MAXIMAL BILINEAR SINGULAR INTEGRAL OPERATORS ASSOCIATED WITH DILATIONS OF PLANAR SETS

GEOFF DIESTEL AND LOUKAS GRAFAKOS

Abstract. We obtain square function estimates and bounds for maximal singular integral operators associated with bilinear multipliers given by characteristic functions of dyadic dilations of certain planar sets. As as consequence, we deduce pointwise almost everywhere convergence for lacunary partial sums of bilinear Fourier series with respect to methods of summation determined by the corresponding planar sets.

1. Introduction

We denote by $\widehat{f}(\xi) = \int f(x)e^{2\pi i\xi\cdot x}dx$ the Fourier transform of a function f on $\mathbb{R}$. We will use the notation $f^\vee(\xi) = \widehat{f}(-\xi)$ for the inverse Fourier transform of f. If $A \subset \mathbb{R}^2$, we denote by χ_A the characteristic function of the set A and by nA the set $\{nx : \ x \in A\}$. For a bounded function m, T_m will denote the bilinear operator with multiplier m, i.e. the operator

$$T_m(f,g)(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{f}(\xi)\widehat{g}(\eta)m(\xi,\eta)e^{2\pi i x(\xi+\eta)}d\xi\, d\eta\,, \qquad x \in \mathbb{R}. \tag{1.1}$$

Bilinear transference results (see Fan and Sato [4], Blasco and Villaroya [1], and Grafakos and Honzík [6]), relate the issue of norm (resp. almost everywhere) convergence as $n \to \infty$ of the bilinear Fourier series

$$\sum_{(j,k)\in(nA)\cap\mathbb{Z}^2} \widehat{f}(j)\widehat{g}(k)e^{2\pi i(j+k)x} \tag{1.2}$$

for 1-periodic functions f and g on the line, to boundedness properties of the bilinear multiplier operator (resp. maximal bilinear multiplier operator) with multiplier χ_A; here A is an open subset of $\mathbb{R}^2$ that contains the origin. An example of interest is given by the quadrilateral

$$A = \{(\xi,\eta) \in \mathbb{R}^2 : \ |\xi + \alpha\eta| < 1, |\eta - \xi| < 1\}, \qquad \alpha \in \mathbb{R} \setminus \{1,-1\},$$

in which case (1.2) converges in $L^r([0,1])$ for functions $f \in L^p([0,1])$ and $g \in L^q([0,1])$ when $1 < p,q \le \infty$ and $0 < 1/r = 1/p + 1/q < 3/2$, as the operator T_m in (1.1) with $m = \chi_A$ is bounded from $L^p(\mathbb{R}) \times L^q(\mathbb{R}) \to L^r(\mathbb{R})$ for this range of p,q,r. This is a consequence of the boundedness of the bilinear Hilbert transform on the line, established by Lacey and Thiele [10], [11]. Another interesting example is provided by the unit disc

$$A = \{(\xi,\eta) \in \mathbb{R}^2 : \ \xi^2 + \eta^2 < 1\},$$

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for which the series in (1.2) converges in $L^r([0,1])$ for $f \in L^p([0,1])$ and $g \in L^q([0,1])$ when $2 \le p, q < \infty$ and $1/2 \le 1/p + 1/q < 1$, as the associated bilinear multiplier χ_A was proved by Grafakos and Li [7] to be bounded from $L^p(\mathbb{R}) \times L^q(\mathbb{R}) \to L^r(\mathbb{R})$ for these indices.

When $A = P \subset \mathbb{R}^2$ is an open polygon containing the origin, the pointwise almost everywhere convergence of the bilinear series in (1.2) can be deduced from a theorem of Muscalu, Tao and Thiele [12] on the maximal bilinear multiplier with symbol the characteristic function of a polygon (a consequence of the boundedness of the bi-Carleson operator). If any part of the boundary of P is parallel to the anti-diagonal $\xi + \eta = 0$, the pointwise almost everywhere convergence holds for $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$, $1 < p, q \le \infty$ and $1/p + 1/q < 1$. Otherwise, the indices can be extended to the range $1/p + 1/q < 3/2$.

Motivated by the study of the convergence of bilinear Fourier series with respect to various types of summation, in this article, we obtain bounds for dyadic (bilinear) maximal operators associated with dilations of certain planar sets. A useful tool in the study of the boundedness of these operators are bilinear square functions, analogs of the classical (linear) Littlewood-Paley square functions, that measure the orthogonality of bilinear operators on certain L^p spaces. The results of this paper can be summarized as follows: In section 2 we discuss some preliminary orthogonality results concerning bilinear operators. In section 3, we give a simple proof of the boundedness of the maximal bilinear multiplier operator formed by lacunary dilations of a polygon in $\mathbb{R}^2$. Our proof avoids using the results in [12] and is based on a vector-valued estimate found in [3] and a geometric observation about dilations of planar polygons. In section 4, we prove estimates for sums of signed bilinear operators with multipliers dyadic dilates of planar polygons, *uniformly* in all choices of signs. In section 5, we give lacunary pointwise almost everywhere convergence and Littlewood-Paley type results in the local L^2 case for planar sets with a smooth boundary; we focus our attention to a specific figure of a square with rounded corners. These results are obtained using the orthogonality estimates in section 2.

2. Preliminary Results

The following vector-valued result provides a multilinear version of a classical theorem of Marcinkiewicz and Zygmund and can be found in Grafakos and Martell [8].

Proposition 1. *Suppose* $0 < p, q, r < \infty$, $1/p + 1/q = 1/r$ *and let*

$$T : L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$$

be a bounded bilinear operator. Then T admits ℓ^2-valued extension. This means that there is a constant $C(p,q) < \infty$ such that for all sequences $f_k \in L^p(\mathbb{R})$ and $g_j \in L^q(\mathbb{R})$ we have

$$\left\| \Big(\sum_k \sum_j \left| T(f_k, g_j) \right|^2 \Big)^{1/2} \right\|_r \le C(p,q) \left\| \Big(\sum_k |f_k|^2 \Big)^{1/2} \right\|_p \left\| \Big(\sum_j |g_j|^2 \Big)^{1/2} \right\|_q \tag{2.3}$$

and in particular

$$\left\| \Big(\sum_j \left| T(f_j, g_j) \right|^2 \Big)^{1/2} \right\|_r \le C(p,q) \left\| \Big(\sum_j |f_j|^2 \Big)^{1/2} \right\|_p \left\| \Big(\sum_j |g_j|^2 \Big)^{1/2} \right\|_q . \tag{2.4}$$

The following corollary was proved in Diestel [3]; it is a consequence of Proposition 1.

Corollary 1. *Let T be a bounded bilinear operator with symbol $m(\xi, \eta)$. If $T_{j,k}$ has symbol $m(\xi - c_j, \eta - d_k)$ for real sequences $\{c_j\}_{j \in \mathbf{Z}}$ and $\{d_k\}_{k \in \mathbf{Z}}$, then there is a constant $C(p,q) < \infty$*

such that

$$\Big\| \Big(\sum_k \sum_j |T_{j,k}(f_k, g_j)|^2 \Big)^{1/2} \Big\|_r \le C(p,q) \Big\| \Big(\sum_k |f_k|^2 \Big)^{1/2} \Big\|_p \Big\| \Big(\sum_j |g_j|^2 \Big)^{1/2} \Big\|_q . \tag{2.5}$$

Moreover, if supp $(m(\xi - c_j, \eta - d_k)) \subset [2^j, 2^{j+1}] \times [2^k, 2^{k+1}]$ *and* $1 < p, q < \infty$ *such that* $1/p + 1/q = 1/r < 2$*, then*

$$\Big\| \Big(\sum_k \sum_j |T_{j,k}(f, g)|^2 \Big)^{1/2} \Big\|_r \le C_{p,q} \|f\|_p \|g\|_q . \tag{2.6}$$

We will also make use of the following orthogonality lemma for the local L^2 case, i.e. the case when $2 \le p, q < \infty$ and $1 < r \le 2$. In the sequel all intervals will be finite. Recall that the Littlewood-Paley operators Δ_j associated with a family of intervals I_j are the operators $\Delta_j(f) = (\widehat{f}\chi_{I_j})^{\vee}$.

Lemma 1. *Let* $2 \le p, q < \infty$, $1 \le r \le 2$ *and* $1/p + 1/q = 1/r$*. Suppose that* $\{T_j\}_{j \in \mathbb{Z}}$ *is a family of uniformly bounded bilinear operators mapping* $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ *into* $L^r(\mathbb{R})$ *such that*

$$T_j(f, g)(x) = T_j(\Delta_j^1(f), \Delta_j^2(g)) \tag{2.7}$$

where Δ_j^1 *are Littlewood-Paley operators associated with a family of pairwise disjoint intervals* $\{A_j\}_j$ *and* Δ_j^2 *are Littlewood-Paley operators associated with a family of pairwise disjoint intervals* $\{B_j\}_j$*. Then, there is a constant* $C(p,q) < \infty$ *such that*

$$\Big\| \Big(\sum_j |T_j(f, g)|^2 \Big)^{1/2} \Big\|_r \le C(p,q) \|f\|_q \|g\|_q \tag{2.8}$$

for all $f \in L^p(\mathbb{R})$ *and* $g \in L^q(\mathbb{R})$*.*

Proof.

$$\begin{aligned}
& \Big\| \Big(\sum_j |T_j(f, g)|^2 \Big)^{1/2} \Big\|_r^r \\
= \;& \int_{\mathbb{R}} \Big(\sum_j |T_j(\Delta_j^1(f), \Delta_j^1(g))(x)|^2 \Big)^{r/2} dx \\
\le \;& \sum_j \int_{\mathbb{R}} |T_j(f, g)(x)|^r dx, \quad (r \le 2) \\
\le \;& C(p,q) \sum_j \|\Delta_j^1(f)\|_p^r \|\Delta_j^2(g)\|_q^r \\
\le \;& C(p,q) \Big(\sum_j \|\Delta_j^1(f)\|_p^p \Big)^{r/p} \Big(\sum_j \|\Delta_j^2(g)\|_q^q \Big)^{r/q} \\
\le \;& C(p,q) \Big\| \Big(\sum_j |\Delta_j^1(f)|^2 \Big)^{1/2} \Big\|_p^r \Big\| \Big(\sum_j |\Delta_j^2(g)|^2 \Big)^{1/2} \Big\|_q^r, \quad (p, q \ge 2) \\
\le \;& C(p,q) \|f\|_p \|g\|_q, \quad \text{(by Rubio de Francia's theorem [13]).}
\end{aligned}$$

□

3. Square Function Estimates and Dyadic Pointwise Convergence I

Suppose that P is a planar polygon. It is an easy geometric observation that there exists a positive integer M so that for $i = 1, 2, \cdots, M$ there are one-sided cones C_i and an appropriate choice of signs $\epsilon_i = \pm 1$ such that

$$\chi_P(\xi, \eta) = \sum_{i=1}^{M} \epsilon_i \chi_{C_i}(\xi, \eta).$$

So to study the characteristic function of a planar polygon as a bilinear multiplier, it will suffice to study multipliers given by characteristic functions of one-sided planar cones. A very interesting geometric property that cones possess is the following: *the dilation (about the origin) of a one-sided cone is another one-side cone that can be obtained from the first one by a translation.* Therefore, we can express the characteristic function of the dilation nP of the planar polygon P as

$$\chi_{nP}(\xi, \eta) = \sum_{i=1}^{M} \epsilon_i \chi_{C_i}(\xi - a_i(n), \eta - b_i(n))$$

for some choice of real numbers $a_i(n)$ and $b_i(n)$ that naturally depend on n.

Figure 1 illlustrates the dilation of a triangle P whose characteristic function can be written as the sum of the characteristic functions of two cones minus the characteristic function of another cone.

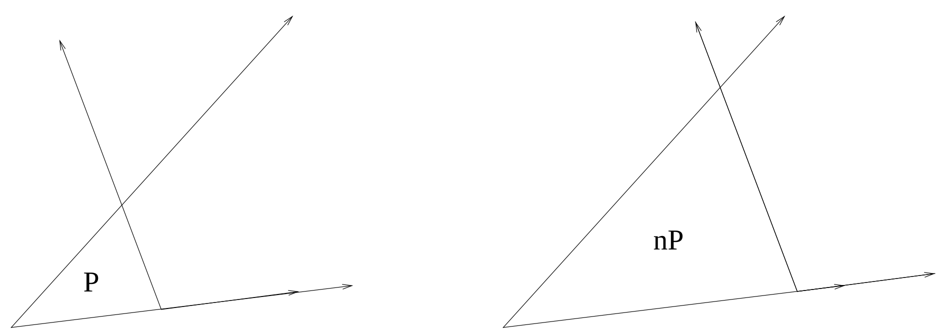

Figure 1. Dilation of a triangle via translations of cones

The idea of identifying dilations with translations of an appropriate set of cones for bilinear polygonal multipliers leads to a lacunary pointwise convergence theorem for bilinear Fourier series associated with lattice points contained in dyadic dilates of polygons. This will be a consequence of bilinear transference and of the following result.

Theorem 1. *Let $1 < p, q < \infty$ and $1/p + 1/q = 1/r < 3/2$. Suppose P is a polygon with no sides parallel to the anti-diagonal $\xi + \eta = 0$ such that $[-1, 1]^2 \subset P \subset [-2, 2]^2$. Then, for all $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ we have the bound*

$$\Big\| \sup_j \big| T_{\chi_{2^j P}}(f, g) \big| \Big\|_r \le C(p, q, P) \|f\|_p \|g\|_q.$$

As a consequence we obtain for $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ that

$$\lim_{n \to \infty} T_{\chi_{2^n P}}(f, g)(x) = f(x)g(x)$$

for almost all $x \in \mathbb{R}$. If $r > 1$, P may have sides parallel to the anti-diagonal $\xi + \eta = 0$.

Proof. Since

$$\sup_j \left|T_{\chi_{[-2^j,2^j]^2}}(f,g)(x)\right| \le \sup_j \left|(\widehat{f}\chi_{[-2^j,2^j]})^\vee(x)\right| \sup_k \left|(\widehat{g}\chi_{[-2^k,2^k]})^\vee(x)\right|,$$

the Carleson-Hunt theorem [2], [9] and Hölder's inequality imply that

$$\left\|\sup_j \left|T_{\chi_{[-2^j,2^j]^2}}(f,g)\right|\right\|_r \le C_0(p,q)\|f\|_p\|g\|_q.$$

Therefore, it suffices to show that

$$\left\|\sup_j \left|(T_{\chi_{2^jP}} - T_{\chi_{[-2^j,2^j]^2}})(f,g)\right|\right\|_r \le C(p,q,M)\|f\|_p\|g\|_q.$$

We can split the operator in question as a sum of four parts as follows:

$$T_{\chi_{2^jP}} - T_{\chi_{[-2^j,2^j]^2}} = \sum_{i=1}^{4} T_j^i,$$

where the multiplier of T_j^i is m_j^i defined below:

$$\begin{aligned}
m_j^1(\xi,\eta) &= \chi_{2^jP}(\xi,\eta)\chi_{[2^j,2^{j+1}]}(\xi)\chi_{[-2^j,2^{j+1}]}(\eta)\\
m_j^2(\xi,\eta) &= \chi_{2^jP}(\xi,\eta)\chi_{[-2^{j+1},2^j]}(\xi)\chi_{[2^j,2^{j+1}]}(\eta)\\
m_j^3(\xi,\eta) &= \chi_{2^jP}(\xi,\eta)\chi_{[-2^{j+1},-2^j]}(\xi)\chi_{[-2^{j+1},2^j]}(\eta)\\
m_j^4(\xi,\eta) &= \chi_{2^jP}(\xi,\eta)\chi_{[-2^j,2^{j+1}]}(\xi)\chi_{[-2^{j+1},-2^j]}(\eta).
\end{aligned}$$

See Figure 2 for a geometric description of this decomposition.

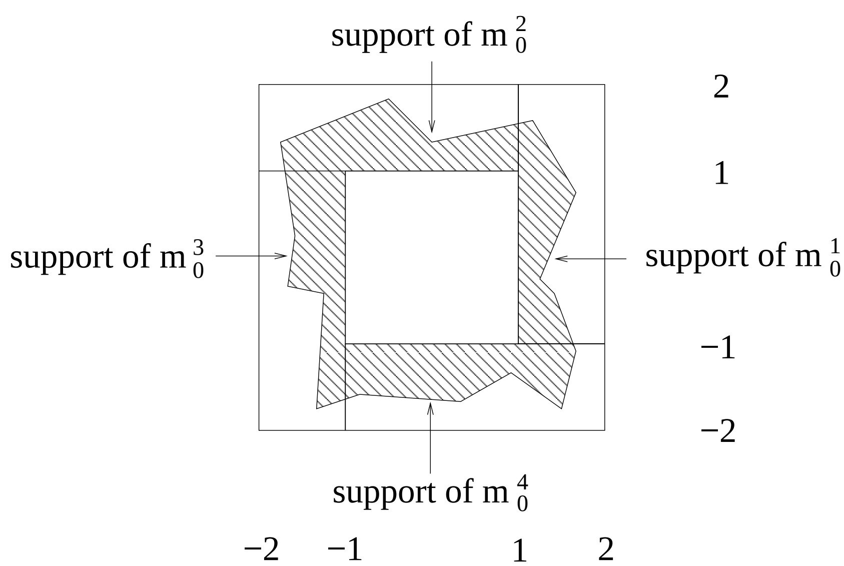

FIGURE 2. Decomposition of $\chi_P - \chi_{[-1,1]^2}$

Now, we must show that

$$\left\|\sup_j \left|T_j^i(f,g)\right|\right\|_r \le C_i(p,q)\|f\|_p\|g\|_q$$

for each i. Since the arguments for different i's are similar, we only consider $i = 1$. Notice that the m_j^1's are characteristic functions of dyadic dilations of a fixed polygon and hence (by the discussion in the introduction of this section) finite sums of characteristic functions of translations of fixed one-sided cones by amounts depending on j. By the results of Lacey and Thiele [10], [11], characteristic functions of one-sided cones are bounded bilinear multipliers. Moreover, letting $\Delta_j(f) = (\widehat{f}\chi_{[2^j,2^{j+1}]})^\vee$, it follows that

$$T_j^1(f,g)(x) = T_j^1(\Delta_j(f),g)(x).$$

Let $g_0 = g$ and $g_k = 0$ for all $k \neq 0$. Now, letting $T_{j,k}^1 = T_j^1$ for all k, we have set up matters in the framework of Corollary 1.

$$\begin{aligned}
&\Big\| \sup_j \big|T_j^1(f,g)\big| \Big\|_r \\
\leq \quad & \Big\| \Big(\sum_j \big|T_j^1(f,g)\big|^2 \Big)^{1/2} \Big\|_r \\
= \quad & \Big\| \Big(\sum_j \big|T_j^1(\Delta_j(f),g)\big|^2 \Big)^{1/2} \Big\|_r \\
= \quad & \Big\| \Big(\sum_j \sum_k \big|T_{j,k}^1(\Delta_j(f),g_k)\big|^2 \Big)^{1/2} \Big\|_r \\
\leq \quad & C_1(p,q,P) \Big\| \Big(\sum_j \big|\Delta_j(f)\big|^2 \Big)^{1/2} \Big\|_p \Big\| \Big(\sum_k \big|g_k\big|^2 \Big)^{1/2} \Big\|_q, \qquad \text{(by Corollary 1)} \\
= \quad & C_1(p,q,P) \Big\| \Big(\sum_j \big|\Delta_j(f)\big|^2 \Big)^{1/2} \Big\|_p \|g\|_q \\
\leq \quad & C_1(p,q,P) \|f\|_p \|g\|_q,
\end{aligned}$$

by the Littlewood-Paley theorem. Putting everything together, it follows that

$$\big\| \sup_j |T_{\chi_{2^jP}}(f,g)| \big\|_r \leq c_r \big(C_0(p,q) + 4 \sup_{1\leq i\leq 4} C_i(p,q,P) \big) \|f\|_p \|g\|_q.$$

Of course, if P contains sides parallel to the anti-diagonal, the same result holds with the added restriction that $r > 1$. □

Using similar arguments to those used in the proof of Theorem 1, the following square function estimate for the tiling of $\mathbb{R}^2$ into the regions $2^jP \setminus 2^{j-1}P$ follows.

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} \big|T_{\chi_{2^jP\setminus 2^{j-1}P}}(f,g)\big|^2 \Big)^{1/2} \Big\|_r \leq C(p,q,P) \|f\|_p \|g\|_q.$$

More generally, the following theorem can be proven.

Theorem 2. *Let $1 < p, q < \infty$, $1/p + 1/q = 1/r < 3/2$, and Q be a polygon with no edges parallel to the anti-diagonal whose boundary is contained in the set $[-2,2]^2 \setminus [-1/2,1/2]^2$. Then for $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ we have*

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} \big|T_{\chi_{2^jQ}}(f,g)\big|^2 \Big)^{1/2} \Big\|_r \leq C(p,q,Q) \|f\|_p \|g\|_q.$$

Moreover, if Q has edges parallel to the anti-diagonal, the same estimate holds for $r > 1$.

Proof. Follow the same reasoning as in the proof of Theorem 1 with few minor modifications; for instance one needs the following definition of the Littlewood-Paley operators $\Delta_j(f) = (\widehat{f}\chi_{[2^{j-1},2^{j+1}]})^\vee$ since ∂Q is a subset of $[-2,2]^2\backslash[-1/2,1/2]^2$ and not of $[-2,2]^2\backslash[-1,1]^2$. □

4. Sums of Certain Families of Bilinear Operators

Consider a family of bounded bilinear operators $\{T_j\}_{j\in\mathbb{Z}}$. What properties must these operators possess to insure that $\sum_j T_j$ is also a bounded bilinear operator? In this section we will consider operators $T_j = T_{\chi_{2^jQ}}$, where Q is a planar polygon. Throughout this section we fix Littlewood-Paley operators Δ_j^1 associated with a family of *disjoint* intervals and another sequence of Littlewood-Paley operators Δ_j^2 associated with another family of disjoint intervals. The following proposition makes use of orthogonality properties of certain polygons.

Proposition 2. *Let $1 < p, q < \infty$, $1/p + 1/q = 1/r$ and Q be a planar polygon such that for all $j \in \mathbb{Z}$*

$$\big\langle T_{\chi_{2^jQ}}(f, g), h\big\rangle = \big\langle T_{\chi_{2^jQ}}(\Delta_j^1(f), g), \Delta_j^2(h)\big\rangle \tag{4.9}$$

or

$$\big\langle T_{\chi_{2^jQ}}(f, g), h\big\rangle = \big\langle T_{\chi_{2^jQ}}(f, \Delta_j^1(g)), \Delta_j^2(h)\big\rangle \tag{4.10}$$

for all Schwartz functions f, g, h. Then for $1 < r < \infty$ we have

$$\sup_{\epsilon_j\in\{-1,1\}} \Big\|\sum_{j\in\mathbb{Z}} \epsilon_j T_{\chi_{2^jQ}}(f, g)\Big\|_r \le C(p, q, Q)\|f\|_p\|g\|_q$$

for all $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$. Moreover, if no edge of Q is parallel to the anti-diagonal and $2/3 < r \le 1$, we have

$$\sup_{\epsilon_j\in\{-1,1\}} \Big\|\sum_{j\in\mathbb{Z}} \epsilon_j T_{\chi_{2^jQ}}(f, g)\Big\|_{H^r(\mathbb{R})} \le C(p, q, Q)\|f\|_p\|g\|_q.$$

Proof. Let $1 < p, q, r < \infty$ and $1/p + 1/q = 1/r$. Assume that (4.9) holds. A variation of the following argument works under assumption (4.10). Let $\epsilon_j \in \{-1, 1\}$. Then

$$\begin{aligned}
&\sup_{\|h\|_{r'}=1} \Big|\Big\langle \sum_j \epsilon_j T_{\chi_{2^jQ}}(f, g), h\Big\rangle\Big| \\
= \ &\sup_{\|h\|_{r'}=1} \Big|\sum_j \big\langle \epsilon_j T_{\chi_{2^jQ}}(\Delta_j^1(f), g), \Delta_j^2(h)\big\rangle\Big| \\
\le \ &\sup_{\|h\|_{r'}=1} \int_{\mathbb{R}} \Big(\sum_j \big|T_{\chi_{2^jQ}}(\Delta_j^1(f), g)(x)\big|^2\Big)^{1/2}\Big(\sum_j \big|\Delta_j^2(h)(x)\big|^2\Big)^{1/2} dx \\
\le \ &C(p, q, Q)\|f\|_p\|g\|_q,
\end{aligned}$$

where the last inequality follows from Hölder's Inequality, Theorem 2, and the Littlewood-Paley theorem. Since ϵ_j were arbitrary, we obtain the desired result.

Using (4.9) one may prove that $\Delta_j^2 T_{2^kQ} = 0$ whenever $j \ne k$ and that $\Delta_k^2 T_{2^kQ} = T_{2^kQ}$ for all $k, j \in \mathbb{Z}$. Indeed, to see the validity of these statements, act these bilinear operators on a pair of functions (f, g) and take the inner product with a third function h. We obtain:

$$\langle\Delta_j^2(T_{2^kQ}(f, g)), h\rangle = \langle T_{2^kQ}(f, g), \Delta_j^2(h)\rangle = \langle T_{2^kQ}(\Delta_k^2(f), g), \Delta_k^2\Delta_j^2(h)\rangle = 0$$

as $\Delta_k^2\Delta_j^2 = 0$ when $j \neq k$. Likewise

$$\begin{aligned}\langle \Delta_k^2(T_{2^kQ}(f,g)), h\rangle &= \langle T_{2^kQ}(f,g), \Delta_k^2(h)\rangle = \langle T_{2^kQ}(\Delta_k^2(f),g), \Delta_k^2\Delta_k^2(h)\rangle \\ &= \langle T_{2^kQ}(\Delta_k^2(f),g), \Delta_k^2(h)\rangle = \langle T_{2^kQ}(f,g), h\rangle.\end{aligned}$$

For $2/3 < r \leq 1$, use the Littlewood-Paley characterization of $H^r(\mathbb{R})$, see [5], to obtain

$$\begin{aligned}\Big\|\sum_{j\in\mathbb{Z}} \epsilon_j T_{\chi_{2^jQ}}(f,g)\Big\|_{H^r(\mathbb{R})} &\approx \bigg\|\Big(\sum_j \big|\Delta_j^2\big(\sum_k \epsilon_k T_{\chi_{2^kQ}}(f,g)\big)(x)\big|^2\Big)^{1/2}\bigg\|_r \\ &= \bigg\|\Big(\sum_j \big|\Delta_j^2\big(\epsilon_j T_{\chi_{2^jQ}}(f,g)\big)(x)\big|^2\Big)^{1/2}\bigg\|_r \\ &= \bigg\|\Big(\sum_j \big|T_{\chi_{2^jQ}}(f,g)(x)\big|^2\Big)^{1/2}\bigg\|_r \\ &\leq C(p,q,Q)\|f\|_p\|g\|_q,\end{aligned}$$

where the last inequality follows from Theorem 2. □

The following corollaries can be proved using duality. Corollary 2 is of particular interest as the operators there have kernels that resemble singular versions of paraproducts.

Corollary 2. *Let $1 < p, q < \infty$. Let Q be a planar polygon such that for all $j \in \mathbb{Z}$*

$$\big\langle T_{\chi_{2^jQ}}(f,g), h\big\rangle = \big\langle T_{\chi_{2^jQ}}(\Delta_j^1(f),g), \Delta_j^2(h)\big\rangle \tag{4.11}$$

for all Schwartz functions f, g, h. Then, if no edge of Q is parallel to the vertical axis in $\mathbb{R}^2$, we have

$$\sup_{\epsilon_j\in\{-1,1\}} \Big\|\sum_{j\in\mathbb{Z}} \epsilon_j T_{\chi_{2^jQ}}(b,g)\Big\|_q \leq C(p,q,Q)\|b\|_{BMO}\|g\|_q$$

for all $b \in BMO(\mathbb{R})$ and $g \in L^q(\mathbb{R})$. Similarly, if Q is a polygon such that for all $j \in \mathbb{Z}$

$$\big\langle T_{\chi_{2^jQ}}(f,g), h\big\rangle = \big\langle T_{\chi_{2^jQ}}(f,\Delta_j^1(g)), \Delta_j^2(h)\big\rangle \tag{4.12}$$

for all Schwartz functions f, g, h, then, if no edge of Q is parallel to the horizontal axis in $\mathbb{R}^2$, we have

$$\sup_{\epsilon_j\in\{-1,1\}} \Big\|\sum_{j\in\mathbb{Z}} \epsilon_j T_{\chi_{2^jQ}}(f,b)\Big\|_p \leq C(p,q,Q)\|f\|_p\|b\|_{BMO}$$

for all $b \in BMO(\mathbb{R})$ and $f \in L^p(\mathbb{R})$.

Proof. The first bilinear adjoint of the operator $T_{\chi_{2^jQ}}(f,g)$ in (4.11) satisfies (4.9) (which is the same identity with the roles of f and h reversed.) Likewise, the second bilinear adjoint of the operator $T_{\chi_{2^jQ}}(f,g)$ in (4.12) satisfies (4.10). Also, the condition that no side of Q is parallel to the anti-diagonal is equivalent to the condition that no side of the transformation of Q under the first adjoint operator is parallel to the vertical axis $\xi = 0$. A similar statement holds for the second adjoint operator and the other coordinate axis. Therefore, the proof of the corollary follows by Proposition 2 and the H^1-BMO norming duality. □

Corollary 3. *Let $1 < p, q, r < \infty$, $1/p + 1/q = 1/r$ and Q be a polygon such that*

$$\langle T_{\chi_{2^jQ}}(f,g), h\rangle = \langle T_{\chi_{2^jQ}}(\Delta_j^1(f), \Delta_j^2(g)), h\rangle$$

for all f, g, h Schwartz functions. Then, for $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ we have

$$\sup_{\epsilon_j \in \{-1,1\}} \Big\| \sum_{j\in\mathbb{Z}} \epsilon_j T_{\chi_{2^jQ}}(f,g) \Big\|_r \leq C(p,q,Q)\|f\|_p \|g\|_q.$$

Moreover, if no edge of Q is parallel to the coordinate axes,

$$\sup_{\epsilon_j \in \{-1,1\}} \Big\| \sum_{j\in\mathbb{Z}} \epsilon_j T_{\chi_{2^jQ}}(f,b) \Big\|_p \leq C(p,Q)\|f\|_p \|b\|_{BMO}$$

and

$$\sup_{\epsilon_j \in \{-1,1\}} \Big\| \sum_{j\in\mathbb{Z}} \epsilon_j T_{\chi_{2^jQ}}(b,g) \Big\|_q \leq C(q,Q)\|g\|_q \|b\|_{BMO}.$$

Proof. As in the proof of Corollary 2, the operators $T_{\chi_{2^jQ}}(f,g)$ are the bilinear transposes of bilinear operators satisfying Proposition 2. Therefore, the proof follows from duality. □

Theorem 2 can now be improved in the range $1 < p, q, r < \infty$.

Theorem 3. *Let $1 < p, q, r < \infty$ and let Q be a polygon whose boundary ∂Q is contained in the set $[-2,2]^2 \setminus [-1,1]^2$. Then for $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ we have*

$$\sup_{\epsilon_j \in \{-1,1\}} \Big\| \sum_{j\in\mathbb{Z}} \epsilon_j T_{\chi_{2^jQ}}(f,g) \Big\|_r \leq C(p,q,Q)\|f\|_p \|g\|_q.$$

Proof. By decomposing $[-2,2]^2 \setminus [-1,1]^2$ into the sets

$$\begin{aligned}
Q_1 &= \big([-2,2]^2 \setminus [-1,1]^2\big) \cap [1/2,2]^2 \\
Q_2 &= \big([-2,2]^2 \setminus [-1,1]^2\big) \cap \big([-1/2,-1/2] \times [1,2]\big) \\
Q_3 &= \big([-2,2]^2 \setminus [-1,1]^2\big) \cap \big([-2,-1/2] \times [1/2,2]\big) \\
Q_4 &= \big([-2,2]^2 \setminus [-1,1]^2 \cap \big([-2,-1] \times [-1/2,1/2]\big) \\
Q_5 &= \big([-2,2]^2 \setminus [-1,1]^2 \cap [-2,-1/2]^2 \\
Q_6 &= \big([-2,2]^2 \setminus [-1,1]^2 \cap \big([-1/2,1/2] \times [-2,-1]\big) \\
Q_7 &= \big([-2,2]^2 \setminus [-1,1]^2 \cap \big([1/2,2] \times [-2,-1/2]\big) \\
Q_8 &= \big([-2,2]^2 \setminus [-1,1]^2 \cap \big([1,2] \times [-1/2,1/2]\big),
\end{aligned}$$

it follows that

$$T_{\chi_{2^jQ}} = \sum_{i=1}^{8} T_{\chi_{2^jQ_i}}.$$

For each i, the family $\{T_{\chi_{2^jQ_i}}\}_{j\in\mathbb{Z}}$ satisfies either Proposition 2 or Corollary 3. Since $1 < p, q, r < \infty$, the proof is complete. □

5. Square Function Estimates and Dyadic Pointwise Convergence II

In this section, we prove results analogous to those in Theorem 1 and Theorem 2 for the characteristic functions of a certain sets with smooth boundaries. Let T_{χ_D} be the bilinear disc operator with multiplier $\chi_D = \chi_{\{\xi^2+\eta^2<1\}}$. Grafakos and Li [7] showed that T_{χ_D} is bounded from $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ into $L^r(\mathbb{R})$ for $2 \le p, q < \infty$, $1 < r \le 2$ and $1/p + 1/q = 1/r$. Using a multilinear transference theorem, Grafakos and Honzík [6] deduced the $L^r([0,1])$ convergence of the bilinear Fourier series

$$\sum_{|(j,k)|<N} \widehat{f}(j)\widehat{g}(k)e^{2\pi i(j+k)x} \to f(x)g(x) \qquad \text{as } N \to \infty$$

for 1-periodic functions $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$.

There are no results known concerning the almost everywhere convergence of bilinear Fourier series with respect to circular summation. However, with the use of Lemma 1, a partial result can be obtained concerning a set $S = P \cup R_1 \cup R_2 \cup R_3 \cup R_4$ obtained by the unit square $(-1, 1)^2$ by rounding off its corners as in Figure 3.

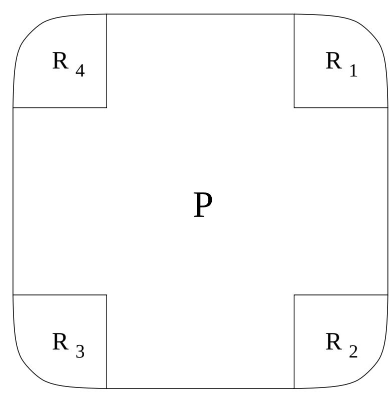

Figure 3. Decomposition of the set $S = P \cup R_1 \cup R_2 \cup R_3 \cup R_4$

As usually, let $T_{\chi_{2^n S}}$ be the bilinear operator with symbol $\chi_{2^n S}$ for $n \in \mathbb{Z}$. Since S is the union of a twelve-sided cross (dodecagon) with sides parallel to the coordinate axes and four quarter-discs with straight sides parallel to the coordinate axes, $\{T_{\chi_{2^n S}}\}_{n\in\mathbb{Z}}$ forms a family of uniformly bounded bilinear operators mapping $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ into $L^r(\mathbb{R})$ for $2 \le p, q < \infty$, $1 < r \le 2$ and $1/p + 1/q = 1/r$. This follows from the boundedness of the bilinear disc operator [7] and the dilation invariance of multipliers. Moreover, the boundary of $2^n S$ is smooth and $2^n S$ converges to $\mathbb{R}^2$ as n tends to infinity. The next theorem concerns the dyadic dilates of the set S.

Theorem 4. *Let $2 \le p, q < \infty$, $1 < r \le 2$ and $1/p+1/q = 1/r$. Then we have the estimate*

$$\Big\| \sup_{n\in\mathbb{Z}^+} \big|T_{\chi_{2^n S}}(f, g)\big| \Big\|_r \le C(p, q)\|f\|_p\|g\|_q$$

for all $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$. Consequently, for such functions we have

$$T_{\chi_{2^n S}}(f, g)(x) \to f(x)g(x)$$

for almost all $x \in \mathbb{R}$ as n tends to infinity.

Proof. Write

$$\chi_S = \chi_P + \sum_{i=1}^{4} m^i,$$

where P is the closed eight-sided polygon with sides parallel to the coordinate axes of Figure 3 and $m^i = \chi_{R_i}$, where R_i are the sets of Figure 3 whose analytic description is

$$\begin{aligned} R_1 &= \{(\xi,\eta)\in\mathbb{R}^2 : (\xi-1/2)^2+(\eta-1/2)^2\le 1/4\}\cap(1/2,1)^2 \\ R_2 &= \{(\xi,\eta)\in\mathbb{R}^2 : (\xi-1/2)^2+(\eta+1/2)^2\le 1/4\}\cap(1/2,1)\times(-1,-1/2) \\ R_3 &= \{(\xi,\eta)\in\mathbb{R}^2 : (\xi+1/2)^2+(\eta+1/2)^2\le 1/4\}\cap(-1,-1/2)^2 \\ R_4 &= \{(\xi,\eta)\in\mathbb{R}^2 : (\xi+1/2)^2+(\eta-1/2)^2\le 1/4\}\cap(-1,-1/2)\times(1/2,1). \end{aligned}$$

Dilating by 2^n we can write,

$$\chi_{2^nS} = \chi_{2^nP} + \sum_{i=1}^{4} m^i(2^n(\cdot)).$$

It follows that

$$\Big\|\sup_{n\in\mathbb{Z}^+}\big|T_{\chi_{2^nS}}(f,g)\big|\Big\|_r \le \Big\|\sup_{n\in\mathbb{Z}^+}\big|T_{\chi_{2^nP}}(f,g)\big|\Big\|_r + \sum_{i=1}^{4}\Big\|\sup_{n\in\mathbb{Z}^+}\big|T_{m^i(2^n(\cdot))}(f,g)\big|\Big\|_r.$$

Using the Carleson-Hunt theorem [2], [9], we obtain that

$$\Big\|\sup_{n\in\mathbb{Z}^+}\big|T_{\chi_{2^nP}}(f,g)\big|\Big\|_r \le C(p,q)\|f\|_p\|g\|_q$$

because 2^nP can be split into five rectangles with sides parallel to the coordinate axes. If it can be shown that for each $i\in\{1,2,3,4\}$ we have

$$\Big\|\sup_{n\in\mathbb{Z}^+}\big|T_{m^i(2^n(\cdot))}(f,g)\big|\Big\|_r \le C(p,q)\|f\|_q\|g\|_q, \tag{5.13}$$

the proof will be complete. Since the families $\{T_{m^i(2^n(\cdot))}\}_{n\in\mathbb{Z}^+}$ satisfy the hypotheses of Lemma 1 for $i=1,2,3$ and 4, we have

$$\Big\|\sup_{n\in\mathbb{Z}^+}\big|T_{m^i(2^n(\cdot))}(f,g)\big|\Big\|_r \le \Big\|\Big(\sum_{n\in\mathbb{Z}^+}\big|T_{m^i(2^n(\cdot))}(f,g)\big|^2\Big)^{1/2}\Big\|_r \le C(p,q,r)\|f\|_q\|g\|_q\,.$$

Thus (5.13) holds and the proof of the theorem is complete. □

Similar techniques yield the following Littlewood-Paley type theorem.

Theorem 5. *Under the same hypotheses as in Theorem 4, we have*

$$\Big\|\Big(\sum_j\big|T_{\chi_{2^jS\setminus 2^{j-1}S}}(f,g)\big|^2\Big)^{1/2}\Big\|_r \le C(p,q)\|f\|_p\|g\|_q.$$

Proof. Just notice that $2^jS\setminus 2^{j-1}S$ is the union of four rectangles, four quarter circles, and four squares with quarter circles removed. The square function of the quarter circles and the four squares with quarter circles removed can be controlled with the use of Lemma 1. The square function related to the each of the four rectangles can be controlled using Proposition 1 by the same methods used in the proof of Theorems 1 and 2. Just use orthogonality and dilate the rectangles by translating the cones used in that construction. □

As a consequence of Theorem 4 and the bilinear transference theorem for maximal operators of Grafakos and Honzík [6], we obtain the following.

Corollary 4. *Let S be the square with rounded corners defined in section 4 and let $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$ be 1-periodic functions where the indices p, q satisfy $2 \le p, q < \infty$ and $1/2 \le 1/p + 1/q < 1$. Then the bilinear Fourier series*

$$\sum_{(j,k)\in(2^n S)\cap\mathbb{Z}^2} \widehat{f}(j)\widehat{g}(k)e^{2\pi i(j+k)x}$$

converges to $f(x)g(x)$ as $n \to \infty$ for almost all $x \in [0,1]$.

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Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: loukas@math.missouri.edu

Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA
E-mail address: diestelg@math.sc.edu