt,$$

where $B(0,1)$ is the unit ball in $\mathbb{R}^n$. By Minkowski's integral inequality, we have

$$\begin{aligned}
&\left(\frac{1}{|B(0,R)|^{1+\lambda q}} \int_{B(0,R)} \left|\frac{1}{\Omega_n}\int_{B(0,1)} f(t|x|)\,dt\right|^q dx\right)^{1/q} \\
\leq\ & \frac{1}{\Omega_n}\int_{B(0,1)} \left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)} |f(|t|x)|^q\,dx\right)^{1/q} dt \\
=\ & \frac{1}{\Omega_n}\int_{B(0,1)} \left(\frac{1}{|B(0,|t|R)|^{1+\lambda q}}\int_{B(0,|t|R)} |f(x)|^q dx\right)^{1/q} |t|^{n\lambda}\,dt \\
\leq\ & \frac{\|f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)}}{\Omega_n}\int_{B(0,1)} |t|^{n\lambda}\,dt \\
\leq\ & \frac{1}{1+\lambda}\|f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)}.
\end{aligned}$$

On the other hand, the function $f_0(x) = |x|^{n\lambda}$ lies in $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ and satisfies

$$\mathcal{H}(f_0) = \frac{1}{1+\lambda} f_0.$$

This yields the desired conclusion.

2. Case $m = 2$.

As before we note that the operator $\mathcal{H}^2$ and its restriction to radial functions have the same operator norm in $\dot{B}^{p,\lambda}$. Taking radial functions f_1 and f_2 we then write

$$\mathcal{H}^2(f_1, f_2)(x) = \frac{1}{\Omega_{2n}} \int_{|(z_1,z_2)|<1} f_1(|x|z_1) f_2(|x|z_2)\,dz_1 dz_2.$$

Using Minkowski's integral inequality and Hölder's inequality, we have

$$
\begin{aligned}
&\left(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}\left|\int_{|(y_1,y_2)|<1} f_1(y_1|x|)f_2(y_2|x|)\,dy_1dy_2\right|^p dx\right)^{1/p}\\
\le\ &\int_{|(y_1,y_2)|<1}\left(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}\left|f_1(|y_1|x)f_2(|y_2|x)\right|^p dx\right)^{1/p}\\
\le\ &\int_{|(y_1,y_2)|<1}\left(\frac{1}{|B(0,R)|^{1+\lambda_1 p_1}}\int_{B(0,R)}\left|f_1(|y_1|x)\right|^{p_1} dx\right)^{1/p_1}\\
&\quad\left(\frac{1}{|B(0,R)|^{1+\lambda_2 p_2}}\int_{B(0,R)}\left|f_2(|y_2|x)\right|^{p_2} dx\right)^{1/p_2} dy_1dy_2\\
=\ &\int_{|(y_1,y_2)|<1}\left(\frac{1}{|B(0,|y_1|R)|^{1+\lambda_1 p_1}}\int_{B(0,|y_1|R)}\left|f_1(x)\right|^{p_1} dx\right)^{1/p_1}\\
&\quad\left(\frac{1}{|B(0,|y_2|R)|^{1+\lambda_2 p_2}}\int_{B(0,|y_2|R)}\left|f_2(x)\right|^{p_2} dx\right)^{1/p_2} |y_1|^{n\lambda_1}|y_2|^{n\lambda_2}\,dy_1dy_2\\
\le\ &\int_{|(y_1,y_2)|<1}|y_1|^{n\lambda_1}|y_2|^{n\lambda_2}\,dy_1dy_2\,\|f_1\|_{\dot B^{p_1,\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot B^{p_2,\lambda_2}(\mathbb{R}^n)}
\end{aligned}
$$

where $\lambda=\lambda_1+\lambda_2$. A calculation yields that the value of the integral is

$$
\int_{|(y_1,y_2)|<1}|y_1|^{n\lambda_1}|y_2|^{n\lambda_2}dy_1dy_2=\frac{{\omega_n}^2}{2n}\frac{1}{2+\lambda}B\Big(\frac{n}{2}(1+\lambda_1),\frac{n}{2}(1+\lambda_2)\Big)\,, \tag{4}
$$

and this proves one direction in the claimed identity.

On the other hand, taking $\tilde f_i(x_i)=|x_i|^{n\lambda_i}$ for $x_i\in\mathbb{R}^n$, $i=1,2$, we obtain

$$
\begin{aligned}
\frac{1}{|B(0,R)|^{1+\lambda_i p_i}}\int_{B(0,R)}\left|\tilde f_i(x_i)dx_1\right|^p dx_i &= \frac{1}{|B(0,R)|^{1+\lambda_i p_i}}\int_{B(0,R)}|x_i|^{n\lambda_i p_i}dx_i\\
&= \frac{\omega_n}{R^{n(1+\lambda_i p_i)}}\int_0^R r^{n\lambda_i p_i+n-1}dr\\
&= \frac{\omega_n}{n(1+\lambda_i p_i)}\,.
\end{aligned}
$$

It is easy to verify that $\tilde f_i\in\dot B^{p_i,\lambda_i}$, $i=1,2$. By a simple calculation, we obtain that

$$
\mathcal{H}^2(\tilde f_1,\tilde f_2)(x)=\tilde f_1(x)\,\tilde f_2(x)\,\frac{1}{\Omega_{2n}}\int_{|(y_1,y_2)|<1}|y_1|^{n\lambda_1}|y_2|^{n\lambda_2}dy_1dy_2\,.
$$

This observation, combined with (4) concludes the proof in the case $m=2$.

3. Case $m\ge 3$

This case presents only notational differences and does not require any new ideas. For brevity we omit the details. □

Next we have the following result concerning best constants on the subspaces of $L^{p_i,\lambda_i}(\mathbb{R}^n)$ consisting of radial functions.

Proposition 1. *Let $m\in\mathbb{N}$, f_i be radial functions in $L^{p_i,\lambda_i}(\mathbb{R}^n)$, $i=1,2,\dots,m$, $1<p_i<\infty$, $-1/p_i\le\lambda_i<0$, $i=1,2,\dots,m$, $1\le p<\infty$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+\dots+\frac{1}{p_m}$, and $\lambda=\lambda_1+\lambda_2+\dots+\lambda_m$. Then*

$$\|\mathcal{H}^m(f_1,f_2,\dots,f_m)\|_{L^{p,\lambda}}\le C_{n,m,\lambda,\lambda_1,\lambda_2,\dots,\lambda_m}\prod_{i=1}^m\|f_i\|_{L^{p_i,\lambda_i}},\tag{5}$$

Moreover, the constant

$$C_{n,m,\lambda,\lambda_1,\lambda_2,\dots,\lambda_m}=\frac{{\omega_n}^m}{\omega_{mn}}\frac{m}{\lambda+m}\frac{1}{2^{m-1}}\frac{\prod_{i=1}^m\Gamma(\frac{n}{2}(\lambda_i+1))}{\Gamma(\frac{n}{2}(m+\lambda))}$$

is the same with that in Theorem 2, and is the best possible in inequality (5) *for radial functions.*

Proof. We first consider the case $m=1$.

As in the proof of Theorem 2, we take $f_0(x)=|x|^{n\lambda}$, $x\in\mathbb{R}^n$ and we only need to prove that $f_0\in L^{q,\lambda}(\mathbb{R}^n)$. In fact, we consider two cases:

(i) if $|a|>2R$, then $|x|>R$. For $-1/q<\lambda<0$, we have

$$\frac{1}{|Q(a,\,R)|^{1+\lambda q}}\int_{Q(a,\,R)}|x|^{n\lambda q}dx\le\frac{1}{|Q(a,\,R)|^{1+\lambda q}}\int_{Q(a,\,R)}R^{n\lambda q}dx=1.$$

(ii) if $|a|<2R$, then $Q(a,\,R)\subset Q(0,\,3R)$, we have

$$\frac{1}{|Q(a,\,R)|^{1+\lambda q}}\int_{Q(a,\,R)}|x|^{n\lambda q}dx\le\frac{1}{|Q(a,\,R)|^{1+\lambda q}}\int_{Q(0,\,3R)}|x|^{n\lambda q}dx=3^{n(1+q\lambda)}.$$

We now turn to the case $m=2$.

Let $f_1\in L^{p_1,\lambda_1}$ and $f_2\in L^{p_2,\lambda_2}$, be radial functions. By Minkowski's integral inequality and Hölder's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{|Q(a,R)|^{1+\lambda p}}\int_{Q(a,R)}\left|\int_{|(y_1,y_2)|<1}f_1(y_1|x|)f_2(y_2|x|)dy_1dy_2\right|^p dx\right)^{1/p}\\
=&\left(\frac{1}{|Q(a,R)|^{1+\lambda p}}\int_{Q(a,R)}\left|\int_{|(y_1,y_2)|<1}f_1(|y_1|x)f_2(|y_2|x)dy_1dy_2\right|^p dx\right)^{1/p}\\
\le&\int_{|(y_1,y_2)|<1}\left(\frac{1}{|Q(a,R)|^{1+\lambda p}}\int_{Q(a,R)}\left|f_1(|y_1|x)f_2(|y_2|x)\right|^p dx\right)^{1/p}dy_1dy_2\\
\le&\int_{|(y_1,y_2)|<1}\left(\frac{1}{|Q(a,R)|^{1+\lambda_1 p_1}}\int_{Q(a,R)}\left|f_1(|y_1|x)\right|^{p_1}dx\right)^{1/p_1}\\
&\quad\left(\frac{1}{|Q(a,R)|^{1+\lambda_2 p_2}}\int_{Q(a,R)}\left|f_2(|y_2|x)\right|^{p_2}dx\right)^{1/p_2}dy_1dy_2
\end{aligned}$$

$$
\begin{aligned}
&= \int_{|(y_1,y_2)|<1} \left(\frac{1}{|Q(a|y_1|,|y_1|R)|^{1+\lambda_1 p_1}} \int_{Q(a|y_1|,|y_1|R)} \Big| f_1(x) \Big|^{p_1} dx \right)^{1/p_1} \\
&\qquad \left(\frac{1}{|Q(a|y_2|,|y_2|R)|^{1+\lambda_2 p_2}} \int_{Q(a|y_2|,|y_2|R)} \Big| f_2(x) \Big|^{p_2} dx \right)^{1/p_2} |y_1|^{n\lambda_1} |y_2|^{n\lambda_2} dy_1 dy_2 \\
&\le \int_{|(y_1,y_2)|<1} |y_1|^{n\lambda_1} |y_2|^{n\lambda_2} dy_1 dy_2 \|f_1\|_{L^{p_1,\lambda_1}} \|f_2\|_{L^{p_2,\lambda_2}} \\
&= C_{n,\lambda,\lambda_1,\lambda_2} \|f_1\|_{L^{p_1,\lambda_1}} \|f_2\|_{L^{p_2,\lambda_2}} .
\end{aligned}
$$

On the other hand, taking

$$\tilde{f}_i(x_i) = |x_i|^{n\lambda_i}, x_i \in \mathbb{R}^n, i = 1, 2,$$

we easily verify that $\tilde{f}_i \in L^{p_i,\lambda_i}$, $i = 1, 2$. by considering the cases $|a| > 2R$ and $|a| < 2R$. Then the desired conclusion follows via the method in the proof of Theorem 2. We omit the details.

At last, the case $m \ge 3$ presents only notational differences and does not require any new ideas; for brevity the details are omitted. □

4. m-linear Hilbert operators

In this section we focus our attention to the positive real numbers $(0, \infty)$ with the usual Lebesgue measure and we let T^m be the m-linear Hilbert operator

$$T^m(f_1, \dots, f_m)(x) = \int_0^\infty \dots \int_0^\infty \frac{f_1(x_1) \dots f_m(x_m)}{(x + x_1 + \dots + x_m)^m} dx_1 \dots dx_m, \tag{6}$$

where $x > 0$. The following is a known sharp estimate

$$\int_0^\infty T^1(f)(x) g(x)\, dx \le \frac{\pi}{\sin(\pi/p)} \|f\|_{L^p(0,\infty)} \|g\|_{L^{p'}(0,\infty)},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, $1 < p < \infty$; see for instance [13], [10], and [6].

We have the following result concerning T^1

Proposition 2. *Let $1 < p < \infty$, $-1 < \alpha < p - 1$. For any function f in $L^p(x^\alpha dx)$, we have*

$$\|T^1(f)\|_{L^p(x^\alpha dx)} \le \frac{\pi}{\sin(\pi(\alpha+1)/p)} \|f\|_{L^p(x^\alpha dx)}.$$

Moreover,

$$\|T^1\|_{L^p(x^\alpha dx) \to L^p(x^\alpha dx)} = \frac{\pi}{\sin(\pi(\alpha+1)/p)} .$$

Proof. By Minkowski's integral inequality, we have

$$\begin{aligned}
\|T^1(f)\|_{L^p(x^\alpha dx)} &= \left(\int_0^\infty \left|\int_0^\infty \frac{f(y)}{x+y}dy\right|^p x^\alpha dx\right)^{\frac{1}{p}} \\
&\leq \left(\int_0^\infty \left(\int_0^\infty \frac{|f(yx)|}{1+y}dy\right)^p x^\alpha dx\right)^{\frac{1}{p}} \\
&\leq \int_0^\infty \left(\int_0^\infty |f(yx)|^p x^\alpha dx\right)^{\frac{1}{p}} \frac{1}{1+y}dy \\
&= \int_0^\infty \left(\int_0^\infty |f(x)|^p x^\alpha dx\right)^{\frac{1}{p}} \frac{y^{-(\alpha+1)/p}}{1+y}dy \\
&= B\Big(1-\frac{\alpha+1}{p}, \frac{\alpha+1}{p}\Big)\|f\|_{L^p(x^\alpha dx)}.
\end{aligned}$$

Thus, one deduces the estimate

$$\|T^1\|_{L^p(x^\alpha dx)\to L^p(x^\alpha dx)} \leq B\Big(1-\frac{\alpha+1}{p}, \frac{\alpha+1}{p}\Big).$$

To obtain a lower bound for the operator norm we take $0<\varepsilon<\min\{1, p-\alpha-1\}$, and define

$$f_\varepsilon(x) = \begin{cases} 0, & x\leq 1, \\ x^{-\frac{1}{p}-\frac{\alpha}{p}-\frac{\varepsilon}{p}}, & |x|>1. \end{cases}$$

A calculation yields that $\|f_\varepsilon\|^p_{L^p(x^\alpha dx)} = \frac{1}{\varepsilon}$. We have

$$\begin{aligned}
\|T^1(f_\varepsilon)\|_{L^p(x^\alpha dx)} &= \left(\int_0^\infty \left(\int_1^\infty \frac{y^{-\frac{1+\alpha+\varepsilon}{p}}}{x+y}dy\right)^p x^\alpha dx\right)^{\frac{1}{p}} \\
&\geq \left(\int_1^\infty \left(\int_1^\infty \frac{y^{-\frac{1+\alpha+\varepsilon}{p}}}{x+y}dy\right)^p x^\alpha dx\right)^{\frac{1}{p}} \\
&= \left(\int_1^\infty \left(\int_{\frac{1}{x}}^\infty \frac{y^{-\frac{1+\alpha+\varepsilon}{p}}}{1+y}dy\right)^p x^{-1-\varepsilon} dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Note that

$$\begin{aligned}
\int_{\frac{1}{x}}^\infty \frac{y^{-\frac{1+\alpha+\varepsilon}{p}}}{1+y}\,dy &= \int_0^\infty \frac{y^{-\frac{1+\alpha+\varepsilon}{p}}}{1+y}dy - \int_0^{\frac{1}{x}} \frac{y^{-\frac{1+\alpha+\varepsilon}{p}}}{1+y}\,dy \\
&\geq B\Big(1-\frac{1+\alpha+\varepsilon}{p}, \frac{1+\alpha+\varepsilon}{p}\Big) - \int_0^{\frac{1}{x}} y^{-\frac{1+\alpha+\varepsilon}{p}}\,dy \\
&= B\Big(1-\frac{1+\alpha+\varepsilon}{p}, \frac{1+\alpha+\varepsilon}{p}\Big) - \frac{x^{-1+\frac{1+\alpha+\varepsilon}{p}}}{1-\frac{1+\alpha+\varepsilon}{p}}.
\end{aligned}$$

It follows that $\|T^1(f_\varepsilon)\|_{L^p(x^\alpha dx)}$ is at least as big as

$$\Big(\int_1^\infty x^{-1-\varepsilon}dx\Big)^{1/p} B\Big(1-\frac{1+\alpha+\varepsilon}{p},\frac{1+\alpha+\varepsilon}{p}\Big)$$

$$-\frac{p}{p-1-\alpha-\varepsilon}\Big(\int_1^\infty x^{(-1+\frac{1+\alpha+\varepsilon}{p})p-1-\varepsilon}dx\Big)^{1/p}$$

$$= \frac{1}{\varepsilon^{1/p}}B\Big(1-\frac{1+\alpha+\varepsilon}{p},\frac{1+\alpha+\varepsilon}{p}\Big)-\frac{p}{p-1-\alpha-\varepsilon}\,\frac{1}{(p-\alpha-1)^{1/p}}\,.$$

Letting $\varepsilon\to 0$, we deduce that

$$\lim_{\varepsilon\to 0}\frac{\|T^1(f_\varepsilon)\|_{L^p(x^\alpha dx)}}{\|f_\varepsilon\|_{L^p(x^\alpha dx)}}\geq B\Big(1-\frac{\alpha+1}{p},\frac{1+\alpha}{p}\Big)\,.$$

Thus

$$\|T^1\|_{L^p(x^\alpha dx)\to L^p(x^\alpha dx)}\geq B\Big(1-\frac{\alpha+1}{p},\frac{1+\alpha}{p}\Big)=\frac{\pi}{\sin(\pi(\alpha+1)/p)}$$

and this concludes the proof of the proposition. □

Next we recall the following result from Bényi and Oh [2].

Theorem. *Let $m\geq 2$, $f_i\in L^{p_i}(0,\infty)$, $i=1,2,\dots,m$, $1<p_i<\infty$, $1<p<\infty$, and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+\dots+\frac{1}{p_m}$. Then*

$$\|T^m\|_{L^{p_1}(0,\infty)\times L^{p_2}(0,\infty)\times\cdots\times L^{p_m}(0,\infty)\to L^p(0,\infty)}=\frac{\prod_{i=1}^m\Gamma(\frac{1}{p_i'})\Gamma(\frac{1}{p})}{\Gamma(m)}.$$

We provide the following weighted extension of this result.

Theorem 3. *Let $m\in\mathbb{N}$, f_i be in $L^{p_i}(x^{\frac{\alpha_i p_i}{p}}dx), i=1,2,\dots,m$, $1<p_i<\infty$, $1\leq p<\infty$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+\dots+\frac{1}{p_m}$, $-(1+1/p_i)p<\alpha_i<p(1-1/p_i)$ and $\alpha=\alpha_1+\alpha_2+\dots+\alpha_m$. Then*

$$\|T^m\|_{L^{p_1}(x^{\frac{\alpha_1p_1}{p}}dx)\times L^{p_2}(x^{\frac{\alpha_2p_2}{p}}dx)\times\cdots\times L^{p_m}(x^{\frac{\alpha_mp_m}{p}}dx)\to L^p(x^\alpha dx)}=\frac{\prod_{i=1}^m\Gamma(\frac{1}{p_i'}-\frac{\alpha_i}{p})\Gamma(\frac{1+\alpha}{p})}{\Gamma(m)}.$$

Proof. For simplicity we only provide the proof in the case $m=2$. (The case $m=1$ is the essence of Proposition 2.) The proof of Theorem 2 yields the stated upper bound. It therefore suffices to show that the constant obtained in this way is also a lower bound. For

$$0<\varepsilon<\min\Big\{1,\frac{p_1}{p_2}(\frac{1}{p_1'}-\frac{\alpha_1}{p}),\frac{1}{p_2'}-\frac{\alpha_2}{p}\Big\},$$

we take

$$f_1^\varepsilon(x_1)=\begin{cases}0, & 0<x_1\leq 1,\\ x_1^{-\frac{1}{p_1}-\frac{\alpha_1}{p}-\frac{p_2\varepsilon}{p_1}}, & x_1>1,\end{cases}$$

and

$$f_2^\varepsilon(x_2)=\begin{cases}0, & 0<x_2\le 1,\\ x_2^{-\frac{1}{p_2}-\frac{\alpha_2}{p}-\varepsilon}, & x_2>1.\end{cases}$$

We have that

$$\|f_1^\varepsilon\|^{p_1}_{L^{p_1}(x_1^{\frac{\alpha_1 p_1}{p}}dx_1)}=\|f_2^\varepsilon\|^{p_2}_{L^{p_2}(x_2^{\frac{\alpha_2 p_2}{p}}dx_2)}=\frac{1}{p_2\varepsilon}.$$

We have the lower estimate

$$\begin{aligned}
&\|T(f_1^\varepsilon,f_2^\varepsilon)\|_{L^p(x^\alpha dx)}\\
=&\left(\int_0^\infty\left(\int_0^\infty\int_0^\infty\frac{f_1^\varepsilon(x_1)f_2^\varepsilon(x_2)}{(x+x_1+x_2)^2}\,dx_1dx_2\right)^p x^\alpha\,dx\right)^{1/p}\\
\ge&\left(\int_1^\infty\left(\int_1^\infty\int_1^\infty\frac{x_1^{-\frac{1}{p_1}-\frac{\alpha_1}{p}-\frac{p_2\varepsilon}{p_1}}x_2^{-\frac{1}{p_2}-\frac{\alpha_2}{p}-\varepsilon}}{(x+x_1+x_2)^2}\,dx_1dx_2\right)^p x^\alpha\,dx\right)^{1/p}\\
=&\left(\int_1^\infty x^{-1-p_2\varepsilon}\left(\int_{1/x}^\infty\int_{1/x}^\infty\frac{x_1^{-\frac{1}{p_1}-\frac{\alpha_1}{p}-\frac{p_2\varepsilon}{p_1}}x_2^{-\frac{1}{p_2}-\frac{\alpha_2}{p}-\varepsilon}}{(1+x_1+x_2)^2}\,dx_1dx_2\right)^p dx\right)^{1/p}\\
\ge&\left(\int_{1/\varepsilon}^\infty x^{-1-p_2\varepsilon}\left(\int_{\varepsilon}^\infty\int_{\varepsilon}^\infty\frac{x_1^{-\frac{1}{p_1}-\frac{\alpha_1}{p}-\frac{p_2\varepsilon}{p_1}}x_2^{-\frac{1}{p_2}-\frac{\alpha_2}{p}-\varepsilon}}{(1+x_1+x_2)^2}\,dx_1dx_2\right)^p dx\right)^{1/p}\\
=&\left(\frac{\varepsilon^{\varepsilon p_2}}{p_2\varepsilon}\right)^{1/p}\int_\varepsilon^\infty\int_\varepsilon^\infty\frac{x_1^{-1/p_1-\alpha_1/p-p_2\varepsilon/p_1}x_2^{-1/p_2-\alpha_2/p-\varepsilon}}{(1+x_1+x_2)^2}\,dx_1dx_2.
\end{aligned}$$

Next, we write

$$\int_\varepsilon^\infty\int_\varepsilon^\infty\frac{x_1^{-1/p_1-\alpha_1/p-p_2\varepsilon/p_1}x_2^{-1/p_2-\alpha_2/p-\varepsilon}}{(1+x_1+x_2)^2}\,dx_1dx_2=I_1-I_2,$$

where

$$\begin{aligned}
I_1&=\int_0^\infty\int_\varepsilon^\infty\frac{x_1^{-1/p_1-\alpha_1/p-p_2\varepsilon/p_1}x_2^{-1/p_2-\alpha_2/p-\varepsilon}}{(1+x_1+x_2)^2}\,dx_1dx_2\\
I_2&=\int_0^\varepsilon\int_\varepsilon^\infty\frac{x_1^{-1/p_1-\alpha_1/p-p_2\varepsilon/p_1}x_2^{-1/p_2-\alpha_2/p-\varepsilon}}{(1+x_1+x_2)^2}\,dx_1dx_2\,.
\end{aligned}$$

We write I_1 as follows:

$$\begin{aligned}
I_1 &= \int_0^\infty \int_\varepsilon^\infty \frac{x_1^{-1/p_1-\alpha_1/p-p_2\varepsilon/p_1} x_2^{-1/p_2-\alpha_2/p-\varepsilon}}{(1+x_1+x_2)^2}\, dx_1 dx_2 \\
&= \int_0^\infty \int_0^\infty \frac{x_1^{-1/p_1-\alpha_1/p-p_2\varepsilon/p_1} x_2^{-1/p_2-\alpha_2/p-\varepsilon}}{(1+x_1+x_2)^2}\, dx_1 dx_2 \\
&\quad - \int_0^\infty \int_0^\varepsilon \frac{x_1^{-1/p_1-\alpha_1/p-p_2\varepsilon/p_1} x_2^{-1/p_2-\alpha_2/p-\varepsilon}}{(1+x_1+x_2)^2}\, dx_1 dx_2 \\
&= B\Big(\frac{1}{p_1'} - \frac{\alpha_1}{p} - \frac{p_2\varepsilon}{p_1}, 1 + \frac{1}{p_1} + \frac{\alpha_1}{p} + \frac{p_2\varepsilon}{p_1}\Big) B\Big(\frac{1}{p_2'} - \frac{\alpha_2}{p} - \varepsilon, \frac{\alpha+1+p_2\varepsilon}{p}\Big) \\
&\quad - \int_0^\varepsilon \int_0^\infty \frac{x_1^{-1/p_1-\alpha_1/p-p_2\varepsilon/p_1} x_2^{-1/p_2-\alpha_2/p-\varepsilon}}{(1+x_1+x_2)^2}\, dx_2 dx_1 \\
&= B\Big(\frac{1}{p_1'} - \frac{\alpha_1}{p} - \frac{p_2\varepsilon}{p_1}, 1 + \frac{1}{p_1} + \frac{\alpha_1}{p} + \frac{p_2\varepsilon}{p_1}\Big) B\Big(\frac{1}{p_2'} - \frac{\alpha_2}{p} - \varepsilon, \frac{\alpha+1+p_2\varepsilon}{p}\Big) \\
&\quad - B\Big(\frac{1}{p_2'} - \frac{\alpha_2}{p_2} - \varepsilon, 1 + \frac{1}{p_2} + \frac{\alpha_2}{p_2} + \varepsilon\Big) \int_0^\varepsilon \frac{x_1^{-\frac{1}{p_1}-\frac{\alpha_1}{p}-\frac{p_2\varepsilon}{p_1}}}{(1+x_1)^{1+\frac{1}{p_2}+\frac{\alpha_2}{p}+\varepsilon}}\, dx_1 \\
&\geq B\Big(\frac{1}{p_1'} - \frac{\alpha_1}{p} - \frac{p_2\varepsilon}{p_1}, 1 + \frac{1}{p_1} + \frac{\alpha_1}{p} + \frac{p_2\varepsilon}{p_1}\Big) B\Big(\frac{1}{p_2'} - \frac{\alpha_2}{p} - \varepsilon, \frac{\alpha+1+p_2\varepsilon}{p}\Big) \\
&\quad - B\Big(\frac{1}{p_2'} - \frac{\alpha_2}{p_2} - \varepsilon, 1 + \frac{1}{p_2} + \frac{\alpha_2}{p_2} + \varepsilon\Big) \frac{\varepsilon^{1-\frac{1}{p_1}-\frac{\alpha_1}{p}-\frac{p_2\varepsilon}{p_1}}}{1-\frac{1}{p_1}-\frac{\alpha_1}{p}-\frac{p_2\varepsilon}{p_1}}.
\end{aligned}$$

We estimate I_2 in the following way:

$$\begin{aligned}
I_2 &= \int_0^\varepsilon \int_\varepsilon^\infty \frac{x_1^{-1/p_1-\alpha_1/p-p_2\varepsilon/p_1} x_2^{-1/p_2-\alpha_2/p-\varepsilon}}{(1+x_1+x_2)^2}\, dx_1 dx_2 \\
&= \int_0^\varepsilon \int_0^\infty \frac{x_1^{-1/p_1-\alpha_1/p-p_2\varepsilon/p_1} x_2^{-1/p_2-\alpha_2/p-\varepsilon}}{(1+x_1+x_2)^2}\, dx_1 dx_2 \\
&\quad - \int_0^\varepsilon \int_0^\varepsilon \frac{x_1^{-1/p_1-\alpha_1/p-p_2\varepsilon/p_1} x_2^{-1/p_2-\alpha_2/p-\varepsilon}}{(1+x_1+x_2)^2}\, dx_1 dx_2 \\
&= B\Big(\frac{1}{p_1'} - \frac{\alpha_1}{p} - \frac{p_2\varepsilon}{p_1}, 1 + \frac{1}{p_1} + \frac{\alpha_1}{p} + \frac{p_2\varepsilon}{p_1}\Big) \int_0^\varepsilon \frac{x_2^{-\frac{1}{p_2}-\frac{\alpha_2}{p}-\varepsilon}}{(1+x_2)^{1+\frac{1}{p_1}+\frac{\alpha_1}{p}+\frac{p_2\varepsilon}{p_1}}}\, dx_2 \\
&\quad - \int_0^\varepsilon \int_0^\varepsilon \frac{x_1^{-1/p_1-\alpha_1/p-p_2\varepsilon/p_1} x_2^{-1/p_2-\alpha_2/p-\varepsilon}}{(1+x_1+x_2)^2}\, dx_1 dx_2 \\
&\leq B\Big(\frac{1}{p_1'} - \frac{\alpha_1}{p} - \frac{p_2\varepsilon}{p_1}, 1 + \frac{1}{p_1} + \frac{\alpha_1}{p} + \frac{p_2\varepsilon}{p_1}\Big) \frac{\varepsilon^{1-\frac{1}{p_2}-\frac{\alpha_2}{p}-\varepsilon}}{1-\frac{1}{p_2}-\frac{\alpha_2}{p}-\varepsilon} \\
&\quad - \int_0^\varepsilon \int_0^\varepsilon \frac{x_1^{-1/p_1-\alpha_1/p-p_2\varepsilon/p_1} x_2^{-1/p_2-\alpha_2/p-\varepsilon}}{(1+x_1+x_2)^2} dx_1 dx_2.
\end{aligned}$$

The previous expressions for I_1 and I_2 allow us to easily compute their limits as $\varepsilon \to 0$. Indeed, letting $\varepsilon \to 0$, we obtain

$$\begin{aligned} & \lim_{\varepsilon\to 0} \frac{\|T(f_1^\varepsilon, f_2^\varepsilon)\|_{L^p(x^\alpha dx)}}{\|f_1^\varepsilon\|_{L^{p_1}(x^{\frac{\alpha_1 p_1}{p}} dx)} \|f_2^\varepsilon\|_{L^{p_2}(x^{\frac{\alpha_2 p_2}{p}} dx)}} \\ \geq \ & B\Big(1-\frac{\alpha_1}{p}-\frac{1}{p_1}, 1+\frac{\alpha_1}{p}+\frac{1}{p_1}\Big) B\Big(1-\frac{\alpha_2}{p}-\frac{1}{p_2}, \frac{1+\alpha}{p}\Big) \\ = \ & \Gamma\Big(1-\frac{\alpha_1}{p}-\frac{1}{p_1}\Big)\Gamma\Big(1-\frac{\alpha_2}{p}-\frac{1}{p_2}\Big)\Gamma\Big(\frac{1+\alpha}{p}\Big). \end{aligned}$$

This estimate provides the reverse norm inequality and finishes the proof of Theorem 3. □

5. Final Remarks

Obviously, both the m-linear Hardy operator (2) and the m-linear Hilbert operator (6) map $L^1 \times \ldots \times L^1$ to weak $L^{1/m}$. It follows by interpolation that they map $L^{p_1} \times \ldots \times L^{p_m}$ to L^p when $1/p_1 + \ldots + 1/p_m = 1/p > 1$. We are uncertain at the moment as to what the norm of these operators are on these spaces when $p < 1$.

We provide some remarks related to the case $m = 2$ and $\frac{1}{2} \leq p < 1$. These easily extend to general $m \geq 2$.

Proposition 3. *Let f_i be in $L^{p_i}(\mathbb{R}^n)$, $i = 1, 2$, $1 < p_i < \infty$, $1/2 < p < 1$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then we have*

$$\|\mathcal{H}^2(f_1, f_2)\|_{L^p} \leq C'_{p,p_1,p_2,n}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}},$$

where $\frac{{\omega_n}^2}{\omega_{2n}} \frac{p}{2p-1} B\big(\frac{n}{2} - \frac{n}{2p_1}, \frac{n}{2} - \frac{n}{2p_2}\big) \leq C'_{p,p_1,p_2,n} \leq \frac{\Omega_n^2}{\Omega_{2n}} \frac{p_1}{p_1-1} \frac{p_2}{p_2-1}$.

Proof. The idea of the proof of Theorem 1 yields a lower bound. For the upper bound, since the condition $|(y_1, y_2)| < |x|$ implies that $|y_1| < |x|$ and $|y_2| < |x|$, we obtain the estimate

$$\begin{aligned} |\mathcal{H}^2(f_1, f_2)(x)| &= \Big|\frac{1}{\Omega_{2n}} \frac{1}{|x|^{2n}} \int_{|(y_1,y_2)|<|x|} f_1(y_1) f_2(y_2)\, dy_1 dy_2\Big| \\ &\leq \frac{1}{\Omega_{2n}} \frac{1}{|x|^{2n}} \int_{|y_1|<|x|} |f_1(y_1)|\, dy_1 \int_{|y_2|<|x|} |f_2(y_2)|\, dy_2 \\ &\leq \frac{\Omega_n^2}{\Omega_{2n}} \mathcal{H}(f_1)(x)\mathcal{H}(f_1)(x). \end{aligned}$$

By Hölder's inequality ($\frac{1}{p_1/p} + \frac{1}{p_2/p} = 1$) and Theorem 1 in [3], we obtain

$$\|\mathcal{H}^2(f_1, f_2)\|_{L^p(\mathbb{R}^n)} \leq \frac{\Omega_n^2}{\Omega_{2n}}\Big(\frac{p_1}{p_1 - 1}\Big)\Big(\frac{p_2}{p_2 - 1}\Big)\|f_1\|_{L^{p_1}(\mathbb{R}^n)}\|f_2\|_{L^{p_2}(\mathbb{R}^n)}.$$

This proves the claimed estimate. □

We have an analogous proposition for the bilinear Hilbert operator T^2.

Proposition 4. *Let f_i be in $L^{p_i}(\mathbb{R}^n)$, $i = 1, 2$, $1 < p_i < \infty$, $1/2 < p < 1$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then we have*

$$\|T^2(f_1, f_2)\|_{L^p} \le C''_{p,p_1,p_2,n}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}},$$

where

$$\Gamma(1/p_1')\Gamma(1/p_2')\Gamma(1/p) \le C''_{p,p_1,p_2} \le \frac{\pi}{\sin \pi/p_1}\,\frac{\pi}{\sin \pi/p_2}.$$

At last, our guess for the sharp bounds in the case $1/m < p < 1$ are the constants obtained by the radial counterexamples.

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References

[1] J. Alvarez, M. Guzmán-Partida and J. Lakey, *Spaces of bounded λ-central mean oscillation, Morrey spaces, and λ-central Carleson measures*, Collect. Math. **51** (2000), 1–47.

[2] Á. Bényi and T. Oh, *Best constants for certain multilinear integral operators*, J. Inequal. Appl. **2006** (2006), Article ID 28582, 1–12.

[3] M. Christ and L. Grafakos, *Best constants for two nonconvolution inequalities*, Proc. Amer. Math. Soc. **123** (1995), 1687–1693.

[4] R. R. Coifman and Y. Meyer, *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. **212** (1975), 315–331.

[5] W. Faris, *Weak Lebesque spaces and quantum mechanical binding*, Duke Math. J. **43** (1976), 365–373.

[6] L. Grafakos, Classical Fourier analysis, Second edition, Graduate Texts in Math., no 249, Springer, New York, 2008.

[7] L. Grafakos and S. Montgomery-Smith, *Best constants for uncentred maximal functions*, Bull. London Math. Soc. **29** (1997), 60–64.

[8] L. Grafakos and R. H. Torres, *A multilinear Schur test and multiplier operators*, J. Func. Anal. **187** (2001), 1–24.

[9] G. H. Hardy, *Note on a theorem of Hilbert*, Math. Zeit. **6** (1920), 314–317.

[10] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed., Cambridge University Press, Cambridge, UK, 1952.

[11] A. Melas, *The best constant for the centered Hardy-Littlewood maximal inequality*, Ann. Math. **157** (2003), 647–488.

[12] C. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126–166.

[13] I. Schur, *Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen*, Journal f. Math. **140** (1911), 1–28.

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