

A SHORT GLIMPSE OF THE GIANT FOOTPRINT OF FOURIER ANALYSIS AND RECENT MULTILINEAR ADVANCES

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ABSTRACT. We provide a quick overview of the genesis and impact of Fourier Analysis in mathematics. We review some important results that have driven research during the last 50 years and we discuss recent advances in multilinear aspects of the theory.

1. INTRODUCTION

Fourier series are special trigonometric expansions named after of Jean-Baptiste Joseph Fourier (1768–1830), who made important contributions to their study. Fourier introduced these series in his attempt to solve the heat equation in a metal plate. He published his initial results in 1807 in his *Mémoire sur la propagation de la chaleur dans les corps solides*. Although important theoretical aspects of Fourier series were later proved by Laplace and Dirichlet among others, Fourier’s *Mémoire* introduced to mathematics the series that carry his name today. Fourier’s research led to the understanding that arbitrary (continuous) functions can be represented as infinite trigonometric series. Fourier’s complete theory appeared in his monograph *Théorie analytique de la chaleur*, published in 1822 by the French Academy.

Fourier accompanied Napoleon on his expedition to Egypt and until the turn of the nineteenth century he was engaged in extensive research on Egyptian antiquities. In view of his prominent career in Egyptology, it came as a surprise that he initiated an extensive study of heat propagation. Fourier began his work on the theory of heat in Grenoble in 1807 and completed it in Paris in 1822. In his work he expressed the conduction of heat in two-dimensional objects (i.e., very thin sheets of material) in terms of the differential equation

$$\frac{\partial u}{\partial t} = k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

in which $u(t, x, y)$ is the temperature at any time t at a point (x, y) of the plane and k is a constant, called the diffusivity of the material.

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The problem is to find the temperature, for example, in a conducting plate, if at time $t = 0$, the temperature is given at the boundary and at the points of the plane. For the solution of this problem Fourier introduced infinite series of sines and cosines.

To better comprehend propagation of heat, consider a square metal plate of side unit length pictured as $[0, 1] \times [0, 1]$ in the (x, y) coordinate system. Suppose that there is no heat source within the plate and that three of its four sides are held at 0 degrees Celsius, while linearly increasing temperature $T(x, 1) = x$, for $x \in (0, 1)$, is applied on its fourth side $y = 1$. Then the stationary heat distribution (the heat distribution as time tends to infinity) is depicted in Figure 1. Notice that there is a concentration of heat near the rightmost quarter of the top boundary of the plate, while heat seems to disperse towards the center of the plate.

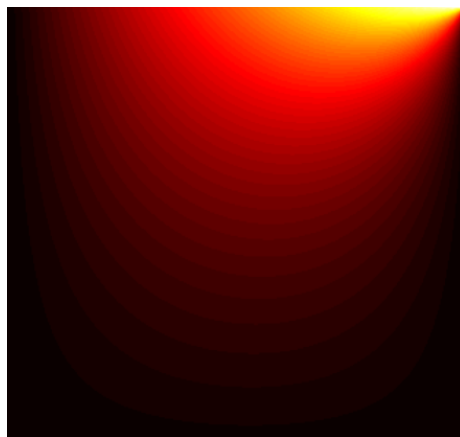


FIGURE 1. Heat distribution in a metal plate obtained using Fourier's method. Colder areas correspond to darker colors. Acknowledgment: Wikipedia.

To provide the precise mathematical expression of Fourier series, we define the m^{th} Fourier coefficient of a 1-periodic function F on the line by

$$\widehat{F}(m) = \int_0^1 F(x)e^{-2\pi i x m} dx,$$

where m is an integer. Then the series

$$(1) \quad \sum_{m \in \mathbb{Z}} \widehat{F}(m)e^{2\pi i x m}$$

is called the Fourier series of F . This series can also be written in terms of sines and cosines: $\sum_{m \in \mathbb{Z}} a_m \cos(2\pi m x) + b_m \sin(2\pi m x)$, where there is a relationship between $\widehat{F}(m)$ and the pair (a_m, b_m) . In two dimensions $n = 2$, F is 1-periodic function on \mathbb{R}^2 (with period 1 in each variable). The $(m_1, m_2)^{\text{th}}$ Fourier coefficient of F is defined as

$$\widehat{F}(m_1, m_2) = \int_0^1 \int_0^1 F(x_1, x_2)e^{-2\pi i(x_1 m_1 + x_2 m_2)} dx_1 dx_2,$$

where m_1, m_2 are integers. Then the Fourier series of F is given by

$$(2) \quad \sum_{m_1 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \widehat{F}(m_1, m_2) e^{2\pi i(x_1 m_1 + x_2 m_2)}.$$

It is a fundamental question in which sense does the series in (1) and (2) converge. An important feature of these series is that if they converge in some sense, they do so back to the function; in other words they provide a representation of the function. This property, called *Fourier inversion*, says (for instance in one dimension) that the identity

$$F(x) = \sum_{m \in \mathbb{Z}} \widehat{F}(m) e^{2\pi i x m}$$

holds in many cases, such as when the 1-periodic function F is smooth. Some of the convergence properties of the series for more singular functions F are discussed in the next historical section.

For numerical implementation, an alternative version of the Fourier transform is used. The *discrete Fourier transform* is an operator defined on sequences $a = \{a(n)\}_{n=1}^N$ as follows:

$$\widehat{a}(n) = \frac{1}{\sqrt{N}} \sum_{k=1}^N a(k) e^{-2\pi i \frac{k}{N} n}$$

and the following inversion identity holds:

$$a(n) = \frac{1}{\sqrt{N}} \sum_{k=1}^N \widehat{a}(k) e^{2\pi i \frac{k}{N} n}.$$

This representation is employed to study a discretized signal via Fourier analysis.

2. A SHORT HISTORICAL ACCOUNT OF THE THEORY OF ONE-DIMENSIONAL FOURIER SERIES

To study convergence of Fourier series we introduce the *partial sum operator*

$$S_N(f)(x) = \sum_{m=-N}^N \widehat{f}(m) e^{2\pi i m x}$$

where N is a positive integer.

Elementary functional analysis shows that convergence of the sequence $S_N(f)$ to f in $L^p([0, 1])$ as $N \rightarrow \infty$ is equivalent to the L^p boundedness of the conjugate function

$$\widetilde{F}(x) = \lim_{\varepsilon \rightarrow 0} \int_{\frac{1}{2} \geq |t| \geq \varepsilon} F(x-t) \cot(\pi t) dt$$

on the circle¹. The boundedness of the conjugate function on the circle and, hence, the L^p convergence of one-dimensional Fourier series was announced by Riesz in [23], but its proof appeared a little later in [24].

Luzin [20] conjectured in 1913 that the Fourier series of continuous functions converge almost everywhere convergence. This conjecture was settled by Carleson [3] in 1965 for the more general class of square-summable functions. Carleson's important contribution was to obtain the L^2 boundedness for the *maximal partial sum operator*, called today *Carleson operator*, which is defined as the supremum of all partial sums of a square-integrable function, i.e.,

$$S_*(f)(x) = \sup_{N>0} |S_N(f)(x)|.$$

Carleson's theorem was later extended by Hunt [11] for the class of L^p functions for all $1 < p < \infty$. Hunt's significant contribution was the derivation of the powerful estimate

$$(3) \quad |\{S_*(\chi_F) > \lambda\}| \leq C|F| \begin{cases} \frac{1}{\lambda} \log \frac{1}{\lambda} & \text{when } \lambda < \frac{1}{2} \\ e^{-c\lambda} & \text{when } \lambda \geq \frac{1}{2} \end{cases}$$

for the distribution function of the Carleson operator acting on the characteristic function of a measurable set F . Making use of (3), Sjölin [25] sharpened this result by showing that the Fourier series of 1-periodic functions f with

$$\int_0^1 |f|(\log^+ |f|)(\log^+ \log^+ |f|) dt < \infty$$

converge almost everywhere. This result was improved by Antonov [1] to functions f with the property that

$$(4) \quad \int_0^1 |f|(\log^+ |f|)(\log^+ \log^+ \log^+ |f|) dt < \infty$$

Counterexamples due to Konyagin [15] show that Fourier series of functions f with $|f|(\log^+ |f|)^{\frac{1}{2}}(\log^+ \log^+ |f|)^{-\frac{1}{2}-\varepsilon}$ integrable over \mathbf{T}^1 may diverge when $\varepsilon > 0$. So at present, there is a gap between this counterexample and condition (4). Examples of continuous functions whose Fourier series diverge exactly on given sets of measure zero are given in Katznelson [14] and Kahane and Katznelson [13].

¹1-periodic functions on the line can also be viewed as functions on the circle.

Fefferman [8] provided an alternative proof of the almost everywhere convergence of one-dimensional Fourier series of square-integrable 1-periodic functions. He achieved this by showing that Carleson's operator S_* maps $L^2 \rightarrow L^{2-\varepsilon}$ on the circle introducing a new technique referred to today as *time-frequency analysis*. In his work he pointed out that his methodology could also yield the almost everywhere convergence of one-dimensional Fourier series of L^p functions. But Fefferman remarked in [8] that his techniques are not powerful enough to recover Hunt's distributional estimate (3).

Lacey and Thiele [18] provided an analogue of Fefferman's proof on the line showing that the maximal Fourier integral operator

$$(5) \quad S_{**}(F)(x) = \sup_{T>0} \left| \int_{-T}^T \widehat{F}(\xi) e^{2\pi i x \xi} d\xi \right|$$

maps $L^2(\mathbb{R})$ to a space larger than L^2 called weak- L^2 . Here $\widehat{F}(\xi) = \int_{\mathbb{R}} F(y) e^{-2\pi i y \xi} dy$ is the Fourier transform of a function F on the line. In doing so, they improved a "technical issue" in Fefferman's proof where weak- L^2 was replaced by $L^{2-\varepsilon}$; it should be noted that for the almost everywhere convergence on the circle, $L^{2-\varepsilon}$ would suffice. However, the estimate from L^2 to $L^{2-\varepsilon}$ is not possible on the line, by homogeneity considerations. Grafakos, Tao, and Terwilleger [10] showed that the maximal operator S_{**} in (5) actually maps $L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$ for all $1 < p < \infty$ and, additionally, recovered Hunt's estimate (3). Moreover, [10] provided a time-frequency analysis proof of Hunt's distributional estimate (3); this actually proves that Fefferman's time-frequency analysis methodology is actually equally powerful as Carleson's original approach, a fact that was not thought possible in [8].

3. APPLICATIONS OF FOURIER ANALYSIS

Fourier Analysis has penetrated many subjects. *Radiotelecommunications, acoustics, oceanography, optics, spectroscopy, crystallography* are only a few applied areas in which Fourier Analysis is used today. But it seems that Fourier Analysis has made its largest impact outside mathematics in *signal and image processing*. In most applications, Fourier expansions of signals are not as useful as *wavelet* expansions. Wavelets are bases of L^2 generated by a single function ψ via translations and dilations; such functions were first constructed in [21], [19]. Precisely, there exists a function ψ on the real line such that the set of functions

$$\psi_{\mu,d} = 2^{-\frac{d}{2}} \psi(2^{-d} - \mu), \quad \mu, d \in \mathbb{Z}$$

forms an orthonormal basis of $L^2(\mathbb{R})$.

A signal f can be expanded in terms of a wavelets series as

$$(6) \quad f(x) = \sum_{\mu, d \in \mathbb{Z}} \langle f, \psi_{\mu, d} \rangle \psi_{\mu, d}(x), \quad x \in \mathbb{R}.$$

This series is analogous to the Fourier series (1) if we notice that the Fourier coefficient $\widehat{F}(m)$ is the inner product of F against the exponential $e^{2\pi imx}$, just as the wavelet coefficient $\langle f, \psi_{\mu, d} \rangle$ is the inner product of f against $\psi_{\mu, d}$. The main difference is that wavelets are localized in both time and frequency whereas the standard Fourier transform is only localized in frequency; in fact the Fourier transform has bad spatial localization and this makes it unsuitable for most applications.

Wavelet decompositions are widely used in signal processing. For instance, large wavelet coefficients can be set to be zero via *low pass filters*, which annihilate high frequencies, and analogously small wavelet coefficients can be set to be zero via *high pass filters*. The advantage of low pass filtering is that it provides a good way to eliminate noise from signals or images. More complicated operations with filters lead to *compression*, *image sharpening*, and *detail recognition*.

A part of signal processing analysis can also be done via *Gabor expansions* given by

$$(7) \quad f(x) = \sum_{\tau, m} c_{\tau, m} \varphi(x - \tau) e^{2\pi i x m}$$

or *wavepacket expansions*

$$(8) \quad f(x) = \sum_{\tau, m} c_{\tau, d, m} 2^{-d/2} \varphi(2^{-d}x - \tau) e^{2\pi i x m},$$

where $c_{\tau, m}$ and $c_{\tau, d, m}$ are the corresponding coefficients of the given signal f in terms of the expansion. The main difference between (6) and (7) is that dilations d in (6) are replaced by modulations in (7). Wavepacket expansions as in (8) are like Gabor expansions but have the advantage that can be adjusted to any scale d . In other words, the parameter d is at our choice in (8) while it set equal to zero in (7). In concrete applications, we often decompose a singular symbol σ as an infinite series of smooth symbols σ_d , each of a different scale d , and apply a wavepacket expansion at each scale d to each σ_d .

4. SUMMATION OF FOURIER SERIES IN HIGHER DIMENSIONS

Suppose that f is a function in \mathbb{R}^2 which is 1-periodic in each variable. Can we sum its two-dimensional Fourier series spherically? In

other words do the partial sums

$$(9) \quad \sum_{|m_1|^2+|m_2|^2 \leq R^2} \widehat{f}(m_1, m_2) e^{2\pi i(x_1 m_1 + x_2 m_2)}$$

converge back to f as $R \rightarrow \infty$? It all depends in the sense convergence is interpreted. It is a fairly easy to verify that if f is square integrable over $[0, 1]^2$, then the two-dimensional spherical Fourier partial sums (9) converge back to f in $L^2([0, 1]^2)$. But Fefferman [7] has shown that there exist functions in $L^p([0, 1]^2)$ for $p \neq 2$ such that (9) diverge in L^p . Thus, L^p convergence in two dimensions holds if and only if $p = 2$. By principles of functional analysis, the L^p convergence of the series in (9) for f in $L^p([0, 1]^2)$ is equivalent to the $L^p(\mathbb{R}^2)$ boundedness of the *ball multiplier operator*

$$g \mapsto \int_{|\xi_1|^2+|\xi_2|^2 \leq 1} \widehat{g}(\xi_1, \xi_2) e^{2\pi i(x_1 \xi_1 + x_2 \xi_2)} d\xi_1 d\xi_2,$$

where $\widehat{g}(\xi_1, \xi_2) = \int_{\mathbb{R}^2} g(y_1, y_2) e^{-2\pi i(y_1 \xi_1 + y_2 \xi_2)} dy_1 dy_2$ is the 2-dimensional Fourier transform of g . It was precisely this operator that was shown in [7] to be bounded on $L^p(\mathbb{R}^2)$ if and only if $p = 2$. And this result holds in all dimensions $n \geq 2$ in view of de Leeuw's theorem [5]. It is noteworthy that one of the greatest problems in analysis remains unsolved: Does the series (9) converge a.e. if f lies in $L^2([0, 1]^2)$?

In view of the lack of convergence of spherical partial sums of Fourier series in higher dimensions, the following modification called the *Bochner-Riesz higher-dimensional spherical summation* was introduced:

$$(10) \quad \sum_{|m_1|^2+|m_2|^2 \leq R^2} \left(1 - \frac{|m_1|^2 + |m_2|^2}{R^2}\right)^\lambda \widehat{f}(m_1, m_2) e^{2\pi i(x_1 m_1 + x_2 m_2)}.$$

In (10) λ is a positive parameter and indicates the smoothness of the means of the modified spherical summation. The smoothness of these means increases as λ increases. In Figure 2 the function $(1 - t^2)_+^\lambda$ is plotted for the values $\lambda = 0$ top left, $\lambda = 0.1$, top right $\lambda = 0.2, \dots, \lambda = 1$ for the last one. For $\lambda = 0$ only boundedness in L^2 is expected to hold, but as soon as λ takes positive values, more values of p near 2 could be included.

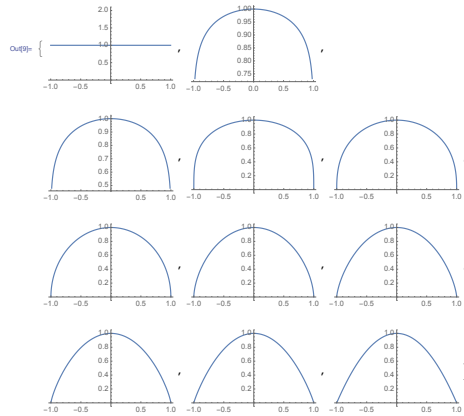


FIGURE 2

The main question now is, given $p \neq 2$ what is the smallest $\lambda_0 \geq 0$ such that for all $\lambda > \lambda_0$ and for all $f \in L^p([0, 1]^2)$ the series in (10) converges in L^p ? The Bochner-Riesz conjecture states that $\lambda_0 = \max(n|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}, 0)$. Unboundedness of the Bochner-Riesz means is known to hold in the grey region of Figure 3, but boundedness in the white region above it is only known in dimension $n = 2$; see Carleson and Sjölin [4].

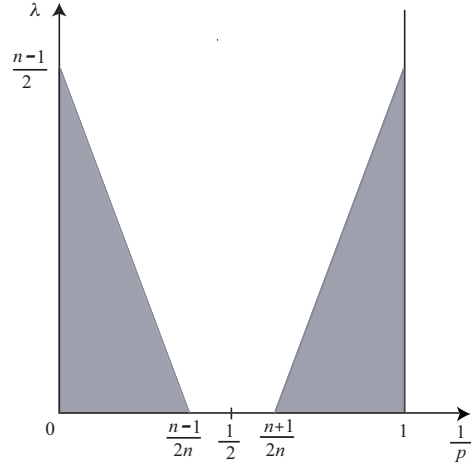


FIGURE 3

As of this writing the *Bochner-Riesz conjecture* remains open in dimensions $n \geq 3$. Several partial results have been obtained since 1972.

5. PRODUCTS OF FOURIER SERIES

Suppose we are given two 1-periodic functions f and g on the line. The question we address is in which sense does the series

$$\sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \widehat{f}(m) \widehat{g}(n) e^{2\pi i(m+n)x}$$

converge back to the product $f(x)g(x)$, for $x \in [0, 1]$. To study this convergence we consider truncated sums and take the limit (in some sense) as the truncation tends to infinity. To make this precise, we introduce *means of convergence* $c_R(m, n)$, which is a compactly supported sequence in \mathbb{Z}^2 (for any $R > 0$) with the property that $c_R(m, n) \rightarrow 1$ as $R \rightarrow \infty$. Then we form the bilinear means

$$\sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} c_R(m, n) \widehat{F}(m) \widehat{G}(n) e^{2\pi i(m+n)x}$$

and ask if they converge to $f(x)g(x)$ in $L^p([0, 1])$ as $R \rightarrow \infty$, where f is a given function in $L^{p_1}([0, 1])$ and g is in $L^{p_2}([0, 1])$, where the indices are related, as in Hölder's inequality, by $1/p = 1/p_1 + 1/p_2$.

Examples of such means are

$$c_R(m, n) = \chi_{|m|^2 + |n|^2 \leq R^2}$$

$$c_R(m, n) = \chi_{RQ}(m, n),$$

where Q is a quadrilateral that contains the origin and

$$RQ = \{R(x_1, x_2) : (x_1, x_2) \in Q\}.$$

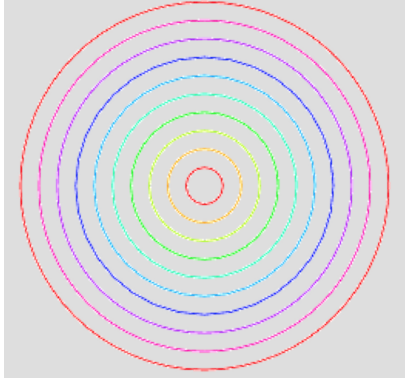


FIGURE 4. Circular summation: pairs (m, n) that appear in the summation are lattice points inside circles whose radii tend to infinity.

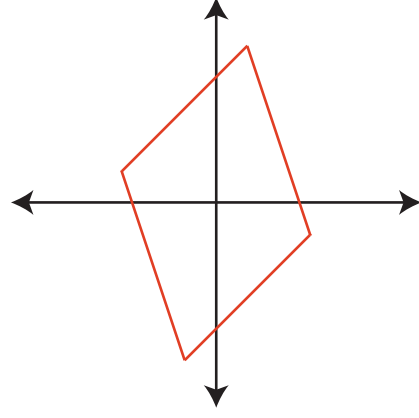


FIGURE 5. Rectangular summation: pairs (m, n) that appear in the summation are lattice points inside a rectangle which is dilated to infinity (about the origin).

These two forms of summation create the following two problems: With limits are taken in the L^p sense, are the assertions

$$(11) \quad \lim_{N \rightarrow \infty} \sum_{\substack{|m-\beta n| \leq N \\ |m-n| \leq N}} \widehat{f}(m) \widehat{g}(n) e^{2\pi i(m+n)x} = f(x)g(x)$$

and

$$(12) \quad \lim_{R \rightarrow \infty} \sum_{\substack{m, n \in \mathbb{Z} \\ m^2 + n^2 \leq R^2}} \widehat{f}(m) \widehat{g}(n) e^{2\pi i(m+n)x} = f(x)g(x)$$

valid if f is a given function in $L^{p_1}([0, 1])$, g is in $L^{p_2}([0, 1])$, and $1/p = 1/p_1 + 1/p_2$? Here β is a real number with $\beta \neq 1$.

There is also a third form of bilinear summation which is inspired by the Bochner-Riesz means of the previous section. Here we take $c_R(m, n) = \left(1 - \frac{|m|^2 + |n|^2}{R^2}\right)_+^\lambda$ for some $\lambda > 0$. The question is whether

$$\sum_{\substack{k, m \in \mathbb{Z} \\ |k|^2 + |m|^2 \leq R^2}} \left(1 - \frac{|k|^2 + |m|^2}{R^2}\right)^\lambda \widehat{f}(k) \widehat{g}(m) e^{2\pi i x(k+m)} \rightarrow f(x)g(x)$$

in $L^p([0, 1])$ for $f \in L^{p_1}([0, 1])$, $g \in L^{p_2}([0, 1])$ and $1/p = 1/p_1 + 1/p_2$.

6. ADVANCES IN MULTILINEAR FOURIER ANALYSIS

In view of some basic functional analysis, the convergence of product Fourier series via quadrilateral summation is equivalent to the $L^p(\mathbb{R})$ boundedness of the *bilinear Hilbert transform*

$$H_\alpha(f, g)(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \int_{|t| \geq \varepsilon} f(x - \alpha t) g(x - t) \frac{dt}{t},$$

where α is a real number. Lacey and Thiele [16], [17] obtained the boundedness of H_α from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$ when $1/p_1 + 1/p_2 = 1/p$, $1 < p_1, p_2 < \infty$ and $p > 2/3$, provided $\alpha \neq 1$. In the exceptional case $\alpha = 1$, the bilinear Hilbert transform $H_1(f, g)$ reduces to $\mathcal{H}(fg)$, where \mathcal{H} is the classical *Hilbert transform* on the line; thus H_1 maps $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$ only when $p > 1$. From this result one concludes that assertion (11) is valid for $p > 2/3$, provided $\beta \notin \{-1, 1\}$. Geometrically interpreting this result, it says that, as long as the quadrilateral Q does not have sides parallel to the antidiagonal $x + y = 0$, the series in (11) converges in L^p for $p > 2/3$, while if the quadrilateral contains sides parallel to the antidiagonal (i.e., β could be -1), then the series in (11) converges in L^p for $p > 1$. There are also corresponding results of Muscalu, Tao, and Thiele [22] related to the a.e. convergence of the partial sums in (11).

Regarding *product circular summation*, the question of the L^p convergence of the series (12) is equivalent to that of the boundedness of the disc multiplier operator

$$T_{disc}(f, g)(x) = \iint_{|\xi|^2 + |\eta|^2 \leq 1} \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x(\xi + \eta)} d\xi d\eta$$

from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$. This equivalence follows from basic functional analysis.

Theorem 6.1 ([9]). *For $2 \leq p_1, p_2, p' < \infty$ (where $p' = p/(p-1)$), T_{disc} maps $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$, whenever $1/p_1 + 1/p_2 = 1/p$.*

As a consequence we obtain that the convergence in (12) is valid on $L^p([0, 1])$ when $f \in L^{p_1}([0, 1])$, $g \in L^{p_2}([0, 1])$, and $2 \leq p_1, p_2, p' < \infty$.

One may wonder if there are analogous results in higher dimensions. For instance is it true that

$$(13) \quad \lim_{R \rightarrow \infty} \sum_{\substack{m, n \in \mathbb{Z}^d \\ |m|^2 + |n|^2 \leq R^d}} \widehat{f}(m) \widehat{g}(n) e^{2\pi i(m+n) \cdot x} = f(x)g(x)$$

valid in the L^p sense, if f is a given function in $L^{p_1}([0, 1]^d)$ and g is in $L^{p_2}([0, 1]^d)$ and $1/p = 1/p_1 + 1/p_2$? The convergence asked in (13)

was shown to be false in dimensions $d \geq 2$ when of the indices p_1, p_2, p' is smaller than 2 by Grafakos and Diestel [6]. The main idea in that proof was the Kakeya counterexample idea introduced in this context in [7].

For the Bochner-Riesz summation for products of Fourier series, there is an optimal result in the $L^2 \times L^2$ to L^1 case:

Theorem 6.2 ([2]). *Let $d \geq 1$, $\lambda > 0$, and $f, g \in L^2([0, 1]^d)$. Then*

$$(14) \quad \sum_{\substack{k, m \in \mathbb{Z}^d \\ |k|^2 + |m|^2 \leq R^2}} \left(1 - \frac{|k|^2 + |m|^2}{R^2}\right)^\lambda \widehat{f}(k) \widehat{g}(m) e^{2\pi i x \cdot (k+m)} \rightarrow f(x)g(x)$$

in $L^1([0, 1]^d)$ as $R \rightarrow \infty$.

This result is sharp in the sense that λ cannot be taken to be zero in (14). The L^p convergence for other indices is also studied in the reference of Bernicot, Grafakos, Song, Yan [2], while improvements were recently obtained by Jeong, Lee, Vargas [12].

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