SHARP INEQUALITIES FOR LINEAR COMBINATIONS OF ORTHOGONAL MARTINGALES

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Abstract. For any two real-valued continuous-path martingales $X = \{X_t\}_{t\geq 0}$ and $Y = \{Y_t\}_{t\geq 0}$, with X and Y being orthogonal and Y being differentially subordinate to X, we obtain sharp L^p inequalities for martingales of the form $aX + bY$ with a, b real numbers. The best L^p constant is equal to the norm of the operator $aI + bH$ from L^p to L^p, where H is the Hilbert transform on the circle or real line. The values of these norms were found by Hollenbeck, Kalton and Verbitsky [15]. We also give applications of our martingale inequalities to Riesz transforms and some discrete operators.

1. Introduction

The research on martingale inequalities was initiated in 1966 by Burkholder [7] and was further pursued in [8], [9] and [10], where techniques for sharp estimates for them were developed. Martingale inequalities nowdays find applications in probability and analysis and their impact is quite far-reaching.

Based on the Burkholder's method, Bañuelos and Wang [4] obtained sharp inequalities for martingales under the assumption of differential subordination and orthogonality, and used them to provide probabilistic proofs to the results of Pichorides [20] concerning the norm of the Hilbert transform on $L^p(\mathbb{R})$ and of Iwaniec and Martin [16] about the norm of the Riesz transforms on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. We refer the reader to [1], [3] and [19] for more on orthogonal martingales and applications.

We describe the pertinent framework for this paper. Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\mathcal{F} = (\mathscr{F}_t)_{t\geq 0}$ be a nondecreasing family of sub-σ-fields of $\mathscr{F}$. Let $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ be two real-valued martingales with respect to $\mathcal{F}$. We say that X is orthogonal to Y if $\langle X, Y\rangle_t = 0$ for all $t \geq 0$, where $\langle X, Y\rangle_t$ is the predictable quadratic covariation between X and Y. We also say that Y is differentially subordinate to X (see [4]) if $|Y_0| \leq |X_0|$ and $\langle X\rangle_t - \langle Y\rangle_t$ is a nondecreasing and nonnegative function of t for $t \geq 0$, where $\langle X\rangle_t$ is the predictable quadratic variation of X. For continuous-path martingales, $\langle X\rangle_t$ is the same as the quadratic variation $[X]_t$. We assume throughout the paper that $Y_0 = 0$, this assumption is natural since all applications to Hilbert transform, Riesz transforms and discrete Hilbert transform fit it (see [4, 2]).

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For continuous-path real-valued martingale $X = (X_t)_{t\geq 0}$, $1 < p < \infty$, define

$$\|X\|_p = \sup_{t\geq 0} \|X_t\|_p,$$

where $\|X_t\|_p = (E|X_t|^p)^{1/p}$. For $1 < p < \infty$, define the constant $n_p = \cot(\pi/(2p^*))$, where $p^* = \max(p, p/(p-1))$. In [4], Bañuelos and Wang obtained the following result:

Theorem A. ([4]) *Let X and Y be two real-valued continuous-path martingales such that X and Y are orthogonal and Y is differentially subordinate to X. Then for $1 < p < \infty$,*

$$\|Y\|_p \leq n_p \|X\|_p \tag{1.1}$$

and

$$\|(X^2 + Y^2)^{1/2}\|_p \leq E_p \|X\|_p, \tag{1.2}$$

where $E_p = (1 + n_p^2)^{1/2}$. The constants are best possible.

The same authors later generalized the previous result to continuous-time martingales which may or may not have continuous paths([5]) under $[X, Y]_t = 0$, which is a stronger condition than orthogonal.

It is well known that the constant n_p is exactly the operator norm of Hilbert transform H on $L^p(\mathbb{R})$ and of the conjugate function $H^{\mathbb{T}}$ on $L^p(\mathbb{T})$, where $\mathbb{T}$ is the unit circle (see Pichorides [20]). These operators are given by

$$Hf(x) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{f(t)}{x-t}dt, \quad H^{\mathbb{T}}f(x) = \frac{1}{2\pi}\text{p.v.}\int_{-\pi}^{\pi} f(x)\cot\frac{x-t}{2}dt.$$

The constant n_p is also the operator norm of Riesz transform R_j on $L^p(\mathbb{R}^n)$ (see Iwaniec and Martin [16]). So the results of Theorem A can be seen as the martingale analogues of the results of Pichorides [20], Iwaniec and Martin [16], and Essén [13].

The main purpose of this paper is to construct sharp inequalities for linear combinations of orthogonal martingales. The linear combination case was initially mentioned in [6] by Birman as a problem on the exact value of the operator norm $\|I - \Pi\|_{L^p(\mathbb{R}+)}$, $1 < p < \infty$, where Π is the re-expansion operator: for $x > 0$

$$\Pi f(x) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}+} \frac{2xf(t)}{x^2 - t^2}dt.$$

This problem has its origin in scattering by unbounded obstacles in the plane which was solved in the work of Hollenbeck, Kalton and Verbitsky [15] by considering the operator norm of $\|I - H\|_{L^p(\mathbb{R})}$.

For $a, b \in \mathbb{R}$, $1 < p < \infty$, define the constant

$$B_p = \max_{x\in\mathbb{R}} \frac{|ax - b + (bx + a)\tan\gamma|^p + |ax - b - (bx + a)\tan\gamma|^p}{|x + \tan\gamma|^p + |x - \tan\gamma|^p}, \tag{1.3}$$

where $\gamma = \frac{\pi}{2p}$. By changing variables, B_p can be equivalently defined as

$$B_p = (a^2 + b^2)^{p/2} \max_{0\leq\theta\leq 2\pi} \frac{|\cos(\theta + \theta_0)|^p + |\cos(\theta + \theta_0 + \frac{\pi}{p})|^p}{|\cos\theta|^p + |\cos(\theta + \frac{\pi}{p})|^p}, \tag{1.4}$$

and

$$B_p = (a^2+b^2)^{p/2} \max_{0\le\vartheta\le 2\pi} \frac{|\cos(\vartheta-\theta_0)|^p + |\cos(\vartheta-\theta_0+\frac{\pi}{p})|^p}{|\cos\vartheta|^p + |\cos(\vartheta+\frac{\pi}{p})|^p}, \tag{1.5}$$

where $\tan\theta_0 = b/a$. These constants appeared in the work of Hollenbeck, Kalton and Verbitsky [15] who showed that the norm of $aI + bH^{\mathbb{T}}$ from $L^p(\mathbb{T})$ to $L^p(\mathbb{T})$ is equal to $B_p^{1/p}$, where I is the identity operator and $H^{\mathbb{T}}$ is the conjugate function operator on the circle. The same assertion is also true for the norm of $aI + bH$ from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$, where H is the Hilbert transform on real line, through a dilation argument known as "blowing up the circle" (see [22], Chapter XVI, Theorem 3.8). Recently, the authors [11] provided a direct proof of the sharp $L^p(\mathbb{R})$ inequality for $aI + bH$ by an argument that uses an explicit formula for a crucial subharmonic majorant.

In this work, we construct sharp inequalities for linear combination martingales $aX + bY$, where X and Y are two real-valued continuous-path martingales with orthogonal and differentially subordinate assumptions and a, b are arbitrary real numbers. Motivated by the usefulness of the explicit formula of the crucial subharmonic majorant G in [11], we find a new property of G in this paper, that is, we derive two alternative explicit expressions for this function centered around two points respectively (Lemma 2.2). This property of G is new for linear combination cases since it did not occur on $a = 0$ (when $a = 0$, two alternative expressions goes to one), and we use it appropriately in the proof of the main estimate (1.6) below.

Theorem 1.1. *Let X and Y be two real-valued continuous-path martingales such that X and Y are orthogonal. Let B_p be given by* (1.5)*. If Y is differentially subordinate to X and $Y_0 = 0$, then for $a, b \in \mathbb{R}$ and $1 < p < \infty$*

$$\|aX + bY\|_p \le B_p^{1/p} \|X\|_p. \tag{1.6}$$

The constant $B_p^{1/p}$ is the best possible in this inequality.

Remarks. Inequality (1.6) is the martingale analogue of that in Hollenbeck et al. [15] for analytic functions in the unit disc. If X and Y are continuous-time martingales which may or may not have continuous paths, this inequality also holds for the same assumption using quadratic variation instead of predictable quadratic variation.

We now turn to the proof of this theorem. Without loss of generality, we assume that $a = \cos\theta_0, b = \sin\theta_0$, so that $a^2 + b^2 = 1$.

2. Some Lemmas

In this section we discuss some crucial lemmas in the proof of the main theorem. The first lemma is a version of Lemma 4.2 in [15], in which we derive an explicit formula for a subharmonic function G that plays a crucial role in the proof.

Lemma 2.1. [11, Lemma 3.2] *Let $1 < p < \infty$, B_p be given by* (1.5)*, $T = \{re^{it} : r > 0, t_0 < t < t_0 + \frac{\pi}{p}\}$, where t_0 is the value that makes right part of* (1.5) *attain its maximum, and take $\varepsilon > 0$ such that $t_0 - \varepsilon < t_0 < t_0 + \pi/p < t_0 + \pi - \varepsilon$. Let*

$z=re^{it}, z_0=re^{it_0}, G(z)=G(re^{it})$ be π-periodic in t and when $t_0-\varepsilon<t<t_0+\pi-\varepsilon$:

$$G(z)=\begin{cases} B_p|\mathrm{Re}z_0|^{p-1}\mathrm{sgn}(\mathrm{Re}z_0)\mathrm{Re}[(\frac{z}{z_0})^p z_0]-|a\mathrm{Re}z_0+b\mathrm{Im}z_0|^{p-1} & \\ \qquad \times\ \mathrm{sgn}(a\mathrm{Re}z_0+b\mathrm{Im}z_0)(a\mathrm{Re}[(\frac{z}{z_0})^p z_0]+b\mathrm{Im}[(\frac{z}{z_0})^p z_0]), & \text{if} \quad z\in T \\ B_p|\mathrm{Re}z|^p-|a\mathrm{Re}z+b\mathrm{Im}z|^p, & \text{if} \quad z\notin T. \end{cases}$$

Then $G(z)$ is subharmonic on $\mathbb{C}$ and satisfies

$$|a\,\mathrm{Re}z+b\,\mathrm{Im}z|^p\le B_p|\mathrm{Re}z|^p-G(z). \tag{2.1}$$

for all $z\in\mathbb{C}$.

In the next lemma, we provide two other explicit formulas for G centered around the points t_0 and $u_0=t_0+\pi/p$, respectively.

Lemma 2.2. *Let $1<p<\infty$, B_p, T and t_0,ε be as in Lemma 2.1. Let $z=re^{it}, z_0=re^{it_0}$. Then for $z=re^{it}\in T$, $G(z)$ in Lemma 2.1 has the following equivalent expressions:*

$$G(z)=r^p\left[B_p\frac{|\cos t_0|^p}{\cos t_0}\cos(p(t-t_0)+t_0)-\frac{|\cos(t_0-\theta_0)|^p}{\cos(t_0-\theta_0)}\cos(p(t-t_0)+t_0-\theta_0)\right] \tag{2.2}$$

and

$$G(z)=r^p\left[B_p\frac{|\cos u_0|^p}{\cos u_0}\cos(p(t-u_0)+u_0)-\frac{|\cos(u_0-\theta_0)|^p}{\cos(u_0-\theta_0)}\cos(p(t-u_0)+u_0-\theta_0)\right], \tag{2.3}$$

where $u_0=t_0+\pi/p$, $\tan\theta_0=b/a$, $G(z)$ is π-periodic in t and $t_0-\varepsilon<t<t_0+\pi-\varepsilon$.

Proof. Expression (2.2) is just the one given in Lemma 3.2 in [11]. We now prove (2.3). In the proof of Lemma 3.2 in [11], using the notation in that reference, we have

$$h(x)=\widetilde{f}(\widetilde{p}\widetilde{t_0})\cos(x-\widetilde{p}\widetilde{t_0})+\widetilde{f}'_+(\widetilde{p}\widetilde{t_0})\sin(x-\widetilde{p}\widetilde{t_0}), \tag{2.4}$$

where $\widetilde{p}=p/2, \widetilde{t_0}=2t_0$ and

$$\widetilde{f}(t)=B_p|\cos(t/p)|^p-|a\cos(t/p)+b\sin(t/p)|^p, \tag{2.5}$$

if we can prove

$$h(x)=\widetilde{f}(\widetilde{p}\widetilde{t_0}+\pi)\cos(x-\widetilde{p}\widetilde{t_0}-\pi)+\widetilde{f}'_+(\widetilde{p}\widetilde{t_0}+\pi)\sin(x-\widetilde{p}\widetilde{t_0}-\pi), \tag{2.6}$$

then following the proof of Lemma 3.2 in [11], we deduce (2.3) when $z\in T$.

To obtain (2.6), in view of (2.4), it is sufficient to show that

$$\widetilde{f}(\widetilde{p}\widetilde{t_0})+\widetilde{f}(\widetilde{p}\widetilde{t_0}+\pi)=0 \tag{2.7}$$

and

$$\widetilde{f}'_+(\widetilde{p}\widetilde{t_0})+\widetilde{f}'_+(\widetilde{p}\widetilde{t_0}+\pi)=0. \tag{2.8}$$

In fact,

$$\widetilde{f}(\widetilde{p}\widetilde{t_0}+\pi)=\widetilde{f}(pt_0+\pi)=B_p|\cos(t_0+\pi/p)|^p-|a\cos(t_0+\pi/p)+b\sin(t_0+\pi/p)|^p,$$

by

$$B_p=(a^2+b^2)^{p/2}\frac{|\cos(t_0-\theta_0)|^p+|\cos(t_0-\theta_0+\frac{\pi}{p})|^p}{|\cos t_0|^p+|\cos(t_0+\frac{\pi}{p})|^p}, \tag{2.9}$$

where $\tan\theta_0 = b/a$, we have

$$\begin{aligned} & B_p|\cos(t_0+\pi/p)|^p - |a\cos(t_0+\pi/p) + b\sin(t_0+\pi/p)|^p \\ = \;& -B_p|\cos t_0|^p + |a\cos t_0 + b\sin t_0|^p = -\widetilde{f}(pt_0) = -\widetilde{f}(\widetilde{pt_0}), \end{aligned}$$

so we get (2.7).

For (2.8), note that for $1 < p < \infty$, $\widetilde{f}(t)$ is $p\pi$-periodic and continuously differentiable. By (1.5), $g(t) = \widetilde{f}(t) + \widetilde{f}(t+\pi) \geq 0$ and $g(t)$ has a minimum at $\widetilde{pt_0}$, so

$$\widetilde{f}'_+(\widetilde{pt_0}) + \widetilde{f}'_+(\widetilde{pt_0}+\pi) = g'(\widetilde{pt_0}) = 0,$$

Thus the lemma is proved. □

The preceding lemma indicates that the function G has some symmetry properties in terms of t_0 and u_0.

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is based on the techniques of Burkholder; also see [4]. We choose the appropriate function for Theorem 1.1 to be the opposite of function G in Lemma 2.1 and use the explicit formulas for G obtained by Lemma 2.2.

For $x, y \in \mathbb{R}$, $1 < p < \infty$, set

$$V(x,y) = |ax+by|^p - B_p|x|^p,$$

where $x = r\cos t, y = r\sin t$, and $0 < t < 2\pi$. Define

$$U(x,y) = -G(x+iy) = -G(z),$$

where $z = re^{it}$ and $G(z)$ is the function in Lemma 2.1. Then by Lemma 2.1, we have

$$V \leq U. \tag{3.1}$$

Denoting by U_{xx}, U_{yy} the second order partial derivatives of $U(x,y)$, we need only to show that for all $h, k \in \mathbb{R}$,

$$U_{xx}(x,y)h^2 + U_{yy}(x,y)k^2 \leq -c(x,y)(h^2-k^2) \tag{3.2}$$

for $(x,y) \in S_i$, where $S_i, i \geq 1$ is a sequence of open connected sets such that the union of the closure of S_i is $\mathbb{R}^2$, and $c(x,y) \geq 0$ that is bounded on $1/\delta \leq r \leq \delta$ for any $\delta > 0$. In fact, for continuous-path martingales we use Proposition 1.2 with Remark 1.1 in [4] (for continuous-time martingales, using Proposition 1 with its Remark in [5]), by (3.2), we can get

$$EV(X_t,Y_t) \leq EU(X_t,Y_t) \leq EU(X_0,Y_0) \leq 0,$$

the last inequality is due to Lemma 4.3 in [15] and $Y_0 = 0$. Thus

$$E|aX_t + bY_t|^p \leq B_pE|X_t|^p,$$

then we get (1.6).

To show (3.2),we split the argument into two cases. First, for $z = x+iy \notin T$ we have

$$U(x,y) = |ax+by|^p - B_p|x|^p,$$

and by a direct calculation we obtain from this that

$$U_{xx}(x,y) = p(p-1)\Big(|ax+by|^{p-2}a^2 - B_p|x|^{p-2}\Big) \tag{3.3}$$

except on the lines $\{z: x=0\}$ and $\{z: ax+by=0\}$, and

$$U_{yy}(x,y) = p(p-1)|ax+by|^{p-2}b^2 \tag{3.4}$$

except on the line $\{z: ax+by=0\}$. Then

$$\begin{aligned} U_{xx}(x,y)h^2 + U_{yy}(x,y)k^2 = \ & p(p-1)\Big(|ax+by|^{p-2} - B_p|x|^{p-2}\Big)h^2 \\ & -p(p-1)|ax+by|^{p-2}b^2(h^2-k^2). \end{aligned} \tag{3.5}$$

By the property of $G(z)$, we have (see [15])

$$|ax+by|^{p-2} \le B_p|x|^{p-2} \tag{3.6}$$

in this region. So by (3.5) and (3.6) we get

$$U_{xx}(x,y)h^2 + U_{yy}(x,y)k^2 \le -p(p-1)|ax+by|^{p-2}b^2(h^2-k^2). \tag{3.7}$$

Then (3.2) holds with obvious choice of $c(x,y)$.

We now consider the second case where $z \in T$. Recall that $t_0 - \varepsilon < t < t_0 + \pi - \varepsilon$. We use the expression (2.2) for $G(z)$, then

$$U(x,y) = r^p\Big[\frac{|\cos(t_0-\theta_0)|^p}{\cos(t_0-\theta_0)}\cos(p(t-t_0)+t_0-\theta_0) - B_p\frac{|\cos t_0|^p}{\cos t_0}\cos(p(t-t_0)+t_0)\Big].$$

Since

$$r_x = \cos t, \quad r_y = \sin t,$$

$$t_x = -\frac{1}{r}\sin t, \quad t_y = \frac{1}{r}\cos t,$$

we get

$$\begin{aligned} U_{xx}(x,y) = p(p-1)r^{p-2} \ & \Big(\frac{|\cos(t_0-\theta_0)|^p}{\cos(t_0-\theta_0)}\cos(2t-p(t-t_0)-(t_0-\theta_0)) \\ & -B_p\frac{|\cos t_0|^p}{\cos t_0}\cos(2t-p(t-t_0)-t_0)\Big), \end{aligned}$$

where $x = r\cos t, y = r\sin t$, $\tan\theta_0 = b/a$, and

$$U_{yy}(x,y) = -U_{xx}(x,y).$$

Then

$$U_{xx}(x,y)h^2 + U_{yy}(x,y)k^2 = U_{xx}(x,y)(h^2-k^2). \tag{3.8}$$

We claim that

$$U_{xx}(x,y) \le 0 \tag{3.9}$$

for $z \in T$, where $z = x+iy$. In fact,

$$U_{xx}(re^{it_0}) = p(p-1)r^{p-2}\Big(\frac{|\cos(t_0-\theta_0)|^p}{\cos(t_0-\theta_0)}\cos(t_0+\theta_0) - B_p\frac{|\cos t_0|^p}{\cos t_0}\cos t_0\Big),$$

we know from [15, p.249] that

$$|a\cos t_0 + b\sin t_0|^{p-2} \le B_p|\cos t_0|^{p-2}, \tag{3.10}$$

which, since $a = \cos\theta_0, b = \sin\theta_0$, is equivalent to

$$|\cos(t_0-\theta_0)|^{p-2} \le B_p|\cos t_0|^{p-2}. \tag{3.11}$$

Combining (3.11) with the fact that

$$\cos(t_0-\theta_0)\cos(t_0+\theta_0)\le\cos^2 t_0, \tag{3.12}$$

we have

$$U_{xx}(re^{it_0})\le 0. \tag{3.13}$$

Now we use the expression (2.3) for $G(z)$ to get

$$U(x,y)=r^p\left[\frac{|\cos(u_0-\theta_0)|^p}{\cos(u_0-\theta_0)}\cos(p(t-u_0)+u_0-\theta_0)-B_p\frac{|\cos u_0|^p}{\cos u_0}\cos(p(t-u_0)+u_0)\right],$$

where $u_0=t_0+\pi/p$, then

$$\begin{aligned}U_{xx}(x,y)=p(p-1)r^{p-2}\quad&\left(\frac{|\cos(u_0-\theta_0)|^p}{\cos(u_0-\theta_0)}\cos(2t-p(t-u_0)-(u_0-\theta_0))\right.\\&\left.-B_p\frac{|\cos u_0|^p}{\cos u_0}\cos(2t-p(t-u_0)-u_0)\right),\end{aligned}$$

where $x=r\cos t, y=r\sin t$, $\tan\theta_0=b/a$, so

$$U_{xx}(re^{iu_0})=p(p-1)r^{p-2}\left(\frac{|\cos(u_0-\theta_0)|^p}{\cos(u_0-\theta_0)}\cos(u_0+\theta_0)-B_p\frac{|\cos u_0|^p}{\cos u_0}\cos u_0\right),$$

where $u_0=t_0+\pi/p$. We know from [15, p.249] that

$$|a\cos u_0+b\sin u_0|^{p-2}\le B_p|\cos u_0|^{p-2}, \tag{3.14}$$

which is equivalent to

$$|\cos(u_0-\theta_0)|^{p-2}\le B_p|\cos u_0|^{p-2}. \tag{3.15}$$

Combining (3.15) with

$$\cos(u_0-\theta_0)\cos(u_0+\theta_0)\le\cos^2 u_0, \tag{3.16}$$

we have

$$U_{xx}(re^{i(t_0+\pi/p)})\le 0. \tag{3.17}$$

Write $U_{xx}(re^{it})=p(p-1)r^{p-2}u(t)$, where

$$u(t)=A\cos|p-2|t+\operatorname{sgn}(2-p)B\sin|p-2|t,$$

and

$$\begin{aligned}A&=\frac{|\cos(t_0-\theta_0)|^p}{\cos(t_0-\theta_0)}\cos\big((p-1)t_0+\theta_0\big)-B_p\frac{|\cos t_0|^p}{\cos t_0}\cos(p-1)t_0,\\B&=B_p\frac{|\cos t_0|^p}{\cos t_0}\sin(p-1)t_0-\frac{|\cos(t_0-\theta_0)|^p}{\cos(t_0-\theta_0)}\sin\big((p-1)t_0+\theta_0\big).\end{aligned}$$

Then $u(t)$ is a $|p-2|$-trigonometric function, thus also $|p-2|$-trigonometrically convex for $t_0<t<t_0+\pi/p$ (see [18, p.54]). We have $U_{xx}(re^{it})=p(p-1)r^{p-2}u(t)$ is harmonic thus subharmonic within the angle $\{z=re^{it}: r>0, t_0<t<t_0+\pi/p\}$ via a direct computation. Then, by (3.13) and (3.17), which are consequences of the two expressions property of G, we can use the Phragmén-Lindelöf theorem for subharmonic functions (see [18, p.49]), to get

$$U_{xx}(x,y)=U_{xx}(re^{it})\le 0 \tag{3.18}$$

for $z \in T$. We can also use the maximum principle for harmonic functions directly to deduce (3.18). This proves (3.9). Then (3.2) holds with $c(x, y) = -U_{xx}(x, y)$. This completes the proof of (1.6). □

Remarks. (a) The key property that G has two expressions centered around two points so that we can use the Phragmén-Lindelöf theorem is new for linear combination cases, which do not occur on $a = 0$ case.

(b) When $a = 0$, $b = 1$, the function $U = -G$ becomes the function $U_2(x, y)$ in [4]($p > 2$) and the function used in [20] and [14]($1 < p < 2$). In $a = 0$ case the function only have one expression and one just need to simply compute its second-order derivatives.

4. The sharpness of the constant $B_p^{1/p}$

To show that the constant B_p is sharp, we apply a similar argument as in [4]. Let $f(z) = u(z) + iv(z)$ be analytic in the unit disc D with $f(0) = 0$ and B_t be Brownian motion in D killed upon leaving D. Consider the martingales $X_t = u(B_t)$ and $Y_t = v(B_t)$, we have $\langle X, Y\rangle_t = 0$ and $\langle X\rangle_t - \langle Y\rangle_t = 0$ (see [12]). So X and Y are orthogonal with equal quadratic variations. Then the inequality in Theorem 1.1 exactly reduces to the inequality in Theorem 4.1 in [15].

Since $B_p^{1/p}$ is already the best constant in Theorem 4.1 of [15], we conclude that the constant $B_p^{1/p}$ cannot be improved in Theorem 1.1.

5. Examples and applications

In this section, we give some applications of Theorem 1.1 to Riesz transforms and operators related to discrete versions of the Hilbert transform.

The Riesz transforms $R_j, j = 1, 2, ..., n$ are defined by (see [21, p.57])

$$R_j f(x) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \text{p.v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y) dy$$

for $f \in L^p(\mathbb{R}^n)$. It has been shown that the L^p norm for R_j is the same as the one for 1-dimensional Hilbert transform, n_p, by using the method of rotations applicable to singular integrals with odd kernels (see [16]). But in the linear combination case $aI + bR_j, a, b \in \mathbb{R}$, the classical method of rotations could not be used to obtain its L^p norm. Our martingale inequalities for linear combination case provide a possible way to solve this problem. We have the following corollary of Theorem 1.1.

Corollary 5.1. *For any $j = 1, 2, ..., n$, $a, b \in \mathbb{R}$ and $1 < p < \infty$,*

$$\|(aI + bR_j)f\|_p \le B_p^{1/p} \|f\|_p, \tag{5.1}$$

where B_p is given by (1.5). The constant $B_p^{1/p}$ is the best possible in this inequality.

We give a sketch proof of Corollary 5.1. Using the same connection between martingale transforms and Riesz transforms in [4, p.592] involving linear combinations with the identity operator, and Theorem 1.1, we can get (5.1). The constant $B_p^{1/p}$ is the best possible follows from the fact that $aI + bR_j$ are extensions, in the Fourier multiplier sense, of $aI + bH$, where H is the Hilbert transform; see the proof of (47) in [16, p.37] for the full details.

Consider the following discrete version of the Hilbert transform

$$(Da)_n := \frac{1}{\pi} \sum_{k \neq 0} \frac{a_{n-k}}{k}, \tag{5.2}$$

where k runs over all the non-zero integers in $\mathbb{Z}$ and $a = (a_n)_n$. Recently, Bañuelos and Kwaśnicki [2] proved that the operator norm of D on $\ell^p(\mathbb{Z})$ is equal to the operator norm of the continuous Hilbert transform H on $L^p(\mathbb{R})$. The proof in [2] is based on Theorem A and uses two auxiliary operators $\mathcal{J}$ [defined in (5.3)] and $\mathcal{K}$ which satisfies $\mathcal{K}\mathcal{J} = D$. We find out some interesting relationship between $\mathcal{K}$ and the identity operator I (see (5.13)) and apply Theorem 1.1 to obtain the following results concerning $\mathcal{J}$ and $\mathcal{K}$.

Proposition 5.1. *Let $e = (e_n)_n$ be a sequence in $\ell^p(\mathbb{Z}), 1 < p < \infty$. Let $(\mathcal{J}e)_n = \sum_{m \in \mathbb{Z}} \mathcal{J}_m e_{n-m}$, where*

$$\mathcal{J}_n = \frac{1}{\pi n} \left(1 + \int_0^\infty \frac{2y^3}{(y^2 + \pi^2 n^2) \sinh^2 y} dy \right) \tag{5.3}$$

for $n \neq 0$, and $\mathcal{J}_0 = 0$. Then for $a, b \in \mathbb{R}$,

$$\|(aI + b\mathcal{J})e\|_p \leq B_p^{1/p} \|e\|_p, \tag{5.4}$$

where B_p is given by (1.5) and I is the identity operator: the convolution with kernel $I_0 = 1, I_n = 0$ for $n \neq 0$. The constant $B_p^{1/p}$ is the best possible in this inequality.

Proof. We use the notation in [2]. We only need to redefine the operator in (2.5) in [2], so that let

$$(J_A e)_n = \mathbb{E}_{(x_0,y_0)} \big(a\|A\| M_{\zeta-} + bA \star M_{\zeta-} | Z_{\zeta-} = (2\pi n, 0) \big). \tag{5.5}$$

Since the conditional expectation is a contraction on $L^p, 1 < p < \infty$, it follows from (1.6) in Theorem 1.1 that

$$\|J_A e\|_p \leq B_p^{1/p} \|A\| \|e\|_p. \tag{5.6}$$

Let

$$H = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

we have

$$\|J_H e\|_p \leq B_p^{1/p} \|e\|_p. \tag{5.7}$$

Notice that $\mathbb{E}_{(x_0,y_0)} \big(M_{\zeta-} | Z_{\zeta-} = (2\pi n, 0) \big) = (Ie)_n$, where I is the identity operator, then following the same proof in [2], we deduce (5.4).

The sharpness of the constant is due to the sharpness of Proposition 5.2, which is new and will be proved later, and the fact that

$$\|(a\mathcal{K} + bD)e\|_p \leq \|(aI + b\mathcal{J})e\|_p \tag{5.8}$$

for any sequence $a \in \ell^p(\mathbb{Z}), 1 < p < \infty$ and $a, b \in \mathbb{R}$. □

Proposition 5.2. *Let $e = (e_n)_n$ be a sequence in $\ell^p(\mathbb{Z}), 1 < p < \infty$, D be defined in (5.2). Let $\mathcal{K}$ be the convolution operator in Section 2.3 in [2] with kernel $(\mathcal{K}_n)$ such that $\mathcal{K}_n \geq 0$ for all n and the sum of all $\mathcal{K}_n$ is equal to 1. Then for $a, b \in \mathbb{R}$,*

$$\|(a\mathcal{K} + bD)e\|_p \leq B_p^{1/p} \|e\|_p, \tag{5.9}$$

where B_p *is given by* (1.5)*. The constant* $B_p^{1/p}$ *is the best possible in this inequality.*

Proof. By Section 2.3 in [2],

$$(De)_n = (\mathcal{K}\mathcal{J}e)_n,$$

then by Proposition 5.1, we have

$$\|(a\mathcal{K} + bD)e\|_p = \|\mathcal{K}(aI + b\mathcal{J})e\|_p \leq B_p^{1/p}\|e\|_p. \tag{5.10}$$

Denote by $\|\cdot\|_{p,p}$ the operator norm from L^p to L^p (or ℓ^p to ℓ^p). To deduce the sharpness, we define the dilation operators T_ε for any $\varepsilon > 0$ and $1 < p < \infty$ by $(T_\varepsilon f)(x) = \varepsilon^{1/p} f(\varepsilon x)$, then $\|T_\varepsilon\|_{p,p} = 1$ for all $\varepsilon > 0$. Notice that $\mathcal{K}$ is a convolution operator with kernel $(\mathcal{K}_n)$ such that $\mathcal{K}_n \geq 0$ for all n and $\sum_{n\in\mathbb{Z}} \mathcal{K}_n = 1$ (see [2]). Because of Theorem 4.2 in [17], now we can work on the real line and replace D and $\mathcal{K}$ by

$$(M_D f)(x) = \text{p.v.}\frac{1}{\pi}\sum_{m\neq 0} \frac{f(x-m)}{m} \tag{5.11}$$

and

$$(M_{\mathcal{K}} f)(x) = \sum_{m\in\mathbb{Z}} \mathcal{K}_m f(x-m), \tag{5.12}$$

respectively. It is known by [17] that

$$\lim_{\varepsilon\to 0}(T_{1/\varepsilon} M_D T_\varepsilon f)(x) = (Hf)(x),$$

where H is the Hilbert transform. We claim that

$$\lim_{\varepsilon\to 0}(T_{1/\varepsilon} M_{\mathcal{K}} T_\varepsilon f)(x) = (If)(x), \tag{5.13}$$

for a.e. $x \in \mathbb{R}$ and $f \in L^p(\mathbb{R})$, where I is the identity operator such that $(If)(x) = f(x)$. In fact, for any $f \in \mathcal{S}(\mathbb{R})$ (Schwartz function), we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}(T_{1/\varepsilon} M_{\mathcal{K}} T_\varepsilon f)(x)\\
=\;& \lim_{\varepsilon\to 0}\textstyle\sum_{|m|\leq N} \mathcal{K}_m f(x-\varepsilon m) + \lim_{\varepsilon\to 0}\sum_{|m|>N} \mathcal{K}_m f(x-\varepsilon m)\\
=\;& \textstyle\sum_{|m|\leq N} \mathcal{K}_m f(x) + \lim_{\varepsilon\to 0}\sum_{|m|>N} \mathcal{K}_m f(x-\varepsilon m)
\end{aligned}$$

for any $N > 0$. Then

$$\begin{aligned}
&\big|f(x) - \lim_{\varepsilon\to 0}(T_{1/\varepsilon} M_{\mathcal{K}} T_\varepsilon f)(x)\big|\\
=\;& \big|\textstyle\sum_{m\in\mathbb{Z}} \mathcal{K}_m f(x) - \sum_{|m|\leq N} \mathcal{K}_m f(x) - \lim_{\varepsilon\to 0}\sum_{|m|>N} \mathcal{K}_m f(x-\varepsilon m)\big|\\
=\;& \big|\textstyle\sum_{|m|>N} \mathcal{K}_m f(x) - \lim_{\varepsilon\to 0}\sum_{|m|>N} \mathcal{K}_m f(x-\varepsilon m)\big|\\
\leq\;& \lim_{\varepsilon\to 0}\textstyle\sum_{|m|>N} \mathcal{K}_m \big|f(x) - f(x-\varepsilon m)\big|\\
\leq\;& C(f)\textstyle\sum_{|m|>N} \mathcal{K}_m
\end{aligned}$$

for any $N > 0$. Since $\mathcal{K}_n \geq 0$ for all n and $\sum_{n\in\mathbb{Z}} \mathcal{K}_n = 1$, letting $N \to \infty$, we get

$$\lim_{\varepsilon\to 0}(T_{1/\varepsilon} M_{\mathcal{K}} T_\varepsilon f)(x) = f(x)$$

for $x \in \mathbb{R}$, $f \in \mathcal{S}(\mathbb{R})$. For $f \in L^p(\mathbb{R})$, we can get (5.13) for a.e. $x \in \mathbb{R}$ by replacing absolute value to L^p norm.

Then, by Fatou's lemma and $\|T_\varepsilon\|_{p,p} = 1$,

$$\|aI + bH\|_{p,p} \le \sup_{\varepsilon} \|T_{1/\varepsilon}(aM_{\mathcal{K}} + bM_D)T_\varepsilon\|_{p,p} \le \|aM_{\mathcal{K}} + bM_D\|_{p,p}.$$

By Theorem 4.2 in [17],

$$\|aM_{\mathcal{K}} + bM_D\|_{p,p} = \|a\mathcal{K} + bD\|_{p,p},$$

so, using (5.10) and the fact that $\|aI + bH\|_{p,p} = B_p^{1/p}$ (see [15]), we get the sharpness

$$\|a\mathcal{K} + bD\|_{p,p} = \|aI + bH\|_{p,p} = B_p^{1/p}.$$

□

From the proof above, we abstract a proposition for convolution operators:

Proposition 5.3. *Let $\mathcal{K}$ be the convolution operator defined by*

$$\mathcal{K}f(n) = \sum_{k\in\mathbb{Z}} \mathcal{K}_k f(n-k)$$

for $f \in \ell^p(\mathbb{Z})$, with kernel $(\mathcal{K}_n)$ such that $\mathcal{K}_n \ge 0$ for all n and $\sum_{n\in\mathbb{Z}} \mathcal{K}_n = 1$. Then $\|\mathcal{K}\|_{p,p} = 1$.

For the operator $aI + bD$, $a, b \in \mathbb{R}$, applying the method in [17, Lemma 4.3], we immediately obtain

$$\|aI + bD\|_{p,p} \ge \|aI + bH\|_{p,p} = B_p^{1/p}.$$

We conjecture that $\|aI+bD\|_{p,p} = \|a\mathcal{K}+bD\|_{p,p} = B_p^{1/p}$. The solution of this conjecture may require additional ideas as I and D are natural projections of nonorthogonal martingales.

Another discrete Hilbert transform is defined by

$$(D_{1/2}a)_n = \text{p.v.}\frac{1}{\pi}\sum_{k\in\mathbb{Z}} \frac{a_{n-k}}{k+1/2}. \tag{5.14}$$

More generally, consider the operator defined for $\alpha \in (0, 1)$ by

$$(D_\alpha a)_n = \text{p.v.}\frac{\sin \pi\alpha}{\pi}\sum_{k\in\mathbb{Z}} \frac{a_{n-k}}{k+\alpha}. \tag{5.15}$$

A natural conjecture is that for $1 < p < \infty$ and $\alpha \in (0, 1)$, the norms $\|D_\alpha\|_{p,p}$ coincide with the norms $\|\cos \pi\alpha I + \sin \pi\alpha H\|_{p,p} = B_p^{1/p}$; see Conjecture 5.7 in [17]. It could be observed that our martingale inequalities have some relationship with this conjecture.

References

[1] R. Bañuelos, *The foundational inequalities of D. L. Burkholder and some of their ramifications*, Illinois J. Math. **54** (2010), 789–868.

[2] R. Bañuelos, M. Kwaśnicki, *On the ℓ^p-norm of the discrete Hilbert transform*, Duke Math. J. **168** (2019), 471–504.

[3] R. Bañuelos, A. Osękowski, *Sharp martingale inequalities and applications to Riesz transforms on manifolds, Lie group and Gauss space*, J. Funct. Anal. **269** (2015), 1652–1713.

[4] R. Bañuelos, G. Wang, *Sharp inequalities for martingales with applications to the Beurling-Ahlfors and Riesz transforms*, Duke Math. J. **80** (1995), 575–600.

[5] R. Bañuelos, G. Wang, *Orthogonal martingales under differential subordination and applications to Riesz transforms*, Illinois J. Math. **40** (1996), 678–691.

[6] M. S. Birman, *Re-expansion operators as objects of spectral analysis*, Linear and Complex Analysis Problem Book, Lecture Notes in Math. 1043, Springer, (1984), 130–134.
[7] D. L. Burkholder, *Martingale transforms*, Ann. Math. Statist. **37** (1966), 1494–1504.
[8] D. L. Burkholder, *Boundary value problems and sharp inequalities for martingale transforms*, Ann. Probab. **12** (1984), 647–702.
[9] D. L. Burkholder, *Sharp inequalities for martingales and stochastic integrals*, Astérisque **157** (1988), 75–94.
[10] D. L. Burkholder, *Explorations in martingale theory and its applications*, Lecture Notes in Mathematics **1464** (1991), 1–66.
[11] Y. Ding, L. Grafakos, K. Zhu, *On the norm of the operator $aI + bH$ on $L^p(\mathbb{R})$*, Bull. Korean Math. Soc. **55** (2018), 1209–1219.
[12] R. Durrett, *Brownian Motion and Martingales in Analysis*, Wadsworth, Belmont, California, (1984).
[13] M. Essén, *A superharmonic proof of the M. Riesz conjugate function theorem*, Ark. Mat. **22** (1984), 281–288.
[14] L. Grafakos, *Best bounds for the Hilbert transform on $L^p(\mathbb{R}^1)$*, Math. Res. Lett. **4** (1997), 469–471.
[15] B. Hollenbeck, N. J. Kalton, I. E. Verbitsky, *Best constants for some operators associated with the Fourier and Hilbert transforms*, Studia Math. **157** (2003), 237–278.
[16] T. Iwaniec, G. Martin, *Riesz transforms and related singular integrals*, J. Reine Angew. Math. **473** (1996), 25–58.
[17] E. Laeng, *Remarks on the Hilbert transform and some families of multiplier operators related to it*, Collect. Math. **58** (2007), 25-44.
[18] B. Ya. Levin, *Lectures on entire functions*, Transl. Math. Monogr. 150, Amer. Math. Soc., Providence, RI, 1996.
[19] A. Osękowski, *Sharp martingale and semimartingale inequalities*, Monografie matematyczne 72, Birkhäuser, Springer, Basel, 2012.
[20] S. K. Pichorides, *On the best values of the constants in the theorems of M. Riesz, Zygmund and Kolmogorov*, Studia Math. **44** (1972), 165–179.
[21] E. M. Stein, *Singular integrals and differentiability properties of functons*, Princeton University Press, Princeton, 1979.
[22] A. Zygmund, *Trigonometric Series*, Volumme II, Cambridge UK, 1968.

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