METHOD OF ROTATIONS FOR BILINEAR SINGULAR INTEGRALS

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Abstract. Suppose that Ω lies in the Hardy space H^1 of the unit circle $\mathbf{S}^1$ in $\mathbf{R}^2$. We use the Calderón–Zygmund method of rotations and the uniform boundedness of the bilinear Hilbert transforms to show that the bilinear singular operator with the rough kernel p.v. $\Omega(x/|x|)|x|^{-2}$ is bounded from $L^p(\mathbf{R}) \times L^q(\mathbf{R})$ to $L^r(\mathbf{R})$, for a large set of indices satisfying $1/p + 1/q = 1/r$. We also provide an example of a function Ω in $L^q(\mathbf{S}^1)$ with mean value zero to show that the singular integral operator given by convolution with p.v. $\Omega(x/|x|)|x|^{-2}$ is not bounded from $L^{p_1}(\mathbf{R}) \times L^{p_2}(\mathbf{R})$ to $L^p(\mathbf{R})$ for $1/2 < p < 1$, $1 < p_1, p_2 < \infty$, $1/p_1 + 1/p_2 = 1/p$, $1 \le q < \infty$, and $1/p + 1/q > 2$.

1. INTRODUCTION AND MAIN RESULTS

Suppose that $\mathbf{S}^{n-1}$ denotes the unit sphere of $\mathbf{R}^n$, equipped with normalized Lebesgue measure $d\sigma(x')$ for some $n \ge 2$. Let Ω be an integrable function on $\mathbf{S}^{n-1}$ that satisfies

$$\int_{\mathbf{S}^{n-1}} \Omega(x')\, d\sigma(x') = 0.$$

We introduce the kernel

$$K_\Omega(x) = \frac{\Omega(x/|x|)}{|x|^n}, \quad x \ne 0,$$

which is homogeneous of degree $-n$, and the distribution W_Ω in $\mathscr{S}'(\mathbf{R}^n)$ by setting

$$< W_\Omega, \phi >= \lim_{\epsilon \to 0} \int_{|x| \ge \epsilon} K_\Omega(x)\phi(x)\, dx = \lim_{\epsilon \to 0} \int_{\epsilon \le |x| \le \epsilon^{-1}} K_\Omega(x)\phi(x)\, dx$$

for ϕ in the Schwartz class $\mathscr{S}(\mathbf{R}^n)$. The singular integral operator T_Ω is defined by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbf{R}^n} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\, dy$$

for $f \in \mathscr{S}(\mathbf{R}^n)$. Calderón and Zygmund [1] first studied the L^p boundedness of the operator T_Ω. They also introduced the "method of rotations" (see [2]) to show that T_Ω is bounded on $L^p(\mathbf{R}^n)$ if the function Ω is in $L \log L(\mathbf{S}^{n-1})$. The basic idea of the method of rotations is to write Ω as a sum of an odd and an even function, reduce the even part to the odd using the Riesz transform identity $-I = R_1^2 + \cdots + R_n^2$, and express the operator corresponding to the odd part of Ω as an average of the directional Hilbert transforms. This result was extended to functions $\Omega \in H^1(\mathbf{S}^{n-1})$ by Connett [7] and independently by Ricci and Weiss [18]. Here $H^1(\mathbf{S}^{n-1})$ denotes the Hardy space on the sphere which, defined

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in terms of its atomic decomposition as in Coifman and Weiss [6]; for a variety of useful characterizations of $H^1(\mathbf{S}^{n-1})$ one may consult the article of Stefanov [20]. We summarize the results concerning T_Ω in two statements:

Theorem A [2] Suppose that Ω has vanishing integral and is an odd function in $L^1(\mathbf{S}^{n-1})$. Then T_Ω is bounded from L^p to itself for $1 < p < \infty$.

Theorem B [18] Suppose that Ω is an even function in $H^1(\mathbf{S}^{n-1})$ (thus it has integral zero). Then T_Ω is bounded from L^p to itself for $1 < p < \infty$.

In this article, we consider bilinear versions of T_Ω. To define these operators we start with a complex-valued integrable function Ω on the unit sphere $\mathbf{S}^{2n-1}$ with mean value zero with respect to the surface measure. The bilinear Calderón-Zygmund singular integral operator T^2_Ω associated with Ω is defined by

$$T^2_\Omega(f,g)(x) = \text{p.v.} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} K_\Omega(x-y_1, x-y_2) f(y_1) g(y_2) dy_1 dy_2, \tag{1.1}$$

for Schwartz functions f, g on $\mathbf{R}^n$, where the kernel K_Ω is given by

$$K_\Omega(x_1,x_2) = K_\Omega(x) = \frac{\Omega(x/|x|)}{|x|^{2n}}, \qquad x = (x_1,x_2).$$

We are going to use a modification of the linear method of rotations to obtain the boundedness of the operator T^2_Ω from $L^p(\mathbf{R}) \times L^q(\mathbf{R})$ to $L^r(\mathbf{R})$, whenever $n = 1$ for a certain range of indices. In the bilinear case, the role of the directional Hilbert transforms is played by the bilinear Hilbert transforms and their uniform boundedness (Grafakos and Li [10], Li [16]) is crucial in our approach. We recall that if the function Ω is smooth then the boundedness of the bilinear Calderón–Zygmund operator follows from the results of Coifman and Meyer [4], [5], Kenig and Stein [13], and Grafakos and Torres [12] for more general multilinear operators. Therefore our results are most relevant in the case the function Ω lacks smoothness.

When $\Omega \in L^1(\mathbf{S}^{2n-1})$ is an odd function, Grafakos and Torres [12] expressed T^2_Ω as an average of the bilinear Hilbert transforms; so, when $n = 1$, the L^p boundedness of T^2_Ω is a consequence of the uniform boundedness of the bilinear Hilbert transforms; see [10]. This provides an analog of Theorem A in the bilinear setting. Motivated by this observation, we pursue a result parallel to Theorem B in the case where Ω is an even function, also in the case $n = 1$.

We denote by $\mathcal{H}$ the set of all triples $(1/p_1, 1/p_2, 1/p)$ such that $1 < p_1, p_2, p < \infty$, $1/p_1 + 1/p_2 = 1/p$, and

$$\Big|\frac{1}{p_1} - \frac{1}{p_2}\Big| < \frac{1}{2}, \qquad \Big|\frac{1}{p_1} - \frac{1}{p'}\Big| < \frac{1}{2}, \qquad \Big|\frac{1}{p_2} - \frac{1}{p'}\Big| < \frac{1}{2}. \tag{1.2}$$

This set is an open hexagon in the $(1/p_1, 1/p_2, 1/p)$ plane which is a proper superset of the set of indices indices $(1/p_1, 1/p_2, 1/p)$ in the local L^2 case, i.e., the case $2 \le p_1, p_2, p' < \infty$ and $1/p_1 + 1/p_2 = 1/p$.

For a function Ω, we set $\Omega_e(x') = \frac{1}{2}\big(\Omega(x') + \Omega(-x')\big)$ and $\Omega_o(x') = \frac{1}{2}\big(\Omega(x') - \Omega(-x')\big)$. Then Ω_e is even, Ω_o is odd and $\Omega = \Omega_e + \Omega_o$. The theorems below are the main results of this article.

Theorem 1.1. *Let Ω be a complex-valued integrable function on the sphere $\mathbf{S}^1$ with mean value zero such that the even part Ω_e lies in $H^1(\mathbf{S}^1)$. Then the operator T^2_Ω is bounded from $L^p(\mathbf{R}) \times L^q(\mathbf{R})$ to $L^r(\mathbf{R})$, whenever $(1/p, 1/q, 1/r) \in \mathcal{H}$.*

We extend a little the set of p and q where the Theorem 1.1 is valid for functions Ω with slightly better integrability.

Proposition 1.1. *Let Ω be a complex-valued function on the sphere $\mathbf{S}^1$ with mean value zero such that $\Omega \in L^s(\mathbf{S}^1)$, $s > 1$. Then there is an $\epsilon > 0$ such that the operator T^2_Ω is bounded from $L^p(\mathbf{R}) \times L^q(\mathbf{R})$ to $L^r(\mathbf{R})$ for indices $(1/p, 1/q, 1/r)$ in some open hexagon $\mathcal{H}'$ with the following properties: (i) the vertices $(1/2, 1/2, 1)$, $(1/2, 0, 1/2)$, $(0, 1/2, 1/2)$ of $\mathcal{H}$ are also vertices of $\mathcal{H}'$; (ii) the remaining three vertices of $\mathcal{H}'$ are ε away from the corresponding vertices of $\mathcal{H}$; (iii) $\mathcal{H}'$ strictly contains $\mathcal{H}$.*

Theorem 1.2. *Let $1/2 < p < 1$, $1 \le q < \infty$, and $2 < \frac{1}{p} + \frac{1}{q}$. Then there is a sequence Ω_ε with mean value 0 such that $\|\Omega_\varepsilon\|_{L^q(\mathbf{S}^1)} = c_q < \infty$ such that $\|T^2_{\Omega_\varepsilon}\|_{L^{p_1}(\mathbf{R})\times L^{p_2}(\mathbf{R}) \to L^p(\mathbf{R})} \to \infty$ as $\varepsilon \to 0$, when $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $1 < p_1, p_2 < \infty$.*

2. PREREQUISITE MATERIAL

We introduce some well known operators. We define the Fourier transform of an integrable function f on $\mathbf{R}^n$ by

$$\widehat{f}(\xi) = \mathscr{F}(f)(\xi) = \int_{\mathbf{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx$$

and by $\mathscr{F}^{-1}(f)(x) = \widehat{f}(-x)$ the inverse Fourier transform.

Definition 2.1. *The bilinear Hilbert transform in the direction (α_1, α_2) is given by*

$$\mathscr{H}_{\alpha_1,\alpha_2}(f_1, f_2)(x) = \text{p.v.}\,\frac{1}{\pi}\int_{\mathbf{R}} f_1(x - \alpha_1 t) f_2(x - \alpha_2 t)\, \frac{dt}{t}, \qquad x, \alpha_1, \alpha_2 \in \mathbf{R}.$$

One may express the bilinear Hilbert transform $\mathscr{H}_{\alpha_1,\alpha_2}$ in multiplier form as

$$\mathscr{H}_{\alpha_1,\alpha_2}(f_1, f_2)(x) = -i \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2) e^{2\pi i(\xi_1+\xi_2)x} \text{sgn}(\alpha_1\xi_1 + \alpha_2\xi_2)\, d\xi_1\xi_2\,,$$

where $\text{sgn}(t) = 1$ for $t > 0$, $\text{sgn}(t) = -1$ for $t < 0$, and $\text{sgn}(0) = 0$.

We recall that Lacey and Thiele [14], [15] proved that this operator is bounded from $L^p(\mathbf{R}) \times L^q(\mathbf{R})$ to $L^r(\mathbf{R})$ whenever $1 < p, q \le \infty$, $2/3 < r < \infty$, and $1/p + 1/q = 1/r$. Grafakos and Li [10] showed that if $2 < p, q, r' < \infty$, the boundedness of this operator is uniform in α_1 and α_2. Li [16] proved the boundedness of $\mathscr{H}_{\alpha_1,\alpha_2}$ from $L^p \times L^q$ to L^r, whenever $1 < p, q < 2$, $2/3 < r < 1$, uniformly in α_1, α_2 satisfying $|\alpha_1/\alpha_2 - 1| \ge c_0$ (with constant depending only on $c_0 > 0$). Interpolation between these these results yields the uniform boundedness of $\mathscr{H}_{\alpha_1,\alpha_2}$ from $L^p \times L^q$ to L^r, whenever $(1/p, 1/q, 1/r) \in \mathcal{H}$. This theorem provides the main tool needed to prove the results of this article.

Theorem 2.1. [16] *Let $(1/p_1, 1/p_2, 1/p) \in \mathcal{H}$. Then there is a constant $C = C(p_1, p_2)$ such that for all f_1, f_2 Schwartz functions on $\mathbf{R}$,*

$$\sup_{\alpha_1,\alpha_2\in\mathbf{R}} \|\mathscr{H}_{\alpha_1,\alpha_2}(f_1, f_2)\|_p \le C\|f_1\|_{p_1}\|f_2\|_{p_2}.$$

The uniform boundedness of $\mathscr{H}_{\alpha_1,\alpha_2}$ in the range of the indices of Theorem 2.1 clearly implies uniform (in α, β) boundedness of the bilinear Fourier projection

$$P_{\alpha,\beta}(f, g)(x) = \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i(\xi+\eta)x} \chi_{\{\arg(\xi,\eta)\in[\alpha,\beta)\}}(\xi, \eta)\, d\xi d\eta, \tag{2.1}$$

on the cone centered at the origin and determined by the angles α and β. Here $\arg(\xi, \eta)$ is the unique number γ in $[0, 2\pi)$ such that $\xi = |(\xi, \eta)| \cos\gamma$ and $\eta = |(\xi, \eta)| \sin\gamma$.

Lemma 2.1. [9] *Let $\Omega \in L^1(\mathbf{S}^1)$ have mean value zero. Then the Fourier transform of the distribution $W_\Omega = \text{p.v.}\, K_\Omega$, is the function*

$$\widehat{W_\Omega}(\nu) = \int_{\mathbf{S}^1} \Omega(\theta) \left(\log \frac{1}{|\nu \cdot \theta|} - \frac{i\pi}{2} \operatorname{sgn}(\nu \cdot \theta) \right) d\theta \,, \tag{2.2}$$

which is a.e. finite.

Using this identity, we may express

$$T^2_\Omega(f,g)(x) = \int_{\mathbf{R}} \int_{\mathbf{R}} \widehat{f}(\eta)\widehat{g}(\xi) e^{2\pi i(\eta+\xi)x}\, \widehat{W_\Omega}(\xi,\eta)\, d\xi d\eta. \tag{2.3}$$

3. PROOF OF THE RESULTS

We prove Theorem 1.1.

Proof. We set $B(\nu) = \widehat{W_\Omega}(\nu)$ and we split this function in two parts: the even part

$$M(\nu) = \int_{\mathbf{S}^1} \Omega(\theta) \log \frac{1}{|\theta \cdot \nu|}\, d\theta = \int_{\mathbf{S}^1} \Omega_e(\theta) \log \frac{1}{|\theta \cdot \nu|}\, d\theta$$

and the odd part

$$N(\nu) = -\frac{i\pi}{2} \int_{\mathbf{S}^1} \Omega(\theta) \operatorname{sgn}(\theta \cdot \nu)\, d\theta = -\frac{i\pi}{2} \int_{\mathbf{S}^1} \Omega_o(\theta) \operatorname{sgn}(\theta \cdot \nu)\, d\theta \,,$$

$\nu \in \mathbf{R}^2$. All symbols B, M, N are homogeneous of degree zero.

The function $\nu \to \arg(\nu)$ maps the unit circle $\mathbf{S}^1$ onto the interval $[0, 2\pi)$. For a function F on circle $\mathbf{S}^1$ we define another function $\widetilde{F}$ on $[0, 2\pi)$ by setting $\widetilde{F}(\alpha) = F(\cos\alpha, \sin\alpha)$. If we have $\theta, \nu \in \mathbf{S}^1$, we obtain $\theta \cdot \nu = \cos(\arg\theta - \arg\nu)$. Therefore, identity (2.2) may be written in convolution form (on the circle group $\mathbf{T} = [0, 2\pi)$) as

$$\widetilde{B}(\arg\nu) = B(\nu) = N(\nu) + M(\nu) = K * \widetilde{\Omega_o}(\arg\nu) + L * \widetilde{\Omega_e}(\arg\nu),$$

where $K(t) = -\frac{i\pi}{2}\operatorname{sgn}(\cos t)$ and $L(t) = -\log(|\cos t|)$. We use this expression to compute the distributional derivative $\partial\widetilde{B}/\partial t$ where $t \in [0, 2\pi)$. We get

$$\frac{\partial \widetilde{B}}{\partial t} = \frac{\partial K}{\partial t} * \widetilde{\Omega_o} + \frac{\partial L}{\partial t} * \widetilde{\Omega_e},$$

where

$$\frac{\partial K}{\partial t} = i\pi(\delta_{\frac{\pi}{2}} - \delta_{\frac{3\pi}{2}})$$

(δ is the Dirac distribution) and

$$\frac{\partial L}{\partial t} = \tan t.$$

The derivative of L is the kernel $\cot t$ of the classical conjugate function operator shifted by $\pi/2$. Since the conjugate function maps $H^1(\mathbf{T})$ to $L^1(\mathbf{T})$, under the assumption of Theorem 1.1 that $\widetilde{\Omega_e}$ lies in the Hardy space $H^1(\mathbf{T})$, we have that $\frac{\partial L}{\partial t} * \widetilde{\Omega_e}$ lies in $L^1(\mathbf{T})$. Obviously, $\frac{\partial K}{\partial t} * \widetilde{\Omega_o}$ is also an integrable function on $[0, 2\pi)$. We conclude that $\frac{\partial \widetilde{B}}{\partial t} \in L^1([0, 2\pi))$.

For $\theta \in \mathbf{R}^2$ we express the function B as

$$B(\theta) = \widetilde{B}(\arg\theta) = \int_0^{\arg\theta} \widetilde{B}'(\alpha)\, d\alpha = \int_0^{2\pi} A_\alpha(\theta)\widetilde{B}'(\alpha)\, d\alpha \,,$$

where

$$A_\alpha = \chi_{\{(\xi,\eta)\,:\,\arg(\xi,\eta)\notin[0,\alpha)\}}.$$

In view of this, we are able to write the bilinear operator T^2_Ω as

$$T^2_\Omega(f,g) = \int_0^{2\pi} \widetilde{B}'(\alpha) R_\alpha(f,g)\, d\alpha\,, \tag{3.1}$$

where

$$R_\alpha(f,g)(x) = \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i(\xi+\eta)x} A_\alpha(\xi,\eta)\, d\xi d\eta$$

is the bilinear multiplier operator whose symbol is the exterior of the cone formed by turning the positive ξ axis counterclockwise and stopping at the half line starting at the origin and passing through the point $(\cos\alpha, \sin\alpha)$. That is, using the notation introduced in (2.1), we have that

$$R_\alpha(f,g) = fg - P_{0,\alpha}(f,g)\,.$$

In view of Theorem 2.1, one obtains the uniform boundedness in α of the operator R_α from $L^p(\mathbf{R}) \times L^q(\mathbf{R})$ to $L^r(\mathbf{R})$ for $(1/p, 1/q, 1/r) \in \mathcal{H}$. The integral formula (3.1) then yileds boundedness for T^2_Ω in the same range of indices. This concludes the proof of Theorem 1.1. □

The key ingredient of the proof of Theorem 1.1 is the uniform boundedness of the bilinear Hilbert transforms. Analysis of the original proof of Lacey and Thiele shows that for $r > 2/3$ and $1/p + 1/q = 1/r$ there is an $M > 0$ and $C > 0$ such that for $f \in L^p$ and $g \in L^q$ we have

$$\|P_\alpha(f,g)\|_r = Cd^{-M}\|f\|_p\|g\|_q,$$

where $d = \operatorname{dist}(\alpha, \{0, \pi/2, 3\pi/4, \pi, 3\pi/2, 7/4\pi\})$. In other words, the norm of the bilinear Hilbert transform blows up polynomialy near the critical directions. This fact together with bilinear interpolation allows us to extend the set of p and q where the Theorem 1.1 is valid a little bit for functions Ω with better integrability.

We now prove Proposition 1.1.

Proof. We use the formula (3.1) again. We observe that since $\Omega \in L^s(\mathbf{S}^1)$, we have $\widetilde{B}'$ lies in $L^s(\mathbf{T})$. Due to the results of Lacey and Thiele we just mentioned, we have

$$\|R_\alpha(f,g)\|_r \le Cd^{-M}\|f\|_p\|g\|_q,$$

whenever $1 < p,q \le \infty$, $2/3 < r < \infty$, and $1/p + 1/q = 1/r$, where $d = \operatorname{dist}(\alpha, D)$, $D = \{0, \pi/2, 3\pi/4, \pi, 3\pi/2, 7/4\pi\}$. We also have

$$\|R_\alpha(f,g)\|_r \le C\|f\|_p\|g\|_q$$

for $(1/p, 1/q, 1/r) \in \mathcal{H}$. We introduce bilinear operators

$$T_0 = \int_{\{\alpha\,:\,\operatorname{dist}(\alpha,D)>1/2\}} \widetilde{B}'(\alpha) R_\alpha\, d\alpha \tag{3.2}$$

and

$$T_j = \int_{\{\alpha\,:\,2^{-j}\ge\operatorname{dist}(\alpha,D)>2^{-j-1}\}} \widetilde{B}'(\alpha) R_\alpha\, d\alpha \tag{3.3}$$

for $j \ge 1$. Using Hölder's inequality, we obtain that

$$\int_{\{\alpha\,:\,2^{-j}\ge\operatorname{dist}(\alpha,D)>2^{-j-1}\}} |\widetilde{B}'(\alpha)|\, d\alpha \le C2^{-j\frac{s-1}{s}}\|\widetilde{B}'\|_s.$$

Therefore for $j \ge 1$, we have the estimates

$$\|T_j(f,g)\|_r \le C2^{-j\frac{s-1}{s}}\|\widetilde{B}'\|_s\|f\|_p\|g\|_q$$

for $(1/p, 1/q, 1/r) \in \mathcal{H}$ and

$$\|T_j(f,g)\|_r \le C2^{-j\frac{s-1}{s}+jM}\|\widetilde{B}'\|_s\|f\|_p\|g\|_q$$

for $1 < p, q < \infty$ and $r > 2/3$. Now we use standard bilinear interpolation between these two estimates to establish the ϵ-improvement claimed by the proposition. □

We end this section by mentioning that an alternate proof of Theorem 1.1 has recently been obtained by Diestel [8] who showed that T^2_Ω is bounded from $L^{p_1} \times L^{p_2} \to L^p$ for an open set of triples $(1/p_1, 1/p_2, 1/p)$, using the assumption that Ω lies in $L\log L(\mathbf{S}^1)$.

4. THE COUNTEREXAMPLE

Calderón and Zygmund obtained weak type $(1,1)$ bounds for the operator T_Ω whenever Ω is sufficiently smooth; Christ and Rubio de Francia [3], and Seeger [19] extended this result to the case $\Omega \in L\log L$; see also Tao [21]. While the Calderón-Zygmund method was successfully extended to the multilinear setting to yield weak type $(1,1,1/2)$ bounds for certain bilinear singular integrals (i.e., bounds from $L^1\times L^1$ to weak $L^{1/2}$), these results only apply to the case where Ω is a Lipschitz function on the sphere; see for instance Grafakos and Torres [12]. It is not clear if bilinear endpoint results of this sort are possible when Ω lacks smoothness. In this section, we describe another limitation, provided by the example claimed in Theorem 1.2.

Proof. Let $B(x,r)$ be the ball with radius r centered at x in $\mathbf{R}^2$. We pick two points $a^+ = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $a^- = -a^+$ on the circle $\mathbf{S}^1$ and we define sets $\mathfrak{S}^+ = \mathbf{S}^1 \bigcap B(a^+, \varepsilon)$ and $\mathfrak{S}^- = \mathbf{S}^1 \bigcap B(a^-, \varepsilon)$. Fix $\varepsilon > 0$ a small number (say smaller than $1/100$) and define the function

$$\Omega_\varepsilon(\nu) = \varepsilon^{-1/q}(\chi_{\mathfrak{S}^+} - \chi_{\mathfrak{S}^-})(\nu), \quad \text{for } \nu \in \mathbf{S}^1.$$

It is obvious that Ω is an odd function with integral zero over $\mathbf{S}^1$ that satisfies: $\|\Omega_\varepsilon\|_q = c_q$. Consider the functions $f = \varepsilon^{-1/p_1}\chi_{B(0,\varepsilon')}$, $g = \varepsilon^{-1/p_2}\chi_{B(0,\varepsilon')}$, where $\varepsilon' = \varepsilon/100$. These functions satisfy $\|f\|_{p_1} = \|g\|_{p_2} = c'$ where c' is a constant.

Let us fix an $x \in \mathbf{R}$ such that $11/10 < x < 12/10$. Then we have

$$|T^2_{\Omega_\varepsilon}(f,g)(x)| \ge \varepsilon^{-\frac{1}{p}} \int_{|y_1|<\varepsilon'} \int_{|y_2|<\varepsilon'} \frac{\Omega\big(\frac{(x-y_1,x-y_2)}{|(x-y_1,x-y_2)|}\big)}{|(x-y_1,x-y_2)|^2}\, dy_1\, dy_2$$

The integral is over the set of all (y_1, y_2) in $(-\varepsilon', \varepsilon') \times (-\varepsilon', \varepsilon')$ such that the projection of the point $(x - y_1, x - y_2)$ onto the circle $\mathbf{S}^1$ lies in $\mathfrak{S}^+$ or $\mathfrak{S}^-$. Since we are considering an x such that $11/10 < x < 12/10$, this projection will only intersect the circular cap $\mathfrak{S}^+$. We obtain

$$|T^2_{\Omega_\varepsilon}(f,g)(x)| \gtrapprox \varepsilon^{-\frac{1}{p}}\varepsilon^{-\frac{1}{q}}\Big|\Big\{(y_1,y_2) \in (-\varepsilon',\varepsilon')^2 :\ \Big|\frac{(x-y_1,x-y_2)}{|(x-y_1,x-y_2)|} - a^+\Big| < \varepsilon\Big\}\Big|$$

Since ε' is small compared to ε, it follows that the set displayed above is the entire cube $(-\varepsilon', \varepsilon')^2$. So, we get

$$|T^2_{\Omega_\varepsilon}(f,g)(x)| \gtrapprox \varepsilon^{-\frac{1}{p}-\frac{1}{q}+2}.$$

It follows that

$$\|T^2_{\Omega_\varepsilon}(f,g)\|_p^p > \int_{11/10<x<12/10} |T^2_{\Omega_\varepsilon}(f,g)(x)|^p\, dx \gtrapprox \epsilon^{(2-1/p-1/q)p}$$

and the latter tends to infinity as $\varepsilon \to 0$ if $2 - 1/p - 1/q < 0$. □

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References

[1] A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. **88** (1952), 85–139.
[2] A. P. Calderón and A. Zygmund, *On singular integrals*, Amer. J. Math. **78** (1956), 289–309.
[3] M. Christ and J.-L. Rubio de Francia, *Weak type* $(1,1)$ *bounds for rough operators II,* Invent. Math. **93** (1988), 225–237.
[4] R. R. Coifman and Y. Meyer, *Commutateurs d' intégrales singuliéres et opérateurs multilinéaires*, Ann. Inst. Fourier (Grenoble) **28** (1978), 177–202.
[5] R. R. Coifman and Y. Meyer, *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. **212** (1975), 315–331.
[6] R. R. Coifman, G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569–645.
[7] W. C. Connett, *Singular integrals near* L^1, Harmonic analysis in Euclidean spaces, Proc. Sympos. Pure Math. (Williams Coll., Williamstown, Mass., 1978), pp. 163–165, Amer. Math. Soc., Providence, RI, 1979.
[8] G. Diestel, A vector-valued imbedding and Fourier multipliers, preprint.
[9] Grafakos, Classical Fourier Analysis, Second Edition, Graduate Texts in Math., No. 249, Springer, New York, 2008.
[10] L. Grafakos and X. Li, *Uniform bounds for the bilinear Hilbert transforms*, I, Ann. of Math. **159** (2004), 889–933.
[11] L. Grafakos and A. Stefanov, *Convolution Calderón-Zygmund singular integral operators with rough kernels*, Analysis of Divergence: Control and Management of Divergent Processes, pp. 119-143, Birkhauser, Boston-Basel-Berlin, 1999.
[12] L. Grafakos and R. Torres, *Multilinear Calderón-Zygmund theory*, Adv. Math. **165** (2002), 124-164.
[13] C. E. Kenig and E. M. Stein, *Multilinear estimates and fractional integration*, Math. Res. Lett. **6** (1999), 1–15.
[14] M. T. Lacey and C. M. Thiele, L^p *bounds for the bilinear Hilbert transform*, $2 < p < \infty$, Ann. Math. **146** (1997), 693–724.
[15] M. T. Lacey and C. M. Thiele, *On Calderón's conjecture*, Ann. Math. **149** (1999), 475–496.
[16] X. Li, *Uniform bounds for the bilinear Hilbert transforms,II*, Rev. Mat. Iberoam. **22** (2006), 1069–1126.
[17] S. Lu, Y. Ding, and D. Yan, Singular Integrals and Related Topics, World Scientific, Hackensack, NJ, 2007.
[18] F. Ricci and G. Weiss, *A characterization of* $H^1(\Sigma_{n-1})$, Harmonic analysis in Euclidean spaces, Proc. Sympos. Pure Math(Williams Coll., Williamstown, Mass., 1978), pp. 289-294, Amer. Math. Soc., Providence, RI, 1979.
[19] A. Seeger, *Singular integral operators with rough convolution kernels*, J. Amer. Math. Soc. **9** (1996), 95–105.
[20] A. Stefanov, *Characterizations of* H^1 *and applications to singular integrals*, Illinois J. Math. **44** (2000), 574–592.
[21] T. Tao, *The weak-type* $(1,1)$ *of* $L \log L$ *homogeneous convolution operators*, Indiana U. Math. J. **48** (1999), 1547–1584.

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