WEIGHTED KATO-PONCE INEQUALITIES FOR MULTIPLE FACTORS

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Abstract. In this paper we establish a weighted Kato-Ponce inequality for m factors in the endpoint case. Furthermore, we extend the validity of the Kato-Ponce inequality from the class of Schwartz functions to the broader class of functions living in a (weighted) fractional Sobolev space.

1. Introduction

Kato-Ponce (KP) inequalities are normed fractional Leibniz rules estimates of the form

$$\|J^s(fg)\|_{L^p} \leqslant C_{n,s,p_1,p_2}\left(\|J^s f\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{p_1}}\|J^s g\|_{L^{p_2}}\right), \tag{1.1}$$

where f, g are Schwartz functions, $s > 0$ and J^s is the inhomogeneous fractional derivative which is given by multiplication by $(1 + |\xi|^2)^{s/2}$ on the Fourier transform side. Here $1 \leqslant p_1, p_2 \leqslant \infty$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, and the range of the smoothness index s is determined by n and p. These types of inequalities initially appeared in the work of Kato and Ponce [19] in connection with the Euler and Navier-Stokes equations. Such estimates were proved using the Coifman-Meyer bilinear multiplier theorem [7] and Stein's complex interpolation theorem [29]. Subsequently many authors have studied KP (and related) inequalities: we mention for instance the works of Kenig, Ponce, and Vega [20], Christ and Weinstein [6], Gulisashvili and Kon [18], Bae and Biswas [1], Muscalu, Pipher, Thiele, and Tao [23], Grafakos and Oh [15], Bernicot, Maldonado, Moen, and Naibo [3], Muscalu and Schlag [24], Cruz-Uribe and Naibo [10], Fujiwara, Georgiev and Ozawa [12], Li [21], Hale and Naibo [25], Douglas and Grafakos [11].

Upon completing this manuscript, we became aware that Wu [33] had recently obtained the L^1 endpoint case for Muckenhoupt weights in the case of two factors. We arrived independently at this result but our work also includes the L^1 endpoint case for m factors. It should be noted that the case of two factors does not imply the one for multiple factors, and this justifies the present study.

We note that (1.1) is valid exactly when $1 \leqslant p_1, p_2 \leqslant \infty$, $1/2 \leqslant p \leqslant \infty$ and $s > \max\{n(\frac{1}{p} - 1), 0)\}$ or $s \in 2\mathbb{N}$; on this see [15] and [24]. The fact that (1.1) is actually valid in the full range of indices $1 \leqslant p_1, p_2 \leqslant \infty$ and $1/2 \leqslant p \leqslant \infty$ makes it rather intriguing in the theory of bilinear operators. However, it should be stressed that techniques based on Calderón-Zygmund theory cannot provide strong type estimates at endpoints when $p_1 = 1$ or $p_2 = 1$ and $p_1 = p_2 = \infty$.

Bourgain and Li [4] obtained (1.1) when $p_1 = p_2 = \infty$ via a new technique; this endpoint case was previously studied in [14]. The three main ingredients of this endpoint case are Bernstein's inequality, an interpolation technique (similar to Lemma 4.6), and the use of a suitable commutator. This commutator enables high-low frequency paraproducts to be treated almost like high-high frequency paraproducts. A refinement of this technique was employed by Oh and Wu [27] to obtain the other endpoint case when one or both of p_1 and p_2 equal 1.

A weighted KP inequality is an estimate of the form

$$\|J^s(fg)\|_{L^p(w)} \leqslant C_{n,s,p_1,p_2,w_1,w_2}\left(\|J^s f\|_{L^{p_1}(w_1)}\|g\|_{L^{p_2}(w_2)} + \|f\|_{L^{p_1}(w_1)}\|J^s g\|_{L^{p_2}(w_2)}\right) \tag{1.2}$$

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where f, g are Schwartz functions, $p_1, p_2 \geqslant 1$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, w_1, w_2 are locally integrable, nonnegative functions and $w = w_1^{p/p_1} w_2^{p/p_2}$. Naibo and Thomson [26] proved a generalized normed Leibniz rule over weighted Triebel-Lizorkin spaces that implies (1.2) when $1 < p_1, p_2 < \infty$, $w_1 \in A_{p_1}$, $w_2 \in A_{p_2}$ and s lies in an optimal range depending on n, p_1, p_2, w_1, w_2. Oh and Wu [28] proved (1.2) when $1 \leqslant p_1, p_2 \leqslant \infty$, and $w_j = (1 + |\cdot|)^{\alpha_j}$, $\alpha_j \geqslant 0$ where $j = 1, 2$, and notably s depends only on n, p.

In this paper we study weighted KP inequalities for several factors building on the results from [27]. As observed in [11] the 3-factor KP inequality may not follow from the 2-factor KP by grouping two terms into one. When $p < 1$, applying the 2-factor inequality, we will unavoidably end up with some Hölder indices that are less than one. For instance, in the 3-factor case let $p_1 = p_2 = 3/2, p_3 = 2$ and observe that if q_1, q_2 are such that $\frac{2}{3} + \frac{2}{3} + \frac{1}{2} = \frac{1}{q_1} + \frac{1}{2} = \frac{2}{3} + \frac{1}{q_2}$, then $q_1 < 1$ and $q_2 < 1$. Then (1.1) can not be applied in this case as it requires the indices on the right to be greater than or equal to one.

We now state the precise formulations of our main results. The A_p classes and the weighted local Hardy space, $h_p(w)$, are defined in the next section. In the sequel we set $\tau_w = \inf\{p : w \in A_p\}$. All norms below are over $\mathbb{R}^n$.

Theorem 1.1. *Let $m \in \mathbb{Z}^+$, $\frac{1}{m} \leqslant p \leqslant \infty$, $1 \leqslant p_1, \ldots, p_m \leqslant \infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Let $w_j \in A_{p_j}$ and $w = w_1^{\frac{p}{p_1}} \cdots w_m^{\frac{p}{p_m}}$. If $s > \max(n(\frac{\tau_w}{p} - 1), 0)$ or $s \in 2\mathbb{N}$, then there exists a constant $C = C(n, m, s, p_1, \ldots, p_m, w_1, \ldots, w_m) < \infty$ such that for all $f_l \in \mathcal{S}(\mathbb{R}^n)$ with $l \in \{1, \ldots, m\}$ we have*

$$\begin{aligned} \|J^s(f_1 \cdots f_m)\|_{L^p(w)} &\leqslant C\big(\|J^s f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)} \cdots \|f_m\|_{L^{p_m}(w_m)} + \cdots \\ &\qquad \cdots + \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)} \cdots \|J^s f_m\|_{L^{p_m}(w_m)}\big). \end{aligned} \tag{1.3}$$

Furthermore, (1.3) holds if J^s is replaced by D^s.

The following theorem extends the KP inequality from the class of Schwartz functions to functions in a fractional Sobolev space $L^p_s(w)$ defined in Section 2. Note that endpoints are not included in this extension.

Theorem 1.2. *Let $m \in \mathbb{Z}^+$, $\frac{1}{m} < p < \infty$, $1 < p_1, \ldots, p_m < \infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. If $s > \max\big(0, n(\frac{1}{p} - 1)\big)$, then there exists a constant $C = C(n, m, s, p_1, \ldots, p_m) < \infty$ such that for all $f_j \in L^{p_j}_s$ with $j \in \{1, \ldots, m\}$ we have*

$$\begin{aligned} \|J^s(f_1 \cdots f_m)\|_{h^p} &\leqslant C\big(\|J^s f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \cdots \|f_m\|_{L^{p_m}} + \cdots \\ &\qquad \cdots + \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \cdots \|J^s f_m\|_{L^{p_m}}\big). \end{aligned} \tag{1.4}$$

We note that in (1.4) any tuple of indices $(p_1, \ldots, p_m)$ that appears in a summand on the right of the inequality can be replaced by any other tuple $(q_1, \ldots, q_m)$ with $\frac{1}{p} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$.

Remark 1.2.1. *The extension to fractional Sobolev spaces in Theorem 1.2 is not straightforward when $p < 1$ due to lack of duality. Furthermore, as discussed in Section 8, Theorem 1.2 can hold in the weighted setting provided the weights, $w_j \in A_{p_j}$ satisfy some additional reverse Hölder conditions, and $p \leqslant \tau_w < p + 1$.*

We note that Theorem 1.1 can also be proved with weights of the form $(1 + |\cdot|)^\alpha$ for $\alpha \geqslant 0$ and s independent of the choice of weights in analogy with Oh and Wu in [28]; though we do not provide details here.

We summarize the contributions of this article in the relevant literature: (a) use of an efficient decomposition that manages the large array of paraproducts inherited by the complexity of m factors. This requires a further decomposition than that in [11] due to the use of a commutator; (b) a dilation argument that allows the derivation of the homogeneous weighted KP inequality from its inhomogeneous counterpart; and (c) an extension of Theorem 1.1 from Schwartz functions to functions living in a weighted fractional Sobolev space (Theorem 1.2), which is new even in the unweighted case.

Overall, our work not only provides multilinear extensions but also contains certain ingredients that add new perspectives to the existing literature on KP inequalities.

2. Notation

For locally integrable function $w > 0$ a.e. and $0 < p < \infty$, the space $L^p(w)$, is defined as the set of Lebesgue measurable functions on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx\right)^{1/p} < \infty.$$

For $p = \infty$, Lebesgue measure and wdx are mutually absolutely continuous, thus the essential supremum with respect to wdx and Lebesgue measure are the same, hence $\|\cdot\|_{L^\infty} = \|\cdot\|_{L^\infty(w)}$. We denote by M the uncentered Hardy-Littlewood maximal function with respect to cubes. For a locally integrable function g and $t > 0$, the maximal operator M_t is given by $M_t(g) := M(|g|^t)^{\frac{1}{t}}$. For real numbers A, B we use $A \lesssim B$ to mean $A \leqslant CB$ for some positive constant C. We also say A and B are comparable, denoted by $A \sim B$, if and only if $A \lesssim B$ and $B \lesssim A$.

For $f \in L^1(\mathbb{R}^n)$ the Fourier transform and inverse Fourier transform are respectively defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(y) e^{-2\pi i y\cdot\xi} dy \qquad \check{f}(\xi) = \int_{\mathbb{R}^n} f(y) e^{2\pi i y\cdot\xi} dy.$$

We also use $\mathcal{F}$ to denote the Fourier transform, that is $\mathcal{F}(f) = \widehat{f}$ and $\mathcal{F}^{-1}(f) = \check{f}$. The space of Schwartz functions is denoted by $\mathcal{S}(\mathbb{R}^n)$. The dual space of $\mathcal{S}(\mathbb{R}^n)$ is the space of tempered distributions and is denoted by $\mathcal{S}'$. We denote by $\widehat{J^s u} := (1 + |\cdot|^2)^{\frac{s}{2}} \widehat{u}$ the fractional Laplacian operator for $u \in \mathcal{S}'$ and by $\widehat{D^s u} := |\cdot|^s \widehat{u}$ its homogeneous counterpart.

Let $\widehat{\Phi}(\xi)$ be a positive radially decreasing $C^\infty(\mathbb{R}^n)$ function on $\mathbb{R}^n$ supported in twice the unit ball and equal to one on the unit ball. Let $\widehat{\Psi}(\xi) = \widehat{\Phi}(\xi) - \widehat{\Phi}(2\xi)$, which is non-negative and supported in the annulus $\frac{1}{2} \leqslant |\xi| \leqslant 2$. The frequency isolation operators Δ_j and S_j are defined to be convolution with $2^{jn}\Psi(2^j\cdot)$ and $2^{jn}\Phi(2^j\cdot)$ respectively. The shifted frequency isolation operators for $\mu \in \mathbb{R}^n$ are denoted by $\Delta_{j,\mu}$ and $S_{j,\mu}$ are given by convolution with $2^{jn}\Psi(2^j \cdot + c_1\mu)$ and $2^{jn}\Phi(2^j \cdot + c_2\mu)$ respectively, where the constants c_1, c_2 are independent of j and μ. By looking on the Fourier transform side we also have the identity $\sum_{j \leqslant j_0} \Delta_j = S_{j_0}$ for any $j_0 \in \mathbb{Z}$. The operator $\sum_{j>k} \Delta_j$ will be denoted by $\Delta_{>k}$.

A Muckenhoupt weight or A_p weight is a non-negative locally integrable function w on $\mathbb{R}^n$ such that $0 < w < \infty$ a.e, and for $1 < p < \infty$ and for all cubes Q in $\mathbb{R}^n$ with sides parallel to the axes, we have

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\, dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\, dx\right)^{p-1} < \infty.$$

We say $w \in A_1$ if

$$[w]_{A_1} := \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\, dx\right) \|w^{-1}\|_{L^\infty(Q)} < \infty.$$

Now for the some basic facts about A_p weights. If $w \in A_1$ then $M(w) \leqslant C_{n,[w]_{A_1}} w$ a.e. It is well known that if $1 < p < \infty$ and $w \in A_p$ then $\|M(f)\|_{L^p(w)} \leqslant C_{p,n,[w]_{A_p}} \|f\|_{L^p(w)}$. If $w \in A_p$ then the measure wdx is doubling, specifically for $\lambda > 0$ we have $w(\lambda Q) \leqslant \lambda^{np}[w]_{A_p} w(Q)$. If $w \in A_p$ for $p > 1$ then w's dual weight $\theta := w^{-\frac{1}{p-1}}$ is in $A_{p'}$. For $w \in A_\infty := \bigcup_{p\in(1,\infty)} A_p$ we denote $\tau_w = \inf\{p : w \in A_p\}$. A non-negative locally integrable function, w, on $\mathbb{R}^n$ satisfies the reverse

Hölder property with $t \geqslant 1$ ($w \in RH_t$) if for all cubes Q in $\mathbb{R}^n$ with sides parallel to the axes, we have

$$\left(\frac{1}{|Q|}\int_Q w^t(x)\,dx\right)^{\frac{1}{t}} \leqslant \frac{C}{|Q|}\int_Q w(x)\,dx.$$

It is well known that A_p weights for $p > 1$ satisfy the reverse Hölder property where $t > 1$ depends on $[w]_{A_p}$.

The weighted fractional Sobolev space $L^p_s(w)$ for $1 \leqslant p < \infty$ and $w \in A_p$ is defined to be the space of tempered distributions, u, such that $J^s(u)$ is a function in $L^p(w)$. The local Hardy space $h^p(w)$ for $0 < p < \infty$, and $w \in A_\infty$ is defined to be the space of tempered distributions, u, such that $\|u\|_{h^p(w)} := \left\|\sup_{0<t<1} |t^{-n}\Phi(t^{-1}\cdot) * u|\right\|_{L^p(w)} < \infty$. It is known that $h^p(w)$ is complete and continuously embeds in the space of tempered distributions. For more information on local Hardy spaces we refer to [13], [2], and [32].

3. Inhomogeneous Decomposition

In this section we decompose the inhomogeneous fractional derivative, $J^s(f_1 \cdots f_m)$, into paraproducts of 4 types. The initial decomposition is the same as that in [11], but we will require a further decomposition for the commutator. Observe for $f_l \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and using that $\sum_{j\in\mathbb{Z}} \widehat{\Psi}(2^{-j}\xi) = 1$ for $\xi \neq 0$ we obtain

$$\begin{aligned}
&J^s(f_1 f_2 \cdots f_m)(x)\\
&= \int_{\mathbb{R}^{mn}} (1 + |\xi_1 + \cdots + \xi_m|^2)^{\frac{s}{2}} \widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\cdots\widehat{f_m}(\xi_m) e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x} d\xi_1 \cdots d\xi_m\\
&= \int_{\mathbb{R}^{mn}} \Big[\sum_{j_1,\dots,j_m\in\mathbb{Z}^m} \widehat{\Psi}(2^{-j_1}\xi_1)\widehat{\Psi}(2^{-j_2}\xi_2)\cdots\widehat{\Psi}(2^{-j_m}\xi_m)\Big]\\
&\qquad \times (1 + |\xi_1 + \cdots + \xi_m|^2)^{\frac{s}{2}} \widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\cdots\widehat{f_m}(\xi_m) e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x} d\xi_1 \cdots d\xi_m.
\end{aligned} \tag{3.1}$$

We now partition $\mathbb{Z}^m$ into 2^m subsets, then breaking up the kernel of (3.1) as a sum over these subsets will provide the desired paraproducts. For $\vec{\eta} = (\eta_1, \dots, \eta_m) \in \{0,1\}^m \backslash \{\vec{0}\}$ let $t_1, \dots, t_b$ be all the indices of η corresponding to a 1; that is, $1 = \eta_{t_1} = \cdots = \eta_{t_b}$ and the remaining entries are zero. Define

$$\begin{aligned}
\mathscr{B}_{\vec{\eta}} :=\{\vec{j} = (j_1, \dots, j_m) \in \mathbb{Z}^m : j_{t_1}, \dots, j_{t_b} &\text{ are equal, strictly positive, and strictly bigger than}\\
&\text{the remaining entries of } \vec{j}\}.
\end{aligned}$$

Furthermore, let $\mathscr{B}_{\vec{0}} := (\mathbb{Z}_{\leqslant 0})^m = \{0, -1, -2, \dots\}^m$.

Let us quickly verify that $\{\mathscr{B}_{\vec{\eta}}\}_{\vec{\eta}\in\{0,1\}^m}$ is a partition of $\mathbb{Z}^m$. Let $\vec{j} = (j_1, \dots, j_m) \in \mathbb{Z}^m$. If $\max_k(j_k) \leqslant 0$ then $j \in \mathscr{B}_{\vec{0}}$, so suppose that $\max_k(j_k) > 0$. Let $j_{t_1} = \ldots = j_{t_b} = \max_k(j_k)$, where the remaining entries of $\vec{j}$ are strictly smaller. Let $\vec{\eta} \in \{0,1\}^m$ be the element with $\eta_{t_k} = 1$ for $k = 1, \dots, b$ and the remaining entries zero; then clearly $\vec{j} \in \mathscr{B}_{\vec{\eta}}$. To see these sets are disjoint suppose $\vec{\eta} \neq \vec{\alpha}$, without loss of generality let $\eta_1 = 1$ and $\alpha_1 = 0$. Then if $\vec{j} \in \mathscr{B}_{\vec{\eta}}$ we have $\max_k(j_k) = j_1$, while if $\vec{j} \in \mathscr{B}_{\vec{\alpha}}$ we have $\max_k(j_k) > j_1$.

It follows the term in the square brackets of (3.1) can be written as

$$\sum_{\vec{j}\in\mathbb{Z}^m} \widehat{\Psi}(2^{-j_1}\xi_1)\widehat{\Psi}(2^{-j_2}\xi_2)\cdots\widehat{\Psi}(2^{-j_m}\xi_m) = \sum_{\vec{\eta}\in\{0,1\}^m} \sum_{j\in\mathscr{B}_{\vec{\eta}}} \widehat{\Psi}(2^{-j_1}\xi_1)\widehat{\Psi}(2^{-j_2}\xi_2)\cdots\widehat{\Psi}(2^{-j_m}\xi_m) \tag{3.2}$$

From (3.2) we see an $\vec{\eta}$ with exactly l_0 ones can be treated similarly up to permutation. Thus it is enough to show the result for an $\vec{\eta}$ where the first l_0 entries are one, specifically let

$$\vec{\eta}_0 = (\underbrace{1, 1, \ldots, 1}_{l_0}, 0, 0, \ldots, 0).$$

Furthermore, since $\sum_{k<j} \Delta_j = S_{j-1}$ on the Fourier transform side we see that the entries of $\vec{\eta}$ with a 1 correspond to a Δ_j operator, while the coordinates with a 0 correspond to a S_{j-1} operator.

Thus to bound $J^s(f_1 \cdots f_m)$ it is sufficient to bound the following two terms

$$J^s\Big((S_0 f_1) \cdots (S_0 f_m)\Big) \tag{3.3}$$

$$J^s\Big(\sum_{j\in\mathbb{N}} (\Delta_j f_1)(\Delta_j f_2) \cdots (\Delta_j f_{l_0})(S_{j-1} f_{l_0+1}) \cdots (S_{j-1} f_m)\Big). \tag{3.4}$$

For notational convenience we define $\vec{f} = (f_1, \ldots, f_m)$ and

$$u_j^{\eta_0}(\vec{f}) = (\Delta_j f_1)(\Delta_j f_2) \cdots (\Delta_j f_{l_0})(S_{j-1} f_{l_0+1}) \cdots (S_{j-1} f_m). \tag{3.5}$$

Notice that $J^s\Big(\sum_{j\in\mathbb{N}} u_j^{\eta_0}(\vec{f})\Big)$ is a well defined function. This follows from Lebesgue dominated convergence theorem, the fact $\widehat{\Psi}$ is supported in an annulus, and that f_k are Schwartz functions. Furthermore, we have $\operatorname{supp}\mathcal{F}(u_j^{\eta_0}(\vec{f})) \subset B(0, 2^{j+m})$.

3.1. **Further decomposition for $l_0 = 1$.** To bound the $L^p(w)$ quasi-norm of (3.4) we use different techniques when $l_0 = 1$ and when $l_0 > 1$. When $l_0 = 1$ (that is we have exactly one Δ_j operator) we need a further decomposition. As we show in this section this reduces to showing the desired bound on terms in (3.12) and (3.16).

When $l_0 = 1$, this corresponds to the paraproduct

$$\sum_{j\in\mathbb{N}} (\Delta_j f_1)(S_{j-1} f_2) \cdots (S_{j-1} f_m). \tag{3.6}$$

Fix $a \in \mathbb{N}$ to be determined later. Observe that (3.6) can be written as

$$\sum_{j\in\mathbb{N}} (\Delta_j f_1)\Big(S_{j-a} f_2 + \sum_{j-a<k<j} \Delta_k f_2\Big) \cdots \Big(S_{j-a} f_m + \sum_{j-a<k<j} \Delta_k f_m\Big). \tag{3.7}$$

Multiplying out the terms in (3.7) we can write (3.6) as

$$\sum_{j\in\mathbb{N}} (\Delta_j f_1)(S_{j-a} f_2)(S_{j-a} f_3) \cdots (S_{j-a} f_m) \tag{3.8}$$

plus finitely many other paraproducts with at least one Δ_k operator where $k \sim j$. These finitely many other paraproducts will behave in the same way as (3.4) for $l_0 > 1$, hence we focus on (3.8). We now pick a large enough so that the Fourier transform of a summand of (3.8) is supported in an annulus. The support of the Fourier transform of $(S_{j-a} f_2)(S_{j-a} f_3) \cdots (S_{j-a} f_m)$ is contained in the ball centered at zero with the radius $(m-1)2^{j-a+1}$. The support of the Fourier transform of $\Delta_j f_1$ is contained in the annulus $2^{j-1} \leqslant |\xi_1| \leqslant 2^{j+1}$. Choosing a to be some integer larger than $\log_2(8m)$ gives $(2m)2^{j-a+1} \leqslant 2^{j-1}$ for all integers j. It follows on the Fourier transform side this choice of a gives $|\xi_l| \leqslant \frac{1}{2m}|\xi_1|$ for $l \in \{2, \ldots, m\}$. Hence,

$$2|\xi_1| \geqslant |\xi_1 + \cdots + \xi_m| \geqslant |\xi_1| - |\xi_2| - \cdots - |\xi_m| \geqslant |\xi_1| - \frac{(m-1)|\xi_1|}{2m} \geqslant \frac{|\xi_1|}{2},$$

thus $|\xi_1| \sim |\xi_1 + \cdots + \xi_m|$.

Now that the Fourier transform of (3.8) is supported in an annulus we further decompose it in terms of a commutator. For operators A, B let $[A, B] = AB - BA$ be their commutator, then the inhomogeneous derivative of (3.8) can be written as

$$\sum_{j\in\mathbb{N}}[J^s, S_{j-a}f_2\cdots S_{j-a}f_m]\Delta_j f_1 \tag{3.9}$$

$$+\sum_{j\in\mathbb{N}}(J^s\Delta_j f_1)(S_{j-a}f_2)(S_{j-a}f_3)\cdots(S_{j-a}f_m). \tag{3.10}$$

Observe for (3.10) we can write

$$\begin{aligned}&\sum_{j\in\mathbb{N}}(J^s\Delta_j f_1)(S_{j-a}f_2)(S_{j-a}f_3)\cdots(S_{j-a}f_m)\\ &=\sum_{j\in\mathbb{N}}(J^s\Delta_j f_1)(f_2-\Delta_{>j-a}f_2)(f_3-\Delta_{>j-a}f_3)\cdots(f_m-\Delta_{>j-a}f_m).\end{aligned} \tag{3.11}$$

Multiplying out the terms in (3.11) one sees it can be written as a finite linear combination of terms of the form

$$\sum_{j\in\mathbb{N}}(J^s\Delta_j f_1)(G_j^2 f_2)(G_j^3 f_3)\cdots(G_j^m f_m), \tag{3.12}$$

where G_j^l is either the identity operator, I, or $\Delta_{>j-a}$.

Expanding the commutator in (3.9) and applying the fundamental theorem of calculus we obtain

$$\begin{aligned}&[J^s, S_{j-a}f_2\cdots S_{j-a}f_m]\Delta_j f_1(x)\\ &=\int_{\mathbb{R}^{nm}}(\langle\xi_1+\cdots+\xi_m\rangle^s-\langle\xi_1\rangle^s)\widehat{\Delta_j f_1}(\xi_1)\widehat{S_{j-a}f_2}(\xi_2)\cdots\widehat{S_{j-a}f_m}(\xi_m)\\ &\qquad\times e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x}\,d\xi_1\cdots d\xi_m\\ &=\int_{\mathbb{R}^{nm}}\int_0^1\frac{d}{dt}\langle\xi_1+t(\xi_2+\cdots+\xi_m)\rangle^s\widehat{\Delta_j f_1}(\xi_1)\widehat{S_{j-a}f_2}(\xi_2)\cdots\widehat{S_{j-a}f_m}(\xi_m)\\ &\qquad\times e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x}\,dt\,d\xi_1\cdots d\xi_m.\end{aligned} \tag{3.13}$$

Now observe that

$$\frac{d}{dt}\langle\xi_1+t(\xi_2+\cdots+\xi_m)\rangle^s = s(\xi_2+\cdots+\xi_m)\cdot(\xi_1+t(\xi_2+\cdots+\xi_m))\langle\xi_1+t(\xi_2+\cdots+\xi_m)\rangle^{s-2}.$$

Plugging this derivative into (3.13) and multiplying out the dot product gives that (3.13) is a linear combination of terms of the form

$$\begin{aligned}&\int_{\mathbb{R}^{nm}}\int_0^1\xi_\kappa^l(\xi_1^l+t(\xi_2^l+\cdots+\xi_m^l))\langle\xi_1+t(\xi_2+\cdots+\xi_m)\rangle^{s-2}\widehat{\Delta_j f_1}(\xi_1)\\ &\qquad\times\widehat{S_{j-a}f_2}(\xi_2)\cdots\widehat{S_{j-a}f_m}(\xi_m)e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x}\,dt\,d\xi_1\cdots d\xi_m,\end{aligned} \tag{3.14}$$

where $\kappa\in\{2,\ldots,m\}$, $l\in\{1,\ldots,m\}$ and $\xi_\kappa=(\xi_\kappa^1,\ldots,\xi_\kappa^m)$; without loss of generality (by symmetry) we will assume that $\kappa=2$ and $l=1$. Recall a was chosen so that

$$\frac{1}{4}\leqslant 2^{-j}\frac{|\xi_1|}{2}\leqslant 2^{-j}|\xi_1+t(\xi_2+\cdots+\xi_m)|\leqslant 2^{-j}2|\xi_1|\leqslant 4.$$

Let $\Lambda(y)$ be a $\mathcal{C}^\infty(\mathbb{R}^n)$ function that is 1 on $4^{-1}\leqslant|y|\leqslant 4$ and supported in $8^{-1}\leqslant|y|\leqslant 8$. Let

$$\sigma_j(y):=y_1(2^{-2j}+|y|^2)^{\frac{s-2}{2}}\Lambda(y)$$

for $y = (y_1, \ldots, y_n)$, which is a smooth compactly supported function. It follows that (3.14) can be written as

$$2^{j(s-1)} \int_0^1 \int_{\mathbb{R}^{nm}} \xi_2^1 \sigma_j(2^{-j}(\xi_1^1 + t(\xi_2^1 + \cdots + \xi_m^1))) \widehat{\Delta_j f_1}(\xi_1) \widehat{S_{j-a} f_2}(\xi_2) \cdots \widehat{S_{j-a} f_m}(\xi_m) \times e^{2\pi i(\xi_1 + \cdots + \xi_m)\cdot x} \, dt \, d\xi_1 \cdots d\xi_m. \tag{3.15}$$

Now using the identity $\widehat{\partial^\alpha \varphi} = (2\pi i \cdot)^\alpha \widehat{\varphi}$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we move ξ_2^1 to the $\widehat{S_{j-a} f_2}$ term and write (3.15) as

$$2^{j(s-1)} \int_0^1 \int_{\mathbb{R}^{nm}} \sigma_j(2^{-j}(\xi_1^1 + t(\xi_2^1 + \cdots + \xi_m^1))) \widehat{\Delta_j f_1}(\xi_1) \frac{1}{2\pi i} \widehat{S_{j-a} \partial_1 f_2}(\xi_2) \cdots \widehat{S_{j-a} f_m}(\xi_m) \times e^{2\pi i(\xi_1 + \cdots + \xi_m)\cdot x} \, dt \, d\xi_1 \cdots d\xi_m. \tag{3.16}$$

In the sequel we will apply Lemma 4.7 to (3.16). We note here that σ_j and all of its partials are uniformly bounded in j due to the support of Λ and since $j \geqslant 0$. This gives that σ_j's Fourier coefficients are uniformly bounded in j.

To bound $J^s(f_1 \cdots f_m)$ we have reduced bounding terms of 4 types; those given by (3.3), (3.4) when $l_0 > 1$, (3.12), and (3.16).

4. Preliminary material

Lemma 4.1. [16] *(Peetre's Lemma) Let $0 < t < \infty$, $u \in C^1(\mathbb{R}^n)$ (that is its partial derivatives are continuous) and suppose its distributional Fourier transform satisfies* $\operatorname{supp}(\widehat{u}) \subset B(0, r)$, *then*

$$\sup_{y \in \mathbb{R}^n} \frac{|u(x-y)|}{(1 + r|y|)^{\frac{n}{t}}} \leqslant C_{n,t} \mathcal{M}_t(u)(x)$$

for every $x \in \mathbb{R}^n$, where the constant is independent of r.

Lemma 4.2 (Lemma 3.1,[26])**.** *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $0 < t < \infty$, $\mu \in \mathbb{R}^n$, $f \in C^1(\mathbb{R}^n)$ and* $\operatorname{supp}(\widehat{f}) \subset B(0, D2^j)$, *then*

$$\left| 2^{jn} \varphi(2^j \cdot + \mu) * f \right| \lesssim (1 + |\mu|)^{\frac{n}{t}} \mathcal{M}_t(f).$$

Proof. Let $x \in \mathbb{R}^n$. Observe that

$$\begin{aligned}
\left| (2^{jn} \varphi(2^j \cdot + \mu) * f)(x) \right| &\lesssim \int \frac{2^{jn}}{(1 + |2^j y + \mu|)^{n + \frac{n}{t} + 1}} |f(x-y)| \, dy \\
&\leqslant \sup_{y \in \mathbb{R}^n} \frac{|f(x-y)|}{(1 + |2^j y + \mu|)^{\frac{n}{t}}} \int \frac{2^{jn}}{(1 + |2^j y + \mu|)^{n+1}} \, dy \\
&\lesssim (1 + |\mu|)^{\frac{n}{t}} \sup_{y \in \mathbb{R}^n} \frac{|f(x-y)|}{(1 + |2^j y|)^{\frac{n}{t}}} \int \frac{2^{jn}}{(1 + |2^j y + \mu|)^{n+1}} \, dy \\
&\lesssim (1 + |\mu|)^{\frac{n}{t}} \mathcal{M}_t(f)(x),
\end{aligned}$$

where in the last inequality we applied Lemma 4.1. □

Lemma 4.3. *Let $1 \leqslant p \leqslant \infty$, let $w \in A_p$ then the operators $J^{-s}(s > 0), \Delta_j, S_j, \sum_{j > k} \Delta_j$ are bounded from $L^p(w)$ to $L^p(w)$.*

Proof. First recall that convolution with an integrable radially decreasing function is controlled by the Hardy-Littlewood maximal function. Now we will show the operators given in the statement of the lemma are bounded in this weighted setting. If $1 < p < \infty$ then we have

$$\left\| S_j f \right\|_{L^p(w)} = \left\| 2^{jn} \Phi(2^j \cdot) * f \right\|_{L^p(w)} \lesssim \| \mathcal{M}(f) \|_{L^p(w)} \lesssim \| f \|_{L^p(w)}.$$

If $p = 1$ then recalling that Φ is radial we have

$$\begin{aligned}\|S_j f\|_{L^1(w)} &= \int \left| \int 2^{jn}\Phi(2^j(x-y))f(y)dy \right| w(x)\,dx \\ &\leqslant \int |f(y)| \int 2^{jn}|\Phi(2^j(x-y))|w(x)\,dxdy \\ &\lesssim \int |f(y)|\mathcal{M}(w)(y)\,dy \\ &\leqslant \|f\|_{L^1(w)}\,.\end{aligned}$$

Lastly, observe when $p = \infty$

$$\|S_j f\|_{L^\infty(w)} = \|S_j f\|_{L^\infty} = \left\|2^{jn}\Phi(2^j\cdot) * f\right\|_{L^\infty} \lesssim \|f\|_{L^\infty} = \|f\|_{L^\infty(w)}\,.$$

It is easily seen that the same proof works for Δ_j and J^{-s}, as they correspond to convolution with an integrable radially decreasing function.

Lastly observe for $1 \leqslant p \leqslant \infty$ we have

$$\left\|\sum_{j>k}\Delta_j f\right\|_{L^p(w)} = \|f - S_k f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}\,,$$

as desired. □

The following lemma provides an estimate, that among others, is required to show that the inhomogenous Kato-Ponce inequality implies the homogeneous version.

Lemma 4.4 (Lemma 1, [15])**.** *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $s > 0$. Then for any $\gamma \in [0,1]$, there exists a constant $C(n,s,f)$ independent of γ, such that*

$$|(\gamma^2 I - \Delta)^{\frac{s}{2}} f(x)| \leqslant C(n,s,f)(1+|x|)^{-n-s}.$$

Proposition 4.5. *(Bernstein's inequalities) Let $0 < p \leqslant \infty$, $w \in A_\infty$, $s \in \mathbb{R}$, and let $\widehat{\psi}(\xi)$ be a $\mathcal{C}^\infty(\mathbb{R}^n)$ function supported in the annulus $\frac{1}{2} \leqslant |\xi| \leqslant 2$. Define $\Delta_j^\psi f$ to be convolution with $2^{jn}\psi(2^j\cdot)$ for $f \in \mathcal{S}(\mathbb{R}^n)$ and $j \geqslant 0$, then*

$$\|J^s\Delta_j^\psi f\|_{L^p(w)} \sim 2^{js}\|\Delta_j^\psi f\|_{L^p(w)}. \tag{4.1}$$

Proof. Let $\widehat{\psi_\star}(\xi)$ be a C^∞ function that is 1 for $\frac{1}{2} \leqslant |\xi| \leqslant 2$ and supported in $\frac{1}{4} \leqslant |\xi| \leqslant 4$. Let the operator $\Delta_j^\star$ be defined by convolution with $2^{jn}\psi_\star(2^j\cdot)$. Let $\sigma_j(\xi) \equiv (2^{-2j} + |\xi|^2)^{\frac{s}{2}}\widehat{\psi_\star}(\xi)$ which is a smooth compactly supported function. Expanding in Fourier series we have

$$\sigma_j(\xi) = \chi_{[-4,4]^n}(\xi)\sum_{\mu\in\mathbb{Z}^n} c_{j,\mu}e^{2\pi i\xi\cdot\frac{\mu}{8}} \tag{4.2}$$

where due to σ_j's smoothness the coefficients decay rapidly in μ. Moreover, the Fourier coefficients decay independently of j since all of the partial derivatives σ_j are uniformly bounded in j due to the support of $\widehat{\psi_\star}$ and the fact that $j \geqslant 0$. Observe,

$$\begin{aligned}J^s\Delta_j^\psi f(x) &= \int (1+|\xi|^2)^{\frac{s}{2}}\widehat{\psi_\star}(2^{-j}\xi)\widehat{\Delta_j^\psi f}(\xi)e^{2\pi i\xi\cdot x}d\xi \\ &= \int 2^{js}(2^{-2j} + |2^{-j}\xi|^2)^{\frac{s}{2}}\widehat{\psi_\star}(2^{-j}\xi)\widehat{\Delta_j^\psi f}(\xi)e^{2\pi i\xi\cdot x}d\xi \\ &= 2^{js}\int \sum_{\mu\in\mathbb{Z}^n} c_{j,\mu}e^{2\pi i\xi\cdot 2^{-j-3}\mu}\widehat{\Delta_j^\psi f}(\xi)e^{2\pi i\xi\cdot x}d\xi\end{aligned}$$

$$= 2^{js} \sum_{\mu \in \mathbb{Z}^n} c_{j,\mu} \Delta^{\psi}_{j,\mu} f(x).$$

Hence we obtain

$$|J^s \Delta^{\psi}_j f(x)| \leqslant 2^{js} \sum_{\mu \in \mathbb{Z}^n} |c_{j,\mu}| |\Delta^{\psi}_{j,\mu} \Delta^{\psi}_j f(x)|.$$

Applying Lemma 4.2 and using that the Fourier coefficients decay rapidly in μ independent of j we obtain

$$\left| J^s \Delta^{\psi}_j f \right| \lesssim 2^{js} \mathcal{M}_t(\Delta^{\psi}_j f). \tag{4.3}$$

Suppose $w \in A_q$ for $q \geqslant 1$, then by choosing t small enough so that $\frac{p}{t} > q$ we see applying the $L^p(w)$-norm to (4.3) we obtain

$$\|J^s \Delta^{\psi}_j f\|_{L^p(w)} \lesssim 2^{js} \|\Delta^{\psi}_j f\|_{L^p(w)}. \tag{4.4}$$

To get the other direction we simply apply (4.4),

$$2^{js} \|\Delta^{\psi}_j f\|_{L^p(w)} = 2^{js} \|J^{-s} \Delta^{\psi}_j J^s f\|_{L^p(w)} \lesssim \|J^s \Delta^{\psi}_j f\|_{L^p(w)}.$$

□

Lemma 4.6 (Lemma 2.4, [27])**.** *If $a_k \lesssim \min(2^{ka} A, 2^{-kb} B)$ for some $a, b, A, B > 0$ and every $k \in \mathbb{Z}$, then for any $u > 0$, we have $\{a_k\}_{k \in \mathbb{Z}} \in \ell^u(\mathbb{Z})$ and*

$$\|\{a_k\}_{k \in \mathbb{Z}}\|_{\ell^u} \lesssim A^{\frac{b}{a+b}} B^{\frac{a}{a+b}}.$$

In particular, if $\|f_k\|_{L^r(w)} \lesssim |a_k|$ for $0 < r \leqslant \infty$, every $k \in \mathbb{Z}$, and a weight w then

$$\left\| \sum_{k \in \mathbb{Z}} f_k \right\|_{L^r(w)} \lesssim A^{\frac{b}{a+b}} B^{\frac{a}{a+b}}.$$

The following lemma is a A_p weighted multifactor variation of a lemma by Oh and Wu [Lemma 2.3, [27]] that will allow us to bound the commutator.

Lemma 4.7. [27] *Let $\frac{1}{m} \leqslant p \leqslant \infty$, $1 \leqslant p_1, \dots, p_m \leqslant \infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Let $w_l \in A_\infty$ for $l \in \{1, \dots, m\}$, and let $w = w_1^{\frac{p}{p_1}} \cdots w_m^{\frac{p}{p_m}}$. If σ is a compactly supported $\mathcal{C}^\infty(\mathbb{R}^n)$ function, then for any $j, b \in \mathbb{N}$ we have for all $f_l \in \mathcal{S}(\mathbb{R}^n)$ with $l \in \{1, \dots, m\}$*

$$\left\| \int_0^1 \int_{\mathbb{R}^{mn}} \sigma(2^{-j}(\xi_1 + t(\xi_2 + \cdots + \xi_m))) \widehat{\Delta_j f_1}(\xi_1) \widehat{S_{j-b} f_2}(\xi_2) \cdots \widehat{S_{j-b} f_m}(\xi_m) e^{2\pi i (\xi_1 + \cdots + \xi_m) \cdot x} d\vec{\xi} dt \right\|_{L^p(w)}$$

$$\lesssim \left\| \Delta_j f_1 \right\|_{L^{p_1}(w_1)} \left\| S_{j-b} f_2 \right\|_{L^{p_2}(w_2)} \left\| S_{j-b} f_3 \right\|_{L^{p_3}(w_3)} \cdots \left\| S_{j-b} f_m \right\|_{L^{p_m}(w_m)},$$

where the implicit constant depends on $p_1, \dots, p_m, m, n, \sigma, b$. (Here $d\vec{\xi} = d\xi_1 \cdots d\xi_m$ and the $L^p(w)$ quasi-norm is taken in the x variable.)

Proof. Let $\operatorname{supp}(\sigma) \subset [-M, M]^n$ where $M \geqslant 2m$. Expanding σ in Fourier series we have

$$\sigma(y) = \chi_{[-M,M]^n}(y) \sum_{\mu \in \mathbb{Z}^n} c_\mu e^{2\pi i y \cdot \frac{1}{2M} \mu}$$

where due to the smoothness of σ, the Fourier coefficients decay rapidly in μ. Now observe that

$$\chi_{[-M,M]^n}(2^{-j}(\xi_1 + t(\xi_2 + \cdots + \xi_m))) = 1$$

on the support of the integrand, that is, the support of

$$\chi_{[0,1]}(t) \widehat{\Psi}(2^{-j} \xi_1) \widehat{\Phi}(2^{-j+b} \xi_2) \cdots \widehat{\Phi}(2^{-j+b} \xi_m). \tag{4.5}$$

To see this, observe that on the support of (4.5) we have

$$|2^{-j}(\xi_1 + t(\xi_2 + \cdots + \xi_m))| \leqslant 2^{-j} m 2^{j+1} = 2m \leqslant M.$$

Thus the integral in the statement of this lemma can be written as

$$\sum_{\mu\in\mathbb{Z}^n}\int_0^1\int_{\mathbb{R}^{mn}} c_\mu e^{2\pi i(2^{-j}(\xi_1+t(\xi_2+\cdots+\xi_m)))\cdot\frac{\mu}{2M}}\widehat{\Delta_j f_1}(\xi_1)\widehat{S_{j-b}f_2}(\xi_2)\cdots\widehat{S_{j-b}f_m}(\xi_m)e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x}d\vec{\xi}dt$$

(4.6)

$$= \sum_{\mu\in\mathbb{Z}^n} c_\mu \Delta_j f_1\Big(x+\frac{\mu}{2^{j+1}M}\Big)\int_0^1 S_{j-b}f_2\Big(x+\frac{t\mu}{2^{j+1}M}\Big)\cdots S_{j-b}f_m\Big(x+\frac{t\mu}{2^{j+1}M}\Big)dt.$$

Using the subadditivity of $\|\cdot\|_{L^p}^{\tilde p}$ where $\tilde p = \min(p,1)$ and applying Hölder's inequality to (4.6) we have

$$\Big\|\sum_{\mu\in\mathbb{Z}^n} c_\mu \Delta_j f_1\Big(x+\frac{\mu}{2^{j+1}M}\Big)\int_0^1 S_{j-b}f_2\Big(x+\frac{t\mu}{2^{j+1}M}\Big)\cdots S_{j-b}f_m\Big(x+\frac{t\mu}{2^{j+1}M}\Big)\,dt\Big\|_{L^p(w)}^{\tilde p}$$

$$\begin{aligned}&\leqslant \sum_{\mu\in\mathbb{Z}^n}|c_\mu|^{\tilde p}\Big\|\Delta_j f_1\Big(x+\frac{\mu}{2^{j+1}M}\Big)\Big\|_{L^{p_1}(w_1)}^{\tilde p}\Big\|\int_0^1\Big|S_{j-b}f_2\Big(x+\frac{t\mu}{2^{j+1}M}\Big)\Big|\,dt\Big\|_{L^{p_2}(w_2)}^{\tilde p}\\ &\times\Big\|\sup_{t\in[0,1]}\Big|S_{j-b}f_3\Big(x+\frac{t\mu}{2^{j+1}M}\Big)\Big|\Big\|_{L^{p_3}(w_3)}^{\tilde p}\cdots\Big\|\sup_{t\in[0,1]}\Big|S_{j-b}f_m\Big(x+\frac{t\mu}{2^{j+1}M}\Big)\Big|\Big\|_{L^{p_m}(w_m)}^{\tilde p}.\end{aligned} \tag{4.7}$$

Without loss of generality we will only bound the p_m term. By Lemma 4.2 and for small enough ρ we have

$$\begin{aligned}\Big\|\sup_{t\in[0,1]}\Big|S_{j+1-b}S_{j-b}f_m\Big(x+\frac{t\mu}{2^{j+1}M}\Big)\Big|\Big\|_{L^{p_m}(w_m)} &\lesssim \Big\|\sup_{t\in[0,1]}(1+|t\mu|)^{\frac{n}{\rho}}\mathcal{M}_\rho(S_{j-b}f_m)\Big\|_{L^{p_m}(w_m)}\\ &\lesssim (1+|\mu|)^{\frac{n}{\rho}}\big\|S_{j-b}f_m\big\|_{L^{p_m}(w_m)}.\end{aligned}$$

Since the Fourier coefficients, c_μ, have rapid decay in μ we have (4.7) is bounded above by a constant multiple of

$$\big\|\Delta_j f_1\big\|_{L^{p_1}(w_1)}\big\|S_{j-b}f_2\big\|_{L^{p_2}(w_2)}\big\|S_{j-b}f_3\big\|_{L^{p_3}(w_3)}\cdots\big\|S_{j-b}f_m\big\|_{L^{p_m}(w_m)},$$

as desired. □

The following lemma is a simple multiplier theorem that will be used in the case s is an even integer.

Lemma 4.8. *Given $p\in(0,\infty]$, let σ be a compactly supported C^∞ function on $\mathbb{R}^n$, and $w\in A_\infty$. Then for any $j\in\mathbb{Z}$,*

$$\Big\|\int_{\mathbb{R}^n}\sigma(2^{-j}\xi)\widehat{S_j h}(\xi)e^{2\pi i\xi\cdot(\cdot)}d\xi\Big\|_{L^p(w)} \lesssim \big\|S_j h\big\|_{L^p(w)}$$

where the implicit constant is independent of j.

Proof. Let $\operatorname{supp}(\sigma)\subset[-M,M]^n$ where $M\gg 1$. Expanding σ in Fourier series we have

$$\sigma(y) = \chi_{[-M,M]^n}(y)\sum_{\mu\in\mathbb{Z}^n} c_\mu e^{2\pi i y\cdot\frac{1}{2M}\mu}$$

where the Fourier coefficients decay rapidly in μ. Hence we have

$$\begin{aligned}\left\|\int_{\mathbb{R}^n}\sigma(2^{-j}\xi)\widehat{S_jh}(\xi)e^{2\pi i\xi\cdot x}d\xi\right\|_{L^p(w)}^{\tilde{p}} &= \left\|\sum_{\mu\in\mathbb{Z}}c_\mu\int_{\mathbb{R}^n}\widehat{S_jh}(\xi)e^{2\pi i\xi\cdot(x+2^{j-1}M^{-1}\mu)}d\xi\right\|_{L^p(w)}^{\tilde{p}}\\ &\leqslant \sum_{\mu\in\mathbb{Z}}|c_\mu|\left\|S_jh\Big(x+\frac{\mu}{2^{j+1}M}\Big)\right\|_{L^p(w)}^{\tilde{p}}\\ &\lesssim \left\|S_jh\right\|_{L^p(w)}^{\tilde{p}},\end{aligned}$$

where in the last line we applied Lemma 4.2 and used the rapid decay of the Fourier coefficients. □

The following useful Theorem of Naibo and Thomson [26] enables us to side step the issue of decay of the Fourier coefficients.

Theorem 4.9 (Theorem 3.2, [26])**.** *Let $\vec{f}\in(\mathscr{S}(\mathbb{R}^n))^m$, $0<p<\infty$, $w\in A_\infty$ and $s>n(\min(1,p/\tau_w)^{-1}-1)$. Then for the inhomogeneous paraproduct* (3.4) *we have*

$$\left\|J^s\Big(\sum_{j\in\mathbb{N}}u_j^{\vec{\eta_0}}(\vec{f})\Big)\right\|_{L^p(w)} \leqslant \left\|J^s\Big(\sum_{j\in\mathbb{N}}u_j^{\vec{\eta_0}}(\vec{f})\Big)\right\|_{h^p(w)} \lesssim \left\|\Big(\sum_{j\in\mathbb{N}}|2^{js}u_j^{\vec{\eta_0}}(\vec{f})|^2\Big)^{\frac{1}{2}}\right\|_{L^p(w)}$$

where the implicit constant depends only on n,s,p,w.

5. Proof of Theorem 1.1

As stated in the introduction this is an adaptation of Oh and Wu's proof in [27] as well as the work of Naibo and Thomson [26]. After applying our paraproduct decomposition many of the techniques emulate the $m=2$ case. For the readers convenience we supply the details. To prove Theorem 1.1 we will show the desired bound for (3.3),(3.4) when $l_0>1$, (3.12) and (3.16).

5.1. **Low Frequency Term.** First we will deal with bounding (3.3), i.e. $J^s\big((S_0f_1)\cdots(S_0f_m)\big)$, which can be written as

$$\int_{\mathbb{R}^{nm}}(1+|\xi_1+\cdots+\xi_m|^2)^{\frac{s}{2}}\widehat{\Phi}(\xi_1)\widehat{f_1}(\xi_1)\cdots\widehat{\Phi}(\xi_m)\widehat{f_m}(\xi_m)e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x}d\xi_1\cdots d\xi_m \tag{5.1}$$

Let $v:=(S_0f_1)\cdots(S_0f_m)$. Note that $\widehat{v}$ is supported in $|\xi|\leqslant 2m$. Thus we have

$$\widehat{v}(\xi)=(\widehat{\Phi}(2^{-m}\xi))^2\widehat{v}(\xi)=\widehat{\Phi}(2^{-m}\xi)\widehat{S_mv}(\xi),$$

since $\widehat{\Phi}(2^{-m}\cdot)$ equals 1 on the support of $\widehat{v}$. It follows that (5.1) can be written as

$$\int_{\mathbb{R}^n}(1+|\xi|^2)^{\frac{s}{2}}\widehat{\Phi}(2^{-m}\xi)\widehat{S_mv}(\xi)e^{2\pi i\xi\cdot x}d\xi. \tag{5.2}$$

Letting $\sigma(\xi)=(1+|\xi|^2)^{\frac{s}{2}}\widehat{\Phi}(2^{-m}\xi)$, which is a smooth function with compact support, and applying Lemma 4.8 we have

$$\begin{aligned}\left\|\int_{\mathbb{R}^n}\sigma(\xi)\widehat{S_mv}(\xi)e^{2\pi i\xi\cdot(\cdot)}d\xi\right\|_{L^p(w)} &\lesssim \left\|S_m\big((S_0f_1)\cdots(S_0f_m)\big)\right\|_{L^p(w)}\\ &= \|(S_0f_1)\cdots(S_0f_m)\|_{L^p(w)}\\ &\leqslant \|S_0f_1\|_{L^{p_1}(w_1)}\cdots\|S_0f_m\|_{L^{p_m}(w_m)}\\ &\lesssim \left\|J^{-s}J^sf_1\right\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}\cdots\|f_m\|_{L^{p_m}(w_m)}\\ &\lesssim \|J^sf_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}\cdots\|f_m\|_{L^{p_m}(w_m)}.\end{aligned}$$

The last line above is justified by the fact that the Bessel potential, J^{-s}, is bounded on $L^{p_1}(w_1)$ by Lemma 4.3.

5.2. $l_0 \geqslant 2$**.** For the term in (3.4) we need to bound

$$\left\| J^s\Big(\sum_{j\in\mathbb{N}}(\Delta_j f_1)\cdots(\Delta_j f_{l_0})(S_{j-1}f_{l_0+1})\cdots(S_{j-1}f_m)\Big)\right\|_{L^p} \tag{5.3}$$

where at least the first two operators are Δ_j. For a natural number j recall

$$u_j^{\vec{\eta_0}}(\vec{f}) = (\Delta_j f_1)\cdots(\Delta_j f_{l_0})(S_{j-1}f_{l_0+1})\cdots(S_{j-1}f_m).$$

First suppose that $s \in 2\mathbb{N}$. Note that $\mathcal{F}(u_j^{\vec{\eta_0}}(\vec{f}))$ is supported in $|\xi| \leqslant m2^{j+1} < 2^{j+m}$. Thus we have

$$\mathcal{F}(u_j^{\vec{\eta_0}}(\vec{f}))(\xi) = (\widehat{\Phi}((2^{j+m})^{-1}\xi))^2\mathcal{F}(u_j^{\vec{\eta_0}}(\vec{f}))(\xi) = \widehat{\Phi}((2^{j+m})^{-1}\xi)\mathcal{F}(S_{j+m}u_j^{\vec{\eta_0}}(\vec{f}))(\xi),$$

since $\widehat{\Phi}((2^{j+m})^{-1}\cdot)$ equals 1 on the support of $\mathcal{F}(u_j^{\vec{\eta_0}}(\vec{f}))$. It follows that $J^s(\sum_{j\in\mathbb{N}} u_j^{\vec{\eta_0}}(\vec{f}))$ can be written as

$$\begin{aligned}\sum_{j\in\mathbb{N}} 2^{js}\int_{\mathbb{R}^n}(2^{-2j}+|2^{-j}\xi|^2)^{\frac{s}{2}}\widehat{\Phi}((2^{j+m})^{-1}\xi)\mathcal{F}(S_{j+m}u_j^{\vec{\eta_0}}(\vec{f}))(\xi)e^{2\pi i\xi\cdot x}d\xi \\ = \sum_{j\in\mathbb{N}} 2^{js}\int_{\mathbb{R}^n}\sigma_j(2^{-j}\xi)\mathcal{F}(S_{j+m}u_j^{\vec{\eta_0}}(\vec{f}))(\xi)e^{2\pi i\xi\cdot x}d\xi\end{aligned}$$

where

$$\sigma_j(\xi) := (2^{-2j}+|\xi|^2)^{\frac{s}{2}}\widehat{\Phi}(2^{-m}\xi)$$

is a smooth function with compact support. Note that the Fourier coefficients of σ_j will be uniformly bounded in j since $j > 0$ and $s/2$ is an integer. Applying Lemma 4.8 we obtain

$$\begin{aligned}&\left\|2^{js}\int_{\mathbb{R}^n}\sigma_j(2^{-j}\xi)\mathcal{F}(S_{j+m}u_j^{\vec{\eta_0}}(\vec{f}))(\xi)e^{2\pi i\xi\cdot(\cdot)}d\xi\right\|_{L^p(w)} \\ &\lesssim 2^{js}\left\|S_{j+m}u_j^{\vec{\eta_0}}(\vec{f})\right\|_{L^p(w)} \\ &= 2^{js}\left\|u_j^{\vec{\eta_0}}(\vec{f})\right\|_{L^p(w)}. \qquad (5.4)\end{aligned}$$

Now suppose that $s > n(\min(1,p/\tau_w)^{-1}-1)$. Then by Theorem 4.9 we obtain (5.3) is also bounded by a constant multiple of

$$\left\|\Big(\sum_{j\in\mathbb{N}}|2^{js}u_j^{\vec{\eta_0}}(\vec{f})|^2\Big)^{\frac{1}{2}}\right\|_{L^p(w)} \leqslant \left\|\sum_{j\in\mathbb{N}}|2^{js}u_j^{\vec{\eta_0}}(\vec{f})|\right\|_{L^p(w)}. \tag{5.5}$$

In view of Lemma 4.6 and the estimates (5.4) and (5.5) it will sufficient to obtain two upper bounds on $2^{js}\left\|u_j^{\vec{\eta_0}}(\vec{f})\right\|_{L^p(w)}$, which will cover both cases of $s \in 2\mathbb{N}$ and $s > n(\min(1,p/\tau_w)^{-1}-1)$. By Hölder's inequality, $2^{js}\left\|u_j^{\vec{\eta_0}}(\vec{f})\right\|_{L^p(w)}$ is bounded above by

$$2^{js}\|\Delta_j f_1\|_{L^{p_1}(w_1)}\cdots\|\Delta_j f_{l_0}\|_{L^{p_{l_0}}(w_{l_0})}\|S_{j-1}f_{l_0+1}\|_{L^{p_{l_0+1}}(w_{l_0+1})}\cdots\|S_{j-1}f_m\|_{L^{p_m}(w_m)} \tag{5.6}$$

$$\lesssim 2^{js}\|f_1\|_{L^{p_1}(w_1)}\cdots\|f_m\|_{L^{p_m}(w_m)}. \tag{5.7}$$

Applying Proposition 4.5 twice on the first two Δ_j operators we obtain that (5.6) is bounded above by a constant multiple of

$$2^{-js}\left\|\Delta_j J^s f_1\right\|_{L^{p_1}(w_1)}\left\|\Delta_j J^s f_2\right\|_{L^{p_2}(w_2)}\cdots\left\|\Delta_j f_{l_0}\right\|_{L^{p_{l_0}}(w_{l_0})} \times\left\|S_{j-1}f_{l_0+1}\right\|_{L^{p_{l_0+1}}(w_{l_0+1})}\cdots\left\|S_{j-1}f_m\right\|_{L^{p_m}(w_m)}$$

$$\lesssim 2^{-js}\|J^s f_1\|_{L^{p_1}(w_1)}\|J^s f_2\|_{L^{p_2}(w_2)}\|f_3\|_{L^{p_3}}\cdots\|f_m\|_{L^{p_m}(w_m)}. \tag{5.8}$$

In view of (5.7), (5.8) and Lemma 4.6 with $a = s$, $b = -s$, and

$$A = \|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}\cdots\|f_m\|_{L^{p_m}(w_m)},$$
$$B = \|J^s f_1\|_{L^{p_1}}\|J^s f_2\|_{L^{p_2}(w_2)}\cdots\left\|f_{l_0}\right\|_{L^{p_{l_0}}(w_{l_0}])}\left\|f_{l_0+1}\right\|_{L^{p_{l_0+1}}(w_{l_0+1})}\cdots\|f_m\|_{L^{p_m}(w_m)},$$

we obtain

$$\left\|J^s\Big(\sum_{j\in\mathbb{N}}(\Delta_j f_1)(\Delta_j f_2)\cdots(\Delta_j f_{l_0})(S_{j-1}f_{l_0+1})\cdots(S_{j-1}f_m)\Big)\right\|_{L^p(w)}$$
$$\lesssim\Big(\|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}\cdots\|f_m\|_{L^{p_m}(w_m)}\|J^s f_1\|_{L^{p_1}(w_1)}\|J^s f_2\|_{L^{p_2}(w_2)}\|f_3\|_{L^{p_3}(w_3)}\cdots\|f_m\|_{L^{p_m}(w_m)}\Big)^{\frac{1}{2}}$$
$$\lesssim\|J^s f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}\cdots\|f_m\|_{L^{p_m}(w_m)}+\|f_1\|_{L^{p_1}(w_1)}\|J^s f_2\|_{L^{p_2}(w_2)}\|f_3\|_{L^{p_3}(w_3)}\cdots\|f_m\|_{L^{p_m}(w_m)},$$

as desired. This finishes the proof for the diagonal term.

5.3. **High Frequency Terms.** We bound terms in (3.12) i.e.,

$$\sum_{j\in\mathbb{N}}(J^s\Delta_j f_1)(G_j^2 f_2)(G_j^3 f_3)\cdots(G_j^m f_m),$$

where G_j^l is either the identity operator I, or $\Delta_{>j-a}$. First if $G_j^l = I$ for all $l\in\{2,\dots,m\}$ then by Lemma 4.3 $\Delta_{>0}$ is a bounded operator from $L^{p_1}(w_1)\to L^{p_1}(w_1)$, so applying Hölder's inequality to $(J^s\Delta_{>0}f_1)f_2\cdots f_m$, gives the desired bound. Now assume that at least one $G_j^l = \Delta_{j-a}$; without loss of generality we will assume $G_j^l = \Delta_{j-a}$ for every $l\in\{1,\dots,m\}$, i.e.

$$\sum_{j\in\mathbb{N}}(J^s\Delta_j f_1)(\Delta_{>j-a}f_2)\cdots(\Delta_{>j-a}f_m). \tag{5.9}$$

We proceed by a similar method used in the previous case. Note that in general for $f\in\mathcal{S}(\mathbb{R}^n)$, $q\geqslant 1$, and $u\in A_q$ by Lemmas 4.3 and 4.5 we have

$$\left\|\Delta_{>j-a}f\right\|_{L^q(u)}=\left\|\sum_{k>j-a}\Delta_k f\right\|_{L^q(u)}\lesssim\sum_{k>j-a}2^{-ks}\|J^s\Delta_k f\|_{L^q(u)}\lesssim 2^{-js}\|J^s f\|_{L^q(u)}. \tag{5.10}$$

Also, using Lemma 4.3 we have

$$\left\|\Delta_{>j-a}f\right\|_{L^q(u)}\lesssim\|f\|_{L^q(u)}. \tag{5.11}$$

Applying the $L^p(w)$ quasi-norm to each summand in (5.9) gives

$$\left\|(J^s\Delta_j f_1)(\Delta_{>j-a}f_2)\cdots(\Delta_{>j-a}f_m)\right\|_{L^p(w)}$$
$$\lesssim\left\|J^s\Delta_j f_1\right\|_{L^{p_1}(w_1)}\left\|\Delta_{>j-a}f_2\right\|_{L^{p_2}(w_2)}\|f_3\|_{L^{p_3}(w_3)}\cdots\|f_m\|_{L^{p_m}(w_m)}. \tag{5.12}$$

It follows by (5.10),(5.11) that (5.12) is bounded above by a constant multiple of both

$$2^{-js}\|J^s f_1\|_{L^{p_1}(w_1)}\|J^s f_2\|_{L^{p_2}(w_2)}\|f_3\|_{L^{p_3}(w_3)}\cdots\|f_m\|_{L^{p_m}(w_m)}$$
$$\text{and } 2^{js}\|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}\|f_3\|_{L^{p_3}(w_3)}\cdots\|f_m\|_{L^{p_m}(w_m)}.$$

Applying Lemma 4.6 with $a = b = s$ and using the AMGM inequality again gives the desired estimate.

5.4. **Commutator.** Lastly this leaves the estimate for the commutator, which recall we reduced to the term in (3.16) i.e.,

$$2^{j(s-1)}\int_0^1\int_{\mathbb{R}^{nm}}\sigma_j(2^{-j}(\xi_1^1+t(\xi_2^1+\cdots+\xi_m^1)))\widehat{\Delta_j f_1}(\xi_1)\frac{1}{2\pi i}\widehat{S_{j-a}\partial_1 f_2}(\xi_2)\cdots\widehat{S_{j-a}f_m}(\xi_m) \times e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x}\,dtd\xi_1\cdots d\xi_m.$$

Taking the $L^p(w)$ quasi-norm and applying Lemma 4.7 yields

$$\left\|[J^s,S_{j-a}f_2\cdots S_{j-a}f_m]\Delta_j f_1\right\|_{L^p(w)} \tag{5.13}$$

$$\lesssim 2^{j(s-1)}\left\|\Delta_j f_1\right\|_{L^{p_1}(w_1)}\left\|S_{j-a}\partial_1 f_2\right\|_{L^{p_2}(w_2)}\|f_3\|_{L^{p_3}(w_3)}\cdots\|f_m\|_{L^{p_m}(w_m)}, \tag{5.14}$$

where the implicit constant is independent of j. Let Δ_k^1 be the operator associated with the Fourier multiplier $2^{-k}\xi_2^1\widehat{\Psi}(2^{-k}\xi_2)$ (here ξ_2^1 is the first coordinate of ξ_2) and let $0<\epsilon<\min\{1,s\}$. Observe,

$$\begin{aligned}\left\|S_{j-a}\partial_1 f_2\right\|_{L^{p_2}(w_2)} &\leqslant \sum_{k\leqslant j-a}\|\Delta_k\partial_1 f_2\|_{L^{p_2}(w_2)}\\ &= \sum_{k\leqslant j-a}2^k\left\|\Delta_k^1 f_2\right\|_{L^{p_2}(w_2)}\\ &= \sum_{k\leqslant j-a}2^{k(1-\epsilon)}2^{\epsilon k}\left\|\Delta_k^1 f_2\right\|_{L^{p_2}(w_2)}.\end{aligned} \tag{5.15}$$

Noting that by Proposition 4.5 we have

$$2^{\epsilon k}\left\|\Delta_k^1 f_2\right\|_{L^{p_2}(w_2)}\lesssim\min\left(2^{\epsilon k}\|f_2\|_{L^{p_2}(w_2)},2^{k(\epsilon-s)}\|J^s f_2\|_{L^{p_2}(w_2)}\right)$$

and taking the geometric mean with respect to $1-\frac{\epsilon}{s},\frac{\epsilon}{s}$ gives

$$2^{\epsilon k}\left\|\Delta_k^1 f_2\right\|_{L^{p_2}(w_2)}\lesssim\|f_2\|_{L^{p_2}(w_2)}^{1-\frac{\epsilon}{s}}\|J^s f_2\|_{L^{p_2}(w_2)}^{\frac{\epsilon}{s}}.$$

Plugging this estimate into (5.15) and using geometric series we obtain

$$\left\|S_{j-a}\partial_1 f_2\right\|_{L^{p_2}(w_2)}\lesssim 2^{j(1-\epsilon)}\|f_2\|_{L^{p_2}(w_2)}^{1-\frac{\epsilon}{s}}\|J^s f_2\|_{L^{p_2}(w_2)}^{\frac{\epsilon}{s}}.$$

Applying this estimate to (5.13) we have $\left\|[J^s,S_{j-a}f_2\cdots S_{j-a}f_m]\Delta_j f_1\right\|_{L^p(w)}$ is bounded above by a constant multiple of

$$2^{-j\epsilon}\|J^s f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}}^{1-\frac{\epsilon}{s}}\|J^s f_2\|_{L^{p_2}(w_2)}^{\frac{\epsilon}{s}}\|f_3\|_{L^{p_3}(w_3)}\cdots\|f_m\|_{L^{p_m}(w_m)}$$

and

$$2^{js}\|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}\|f_3\|_{L^{p_3}(w_3)}\cdots\|f_m\|_{L^{p_m}(w_m)}.$$

Applying Lemma 4.6 with $a = s$ and $b = \epsilon$ gives

$$\begin{aligned}&\left\|\sum_{j\in\mathbb{Z}^n}[J^s,S_{j-a}f_2\cdots S_{j-a}f_m]\Delta_j f_1\right\|_{L^p(w)}\\ &\lesssim\left(\|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}\|f_3\|_{L^{p_3}(w_3)}\cdots\|f_m\|_{L^{p_m}(w_m)}\right)^{\frac{\epsilon}{s+\epsilon}}\\ &\qquad\times\left(\|J^s f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}^{1-\frac{\epsilon}{s}}\|J^s f_2\|_{L^{p_2}(w_2)}^{\frac{\epsilon}{s}}\|f_3\|_{L^{p_3}(w_3)}\cdots\|f_m\|_{L^{p_m}(w_3)}\right)^{\frac{s}{s+\epsilon}}\end{aligned}$$

$$
\begin{aligned}
&= \Big(\|f_1\|_{L^{p_1}(w_1)} \|J^s f_2\|_{L^{p_2}(w_2)} \|f_3\|_{L^{p_3}(w_3)} \cdots \|f_m\|_{L^{p_m}(w_m)} \Big)^{\frac{\epsilon}{s+\epsilon}} \\
&\qquad \times \Big(\|J^s f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)} \|f_3\|_{L^{p_3}(w_3)} \cdots \|f_m\|_{L^{p_m}(w_m)} \Big)^{\frac{s}{s+\epsilon}} \\
&\lesssim \|f_1\|_{L^{p_1}(w_1)} \|J^s f_2\|_{L^{p_2}(w_2)} \|f_3\|_{L^{p_3}(w_3)} \cdots \|f_m\|_{L^{p_m}(w_m)} \\
&\qquad + \|J^s f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)} \|f_3\|_{L^{p_3}(w_3)} \cdots \|f_m\|_{L^{p_m}(w_m)}
\end{aligned}
$$

where the last line follows from concavity of the natural logarithm. This completes the proof of the inhomogeneous estimate in (1.3). □

6. About Theorem 1.2

With the paraproducts in (3.3) and (3.4) the proof of Theorem 1.2 can be quickly handled for Schwartz functions by Theorem 4.9; recall Theorem 1.2 does not include the L^1 endpoints. In fact, it will be enough to show the desired bound for (3.3) and (3.4). Recall $p_l > 1$ for $l \in \{1, \ldots, m\}$. First assume that $f_1, \ldots, f_m$ are Schwartz functions. The bound for (3.3), i.e. $J^s\big((S_0 f_1) \cdots (S_0 f_m)\big)$ was given in the proof of Theorem 1.1. So we focus our attention on (3.4), i.e. $J^s\big(\sum_{j\in\mathbb{N}} u_j^{\vec{\eta}_0}(\vec{f})\big)$. By Theorem 4.9 we obtain

$$
\begin{aligned}
\Big\| J^s\Big(\sum_{j\in\mathbb{N}} u_j^{\vec{\eta}_0}(\vec{f}) \Big) \Big\|_{h^p(w)} &\lesssim \Big\| \Big(\sum_{j\in\mathbb{N}} |2^{js} u_j^{\vec{\eta}_0}(\vec{f})|^2 \Big)^{\frac12} \Big\|_{L^p(w)} \\
&= \Big\| \Big(\sum_{j\in\mathbb{N}} |2^{js} (\Delta_j f_1)(\Delta_j f_2) \cdots (\Delta_j f_{l_0})(S_{j-1} f_{l_0+1}) \cdots (S_{j-1} f_m)|^2 \Big)^{\frac12} \Big\|_{L^p(w)} \\
&\leqslant \Big\| \Big(\sum_{j\in\mathbb{N}} |2^{js} \Delta_j f_1|^2 \Big)^{\frac12} \Big\|_{L^{p_1}(w_1)} \Big\| \sup_j |\Delta_j(f_2)| \Big\|_{L^{p_2}(w_2)} \cdots \Big\| \sup_j |\Delta_j(f_{l_0})| \Big\|_{L^{p_{l_0}}(w_{l_0})} \\
&\qquad \times \Big\| \sup_j |S_j(f_{l_0+1})| \Big\|_{L^{p_{l_0+1}}(w_{l_0+1})} \cdots \Big\| \sup_j |S_j(f_m)| \Big\|_{L^{p_m}(w_m)} \\
&\lesssim \|J^s f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)} \cdots \|f_m\|_{L^{p_m}(w_m)}
\end{aligned}
$$

where in the last line we used the well known equivalence between the Triebel-Lizorkin norm and Lebesgue norm for $p_1 > 1$, and the other operators are dominated by the Hardy-Littlewood maximal operator. This proves Theorem 1.2 for Schwartz functions, in Section 8 we extend this to weighted fractional Sobolev spaces. □

7. Inhomogeneous weighted KP implies homogeneous weighted KP

We now show that (1.3) implies the homogeneous variant. While it is possible to directly verify the homogeneous KP inequality, it is worth noting that the process is slightly more delicate since it requires a different paraproduct decomposition, and the sums are over $j \in \mathbb{Z}$ rather than $j \in \mathbb{N}$. The forthcoming method appears to offer a more intuitive pathway for achieving the homogeneous version. Though the unweighted version of this technique is mentioned in the literature [8], [18] it does not appear to be adapted in more recent publications.

Proposition 7.1. *Let* $0 < p \leqslant \infty$, $f \in \mathcal{S}(\mathbb{R}^n)$, $J_R^s f := \mathcal{F}^{-1}((R^{-2} + |\cdot|^2)^{\frac{s}{2}} \widehat{f})$, $w \in A_\infty$, *and* $s > \max(0, n(\tau_w/p - 1))$, *then*

$$\lim_{R\to\infty} \|J_R^s f\|_{L^p(w)} = \|D^s f\|_{L^p(w)}.$$

Proof. Let ϵ be small enough so that $\tau = \tau_w + \epsilon$ and $s > \max(0, n(\tau/p-1))$; notice $w \in A_\tau$. First let $p < \infty$. By Lebesgue dominated convergence theorem $J_R^s f$ converges pointwise to $D^s f$. By Lemma 4.4 we have the estimate $|J_R^s f(x)|^p \lesssim (1+|x|)^{-(n+s)p}$ where the implicit constant is independent of R. Now observe

$$\begin{aligned}\int_{\mathbb{R}^n} (1+|x|)^{-(n+s)p} w(x)\,dx &\lesssim c_{w,s,p,n} + \int_{|x|\geqslant 1} (1+|x|)^{-(n+s)p} w(x)\,dx \\ &\leqslant c_{w,s,p,n} + \sum_{j\geqslant 0} \int_{2^j \leqslant |x| < 2^{j+1}} 2^{-j(n+s)p} w(x)\,dx \\ &\lesssim c_{w,s,p,n} + \sum_{j\geqslant 0} 2^{-j(n+s)p} 2^{jn\tau}\end{aligned}$$

which is finite due to the relationship between the indices. Here we used the fact that Muckenhoupt weights are doubling, i.e. if $w \in A_p$ then $w(\lambda Q) \lesssim \lambda^{np} w(Q)$. Thus by Lebesgue dominated convergence theorem again we have $\lim_{R\to\infty} \left\| J_R^s f \right\|_{L^p(w)} = \|D^s f\|_{L^p(w)}$. Now suppose $p = \infty$. Observe that

$$\begin{aligned}|(J_R^s f - D^s f)(x)| &= \Big| \int_{\mathbb{R}^n} ((R^{-2} + |\xi|^2)^{\frac{s}{2}} - |\xi|^s) \widehat{f}(\xi) e^{2\pi i x\cdot\xi}\,d\xi \Big| \\ &\leqslant \int_{\mathbb{R}^n} ((R^{-2} + |\xi|^2)^{\frac{s}{2}} - |\xi|^s) |\widehat{f}(\xi)| d\xi.\end{aligned}$$

Notice that $((1+|\xi|^2)^{\frac{s}{2}} - |\xi|^s)|\widehat{f}(\xi)|$ is a uniform upper bound. Thus by Lebesgue dominated convergence theorem we can bring the limit inside, giving the desired equality. $\square$

To prove the homogeneous case from the inhomogeneous case we will use a dilation argument. For $f \in \mathcal{S}(\mathbb{R}^n)$ let $f^{(R)} := f(R\cdot)$. Observe,

$$\begin{aligned}J^s(f^{(R)})(\xi) &= \int_{\mathbb{R}^n} (1+|y|^2)^{\frac{s}{2}} R^{-n} \widehat{f}(R^{-1}y) e^{2\pi i y\cdot\xi} dy \\ &= \int_{\mathbb{R}^n} (1+|Ry|^2)^{\frac{s}{2}} \widehat{f}(y) e^{2\pi i y\cdot R\xi} dy \\ &= R^s \int_{\mathbb{R}^n} (R^{-2}+|y|^2)^{\frac{s}{2}} \widehat{f}(y) e^{2\pi i y\cdot R\xi} dy \\ &= R^s J_R^s(f)(R\xi)\end{aligned}$$

thus,

$$J^s(f^{(R)})(\xi) = R^s J_R^s(f)(R\xi). \tag{7.1}$$

It follows applying the inhomogeneous KP inequality to $(f_1 \cdots f_m)^{(R)} = (f_1)^{(R)} \cdots (f_m)^{(R)}$ with dilated weights $w_j(R\cdot)$ and using (7.1) gives

$$\begin{aligned}&\|J_R^s(f_1\cdots f_m)(R\cdot)\|_{L^p(w(R\cdot))} \\ &\leqslant C\Big(\|J_R^s f_1(R\cdot)\|_{L^{p_1}(w_1(R\cdot))} \|f_2(R\cdot)\|_{L^{p_2}(w_2(R\cdot))} \cdots \|f_m(R\cdot)\|_{L^{p_m}(w_m(R\cdot))} + \\ &\qquad \cdots + \|f_1(R\cdot)\|_{L^{p_1}(w_1(R\cdot))} \|f_2(R\cdot)\|_{L^{p_2}(w_2(R\cdot))} \cdots \|J_R^s f_m(R\cdot)\|_{L^{p_m}(w_m(R\cdot))} \Big)\end{aligned} \tag{7.2}$$

where the R^s term cancels from both sides. By a change of variables on both sides of (7.2) we note that the factor $R^{-\frac{n}{p}}$ cancels from both sides and thus we obtain

$$\begin{aligned}\|J_R^s(f_1\cdots f_m)\|_{L^p(w)} \leqslant C\Big(& \|J_R^s f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)} \cdots \|f_m\|_{L^{p_m}(w_m)} + \\ &\cdots + \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)} \cdots \|J_R^s f_m\|_{L^{p_m}(w_m)} \Big).\end{aligned}$$

The constant C in this inequality is a function of $[w_j(R\cdot)]_{A_{p_j}}$, but it is easy to see from the definition of Muckenhoupt weight that $[w_j(R\cdot)]_{A_{p_j}} = [w_j]_{A_{p_j}}$. Thus the constant C is independent of R. The homogeneous Kato-Ponce inequality is then attained by letting $R \to \infty$ and using Proposition 7.1. □

8. Density and completion of Theorem 1.2

The existing literature regarding KP inequalities with an integrability index $p < 1$ has primarily focused on Schwartz functions. In this section, we present a density argument that extends these results to fractional Sobolev spaces within the framework of Muckenhoupt weights that satisfy certain reverse Hölder conditions.

It's important to note that if f_j are Schwartz functions, then $J^s(f_1 \cdots f_m)$ is a well-defined function, allowing us to compute its weighted L^p-norm. However, when dealing with general functions $f_j \in L^{p_j}_s(w_j)$, $J^s(f_1, \ldots, f_m)$ is defined solely as a tempered distribution. Consequently, we cannot directly employ $L^p(w)$ on the left-hand side of equation (1.3). To proceed with the proof, we must first establish the well-defined nature of $J^s(f_1 \cdots f_m)$ as a tempered distribution. To prove this we need the following statements about weights.

Proposition 8.1. *Let $g \in L^q(w)$, $1 < q < \infty$ where $w \in A_q$, then g is a well defined tempered distribution.*

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and $\theta = w^{-\frac{q'}{q}}$ which is the dual weight of $w \in A_q$. Observe

$$
\begin{aligned}
|\langle g, \varphi\rangle| &\leqslant \int_{\mathbb{R}^n} |g||\varphi| w^{\frac{1}{q}} w^{-\frac{1}{q}} (1+|x|)^{n+1}(1+|x|)^{-(n+1)} dx \\
&\leqslant \|g\|_{L^q(w)} \left\|(1+|\cdot|)^{-(n+1)}\right\|_{L^{q'}(\theta)} \sup_{x\in\mathbb{R}^n} (1+|x|)^{n+1}|\varphi(x)| \\
&\lesssim \|g\|_{L^q(w)} \left\|(1+|\cdot|)^{-(n+1)}\right\|_{L^{q'}(\theta)} \sum_{|\alpha|\leqslant n+1} \sup_{x\in\mathbb{R}^n} |x|^{\alpha}|\varphi(x)|. \qquad (8.1)
\end{aligned}
$$

The result will follow from (8.1) once we show $\left\|(1+|\cdot|)^{-(n+1)}\right\|_{L^{q'}(\theta)}$ is finite. To see this observe,

$$
\begin{aligned}
\left\|(1+|\cdot|)^{-(n+1)}\right\|^{q'}_{L^{q'}(\theta)} &\lesssim \int_{|x|\leqslant 1} (1+|x|)^{-(n+1)q'} \theta(x)\, dx + \sum_{j\geqslant 0} \int_{2^j \leqslant |x| \leqslant 2^{j+1}} 2^{-j(n+1)q'} \theta(x)\, dx \\
&\lesssim C_\theta + \sum_{j\geqslant 0} 2^{-j(n+1)q'} \int_{|x|\leqslant 2^{j+1}} \theta(x)\, dx \\
&\lesssim C_\theta + \sum_{j\geqslant 0} 2^{-j(n+1)q'} 2^{jnq'} \qquad (8.2) \\
&< \infty,
\end{aligned}
$$

where in (8.2) we used that θdx is a doubling measure. □

Let $Q_{\nu,m} \subset \mathbb{R}^n$ denote, for $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, the n-dimensional cube with sides parallel to the coordinate axes, centered at $2^{-\nu}m$, and with side length $2^{-\nu}$. Furthermore, let $w(Q) = \int_Q w(x)\, dx$ for a weight w and a cube Q. We will need the following weighted Sobolev embedding theorem.

Theorem 8.2. [22] *Let $s > 0$, $1 < p \leqslant q < \infty$, $w_0 \in A_p$, and $w_1 \in A_q$. Then $L^p_s(w_0) \hookrightarrow L^q(w_1)$ if and only if*

$$
\sup_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n} 2^{-\nu s} w_0(Q_{\nu,m})^{-\frac{1}{p}} w_1(Q_{\nu,m})^{\frac{1}{q}} < \infty. \qquad (8.3)
$$

To obtain (8.3) in the sequel we will use weights living in $A_q \bigcap RH_\delta$ for some q and δ. In doing this we do not use any information about the position of the weight, i.e. the $m \in \mathbb{Z}^n$ in Theorem 8.2. We remark that if one instead opts to use power weights, i.e. for $\beta, \alpha > -n$, let $w_{\beta,\alpha} \in A_\infty$ be defined by

$$w_{\beta,\alpha}(x) = \begin{cases} |x|^\beta & \text{if } |x| \leqslant 1 \\ |x|^\alpha & \text{if } |x| > 1, \end{cases}$$

then the position information can also be used. Indeed, [[22], Proposition 4.1] gives a complete characterization of power weights that satisfy (8.3).

We will need the following two lemmas connecting Muckenhoupt and reverse Hölder classes.

Lemma 8.3. [9] *Let $1 < t < \infty$, then $w \in RH_t$ if and only if $w^t \in A_\infty$*

Lemma 8.4. [9] *Let $1 < t < \infty$, then $w \in A_p \bigcap RH_t$ if and only if $w^t \in A_q$, where $q = t(p-1)+1$.*

Let $w_j \in A_{p_j}$, $w = w_1^{\frac{p}{p_1}} \cdots w_m^{\frac{p}{p_m}}$ and recall if $\frac{\tau_w}{p} > 1$ then the lower bound on the smoothness index is $\frac{\tau_w}{p} - 1 < \frac{s}{n}$. Now suppose that $p \leqslant \tau_w < p+1$ then $(p+1-\tau_w)^{-1} \geqslant 1$. Let $\tau = \tau_w + \epsilon$ where $\epsilon > 0$ is small enough that

$$\delta := \frac{1}{p+1-\tau} > 1 \text{ and } \frac{\tau}{p} - 1 < \frac{s}{n}.$$

Pick $\delta_j \geqslant 1$ such that

$$\sum_{j=1}^m \frac{1}{\delta_j p_j} = \frac{1}{\delta p}. \tag{8.4}$$

For example, we could choose $\delta_j = \delta$ for $j = 1, \ldots, m$. Note that the δ_j depend on $p_1, \ldots, p_m$, $w_1, \ldots, w_m, s$.

Definition 8.5. *Let $w_j \in A_{p_j}$, $w = w_1^{\frac{p}{p_1}} \cdots w_m^{\frac{p}{p_m}}$ and suppose that $p \leqslant \tau_w < p+1$. Then we say $(w_1, \ldots, w_m)$ satisfy the joint reverse Hölder condition if $w_j \in RH_{\delta_j}$, where δ_j satisfy* (8.4).

Theorem 1.2 was stated in the unweighted case for simplicity, we will now prove a weighted version of Theorem 1.2 that implies the unweighted case.

Theorem 8.6. *Let $m \in \mathbb{Z}^+$, $\frac{1}{m} < p < \infty$, $1 < p_1, \ldots, p_m < \infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Let $w_j \in A_{p_j}$, $w = w_1^{\frac{p}{p_1}} \cdots w_m^{\frac{p}{p_m}}$ and that $s > \max\big(0, n(\frac{\tau_w}{p} - 1)\big)$. If $\frac{\tau_w}{p} < 1$ then there exists a constant $C = C(n, m, s, p_1, \ldots, p_m, w_1, \ldots, w_m) < \infty$ such that for all $f_j \in L_s^{p_j}(w_j)$ with $j \in \{1, \ldots, m\}$ we have*

$$\begin{aligned} \|J^s(f_1 \cdots f_m)\|_{L^p(w)} \leqslant C\big(&\|J^s f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)} \cdots \|f_m\|_{L^{p_m}(w_m)} + \cdots \\ &\cdots + \|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)} \cdots \|J^s f_m\|_{L^{p_m}(w_m)}\big). \end{aligned} \tag{8.5}$$

Furthermore, if $p \leqslant \tau_w < p+1$ and $(w_1, \ldots, w_m)$ satisfy the joint reverse Hölder condition then (8.5) *holds with $h^p(w)$ in place of $L^p(w)$. We note that in* (8.5) *any tuple of indices $(p_1, \ldots, p_m)$ that appears in a summand on the right of the inequality can be replaced by any other tuple $(q_1, \ldots, q_m)$ with $\frac{1}{p} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$.*

8.1. $J^s(f_1 \cdots f_m)$ **is a well-defined tempered distribution.**

In this subsection we show that if $f_j \in L_s^{p_j}(w_j)$ and $p \leqslant \tau_w < p+1$ and $(w_1, \ldots, w_m)$ satisfy the joint reverse Hölder condition then $J^s(f_1 \cdots f_m)$ is a well defined tempered distribution.

Since $w_j \in A_{p_j} \bigcap RH_{\delta_j}$ we have that $w^{\delta_j} \in A_{\delta_j p_j}$. Indeed by Lemma 8.4 $w_j \in A_t$ where

$$t = \delta_j(p_j - 1) + 1 < \delta_j p_j.$$

Let $v_j = w_j^{\delta_j}$, $\delta_j p_j = q_j$, and $q = \delta p$. Notice that $q > 1$. Observe that

$$v := \prod_{j=1}^m v_j^{\frac{q}{q_j}} = \Big(\prod_{j=1}^m w_j^{\frac{p}{p_j}}\Big)^\delta = w^\delta.$$

Since w^δ is a geometric average of A_∞ weights we have $w^\delta \in A_\infty$. Hence by Lemma 8.3, $w \in RH_\delta$. Thus $w \in A_\tau \bigcap RH_\delta$, furthermore observe

$$\delta(\tau-1)+1 = \frac{\tau-1}{p+1-\tau}+1 = p\delta = q.$$

Hence, by Lemma 8.4 $v \in A_q$.

Applying the $L^q(v)$ norm we obtain

$$\|f_1\cdots f_m\|_{L^q(v)} \leqslant \Big(\int_{\mathbb{R}^n}|f_1|^{q_1}v_1dx\Big)^{\frac{1}{q_1}}\cdots\Big(\int_{\mathbb{R}^n}|f_m|^{q_m}v_mdx\Big)^{\frac{1}{q_m}}.$$

We now show by using Theorem 8.2 that

$$\Big(\int_{\mathbb{R}^n}|f_j|^{q_j}v_j\,dx\Big)^{\frac{1}{q_j}} \lesssim \Big(\int_{\mathbb{R}^n}|J^sf_j|^{p_j}w_j\,dx\Big)^{\frac{1}{p_j}}.$$

Without loss of generality we will assume that $j=1$. For a cube Q with side length $2^{-\nu}$ we obtain

$$\begin{aligned}
2^{-\nu s}w_1(Q)^{-\frac{1}{p_1}}v_1(Q)^{\frac{1}{q_1}} &\lesssim 2^{-\nu s}2^{\nu n\frac{1}{p_1}}2^{-\nu n\frac{1}{q_1}}\left(\frac{1}{|Q|}\int_Q w_1\,dx\right)^{-\frac{1}{p_1}}\left(\frac{1}{|Q|}\int_Q w_1^{\delta_1}\,dx\right)^{\frac{1}{\delta_1 p_1}}\\
&\lesssim 2^{-\nu s}2^{\nu n\frac{1}{p_1}}2^{-\nu n\frac{1}{q_1}}\left(\frac{1}{|Q|}\int_Q w_1\,dx\right)^{-\frac{1}{p_1}}\left(\frac{1}{|Q|}\int_Q w_1\,dx\right)^{\frac{1}{p_1}} \qquad (8.6)\\
&= 2^{\nu(-s+\frac{n}{p_1}-\frac{n}{q_1})}
\end{aligned}$$

where in (8.6) we applied the reverse Hölder inequality. It follows we need that

$$\frac{1}{p_1}-\frac{1}{q_1} < \frac{s}{n}.$$

But this holds since

$$\frac{1}{p_1}-\frac{1}{q_1} \leqslant \sum_{j=1}^m \frac{1}{p_j}-\frac{1}{q_j} = \frac{1}{p}-\frac{1}{\delta p} = \frac{1}{p}(1-(p+1-\tau)) = \frac{\tau}{p}-1 < \frac{s}{n}.$$

By Proposition 8.1 $f_1\cdots f_m$ is a well defined tempered distribution, hence so is $J^s(f_1\cdots f_m)$.

8.2. **Density Argument.** First suppose $p \leqslant \tau_w < p+1$. Recall $q_i = \delta_i p_i$ and $\frac{1}{q} = \frac{1}{q_1}+\cdots+\frac{1}{q_m}$. Pick Schwartz functions f_i^j, for $i \in \{1,\dots,m\}$ converging to f_i respectively in $L_s^{p_i}(w_i)$ as $j \to \infty$. Then f_i^j also converge to f_i respectively in $L^{q_i}(v_i)$ as $j \to \infty$ by Theorem 8.2. We will show this implies $f_1^j\cdots f_m^j$ converge to $f_1\cdots f_m$ in $L^q(v)$ as $j\to\infty$. By induction (adding and subtracting mixed terms) one can show for $a_j, b_j \in \mathbb{C}$ that

$$|a_1\cdots a_m - b_1\cdots b_m| \leqslant \sum_{j=1}^m |a_j-b_j|\,|d_1^j|\cdots|d_{j-1}^j||d_{j+1}^j|\cdots|d_m^j| \qquad (8.7)$$

where d_k^j is either a_k or b_k. Applying (8.7) to $\Big|f_1\cdots f_m - f_1^j\cdots f_m^j\Big|$ we will without loss of generality only consider a summand of the form $|f_1-f_1^j||g_2|\cdots|g_m|$, where g_i is either f_i or f_i^j. We now proceed by bounding

$$\Big\| |f_1 - f_1^j||g_2|\cdots|g_m| \Big\|_{L^q(v)} = \left(\int_{\mathbb{R}^n} \Big(|f_1 - f_1^j||g_2|\cdots|g_m| \Big)^q v\, dx \right)^{\frac{1}{q}} \tag{8.8}$$

$$\leqslant \Big\| f_1 - f_1^j \Big\|_{L^{q_1}(v_1)} \|g_2\|_{L^{q_2}(v_2)} \cdots \|g_m\|_{L^{q_m}(v_m)}$$

letting $j \to \infty$ gives the desired result.

Since $f_1^j, \cdots f_m^j$ converges to $f_1 \cdots f_m$ in $L^q(v)$ this implies convergence in $\mathcal{S}'$. To see this let $g_j \to g$ in $L^q(v)$ and let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} |g - g_j||\varphi|\, dx = \int_{\mathbb{R}^n} |g - g_k||\varphi| v^{\frac{1}{q}} v^{-\frac{1}{q}} dx \leqslant \big\| g - g_j \big\|_{L^q(v)} \|\varphi\|_{L^{q'}(v^{-\frac{q'}{q}})} \tag{8.9}$$

which goes to zero as $j \to \infty$ in view of the fact that the dual weight of $v \in A_q$ is $v^{-\frac{q'}{q}}$.

Convergence of $f_1^j \cdots f_m^j$ to $f_1 \cdots f_m$ in $\mathcal{S}'$ then implies $J^s(f_1^j, \ldots, f_m^j)$ converges to $J^s(f_1, \ldots, f_m)$ in $\mathcal{S}'$. Also, by the KP inequality proved in Section 6 for Schwartz functions the sequence $J^s(f_1^j, \ldots, f_m^j)$ is Cauchy in $h^p(w)$, and thus it converges to G in $h^p(w)$, hence it converges to G in $\mathcal{S}'$. By the uniqueness of the limit in $\mathcal{S}'$, we have that $G = J^s(f_1, \ldots, f_m)$. We conclude that (8.5) holds.

8.3. Now suppose $\frac{\tau_w}{p} < 1$. For this case we may work with general Muckenhoupt weights, that is assume $w_j \in A_{p_j}$. Furthermore, notice that $w \in A_p$ since $\tau_w < p$. Observe,

$$\begin{aligned} \|f_1 \cdots f_m\|_{L^p(w)} &\lesssim \left(\int_{\mathbb{R}^n} \Big(|f_1| \cdots |f_m| w_1^{\frac{1}{p_1}} \cdots w_m^{\frac{1}{p_m}} \Big)^p dx \right)^{\frac{1}{p}} \\ &\leqslant \Big(\int_{\mathbb{R}^n} |f_1|^{p_1} w_1\, dx \Big)^{\frac{1}{p_1}} \cdots \Big(\int_{\mathbb{R}^n} |f_m|^{p_m} w_m\, dx \Big)^{\frac{1}{p_m}} \\ &< \infty \end{aligned}$$

where in the last line we applied Theorem 8.2. By Proposition 8.2 it follows that $J^s(f_1 \cdots f_m)$ is a well defined tempered distribution. The same density argument now works with with $L^p(w)$ in place of $h^p(w)$.

Combining these facts we conclude the proof of Theorem 8.6.

Remark: Suppose that $\{g_j\}_{j\in\mathbb{N}}$ is a sequence of Schwartz functions that is Cauchy in $h^p(\mathbb{R}^n)$ for some $p < 1$. (In our context, $g_j = f_1^j \cdots f_m^j$.) Then as $h^p(\mathbb{R}^n)$ is a complete space, the sequence g_j converges in $h^p(\mathbb{R}^n)$ as $j \to \infty$ to a tempered distribution g_d. But for locally integrable functions, the h^p quasi-norm controls the L^p quasi-norm, so the sequence g_j is also Cauchy in $L^p(\mathbb{R}^n)$. Hence, g_j converges in $L^p(\mathbb{R}^n)$ as $j \to \infty$ to a function $g \in L^p(\mathbb{R}^n)$. A natural question is then how do the function g and the tempered distribution g_d relate. Let P_t be the Poisson kernel. We claim that the a.e. nontangential limit of $P_t * g_d$ as $t \to 0^+$, which we call g_0, is equal to g a.e.

To verify this assertion we make some remarks. The characterizations of local hardy spaces in terms of truncated maximal averages and truncated maximal Poisson averages (see Goldberg [13], Stein [31, Chapter III, 5.17], and Wang, Yang, and Yang [32, Def 4.2 and Lemma 4.3]) provide the necessary ingredients to conclude that Calderón's theorem [5], [30, Chapter 7] on nontangential limits of $P_t * u$ as $t \to 0^+$ for u in H^p is also valid for u in h^p.

Then to verify that $g_0 = g$ a.e. we argue as follows: Suppose that $P_t * g_d \to g_0$ for all $x \in \mathbb{R}^n \backslash E$, where $|E| = 0$. For points in $\mathbb{R}^n \backslash E$ we write

$$g_0 - g = (g_0 - P_t * g_d) + P_t * (g_d - g_j) + (P_t * g_j - g_j) + (g_j - g).$$

For such points we take the pointwise limit as $t \to 0^+$ to obtain

$$|g_0 - g| \leqslant \limsup_{t\to 0^+} |P_t * (g_d - g_j)| + |g_j - g| \leqslant \sup_{0<t<1} |P_t * (g_d - g_j)| + |g_j - g|.$$

Raising to the power p and integrating over $\mathbb{R}^n \backslash E$ yields,

$$\|g_0 - g\|_{L^p}^p \leqslant C\|g_d - g_j\|_{h^p}^p + \|g_j - g\|_{L^p}^p$$

so letting $j \to \infty$ we obtain that $g_0 = g$ a.e.

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