REMARKS ON ALMOST EVERYWHERE CONVERGENCE AND APPROXIMATE IDENTITIES

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Abstract. We prove almost everywhere convergence for convolutions of locally integrable functions with shrinking L^1 dilations of a fixed integrable kernel with an integrable radially decreasing majorant. The set on which the convergence holds is an explicit subset of the Lebesgue set of the locally integrable function of full measure. This result can be viewed as an extension of the Lebesgue differentiation theorem in which the characteristic function of the unit ball is replaced by a more general kernel. We obtain a similar result for multilinear convolutions.

1. Introduction

Almost everywhere convergence for sequences (or families) of functions is an intricate topic that can be especially complicated in its study. For instance the almost everywhere convergence of Fourier series of square integrable functions is notorious for its difficulty; see [4], [7], [8]. But many other topics on almost everywhere convergence can be delicate and involved. By a general theorem [16], in most important cases, almost everywhere convergence is equivalent to the boundedness of an associated maximal operator. And such boundedness is, in many situations, particularly difficult to obtain. For an elegant presentation of general issues related to almost everywhere convergence one may consult the monograph [9] which exhibits a probabilistic viewpoint.

The use of approximate identities is of paramount importance in harmonic analysis. As Dirac mass at the origin is not an integrable function, the best way to approximate the unit element in the algebra of integrable functions is to consider approximate identities. But as convolution is a smoothing operation, convolution with approximate identities therefore allows the approximation of rough integrable functions by smooth ones. Such convolutions may converge in norm

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and almost everywhere. The topic of almost everywhere convergence for families of convolutions with approximate identities has been thoroughly studied by many authors. We mention for instance the following works: [2], [15], [13], [14], [1], [3], [11], [12], [5], [6], [18], noting that this list is by no means exhaustive but only representative of the different angles and aspects of the theory.

In this article we focus on almost everywhere convergence for families given by convolutions with specific approximate identities, formed by a single kernel via L^1 dilations. And our goal is to find relatively weak conditions on the kernel for almost everywhere convergence to hold.

2. Preliminaries

We consider families of approximate identities formed by dilations of a single integrable function on $\mathbf{R}^n$. Such families have natural and useful properties in terms of norm convergence. Precisely, if $K \in L^1(\mathbf{R}^n)$ has integral equal to 1 and

$$K_t(x) = t^{-n} K(x/t), \qquad t > 0,$$

are the L^1 dilations of K, then for any $f \in L^p(\mathbf{R}^n)$, $1 \le p < \infty$ we have that

$$K_t * f \to f \qquad \text{in } L^p(\mathbf{R}^n)$$

as $t \to 0^+$. Moreover, there is an analogous conclusion when $p = \infty$, if $f \in L^\infty(\mathbf{R}^n)$ is assumed to be uniformly continuous.

But the almost everywhere convergence (henceforth abbreviated as a.e.) of the family $f * K_t$ to f as $t \to 0^+$ is a more delicate matter. Usually certain control on the decay of K at infinity and its blowup near zero are required in order to obtain such convergence. In this work we study certain conditions on K suitable for the almost everywhere convergence of the family $f * K_t$ for general functions f on $\mathbf{R}^n$.

A typical result along these lines is the following: Let K in $L^1(\mathbf{R}^n)$ have integral equal to 1 and satisfy $|K(x)| \le A\,|x|^{-n} \min(|x|^\gamma, |x|^{-\gamma})$, where $A, \gamma > 0$. Then given $1 \le p < \infty$ and $f \in L^p(\mathbf{R}^n)$ we have

$$\lim_{t\to 0} K_t * f = f \qquad \text{a.e.}$$

Such a result can be proved by considering the oscillation

$$O_f = \limsup_{t\to 0} |K_t * f - f|$$

of a function f in $\cup_{1\le p<\infty} L^p(\mathbf{R}^n)$. The oscillation is zero a.e. for a dense subclass of L^p (such as smooth functions with compact support;

note here we use $p < \infty$). Thus for $f \in L^p(\mathbf{R}^n)$, we have $O_f = O_{f-\varphi}$ where φ lies in such a dense subclass. Moreover,

$$O_f \le C_{n,\gamma}\, A\, \mathcal{M}(f) + |f|, \tag{1}$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator. This allows one to show that for any $\delta > 0$

$$\big\{x \in \mathbf{R}^n :\ O_f(x) > \delta\big\} = \big\{x \in \mathbf{R}^n :\ O_{f-\varphi}(x) > \delta\big\}. \tag{2}$$

As we will see in more detail in Section 6, by (1) and (2) in conjuction with the weak type $(1,1)$ property of the Hardy-Littlewood maximal operator we obtain (2) is of Lebesgue measure zero and consequently $O_f = 0$ a.e.

This approach does not cover the case of $p = \infty$, in view of the lack of a nice dense subspace of L^∞. But it has the more serious drawback that it does not provide any information about the set of measure zero on which the pointwise convergence fails. In this note we discuss a method that can handle bounded functions and even functions that grow at infinity, provided there is a certain compatibility with the kernel [precisely, property (4)]. But the most important feature of this method is that it precisely describes the set on which the pointwise convergence holds.

We recall that a locally integrable function on $\mathbf{R}^n$ is integrable over all compact subsets of $\mathbf{R}^n$. The space of all locally integrable functions is denoted by $L^1_{\mathrm{loc}}(\mathbf{R}^n)$. The Lebesgue differentiation theorem says that for every locally integrable function f on $\mathbf{R}^n$ there is a measurable subset $\mathcal{L}_f$ of $\mathbf{R}^n$ (called the Lebesgue set of f) with the properties $|\mathbf{R}^n \setminus \mathcal{L}_f| = 0$ and

$$\lim_{t\to 0} \frac{1}{v_n t^n} \int_{|y|\le t} |f(x-y) - f(x)|\, dy = 0 \tag{3}$$

for every $x \in \mathcal{L}_f$. Here v_n is the volume of the unit ball $B(0,1)$ in $\mathbf{R}^n$. The purpose of this article is to investigate how general can an integrable function K be in order for $f * K_t$ to converge a.e. to f. The Lebesgue differentiation theorem says that K can be the characteristic function of the unit ball divided by v_n. The question we examine is for what other integrable functions K such a convergence result is possible. In Theorem 2.1 we show that the existence of a radially decreasing integrable majorant of K is sufficient for almost everywhere convergence to hold. A radially decreasing majorant of K is a radial function which majorizes $|K|$ and is decreasing as a function of its modulus.

A function L on $(0,\infty)$ is called piecewise absolutely continuous if there is a sequence of points

$$0 \leftarrow a_{-N} < \cdots < a_{-2} < a_{-1} < a_0 < a_1 < \cdots < a_N < a_{N+1} \to \infty$$

such that L is absolutely continuous on each interval $[a_j, a_{j+1}]$.

An example of a piecewise $\mathcal{C}^1$ (hence absolutely continuous) decreasing, and continuous function is $L(s) = s^{-1}\min(s^{1/4}, s^{-1/4})$ which fails to be differentiable at $s = 1$. Another more interesting example is the following. Let $g_k \in L^1(\mathbb{R})$ such that g_k is nonnegative and supported in $[a_{k-1}, a_k]$. Furthermore, suppose that $\sum_{k\in\mathbb{Z}} \|g_k\|_{L^1(\mathbf{R}^n)} < \infty$. Define

$$h_j(x) = \Big(\sum_{k=j}^{\infty} \|g_k\|_{L^1(\mathbb{R})} + \int_x^{a_j} g_{j-1}(t)dt\Big)\chi_{[a_{j-1},a_j]}(x)$$

and let $L(x) = \sum_{j\in\mathbb{Z}} h_j(|x|)$. Then on each interval $[a_{j-1}, a_j]$ the function L is absolutely continuous and decreasing, moreover L is continuous at each a_j for $j \in \mathbb{Z}$. To see that L is globally decreasing observe if $j' < j$ and $x \in [a_{j-1}, a_j]$, $x' \in [a_{j'-1}, a_{j'}]$, then

$$L(x) \le \sum_{k=j-1}^{\infty} \|g_k\|_{L^1(\mathbf{R}^n)} \le \sum_{k=j'}^{\infty} \|g_k\|_{L^1(\mathbf{R}^n)} \le L(x').$$

We consider kernels whose absolute value is majorized by decreasing, continuous, and piecewise absolutely continuous functions of the modulus. For such kernels K we obtain an almost everywhere convergence theorem for the convolutions $f * K_t$ as $t \to 0^+$ for general locally integrable functions.

Our main result is as follows:

Theorem 2.1. *Let K be a function on $\mathbf{R}^n$ and let $L : (0,\infty) \to [0,\infty)$ be a decreasing, continuous, and piecewise absolutely continuous function. Assume that*

(A) $|K(x)| \le L(|x|)$ for all $x \in \mathbf{R}^n$.

(B) The function $L(|\cdot|)$ lies in $L^1(\mathbf{R}^n)$.

Let $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$. Suppose that there is a set E_f of measure zero such that for every $0 < \theta \le 1$ and $x \in \mathbf{R}^n \setminus E_f$ we have

$$\lim_{t\to 0^+} \int_{|y|\ge\theta} |f(x-y)|\,|K_t(y)|\,dy = 0. \tag{4}$$

Then there is another set of measure zero D_f such that for all points x in $\mathbf{R}^n \setminus (D_f \cup E_f)$ we have

$$(|f| * |K_t|)(x) < \infty \tag{5}$$

for sufficiently small t (depending on x), and for all x in $\mathcal{L}_f \setminus (D_f \cup E_f)$ we have

$$\lim_{t\to 0^+} (K_t * f)(x) = cf(x), \tag{6}$$

where $c = \int_{\mathbf{R}^n} K(x)\,dx$. Thus, $K_t * f \to cf$ a.e. as $t \to 0^+$.

In Section 5 we provide examples of pairs (f, K) satisfying condition (4); these include L^p functions for $1 \le p \le \infty$ and L^1_{loc} functions if K has compact support. Thus, condition (4) is very natural in this context.

3. Properties of the function L

We begin with the following observations about the function L:

$$\lim_{s\to +\infty} s^n L(s) = 0, \tag{7}$$

and

$$\lim_{s\to 0^+} s^n L(s) = 0. \tag{8}$$

To verify these assertions, we note that by assumption (B) we have

$$I = \int_0^\infty u^{n-1} L(u)\,du < \infty.$$

Then for $s > 0$ we have

$$\int_{s/2}^s u^{n-1} L(u)\,du = \int_{s/2}^\infty u^{n-1} L(u)\,du - \int_s^\infty u^{n-1} L(u)\,du$$

and this converges to 0 as $s \to \infty$ being the difference of two tails of an integrable function and also converges to $I - I = 0$ as $s \to 0^+$. Since L is decreasing we obtain

$$\int_{s/2}^s u^{n-1} L(u)\,du \ge L(s)(s/2)^{n-1}(s - s/2) = \frac{1}{2^n} L(s) s^n,$$

for all $s > 0$, and from this we derive (7) and (8).

Next we focus on the following integration by parts lemma:

Lemma 3.1. *Let $b > 0$. Let L be as in the statement of Theorem 2.1. Then for any nonnegative absolutely continuous function ϕ on $[0, b]$ that satisfies*

$$\phi(r) \le Cr^n \qquad \text{for some } C > 0 \text{ and all } r \in [0, b] \tag{9}$$

and

$$\int_0^b L\Big(\frac{s}{t}\Big) |\phi'(s)|\,ds < \infty \qquad \text{for some } t > 0, \tag{10}$$

the integration by parts formula

$$\int_0^b L\Big(\frac{r}{t}\Big)\phi'(r)dr = L\Big(\frac{b}{t}\Big)\phi(b) - \int_0^b \frac{1}{t}L'\Big(\frac{r}{t}\Big)\phi(r)dr \tag{11}$$

is valid.

Proof. First we note that replacing b by tb, $\phi(r)$ by $\phi(tr)$, C by $t^{-n}C$, matters reduce to the case $t=1$. So we prove (11) when $t=1$.

We consider the largest point a_N such that $a_N < b$. Then $b \le a_{N+1}$ and we apply integration by parts on each interval $[a_i, a_{i+1}]$ for $i < N$ and also on $[a_N, b]$; note that L is absolutely continuous on these intervals.

By (8) there is an $\epsilon_0 > 0$ and such that

$$0 < \epsilon < \epsilon_0 \implies L(\epsilon)\epsilon^n \le 1. \tag{12}$$

For $\epsilon < \min(\epsilon_0, a_{N-2})$ we pick $M < N-2$ such that $a_M < \epsilon \le a_{M+1}$ and we write

$$\begin{aligned}
&\int_\epsilon^b L(r)\phi'(r)dr \\
&= \sum_{i=M+1}^{N-1} \Big(L(a_{i+1})\phi(a_{i+1}) - L(a_i)\phi(a_i)\Big) - \int_{a_i}^{a_{i+1}} L'(r)\phi(r)dr \\
&\qquad + L(b)\phi(b) - L(a_N)\phi(a_N) - \int_{a_N}^b L'(r)\phi(r)dr \\
&\qquad + L(a_{M+1})\phi(a_{M+1}) - L(\epsilon)\phi(\epsilon) - \int_\epsilon^{a_{M+1}} L'(r)\phi(r)dr
\end{aligned}$$

using the classical integration by parts identity, which is justified from the fact that L and ϕ are absolutely continuous on each closed interval that appears.

The sums are telescoping and thus summing them yields

$$\int_\epsilon^b L(r)\phi'(r)dr = L(b)\phi(b) - L(\epsilon)\phi(\epsilon) - \int_\epsilon^b L'(r)\phi(r)dr. \tag{13}$$

At this point we need to let $\epsilon \to 0$. Assume momentarily that

$$\int_0^b |L'(r)|\phi(r)dr < \infty. \tag{14}$$

Then we let $\epsilon \to 0^+$ applying the Lebesgue dominated convergence theorem, whose use is justified by (10) and (14). We obtain

$$\int_0^b L(r)\phi'(r)dr = L(b)\phi(b) - \int_0^b L'(r)\phi(r)dr,$$

noting that

$$L(\epsilon)\phi(\epsilon) = \underbrace{L(\epsilon)\epsilon^n}_{\text{tends to 0}}\ \underbrace{\phi(\epsilon)\epsilon^{-n}}_{\text{bounded}} \to 0$$

in view of condition (8) and (9). It remains to prove (14). Using (9) and the fact that $L' \leq 0$ (whenever it is defined) we need to show that

$$-\int_0^b L'(r)r^n\,dr < \infty.$$

We just repeat the argument leading to (13) with $\phi(r) = r^n$ to obtain

$$\int_\epsilon^b L(r)nr^{n-1}dr = L(b)b^n - L(\epsilon)\epsilon^n - \int_\epsilon^b L'(r)r^n dr, \tag{15}$$

and from this and (12), since $\epsilon < \epsilon_0$, we deduce

$$\begin{aligned} -\int_\epsilon^b L'(r)r^n dr &\leq -L(b)b^n + 1 + \int_0^b L(r)nr^{n-1}dr \\ &= \ -L(b)b^n + 1 + \frac{n}{\omega_{n-1}}\|L(|\cdot|)\|_{L^1(\mathbf{R}^n)}, \end{aligned}$$

where ω_{n-1} is the surface area of the unit sphere $\mathbf{S}^{n-1}$ and $\|L(|\cdot|)\|_{L^1(\mathbf{R}^n)}$ is the L^1 norm of the function $x \to L(|x|)$ on $\mathbf{R}^n$. Letting $\epsilon \to 0^+$ and using the Lebesgue monotone convergence theorem proves (14) and completes the proof of (11). □

4. The proof of Theorem 2.1

Proof. We fix $f \in L^1_{\text{loc}}(\mathbf{R}^n)$ and K, L as in the statement of the theorem. For every $N \in \mathbf{Z}^+$ we have

$$\begin{aligned} &\int_{|x|\leq N} \left[\int_{|y|\leq 1} |f(x-y)|\, L(|y|)\, dy\right] dx \\ &\leq \ \Big(\int_{|y|\leq 1} L(|y|)\, dy\Big) \int_{|x'|\leq N+1} |f(x')|\, dx' \\ &\leq \ \big\|L(|\cdot|)|\big\|_{L^1(\mathbf{R}^n)} \int_{|x|\leq N+1} |f(x)|\, dx \\ &< \ \infty. \end{aligned}$$

It follows from the preceding discussion and the fact that L is a decreasing function that for every $N \in \mathbf{Z}^+$, there is a set of measure zero D_N such that $x \in \overline{B(0,N)} \setminus D_N$ implies

$$\int_{|y|\leq 1} |f(x-y)|\, \frac{1}{t^n} L\Big(\frac{|y|}{t}\Big)\, dy < \infty \quad \text{for all } 0 < t \leq 1. \tag{16}$$

Since we are considering $t \to 0^+$ we may suppose for the rest of the proof that $0 < t \le 1$. Here and throughout $B(x,r)$ is the open ball in $\mathbf{R}^n$ of radius $r > 0$ centered at x. Let

$$D_f = \bigcup_{N=1}^{\infty} D_N.$$

Then for all $x \in \mathbf{R}^n \setminus D_f$ we have

$$\int_{|y|\le 1} |f(x-y)|\,|K_t(y)|\,dy \le \int_{|y|\le 1} |f(x-y)|\,\frac{1}{t^n} L\Big(\frac{|y|}{t}\Big)\,dy < \infty. \tag{17}$$

Now for a given $x \in \mathbf{R}^n \setminus E_f$, (4) with $\theta = 1$ implies that there is a $t_{x,1} > 0$ such that for all t satisfying $0 < t < t_{x,1}$ we have

$$\int_{|y|\ge 1} |f(x-y)|\,|K_t(y)|\,dy < 10. \tag{18}$$

Combining this fact with (17) we obtain that when $x \in \mathbf{R}^n \setminus (D_f \cup E_f)$ and $0 < t < \min\{1, t_{x,1}\}$ we have

$$\int_{\mathbf{R}^n} |f(x-y)|\,|K_t(y)|\,dy < \infty.$$

This proves (5).

Let $\mathcal{L}_f$ be the Lebesgue set of f. We fix a $x_0 \in \mathcal{L}_f \setminus (D_f \cup E_f)$ which has full measure and we prove (6) for $x = x_0$.

Let $v_n = |B(0,1)|$ be the volume of the unit ball in $\mathbf{R}^n$ and

$$I_L = \int_{\mathbf{R}^n} L(|x|)dx = n\,v_n \int_0^\infty s^{n-1} L(s)\,ds < \infty.$$

Given $\varepsilon > 0$, as $x_0 \in \mathcal{L}_f$, there is a $\delta_0 > 0$ (we may assume that $\delta_0 < 1$) such that

$$0 < r \le \delta_0 \implies \frac{1}{v_n r^n} \int_{|y|<r} |f(x_0-y) - f(x_0)|\,dy < \frac{\varepsilon}{I_L}. \tag{19}$$

For $t > 0$ and $t < \min\{1, t_{x_0,1}\}$ we write

$$\begin{aligned}
&\Big|(K_t * f)(x_0) - cf(x_0)\Big| \\
&\qquad = \Big| \int_{\mathbf{R}^n} f(x_0 - y) K_t(y)\, dy - \Big(\int_{\mathbf{R}^n} K_t(y)\, dy \Big) f(x_0) \Big| \\
&\qquad \le \int_{\mathbf{R}^n} \big| f(x_0 - y) - f(x_0) \big|\, |K_t(y)|\, dy \\
&\qquad \le \int_{|y| \ge \delta_0} \big| f(x_0 - y) - f(x_0) \big|\, |K_t(y)|\, dy \qquad (20) \\
&\qquad\qquad + \int_{|y| < \delta_0} \big| f(x_0 - y) - f(x_0) \big|\, |K_t(y)|\, dy. \qquad (21)
\end{aligned}$$

We begin with term (20). We have

$$\begin{aligned}
&\int_{|y| \ge \delta_0} \big| f(x_0 - y) - f(x_0) \big|\, |K_t(y)|\, dy \\
&\quad \le \int_{|y| \ge \delta_0} |f(x_0 - y)|\, |K_t(y)|\, dy + |f(x_0)| \int_{|y| \ge \delta_0} |K_t(y)|\, dy \\
&\quad = \int_{|y| \ge \delta_0} |f(x_0 - y)|\, |K_t(y)|\, dy + |f(x_0)| \int_{|y| \ge \delta_0/t} |K(y)|\, dy.
\end{aligned}$$

By assumption (4) there is a positive constant t_{x_0,δ_0} such that for all t satisfying $0 < t < t_{x_0,\delta_0}$ we have

$$\int_{|y| \ge \delta_0} |f(x_0 - y)|\, |K_t(y)|\, dy < \varepsilon.$$

Moreover there is a $t^*_{x_0,\delta_0} > 0$ such that for $0 < t < t^*_{x_0,\delta_0}$ we have

$$|f(x_0)| \int_{|y| \ge \delta_0/t} |K(y)|\, dy < \varepsilon$$

as the integral above is the tail of an integrable function. Combining these facts we obtain

$$\int_{|y| \ge \delta_0} \big| f(x_0 - y) - f(x_0) \big|\, |K_t(y)|\, dy < 2\varepsilon \qquad (22)$$

whenever

$$0 < t < \min\big\{ t_{x_0,1}, t_{x_0,\delta_0}, t^*_{x_0,\delta_0} \big\}. \qquad (23)$$

To handle the term in (21) for every $r>0$ we use polar coordinates to write

$$\begin{aligned}\int_{|y|<r} &|f(x_0-y)-f(x_0)|\,dy\\ &=\int_0^r \rho^{n-1}\int_{\mathbf{S}^{n-1}} |f(x_0-\rho\theta)-f(x_0)|\,d\theta\,d\rho\\ &=\int_0^r F(\rho)\,d\rho,\end{aligned}$$

where we set

$$F(\rho)=\rho^{n-1}\int_{\mathbf{S}^{n-1}} |f(x_0-\rho\theta)-f(x_0)|\,d\theta.$$

Note that by Fubini's theorem, F is defined for almost every $\rho>0$ as $|f(x_0-\cdot)-f(x_0)|$ is integrable over the ball $B(0,r)$.

We now write for any $0<t\le 1$,

$$\begin{aligned}\int_{|y|<\delta_0} &|f(x_0-y)-f(x_0)|\,|K_t(y)|\,dy\\ &\le \int_{|y|<\delta_0} |f(x_0-y)-f(x_0)|\,\frac{1}{t^n}L\Big(\frac{|y|}{t}\Big)\,dy<\infty \end{aligned}\tag{24}$$

and the expression on the right is finite in view of (16) since x_0 does not lie in $D_f\cup E_f$ and $\delta_0<1$. We now write

$$\begin{aligned}\int_{|y|<\delta_0} &|f(x_0-y)-f(x_0)|\,\frac{1}{t^n}L\Big(\frac{|y|}{t}\Big)\,dy\\ &=\int_0^{\delta_0}\frac{d}{dr}\Big[\int_0^r F(\rho)\,d\rho\Big]\,\frac{1}{t^n}L\Big(\frac{r}{t}\Big)\,dr\\ &=\Big(\int_0^{\delta_0}F(\rho)\,d\rho\Big)\frac{1}{t^n}L\Big(\frac{\delta_0}{t}\Big)-\int_0^{\delta_0}\Big(\int_0^r F(\rho)\,d\rho\Big)\frac{1}{t^n}\frac{1}{t}L'\Big(\frac{r}{t}\Big)dr,\end{aligned}$$

where we used the integration by parts formula (11) with $b=\delta_0$ and

$$\phi(r)=\int_0^r F(\rho)\,d\rho,$$

which is absolutely continuous. Note that ϕ satisfies condition (9) with $C=\varepsilon v_n/I_L$ and condition (10) since

$$\int_0^{\delta_0}F(r)L\Big(\frac{r}{t}\Big)dr\le\int_{|y|\le\delta_0}|f(x_0-y)-f(x_0)|\,L\Big(\frac{|y|}{t}\Big)dy<\infty$$

in view of (24).

We now return to estimating

$$\left(\int_0^{\delta_0} F(\rho)\,d\rho\right)\frac{1}{t^n}L\Big(\frac{\delta_0}{t}\Big)-\int_0^{\delta_0}\left(\int_0^r F(\rho)\,d\rho\right)\frac{1}{t^n}\frac{1}{t}L'\Big(\frac{r}{t}\Big)dr$$

which we write as follows:

$$\left(\frac{1}{\delta_0^n}\int_0^{\delta_0} F(\rho)d\rho\right)\frac{\delta_0^n}{t^n}L\Big(\frac{\delta_0}{t}\Big)-\int_0^{\delta_0}\left(\frac{1}{r^n}\int_0^r F(\rho)\,d\rho\right)\frac{r^n}{t^n}\frac{1}{t}L'\Big(\frac{r}{t}\Big)dr. \tag{25}$$

Using (19) and the fact that $-L'\ge 0$ a.e. we estimate (25) by

$$\begin{aligned}
&\frac{\varepsilon}{I_L}\,v_n\left[\frac{\delta_0^n}{t^n}L\Big(\frac{\delta_0}{t}\Big)-\int_0^{\delta_0}\frac{r^n}{t^n}\frac{1}{t}L'\Big(\frac{r}{t}\Big)\,dr\right]\\
&=\frac{\varepsilon}{I_L}\,v_n\left[\frac{\delta_0^n}{t^n}L\Big(\frac{\delta_0}{t}\Big)-\int_0^{\delta_0/t} r^nL'(r)\,dr\right]\\
&=\frac{\varepsilon}{I_L}\,n\,v_n\left[\int_0^{\delta_0/t} r^{n-1}L(r)\,dr\right]\\
&\le\frac{\varepsilon}{I_L}\,n\,v_n\,\frac{1}{\omega_{n-1}}\int_{\mathbf{R}^n}L(|x|)\,dx\\
&=\varepsilon.
\end{aligned}$$

Here we used the integration by parts identity (11) again with the function $\phi(r)=r^n$ and the fact that $nv_n=\omega_{n-1}$. Combining the inequality just proved with (20), (21), and (22) we obtain

$$|(K_t*f)(x_0)-cf(x_0)|<2\varepsilon+\varepsilon=3\varepsilon,$$

whenever (23) holds. This proves (6). □

5. Comments and remarks

Several remarks on Theorem 2.1 are in order.

Remark 5.1. *Given $f\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, if it is the case that for all $x\in\mathbf{R}^n$*

$$\int_{|y|\le 1}|f(x-y)|\,L(|y|)\,dy<\infty \tag{26}$$

then $D_f=\emptyset$. Additionally if $E_f=\emptyset$, then (6) is valid for every point in the Lebesgue set of f. In this case, the convergence of $f*K_t$ (as $t\to 0^+$) holds on the Lebesgue set $\mathcal{L}_f$ of f.

Remark 5.2. *If the function K in Theorem 2.1 has compact support, then condition (4) holds for any locally integrable function f. Indeed, if K is supported in a ball $B(0,M)$, then the integral in (4) is over the set $\theta\le|y|\le Mt$ and this set becomes empty when $t<\theta/M$.*

Remark 5.3. *If $L(s) = s^{-n}\min(s^{\gamma}, s^{-\gamma})$ for some $\gamma > 0$, then condition (4) can be derived from*

$$\int_{\mathbf{R}^n} \frac{|f(z)|}{(1+|z|)^{n+\gamma}}\,dz < \infty. \tag{27}$$

Indeed, assuming (27), for any $x \in \mathbf{R}^n$, we obtain

$$\int_{\mathbf{R}^n} \frac{|f(z)|}{(1+|x-z|)^{n+\gamma}}\,dz < \infty \tag{28}$$

by splitting the z integral above into the regions $|z| \le 2|x|$ and $|z| \ge 2|x|$. The integral over the region $|z| \le 2|x|$ is finite as f is locally integrable. Also, in the case $|z| \ge 2|x|$ we have $\frac{|z|}{2} \le |z-x| \le \frac{3}{2}|z|$, so we obtain the finiteness of (28) from (27). Then for $|y| \ge \theta$ and $t > 0$ we have

$$|K_t(y)| \le \frac{A}{t^n}\Big|\frac{y}{t}\Big|^{-n}\Big|\frac{y}{t}\Big|^{-\gamma} = A\frac{t^{\gamma}}{|y|^{n+\gamma}} \le A\,t^{\gamma}\Big(\frac{\theta+1}{\theta}\Big)^{n+\gamma}\frac{1}{(1+|y|)^{n+\gamma}}.$$

Combining this estimate with (28) we deduce (4).

Remark 5.4. *We note that condition (4) holds if f lies in L^p where $1 \le p \le \infty$. Indeed, in this case we make use of the bound*

$$\int_{|y|\ge\theta} |f(x-y)|\,|K_t(y)|\,dy \le \|f\|_{L^p(\mathbf{R}^n)}\|K_t\|_{L^{p'}(\mathbf{R}^n\setminus B(0,\theta))},$$

by Hölder's inequality. Here p' is the dual exponent of p which satisfies $1/p + 1/p' = 1$. If $p' < \infty$, then

$$\|K_t\|_{L^{p'}(\mathbf{R}^n\setminus B(0,\theta))} = \Big(\int_{|y|\ge\theta/t} |K(y)|^{p'}\,dy\Big)^{\frac{1}{p'}} \le \Big(\int_{\theta/t}^{\infty} L(s)^{p'}s^{n-1}\,ds\Big)^{\frac{1}{p'}}.$$

Since L is decreasing on $(0,\infty)$ it has a limit as $s \to \infty$ and this cannot be a positive number, otherwise $L(s)s^{n-1}$ would not be integrable over $(0,\infty)$. Let us pick an $s_0 > 0$ such that $L(s) \le 1$ for all $s \ge s_0$. Then pick t_0 such that $t < t_0$ implies $\theta/t > s_0$. It follows if $t < t_0$, then for $s \ge \theta/t$ we have $L(s)^{p'} \le L(s)$ hence

$$\Big(\int_{\theta/t}^{\infty} L(s)^{p'}s^{n-1}\,ds\Big)^{\frac{1}{p'}} \le \Big(\int_{\theta/t}^{\infty} L(s)s^{n-1}\,ds\Big)^{\frac{1}{p'}}$$

and this converges to zero as it is the tail of an integrable function. Hence condition (4) holds in this case.

We now turn to the case $p = 1$ or $p' = \infty$. We apply Hölder's inequality as in the case $p > 1$, but we note that

$$\|K_t\|_{L^{\infty}(\mathbf{R}^n\setminus B(0,\theta))} \le \sup_{|y|\ge\theta}\frac{1}{t^n}L\Big(\frac{|y|}{t}\Big) \le \frac{1}{t^n}L\Big(\frac{\theta}{t}\Big) = \frac{1}{\theta^n}\frac{\theta^n}{t^n}L\Big(\frac{\theta}{t}\Big).$$

Now letting $t \to 0$ and using (7) we obtain that the preceding expression tends to zero and thus condition (4) also holds in this case.

Example 5.1. *Three examples of functions L that satisfy the hypotheses of Theorem 2.1 are the following:*

$$L_1(s) = s^{-n} \begin{cases} s^{\gamma} & \text{when } s < 1 \\ s^{-\gamma} & \text{when } s \geq 1 \end{cases}$$

$$L_2(s) = s^{-n} \begin{cases} (1 + \ln \frac{1}{s})^{-\delta} & \text{when } s < 1 \\ (1 + \ln s)^{-\delta} & \text{when } s \geq 1 \end{cases}$$

$$L_3(s) = s^{-n} \begin{cases} (1 + \ln \frac{1}{s})^{-1}(1 + \ln(1 + \ln \frac{1}{s}))^{-\delta} & \text{when } s < 1 \\ (1 + \ln s)^{-1}(1 + \ln(1 + \ln s))^{-\delta} & \text{when } s \geq 1, \end{cases}$$

where $\gamma > 0$ and $\delta > 1$.

6. A different approach with a nonexplicit set of convergence

In this section we provide another proof of the a.e. convergence claimed in Theorem 2.1 which has the shortcoming that it does not relate the set of a.e. convergence with the Lebesgue set of f.

For simplicity we denote by B_M the closed ball $\overline{B(0, M)}$ for $M > 0$. We fix a locally integrable function f on $\mathbf{R}^n$ and also fix $M > 0$. We work with points x in the closed ball B_M for which $|f(x)| < \infty$ and $(|f| * |K_t|)(x) < \infty$ for all $0 < t \leq 1$. (Almost all points x satisfy $|f(x)| < \infty$, and in Section 4 it was shown that almost all points also satisfy $(|f| * |K_t|)(x) < \infty$ for all $0 < t \leq 1$.) For such points x we clearly have the estimate

$$\begin{aligned} |(f * K_t)(x) - c f(x)| &\leq \int_{|y| \leq 1} |f(x-y)|\,|K_t(y)|\,dy \\ &\quad + |c|\,|f(x)| \\ &\quad + \int_{|y| \geq 1} |f(x-y)|\,|K_t(y)|\,dy \\ &\quad + |f(x)| \int_{|y| \geq 1} |K_t(y)|\,dy. \end{aligned}$$

Taking the $\limsup$ as $t \to 0^+$ and using (4) and the fact that the integral of $|K|$ over the region $|y| \geq 1/t$ tends to zero as $t \to 0^+$ we obtain

$$\begin{aligned}
&\limsup_{t\to 0^+}|(f*K_t)(x)-cf(x)| \\
&\qquad \le \limsup_{t\to 0^+}\Big(\int_{\mathbf{R}^n}|f(x-y)|\chi_{|x-y|\le M+1}\,|K_t(y)|\,dy+|c|\,|f(x)|\Big)\\
&\qquad = \limsup_{t\to 0^+}(|f|\chi_{B_{M+1}}*|K_t|)(x)+|c|\,|f(x)|)\\
&\qquad \le \|L(|\cdot|)\|_{L^1}M(|f|\chi_{B_{M+1}})(x)+|c|\,|f(x)|\\
&\qquad \le C'\,M(|f|\chi_{B_{M+1}})(x).
\end{aligned}$$

Here $C'=\|L(|\cdot|)\|_{L^1}+|c|$, M is the Hardy-Littlewood maximal function, and we used the fact that $|g|*|K_t|$ is pointwise bounded by $M(g)$ times the L^1 norm of an integrable radially decreasing majorant of K (cf. [10, Theorem 2.1.10]). We also used that $|g|\le M(g)$ for any locally integrable function g.

Let $\varepsilon>0$. For our fixed f and M pick a smooth function φ with support inside B_{M+1} such that

$$\|f\chi_{B_{M+1}}-\varphi\|_{L^1}<\varepsilon. \tag{29}$$

We set

$$O_f(x)=\limsup_{t\to 0^+}|(f*K_t)(x)-cf(x)|$$

and we call O_f the oscillation of f. A simple argument shows that $O_\varphi=0$ everywhere and that $O_f=O_{f-\varphi}$ on the set on which $O_f<\infty$ (which is a set of full measure as observed earlier).

Then for a given $\delta>0$ we have

$$\begin{aligned}
\Big|\Big\{x\in B_M:\ O_f(x)>\delta\Big\}\Big| &= \Big|\Big\{x\in B_M:\ O_{f-\varphi}(x)>\delta\Big\}\Big|\\
&\le \Big|\Big\{x\in B_M:\ C'\,M(|f-\varphi|\chi_{B_{M+1}})(x)>\delta\Big\}\Big|\\
&\le \Big|\Big\{x\in \mathbf{R}^n:\ C'\,M(|f-\varphi|\chi_{B_{M+1}})(x)>\delta\Big\}\Big|\\
&\le \frac{3^nC'}{\delta}\|f\chi_{B_{M+1}}-\varphi\|_{L^1}
\end{aligned}$$

by the weak type $(1,1)$ property of the Hardy-Littlewood maximal operator (which holds with constant 3^n). Using (29) we obtain that

$$\Big|\Big\{x\in B_M:\ O_f(x)>\delta\Big\}\Big|\le\frac{3^nC'}{\delta}\varepsilon$$

and letting $\varepsilon\to 0$ we deduce

$$\Big|\Big\{x\in B_M:\ O_f(x)>\delta\Big\}\Big|=0$$

for any $\delta > 0$, hence $O_f = 0$ a.e. on B_M. Now letting $M \to \infty$ through the positive integers we conclude that $O_f = 0$ a.e. on $\mathbf{R}^n$. This yields the a.e. convergence of Theorem 2.1 but, unfortunately, there is no explicit relation to the Lebesgue set of f.

7. Almost everywhere convergence for multilinear convolutions

In this section we generalize Theorem 2.1 to multilinear convolutions. We say $f \in L^p_{\text{loc}}(\mathbf{R}^n)$ if only if $|f|^p \in L^1_{\text{loc}}(\mathbf{R}^n)$ for $1 \le p < \infty$. Also, $f \in L^\infty_{\text{loc}}(\mathbf{R}^n)$ means $\|\chi_K f\|_{L^\infty(\mathbf{R}^n)}$ is finite for any compact set $K \subseteq \mathbf{R}^n$. Notice as a consequence of Hölder's inequality $L^p_{\text{loc}}(\mathbf{R}^n) \subseteq L^1_{\text{loc}}(\mathbf{R}^n)$ for $1 \le p \le \infty$.

We review the notion of multilinear convolutions. Throughout this section we fix a positive integer $m \ge 2$. Suppose that $K(y_1, \dots, y_m)$ is a measurable function on $(\mathbf{R}^n)^m$, where each variable y_j lies in $\mathbf{R}^n$. Let f_j be measurable functions on $\mathbf{R}^n$. If for some $x \in \mathbf{R}^n$ the following integral

$$\int_{(\mathbf{R}^n)^m} |f_1(x-y_1)| \cdots |f_m(x-y_m)| \, |K(y_1, \dots, y_m)| \, dy_1 \cdots dy_m < \infty$$

converges absolutely, then we say that the multilinear convolution of K with the m-tuple $(f_1, \dots, f_m)$ exists at x and equals

$$\int_{(\mathbf{R}^n)^m} f_1(x-y_1) \cdots f_m(x-y_m) \, K(y_1, \dots, y_m) \, dy_1 \cdots dy_m.$$

For notational simplicity we write

$$\vec{y} = (y_1, \dots, y_m)$$

and

$$d\vec{y} = dy_1 \cdots dy_m,$$

and we introduce the tensor function

$$\otimes \vec{f}(\vec{y}) = (f_1 \otimes \cdots \otimes f_m)(y_1, \dots, y_m) = f_1(y_1) \cdots f_m(y_m)$$

for y_j on $\mathbf{R}^n$. The m-tuple $(f_1, \dots, f_m)$ of functions on $\mathbf{R}^n$ provides a function on $(\mathbf{R}^n)^m$. Then the multilinear convolution of K with the m-tuple $(f_1, \dots, f_m)$ at the point $x \in \mathbf{R}^n$ coincides with the regular convolution of $f_1 \otimes \cdots \otimes f_m$ with K at the point $(x, \dots, x) \in (\mathbf{R}^n)^m$.

The L^1 dilation of K is defined as

$$K_t(y_1, \dots, y_m) = \frac{1}{t^{mn}} K\Big(\frac{y_1}{t}, \dots, \frac{y_m}{t}\Big)$$

when $t > 0$.

For the sake of simplicity in notation we denote the multilinear convolution of K_t with the m-tuple $(f_1, \dots, f_m)$ at the point x by

$$(\otimes \vec{f} * K_t)(x).$$

Theorem 7.1. *Let K be a function on $(\mathbf{R}^n)^m$ and let $L : (0, \infty) \to [0, \infty)$ be a decreasing and piecewise absolutely continuous function. Assume that*

(A) $|K(x)| \le L(|x|)$ *for all* $x \in (\mathbf{R}^n)^m$.

(B) $L(|\cdot|)$ *lies in* $L^1((\mathbf{R}^n)^m)$.

Let $f_k \in L^{p_j}_{\text{loc}}(\mathbf{R}^n)$ such that $1 \le p_j \le \infty$ and $1 = \sum_{k=1}^m \frac{1}{p_j}$. Suppose that there is a set $E_{f_1,\dots,f_m}$ of measure zero such that for every $0 < \theta \le 1$ and $x \in \mathbf{R}^n \setminus E_{f_1,\dots,f_m}$ we have

$$\lim_{t \to 0^+} \int_{|\vec{y}| \ge \theta} |f_1(x - y_1)| \cdots |f_m(x - y_m)| \, |K_t(\vec{y})| \, d\vec{y} = 0. \tag{30}$$

Then there is another set of measure zero $D_{f_1,\dots,f_m}$ such that for each x in $\mathbf{R}^n \setminus (D_{f_1,\dots,f_m} \cup E_{f_1,\dots,f_m})$ there is a $t_x > 0$ such that

$$\int_{(\mathbf{R}^n)^m} |f_1(x - y_1) \cdots f_m(x - y_m)| \, |K_t(\vec{y})| \, d\vec{y} < \infty. \tag{31}$$

for all $0 < t < t_x$. Moreover, for all

$$x \in (\mathcal{L}_{f_1} \cap \cdots \cap \mathcal{L}_{f_m}) \setminus (D_{f_1,\dots,f_m} \cup E_{f_1,\dots,f_m})$$

we have

$$\lim_{t \to 0^+} (\otimes \vec{f} * K_t)(x) = c f_1(x) \cdots f_m(x), \tag{32}$$

where

$$c = \int_{(\mathbf{R}^n)^m} K(x) \, dx.$$

Proof. We fix $f_j \in L^{p_j}_{\text{loc}}(\mathbf{R}^n)$ for $j \in \{1, \dots, m\}$ as in the statement of the theorem. For every $N \in \mathbf{Z}^+$ we have

$$\begin{aligned}
&\int_{B_N} \Big[\int_{|\vec{y}\,|\le 1} |f_1(x-y_1)| \cdots |f_m(x-y_m)| L(|\vec{y}\,|)\, d\vec{y} \Big] dx \\
&= \int_{|\vec{y}\,|\le 1} L(|\vec{y}\,|) \int_{B_N} |f_1(x-y_1)| \cdots |f_m(x-y_m)|\, dx d\vec{y} \\
&\le \int_{|\vec{y}\,|\le 1} L(|\vec{y}\,|) \prod_{j=1}^m \Big(\int_{B_N} |f_j(x-y_j)|^{p_j} dx \Big)^{\frac{1}{p_j}} d\vec{y} \\
&\le \int_{|\vec{y}\,|\le 1} L(|\vec{y}\,|) d\vec{y} \prod_{j=1}^m \Big(\int_{B_{N+1}} |f_j(x)|^{p_j} dx \Big)^{\frac{1}{p_j}}
\end{aligned}$$

which is a finite quantity since $f_j \in L^{p_j}_{\text{loc}}(\mathbf{R}^n)$ and L is integrable; here we do the obvious modification if $p_j = \infty$. Since L is decreasing it follows that for every $N \in \mathbf{Z}^+$, there is a set of measure zero D_N in B_N such that

$$x \in B_N \backslash D_N \implies \int_{|\vec{y}\,|\le 1} |f_1(x-y_1)| \cdots |f_m(x-y_m)| \frac{1}{t^{mn}} L\Big(\frac{|\vec{y}\,|}{t}\Big)\, d\vec{y} < \infty$$

for all $0 < t \le 1$. Since we are considering $t \to 0^+$ we may suppose for the rest of the proof that $0 < t \le 1$. Setting

$$D_{f_1,\dots,f_m} = \bigcup_{N=1}^{\infty} D_N,$$

then for all $x \in \mathbf{R}^n \setminus D_{f_1,\dots,f_m}$ we have

$$\int_{|\vec{y}\,|\le 1} |f_1(x-y_1)| \cdots |f_m(x-y_m)| \frac{1}{t^{mn}} L\Big(\frac{|\vec{y}\,|}{t}\Big)\, d\vec{y} < \infty \tag{33}$$

and the same is true with $|K_t(\vec{y}\,)|$ in place of $\frac{1}{t^{mn}} L(\frac{|\vec{y}\,|}{t})$.

Now for a given $x \in \mathbf{R}^n \setminus E_{f_1,\dots,f_m}$, (30) with $\theta = 1$ gives that for some $t_{x,1} > 0$ and all t satisfying $0 < t < t_{x,1}$ we have

$$\int_{|\vec{y}\,|\ge 1} |f_1(x-y_1)| \cdots |f_m(x-y_m)|\, |K_t(\vec{y}\,)|\, d\vec{y} < 100. \tag{34}$$

Combining this fact with (33) we obtain that for

$$x \in \mathbf{R}^n \setminus (D_{f_1,\dots,f_m} \cup E_{f_1,\dots,f_m})$$

and $0 < t < \min\{1, t_{x,1}\}$ we have

$$\int_{(\mathbf{R}^n)^m} |f_1(x-y_1)| \cdots |f_m(x-y_m)|\, |K_t(\vec{y}\,)|\, d\vec{y} < \infty.$$

This yields (31) but also yields the slightly stronger estimate

$$\int_{|\vec{y}|\le 1} |f_1(x-y_1)|\cdots|f_m(x-y_m)|\,\frac{1}{t^n}L\Big(\frac{|\vec{y}\,|}{t}\Big)\,d\vec{y}<\infty \tag{35}$$

for all $0<t\le 1$ whenever $x\in\mathbf{R}^n\setminus(D_{f_1,\dots,f_m}\cup E_{f_1,\dots,f_m})$.

We now fix a point

$$x_0\in(\mathcal{L}_{f_1}\cap\cdots\cap\mathcal{L}_{f_m})\setminus(D_{f_1,\dots,f_m}\cup E_{f_1,\dots,f_m}).$$

We will prove (32) for $x=x_0$.

Denote

$$I_L=\int_{(\mathbf{R}^n)^m} L(|\vec{y}\,|)\,d\vec{y}<\infty.$$

Let $\varepsilon>0$. Without harm assume that $\varepsilon<1$. As $x_0\in\mathcal{L}_{f_1}\cap\cdots\cap\mathcal{L}_{f_m}$, there is a $\delta_0\in(0,1)$ such that

$$0<r\le\delta_0 \implies \frac{1}{v_n r^n}\int_{|y|<r}|f_j(x_0-y)-f_j(x_0)|\,dy<\varepsilon. \tag{36}$$

Now we use the identity

$$a_1a_2\cdots a_m-b_1b_2\cdots b_m=\sum_{i=1}^m b_1\cdots b_{i-1}(a_i-b_i)a_{i+1}\cdots a_m$$

(with the obvious modification when $i=1$ or $i=m$) to estimate

$$\frac{1}{(v_nr^n)^m}\int_{|y_1|<r}\cdots\int_{|y_m|<r}\Big|\prod_{j=1}^m f_j(x_0-y_j)-\prod_{j=1}^m f_j(x_0)\Big|d\vec{y} \tag{37}$$

by

$$\sum_{i=1}^m\Bigg[\prod_{j=1}^{i-1}\frac{1}{v_nr^n}\int_{|y_j|<r}|f_j(x_0-y_j)|dy_j\Bigg]\\ \Bigg[\frac{1}{v_nr^n}\int_{|y_i|<r}|f_i(x_0-y_i)-f_i(x_0)|\,dy_i\Bigg]\Bigg[\prod_{j=i+1}^m|f_i(x_0)|\Bigg].$$

But the preceding expression is bounded by

$$\varepsilon\sum_{i=1}^m\Bigg[\prod_{j=1}^{i-1}\big(|f_j(x_0)|+\varepsilon\big)\Bigg]\Bigg[\prod_{j=i+1}^m|f_i(x_0)|\Bigg]$$

when $r<\delta_0$ in view of (36). So we proved that when $0<r<\delta_0$ we have

$$\frac{1}{(v_nr^n)^m}\int_{|y_1|<r}\cdots\int_{|y_m|<r}\Big|\prod_{j=1}^m f_j(x_0-y_j)-\prod_{j=1}^m f_j(x_0)\Big|d\vec{y}\le\varepsilon\,C_{f_1,\dots,f_m}(x_0)$$

with

$$C_{f_1,\dots,f_m}(x_0) = \sum_{i=1}^{m} \Big[\prod_{j=1}^{i-1} \big(|f_j(x_0)| + 1\big) \Big] \Big[\prod_{j=i+1}^{m} |f_i(x_0)| \Big].$$

For $t > 0$ and $t < \min\{1, t_{x_0,1}\}$ we write

$$\begin{aligned}
&\Big|(\otimes \vec{f} * K_t)(x_0) - c f_1(x_0) \cdots f_m(x_0)\Big| \\
&\qquad = \Big| \int_{(\mathbf{R}^n)^m} \prod_{j=1}^{m} f_j(x_0 - y_j) K_t(\vec{y}\,)\, d\vec{y} - \Big(\int_{(\mathbf{R}^n)^m} K_t(\vec{y}\,) d\vec{y} \Big) \prod_{j=1}^{m} f_j(x_0) \Big| \\
&\qquad \le \int_{(\mathbf{R}^n)^m} \Big| \prod_{j=1}^{m} f_j(x_0 - y_j) - \prod_{j=1}^{m} f_j(x_0) \Big| \, |K_t(\vec{y}\,)|\, d\vec{y} \\
&\qquad \le \int_{|\vec{y}\,| \ge \delta_0} \Big| \prod_{j=1}^{m} f_j(x_0 - y_j) - \prod_{j=1}^{m} f_j(x_0) \Big| \, |K_t(\vec{y}\,)|\, d\vec{y} \qquad (38) \\
&\qquad\qquad + \int_{|\vec{y}\,| < \delta_0} \Big| \prod_{j=1}^{m} f_j(x_0 - y_j) - \prod_{j=1}^{m} f_j(x_0) \Big| \, |K_t(\vec{y}\,)|\, d\vec{y}. \qquad (39)
\end{aligned}$$

To estimate (38) we write

$$\begin{aligned}
&\int_{|\vec{y}\,| \ge \delta_0} \Big| \prod_{j=1}^{m} f_j(x_0 - y_j) - \prod_{j=1}^{m} f_j(x_0) \Big| \, |K_t(\vec{y}\,)|\, d\vec{y} \\
&\qquad \le \int_{|\vec{y}\,| \ge \delta_0} \Big(\prod_{j=1}^{m} \big| f_j(x_0 - y_j) \big| \Big) |K_t(\vec{y}\,)|\, d\vec{y} \\
&\qquad\qquad + \Big(\prod_{j=1}^{m} \big| f_j(x_0) \big| \Big) \int_{|\vec{y}\,| \ge \delta_0/t} |K(\vec{y}\,)|\, d\vec{y}.
\end{aligned}$$

By assumption (30) there is a positive constant t_{x_0,δ_0} such that for all t satisfying $0 < t < t_{x_0,\delta_0}$ we have

$$\int_{|\vec{y}\,| \ge \delta_0} \Big(\prod_{j=1}^{m} \big| f_j(x_0 - y_j) \big| \Big) |K_t(\vec{y}\,)|\, d\vec{y} < \varepsilon.$$

Moreover there is a $t^*_{x_0,\delta_0} > 0$ such that for $0 < t < t^*_{x_0,\delta_0}$ we have

$$\int_{|\vec{y}\,| \ge \delta_0} \Big(\prod_{j=1}^{m} \big| f_j(x_0 - y_j) \big| \Big) |K_t(\vec{y}\,)|\, d\vec{y} < \varepsilon.$$

Combining these facts we obtain that

$$\int_{|\vec{y}|\geq\delta_0}\Big|\prod_{j=1}^m f_j(x_0-y_j)-\prod_{j=1}^m f_j(x_0)\Big|\,|K_t(\vec{y})|\,d\vec{y}\leq 2\varepsilon \tag{40}$$

whenever

$$0<t<\min\big\{t_{x_0,1},t_{x_0,\delta_0},t^*_{x_0,\delta_0}\big\}. \tag{41}$$

We now examine (39). For every $r>0$ we use polar coordinates to write

$$\begin{aligned}\int_{|\vec{y}|<r}&\Big|\prod_{j=1}^m f_j(x_0-y_j)-\prod_{j=1}^m f_j(x_0)\Big|d\vec{y}\\ &=\int_0^r\rho^{mn-1}\int_{\mathbf{S}^{mn-1}}\Big|\prod_{j=1}^m f_j(x_0-\rho\theta_j)-\prod_{j=1}^m f_j(x_0)\Big|d\vec{\theta}\,d\rho\\ &=\int_0^r F(\rho)\,d\rho,\end{aligned}$$

where we set

$$F(\rho)=\rho^{mn-1}\int_{\mathbf{S}^{mn-1}}\Big|\prod_{j=1}^m f_j(x_0-\rho\theta_j)-\prod_{j=1}^m f_j(x_0)\Big|d\vec{\theta}.$$

By Fubini's theorem, F is defined for almost every $\rho>0$.

At this point we treat (39) in a way that is completely analogous to that (21) was handled. By the same reasoning (based on the identity (11)) we obtain that for all $t>0$, when $r<\delta_0$ we have

$$\int_{|\vec{y}|<\delta_0}\Big|\prod_{j=1}^m f_j(x_0-y_j)-\prod_{j=1}^m f_j(x_0)\Big|\,|K_t(\vec{y})|\,d\vec{y}\leq\varepsilon\,C_{f_1,\dots,f_m}(x_0)I_L,$$

where $I_L=\|L(|\cdot|)\|_{L^1((\mathbf{R}^n)^m)}$. Combining this inequality with (40) we finally obtain

$$\Big|(\otimes\vec{f}*K_t)(x_0)-c\prod_{j=1}^m f_j(x_0)\Big|<\big(2+C_{f_1,\dots,f_m}(x_0)I_L\big)\varepsilon$$

whenever (41) is valid. This proves (32). □

8. Examples

We now consider examples of functions f that may grow at infinity for which Theorem 2.1 applies.

Example 8.1. *Let* $0 < \gamma < n$, $|f(x)| \le C(1+|x|)^\tau$ *for* $0 \le \tau < \gamma$ *and* $K(x) = |x|^{-n}\min(|x|^\gamma, |x|^{-\gamma})$.

In this case the observation in Remark 5.3 applies. Then notice that

$$\int_{\mathbf{R}^n} \frac{|f(y)|}{(1+|y|)^{n+\gamma}} dy \le \int_{\mathbf{R}^n} \frac{C(1+|x|)^\tau}{(1+|y|)^{n+\gamma}} dy < \infty,$$

since $\tau < \gamma$, *so condition* (4) *is valid.*

A direct proof of condition (4) *can also be given by changing variables. Then matters reduce to showing that*

$$\int_{|y|\ge\theta/t} (1+|x-ty|)^\tau \, |y|^{-n}\min(|y|^\gamma, |y|^{-\gamma})\, dy \tag{42}$$

tend to zero as $t \to 0^+$. *But for* $t < 1$ *we have*

$$(1+|x-ty|)^\tau \le (1+|x|)^\tau(1+|y|)^\tau,$$

so inserting this in (42) *we obtain the tail of a convergent integral which tends to zero; thus* (4) *is valid in this case.*

Example 8.2. *Let* $|f(x)| \le Ce^{|x|^p}$ *for* $0 \le p < q < \infty$ *and* $K(x) = e^{-\pi|x|^q}$. *We only verify condition* (4). *Let* $\theta > 0$. *By changing variables matters reduce to showing that*

$$\int_{|y|\ge\theta/t} e^{|x-ty|^p}\, e^{-\pi|y|^q} dy \tag{43}$$

tend to zero as $t \to 0^+$. *But this assertion is valid, since for* $t < 1$

$$|x-ty|^p \le c_p(|x|^p + |ty|^p) \le c_p(|x|^p + |y|^p).$$

Inserting this estimate in (43) *and using that* $p < q$, *we obtain that* (43) *tends to zero as* $t \to 0^+$, *being the tails of an integrable function. This yields that* (4) *is satisfied for all* $x \in \mathbf{R}^n$.

We note that in both Examples 8.1 and 8.2 the a.e. convergence is in fact everywhere convergence as the sets E_f and D_f are empty and $\mathcal{L}_f$ is $\mathbf{R}^n$.

Example 8.3. *Many texts discussing approximate identities (for instance* [17], [10]*), such as the one in* (6), *prove pointwise convergence for points at which the underlying function* f *is continuous. In this example we apply Theorem 2.1 to a function that is not continuous at any point.*

Let $\{a_k\}$ *be a positive sequence such that* $\sum_{k=1}^\infty a_k < \infty$. *Define*

$$g(x) = \sum_{j=1}^\infty \frac{a_j}{|x-r_j|^{\frac{1}{2}}} \chi_{[0,1]}(x),$$

where $\{r_j\}$ *is an enumeration of* $\mathbb{Q}\bigcap[0,1]$*. Note that* g *is integrable by Lebesgue monotone convergence theorem (hence finite a.e.), is not continuous at any point in* $[0,1]$ *and is unbounded on every interval of* $[0,1]$*. Now define*

$$f(x)=\sum_{k\in\mathbb{Z}} g(x-k), \qquad x\in\mathbb{R}$$

which provides a periodic extension of g *to* $\mathbb{R}$*. Let* K *be a positive compactly supported function as stated in Theorem 2.1. By Remark 5.2 we have* $E_f=\emptyset$*. We now determine* D_f*. Let* $x\in\mathcal{L}_f$ *and* k' *be the largest integer less than or equal to* x*. Observe,*

$$\begin{aligned}&\int_{|y|\le 1} f(x-y)K_t(y)dy\\ &\quad\le \sum_{k=k'-1}^{k'+1}\sum_{j=1}^{\infty} a_j\int_{|y|\le 3}|x-k-y-r_j|^{-\frac{1}{2}}\chi_{[0,1]}(x-k-y)K_t(y)dy,\end{aligned}$$

where we extended the domain of integration to $|y|\le 3$ *so that every summand over* j *and* k *has an integral with a singularity; to avoid considering cases. Moving forward we only consider the summands relating to* $k=k'$*, as the others follow by the same argument. Thus we continue by bounding this summand,*

$$\begin{aligned}&\sum_{j=1}^{\infty} a_j\int_{|y|\le 3}|x-k'-y-r_j|^{-\frac{1}{2}}K_t(y)dy\\ &\quad\le \sum_{j=1}^{\infty} a_j\Big(\int_{|y|\le 3}|x-k'-y-r_j|^{-\frac{3}{4}}\,dy\Big)^{\frac{2}{3}}\|K_t\|_{L^3(\mathbb{R})}\\ &\quad\le \sum_{j=1}^{\infty} a_j\Big(\int_{|y|\le 5}|y|^{-\frac{3}{4}}\,dy\Big)^{\frac{2}{3}}\|K_t\|_{L^3(\mathbb{R})}\end{aligned}$$

which is finite as K *has compact support, and the series is finite. It follows that* $D_f=\emptyset$ *as well, thus* (6) *holds on* $\mathcal{L}_f$*.*

Example 8.4. *We consider a bilinear convolution which resembles the previous example. Define the functions*

$$h_1(x)=\sum_{k\in\mathbf{Z}}|x-k|^{-1/3}\chi_{|x-k|\le 1/2}$$

and

$$h_2(x)=\sum_{k\in\mathbf{Z}}\big|x-k-1/2\big|^{-1/3}\chi_{|x-k-1/2|\le 1/2}$$

on the real line. Note that $h_1, h_2 \in L^2_{\mathrm{loc}}(\mathbf{R})$*. The Lebesgue sets of* h_1 *and* h_2 *are* $\mathcal{L}_{h_1} = \mathbf{R} \setminus \mathbf{Z}$ *and* $\mathcal{L}_{h_2} = \mathbf{R} \setminus (\frac{1}{2} + \mathbf{Z})$*, as these are the points of continuity of the respective functions.*

Consider the function on $\mathbb{R}^2$ *given by*

$$K(x) = |x|^{-4/3}\chi_{|x|\leq 1}.$$

Then $(h_1 \otimes h_2) * K_t(x, x)$ *converges to* $h_1(x)h_2(x)$ *for* $x \in \mathcal{L}_{h_1} \cap \mathcal{L}_{h_2}$ *as* $t \to 0^+$*. To see this first note that* $E_f = \emptyset$ *again due to the compact support of* K*. Let* $x \in \mathcal{L}_{h_1} \cap \mathcal{L}_{h_2}$ *and* k' *be the unique integer such that* $|x - k'| < 1/2$*. Observe,*

$$\begin{aligned}\int_{|\vec{y}|\leq 1} & h_1(x-y_1)h_2(x-y_2)K_t(y_1,y_2)dy_1dy_2 \\ \leq \sum_{k=k'-1}^{k'+1}\sum_{j=k'-2}^{k'+1} & \int_{|y_1|\leq 2}\int_{|y_2|\leq 3} |x-k-y_1|^{-\frac{1}{3}} \\ & \times |x-j-1/2-y_2|^{-\frac{1}{3}} t^{-2}(|y_1|/t + |y_2|/t)^{-\frac{4}{3}} dy_2 dy_1\end{aligned}$$

where we extended the range of integration so that for each summand over j *and* k *the integral has a singularity. Now without loss of generality we only consider the summand where* $k = j = k'$*, which is bounded by*

$$\begin{aligned}\int_{|y_1|\leq 2} & |x-k'-y_1|^{-\frac{1}{3}} t^{-1} |y_1/t|^{-\frac{2}{3}} dy_1 \\ & \times \int_{|y_2|\leq 3} |x-k'-1/2-y_2|^{-\frac{1}{3}} t^{-1}|y_2/t|^{-\frac{2}{3}} dy_2. \end{aligned} \tag{44}$$

The first integral in (44) *is finite by considering* y_1 *near zero and* y_1 *near* $x - k'$*; the only potential problem is when* $x - k'$ *is zero, but this is not possible as* $x \in \mathcal{L}_{h_1} = \mathbf{R} \setminus \mathbf{Z}$*. The second integral follows for the same reason.*

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