CORRIGENDUM TO "SHARP HARDY SPACE ESTIMATES FOR MULTIPLIERS " (INT. MATH. RES. NOT. 2022 (2022), 10403-10430)

LOUKAS GRAFAKOS AND BAE JUN PARK

Abstract. We correct an error in Lemma 4.1 [pages 10420 - 10422] in our article listed in the title. This concerns the cases $s = n$ and $\gamma \leqslant 2$ below but it does not affect the main results of the article, i.e., Theorems 1.1 and 1.2, as only the case $s < n$ is used in the paper.

Lemma 4.1. Let $0 < s, \gamma < \infty$ and define the function on $\mathbb{R}^n$

$$\mathcal{H}^{(s,\gamma)}(x) := \frac{1}{(1+4\pi^2|x|^2)^{\frac{s}{2}}} \frac{1}{(1+\ln(1+4\pi^2|x|^2))^{\frac{\gamma}{2}}}. \tag{0.1}$$

Then $\widehat{\mathcal{H}^{(s,\gamma)}}$ is a positive radial function and there exist $c_{s,\gamma,n}, d_{s,\gamma,n} > 0$ such that

$$\widehat{\mathcal{H}^{(s,\gamma)}}(\xi) \leqslant c_{s,\gamma,n} e^{-\frac{|\xi|}{2}} \qquad \text{when} \quad |\xi| \geqslant 1 \tag{0.2}$$

and

$$\frac{1}{d_{s,\gamma,n}} \leqslant \frac{\widehat{\mathcal{H}^{(s,\gamma)}}(\xi)}{\mathfrak{T}^{(s,\gamma)}(\xi)} \leqslant d_{s,\gamma,n} \qquad \text{when} \quad |\xi| \leqslant 1 \tag{0.3}$$

where

$$\mathfrak{T}^{(s,\gamma)}(\xi) := \begin{cases} |\xi|^{-(n-s)}(1+2\ln|\xi|^{-1})^{-\frac{\gamma}{2}} & \text{for } \ 0 < s < n \\ \big(1+2\ln|\xi|^{-1}\big)^{1-\frac{\gamma}{2}} & \text{for } \ s = n, 0 < \gamma < 2 \\ \ln\big(1+2\ln|\xi|^{-1}\big) & \text{for } \ s = n, \gamma = 2 \\ 1 & \text{for } \ s = n, \gamma > 2 \\ 1 & \text{for } \ s > n. \end{cases}$$

Proof. It is known that the Fourier transform of $(1+4\pi^2|x|^2)^{-\frac{s}{2}}$ is the Bessel potential $G_s(\xi)$. Recall in [1, Ch. 1.2.2] that

$$G_s(\xi) = \frac{1}{\Gamma(\gamma/2)} \frac{1}{(2\sqrt{\pi})^n} \int_0^\infty e^{-t} e^{-\frac{|\xi|^2}{4t}} t^{\frac{s-n}{2}} \, \frac{dt}{t} \tag{0.4}$$

is a positive radial function and $\|G_s\|_{L^1(\mathbb{R}^n)} = 1$.

Using the identity

$$A^{-\frac{\gamma}{2}} = \frac{1}{\Gamma(\gamma/2)} \int_0^\infty e^{-tA} t^{\frac{\gamma}{2}} \, \frac{dt}{t},$$

2010 *Mathematics Subject Classification.* Primary 42B15, 42B25, 42B30.

which is valid for $A > 0$, we write

$$\big(1+\ln(1+4\pi^2|x|^2)\big)^{-\frac{\gamma}{2}} = \frac{1}{\Gamma(\gamma/2)}\int_0^\infty e^{-t}e^{-t\ln(1+4\pi^2|x|^2)}t^{\frac{\gamma}{2}}\,\frac{dt}{t}$$
$$= \frac{1}{\Gamma(\gamma/2)}\int_0^\infty e^{-t}\frac{1}{(1+4\pi^2|x|^2)^t}t^{\frac{\gamma}{2}}\,\frac{dt}{t}.$$

We obtain from this that the Fourier transform of $\big(1+\ln(1+4\pi^2|x|^2)\big)^{-\frac{\gamma}{2}}$ is

$$\frac{1}{\Gamma(\gamma/2)}\int_0^\infty e^{-t}G_{2t}(\xi)t^{\frac{\gamma}{2}}\,\frac{dt}{t}$$

and consequently,

$$\widehat{\mathcal{H}^{(s,\gamma)}}(\xi) = G_s * \Big(\frac{1}{\Gamma(\gamma/2)}\int_0^\infty e^{-t}G_{2t}(\cdot)t^{\frac{\gamma}{2}}\,\frac{dt}{t}\Big)(\xi) = \frac{1}{\Gamma(\gamma/2)}\int_0^\infty e^{-t}G_{2t+s}(\xi)t^{\frac{\gamma}{2}}\,\frac{dt}{t}.$$

Clearly, $\widehat{\mathcal{H}^{(s,\gamma)}}$ is a positive radial function since so is G_{2t+s}.

Proof of (0.2)

If $|\xi| \geqslant 1$, then $t + \frac{|\xi|^2}{4t} \geqslant t + \frac{1}{4t}$ and also $t + \frac{|\xi|^2}{4t} \geqslant |\xi|$. This implies that

$$t + \frac{|\xi|^2}{4t} \geqslant \frac{t}{2} + \frac{1}{8t} + \frac{|\xi|}{2} \quad \text{for } \ |\xi| \geqslant 1.$$

Therefore,

$$G_s(\xi) \lesssim_n e^{-\frac{|\xi|}{2}}\frac{1}{\Gamma(s/2)}\int_0^\infty e^{-\frac{t}{2}-\frac{1}{8t}}t^{\frac{s-n}{2}}\frac{dt}{t} \qquad \text{uniformly in } \ s > 0 \tag{0.5}$$

If $0 < s \leqslant n$, then $\frac{1}{\Gamma(s/2)} \lesssim_n 1$ and

$$\int_0^\infty e^{-\frac{t}{2}-\frac{1}{8t}}t^{\frac{s-n}{2}}\,\frac{dt}{t} \leqslant \int_0^1 e^{-\frac{1}{8t}}\frac{1}{t^{\frac{n}{2}+1}}dt + \int_1^\infty e^{-\frac{t}{2}}\,dt \lesssim_n 1,$$

which implies that

$$G_s(\xi) \lesssim_n e^{-\frac{|\xi|}{2}} \quad \text{uniformly in } \ 0 < s \leqslant n$$

When $s > n$,

$$\int_0^1 e^{-\frac{t}{2}-\frac{1}{8t}}t^{\frac{s-n}{2}}\,\frac{dt}{t} \leqslant \int_0^1 e^{-\frac{1}{8t}}\,\frac{dt}{t} \lesssim 1 \quad \text{uniformly in } \ s > n,$$

and by using a change of variables

$$\int_1^\infty e^{-\frac{t}{2}-\frac{1}{8t}}t^{\frac{s-n}{2}}\,\frac{dt}{t} \leqslant \int_1^\infty e^{-\frac{t}{2}}t^{\frac{s-n}{2}}\,\frac{dt}{t} = 2^{\frac{s-n}{2}}\int_{\frac{1}{2}}^\infty e^{-u}u^{\frac{s-n}{2}}\,\frac{du}{u}$$
$$\leqslant 2^{\frac{s}{2}}\int_{\frac{1}{2}}^\infty e^{-u}u^{\frac{s}{2}}\,\frac{du}{u} \leqslant 2^{\frac{s}{2}}\Gamma(s/2).$$

This proves that

$$G_s(\xi) \lesssim 2^{\frac{s}{2}}e^{-\frac{|\xi|}{2}} \quad \text{uniformly in } \ s > n.$$

Combining all together,

$$G_s(\xi) \lesssim_n 2^{\frac{s}{2}} e^{-\frac{|\xi|}{2}} \quad \text{uniformly in } \; s > 0.$$

Finally, we have

$$\widehat{\mathcal{H}^{(s,\gamma)}}(\xi) \lesssim_n 2^{\frac{s}{2}} e^{-\frac{|\xi|}{2}} \frac{1}{\Gamma(\gamma/2)} \int_0^\infty e^{-t(1-\ln 2)} t^{\frac{\gamma}{2}} \, \frac{dt}{t} \lesssim_{\gamma,n,s} e^{-\frac{|\xi|}{2}},$$

which completes the proof of (0.2).

Proof of (0.3)
Let us assume that $|\xi| \leqslant 1$. Using (0.4), we write

$$\begin{aligned}
\widehat{\mathcal{H}^{(s,\gamma)}}(\xi) &= \frac{1}{\Gamma(\gamma/2)} \frac{1}{(2\sqrt{\pi})^n} \int_0^\infty e^{-t} \frac{1}{\Gamma(t+s/2)} t^{\frac{\gamma}{2}} \Big(\int_0^\infty e^{-u} e^{-\frac{|\xi|^2}{4u}} u^{\frac{s-n}{2}} u^t \, \frac{du}{u} \Big) \, \frac{dt}{t} \\
&\sim_{\gamma,n} \int_0^\infty e^{-t} \frac{1}{\Gamma(t+s/2)} t^{\frac{\gamma}{2}} \Big(\int_1^\infty e^{-u} u^{\frac{s-n}{2}} u^t \, \frac{du}{u} \Big) \, \frac{dt}{t} \\
&\qquad + \int_0^1 e^{-\frac{|\xi|^2}{4u}} u^{\frac{s-n}{2}} \Big(\int_0^\infty e^{-t} \frac{1}{\Gamma(t+s/2)} u^t t^{\frac{\gamma}{2}} \, \frac{dt}{t} \Big) \, \frac{du}{u} \\
&=: \mathfrak{H}_1^{(s,\gamma)}(\xi) + \mathfrak{H}_2^{(s,\gamma)}(\xi).
\end{aligned}$$

where the equivalence

$$\begin{cases} e^{-\frac{|\xi|^2}{4u}} \sim 1 & u > 1, \; |\xi| \leqslant 1 \\ e^{-u} \sim 1 & 0 < u < 1 \end{cases}$$

is applied.

We first note that

$$\int_1^\infty e^{-u} u^{\frac{s-n}{2}} u^t \, \frac{du}{u} \leqslant \int_1^\infty e^{-u} u^{t+\frac{s}{2}} \, \frac{du}{u} \leqslant \Gamma(t+s/2),$$

and thus

$$0 \leqslant \mathfrak{H}_1^{(s,\gamma)}(\xi) \leqslant \int_0^\infty e^{-t} t^{\frac{\gamma}{2}} \, \frac{dt}{t} = \Gamma(\gamma/2) \lesssim_\gamma 1.$$

Therefore, it suffices to show that for all $s, \gamma > 0$,

$$\mathfrak{H}_2^{(s,\gamma)}(\xi) \sim_{\gamma,n,s} \mathfrak{T}^{(s,\gamma)}(\xi). \tag{0.6}$$

Proof of (0.6) for $s < n$. In this case, we need the following estimate:

$$\int_0^\infty e^{-t} \frac{1}{\Gamma(t+s/2)} u^t t^{\frac{\gamma}{2}} \, \frac{dt}{t} \sim_{\gamma,s,n} \frac{1}{(1+\ln u^{-1})^{\frac{\gamma}{2}}} \tag{0.7}$$

Observe that $\Gamma(t+s/2)\sim_{s,n} 1$ for $0<t<\frac{n-s}{2}$ and thus

$$\int_0^{\frac{n-s}{2}} e^{-t}\frac{1}{\Gamma(t+s/2)}u^t t^{\frac{\gamma}{2}}\ \frac{dt}{t}\sim_{s,n}\int_0^{\frac{n-s}{2}} e^{-t}u^t t^{\frac{\gamma}{2}}\ \frac{dt}{t}=\int_0^{\frac{n-s}{2}} e^{-t(1+\ln u^{-1})}t^{\frac{\gamma}{2}}\ \frac{dt}{t}$$
$$=\frac{1}{(1+\ln u^{-1})^{\frac{\gamma}{2}}}\int_0^{\frac{n-s}{2(1+\ln u^{-1})}} e^{-v}v^{\frac{\gamma}{2}}\ \frac{dv}{v}$$
$$\sim_{\gamma,s,n}\frac{1}{(1+\ln u^{-1})^{\frac{\gamma}{2}}}$$

as

$$1\sim_{s,n}\int_0^{\frac{n-s}{2}} e^{-v}v^{\frac{\gamma}{2}}\ \frac{dv}{v}\leqslant\int_0^{\frac{n-s}{2(1+\ln u^{-1})}} e^{-v}v^{\frac{\gamma}{2}}\ \frac{dv}{v}\leqslant\Gamma(\gamma/2)\sim_\gamma 1.$$

Moreover, if $t>\frac{n-s}{2}$, then $\Gamma(t+s/2)\gtrsim_n\Gamma(n/2)\gtrsim_n 1$. This implies that

$$0\leqslant\int_{\frac{n-s}{2}}^\infty e^{-t}\frac{1}{\Gamma(t+s/2)}u^t t^{\frac{\gamma}{2}}\ \frac{dt}{t}\lesssim_n\int_{\frac{n-s}{2}}^\infty e^{-t(1+\ln u^{-1})}t^{\frac{\gamma}{2}}\ \frac{dt}{t}$$
$$\lesssim\frac{1}{(1+\ln u^{-1})^{\frac{\gamma}{2}}}\Gamma(\gamma/2)\lesssim_\gamma\frac{1}{(1+\ln u^{-1})^{\frac{\gamma}{2}}},$$

which completes the proof of (0.7). Now by using (0.7), we have

$$\mathfrak{H}_2^{(s,\gamma)}(\xi)\sim_{\gamma,s,n}\int_0^1 e^{-\frac{|\xi|^2}{4u}}u^{-\frac{n-s}{2}}\frac{1}{(1+\ln u^{-1})^{\frac{\gamma}{2}}}\ \frac{du}{u}$$
$$=\int_0^{|\xi|^2} e^{-\frac{|\xi|^2}{4u}}u^{-\frac{n-s}{2}}\frac{1}{(1+\ln u^{-1})^{\frac{\gamma}{2}}}\ \frac{du}{u}+\int_{|\xi|^2}^1 e^{-\frac{|\xi|^2}{4u}}u^{-\frac{n-s}{2}}\frac{1}{(1+\ln u^{-1})^{\frac{\gamma}{2}}}\ \frac{du}{u}.$$

By applying a change of variable with $u=r|\xi|^2$, the first integral is equal to

$$\frac{1}{|\xi|^{n-s}}\int_0^1 e^{-\frac{1}{4r}}r^{-\frac{n-s}{2}}\frac{1}{(1+\ln r^{-1}+2\ln|\xi|^{-1})^{\frac{\gamma}{2}}}\ \frac{dr}{r}$$

and then this is comparable to

$$\frac{1}{|\xi|^{n-s}}\frac{1}{(1+2\ln|\xi|^{-1})^{\frac{\gamma}{2}}}$$

because

$$\frac{1}{(1+\ln r^{-1})(1+2\ln|\xi|^{-1})}\leqslant\frac{1}{(1+\ln r^{-1}+2\ln|\xi|^{-1})}\leqslant\frac{(1+\ln r^{-1})}{(1+2\ln|\xi|^{-1})},$$

and

$$\int_0^1 e^{-\frac{1}{4r}}r^{\frac{s-n}{2}}\frac{1}{(1+\ln r^{-1})^{\frac{\gamma}{2}}}\sim_{\gamma,s,n}\int_0^1 e^{-\frac{1}{4r}}r^{\frac{s-n}{2}}(1+\ln r^{-1})^{\frac{\gamma}{2}}\sim_{s,\gamma,n}1.$$

For the other one, we write

$$\int_{|\xi|^2}^1 e^{-\frac{|\xi|^2}{4u}}u^{-\frac{n-s}{2}}\frac{1}{(1+\ln u^{-1})^{\frac{\gamma}{2}}}\ \frac{du}{u}\sim\int_{|\xi|^2}^1 u^{-\frac{n-s}{2}}\frac{1}{(1+\ln u^{-1})^{\frac{\gamma}{2}}}\ \frac{du}{u}$$

as $e^{-\frac{|\xi|^2}{4u}} \sim 1$ for $|\xi|^2 < u < 1$. By applying a change of variable with $v = 1 + \ln u^{-1}$, this is equal to

$$\int_1^{1+2\ln|\xi|^{-1}} e^{\frac{(v-1)(n-s)}{2}} v^{-\frac{\gamma}{2}}\, dv \sim_{s,n} \int_1^{1+2\ln|\xi|^{-1}} e^{\frac{v(n-s)}{2}} v^{-\frac{\gamma}{2}}\, dv.$$

Now we perform an integration by parts to bound the last expression by

$$\frac{2}{n-s} e^{\frac{n-s}{2}} \Big(\frac{1}{|\xi|^{n-s}} \frac{1}{(1+2\ln|\xi|^{-1})^{\frac{\gamma}{2}}} - 1 \Big) + \frac{\gamma}{n-s} \int_1^{1+2\ln|\xi|^{-1}} e^{\frac{v(n-s)}{2}} v^{-\frac{\gamma}{2}-1}\, dv$$

and this is clearly controlled by a constant, depending on γ, s, and n, times

$$\frac{1}{|\xi|^{n-s}} \frac{1}{(1+2\ln|\xi|^{-1})^{\frac{\gamma}{2}}}$$

since for $1 < v < 1 + 2\ln|\xi|^{-1}$

$$e^{\frac{v(n-s)}{2}} v^{-\frac{\gamma}{2}-1} \lesssim_{s,\gamma,n} \frac{1}{|\xi|^{n-s}} \frac{1}{(1+2\ln|\xi|^{-1})^{\frac{\gamma}{2}+1}} \quad \text{uniformly in} \quad |\xi| \leqslant 1.$$

This finishes the proof of (0.6) for $s < n$.

Proof of (0.6) for $s > n$. Since $\frac{1}{\Gamma(t+s/2)} \lesssim_s 1$ and $u^t \leqslant 1$ for all $t > 0$,

$$\begin{aligned}
\mathfrak{H}_2^{(s,\gamma)}(\xi) &\lesssim_s \int_0^1 e^{-\frac{|\xi|^2}{4u}} u^{\frac{s-n}{2}} \Big(\int_0^\infty e^{-t} t^{\frac{\gamma}{2}} \frac{dt}{t} \Big) \frac{du}{u} \\
&= \Gamma(\gamma/2) \int_0^1 e^{-\frac{|\xi|^2}{4u}} u^{\frac{s-n}{2}} \frac{du}{u} \\
&\sim_\gamma \int_0^{|\xi|^2} e^{-\frac{|\xi|^2}{4u}} u^{\frac{s-n}{2}} \frac{du}{u} + \int_{|\xi|^2}^1 e^{-\frac{|\xi|^2}{4u}} u^{\frac{s-n}{2}} \frac{du}{u} \\
&\leqslant \int_0^1 e^{-\frac{1}{4r}} r^{\frac{s-n}{2}} \frac{dr}{r} + \int_{|\xi|^2}^1 u^{\frac{s-n}{2}} \frac{du}{u} \\
&\lesssim_{s,n} 1
\end{aligned}$$

where we applied a change of variables and $|\xi| \leqslant 1$ in the penultimate inequality. On the other hand, since $\Gamma(t + s/2) \lesssim_s 1$ for $0 < t < 1$,

$$\begin{aligned}
\mathfrak{H}_2^{(s,\gamma)}(\xi) &\geqslant \int_{\frac{1}{2}}^1 e^{-\frac{|\xi|^2}{4u}} u^{\frac{s-n}{2}} \Big(\int_0^1 e^{-t} \frac{1}{\Gamma(t+s/2)} u^t t^{\frac{\gamma}{2}} \frac{dt}{t} \Big) \frac{du}{u} \\
&\gtrsim_{s,n} \int_{\frac{1}{2}}^1 \Big(\int_0^1 t^{\frac{\gamma}{2}} \frac{dt}{t} \Big) \frac{du}{u} \sim_\gamma 1, \qquad (0.8)
\end{aligned}$$

which proves (0.6) for $s > n$.

Proof of (0.6) for $s = n$. We note that (0.8) also holds for $s = n$ so that

$$\mathfrak{H}_2^{(n,\gamma)}(\xi) \gtrsim_{n,\gamma} 1.$$

For the remaining estimates, we express

$$\begin{aligned}\mathfrak{H}_2^{(n,\gamma)}(\xi) \sim \int_0^{|\xi|^2} e^{-\frac{|\xi|^2}{4u}} \Big(\int_0^\infty e^{-t} \frac{1}{\Gamma(t+n/2)} u^t t^{\frac{\gamma}{2}} \, \frac{dt}{t} \Big) \, \frac{du}{u} \\ + \int_{|\xi|^2}^1 \Big(\int_0^\infty e^{-t} \frac{1}{\Gamma(t+n/2)} u^t t^{\frac{\gamma}{2}} \, \frac{dt}{t} \Big) \, \frac{du}{u}\end{aligned} \tag{0.9}$$

as we have $e^{-\frac{|\xi|^2}{4u}} \sim 1$ in the second integral. The first integral is clearly dominated by a constant, which depends on n, times

$$\int_0^{|\xi|^2} e^{-\frac{|\xi|^2}{4u}} \Big(\int_0^\infty e^{-t} t^{\frac{\gamma}{2}} \, \frac{dt}{t} \Big) \, \frac{du}{u} = \Gamma(\gamma/2) \int_0^1 e^{-\frac{1}{4r}} \, \frac{dr}{r} \sim_\gamma 1 \quad \text{for all} \;\; 0 < \gamma < \infty.$$

When $\gamma > 2$,

$$\begin{aligned} 0 \leqslant \int_{|\xi|^2}^1 \Big(\int_0^\infty e^{-t} \frac{1}{\Gamma(t+n/2)} u^t t^{\frac{\gamma}{2}} \, \frac{dt}{t} \Big) \, \frac{du}{u} &= \int_0^\infty e^{-t} \frac{1}{\Gamma(t+n/2)} t^{\frac{\gamma}{2}} \Big(\int_{|\xi|^2}^1 u^t \, \frac{du}{u} \Big) \, \frac{dt}{t} \\ &\lesssim_n \int_0^\infty e^{-t} t^{\frac{\gamma}{2}-1} \, \frac{dt}{t} = \Gamma(\gamma/2-1) \sim_\gamma 1, \end{aligned}$$

which finally yields that

$$\mathfrak{H}_2^{(n,\gamma)}(\xi) \sim_{n,s} 1 \quad \text{for} \;\; \gamma > 2.$$

Now, in order to conclude (0.6), it remains to establish

$$\int_{|\xi|^2}^1 \Big(\int_0^\infty e^{-t} \frac{1}{\Gamma(t+n/2)} u^t t^{\frac{\gamma}{2}} \, \frac{dt}{t} \Big) \, \frac{du}{u} \sim_{\gamma,n} \mathfrak{T}^{(n,\gamma)}(\xi) \quad \text{for} \;\; 0 < \gamma \leqslant 2. \tag{0.10}$$

For this one, we see that for $0 < u < 1$,

$$\int_0^\infty e^{-t} \frac{1}{\Gamma(t+n/2)} u^t t^{\frac{\gamma}{2}} \, \frac{dt}{t} \sim_\gamma \frac{1}{(1+\ln u^{-1})^{\frac{\gamma}{2}}}. \tag{0.11}$$

Indeed, by a change of variables,

$$\int_0^\infty e^{-t} \frac{1}{\Gamma(t+n/2)} u^t t^{\frac{\gamma}{2}} \, \frac{dt}{t} \lesssim_n \int_0^\infty e^{-t(1+\ln u^{-1})} t^{\frac{\gamma}{2}} \, \frac{dt}{t} = \frac{1}{(1+\ln u^{-1})^{\frac{\gamma}{2}}} \Gamma(\gamma/2)$$

and

$$\begin{aligned}\int_0^\infty e^{-t}\frac{1}{\Gamma(t+n/2)}u^t t^{\frac{\gamma}{2}}\ \frac{dt}{t} &\geqslant \int_0^1 e^{-t(1+\ln u^{-1})}\frac{1}{\Gamma(t+n/2)}t^{\frac{\gamma}{2}}\ \frac{dt}{t}\\ &\gtrsim_n \int_0^1 e^{-t(1+\ln u^{-1})}t^{\frac{\gamma}{2}}\ \frac{dt}{t}\\ &= \frac{1}{(1+\ln u^{-1})^{\frac{\gamma}{2}}}\int_0^{1+\ln u^{-1}} e^{-v}v^{\frac{\gamma}{2}}\ \frac{dv}{v}\\ &\geqslant \frac{1}{(1+\ln u^{-1})^{\frac{\gamma}{2}}}\int_0^1 e^{-v}v^{\frac{\gamma}{2}}\ \frac{dv}{v} \sim_\gamma \frac{1}{(1+\ln u^{-1})^{\frac{\gamma}{2}}}.\end{aligned}$$

Due to (0.11), the left-hand side of (0.10) is comparable to

$$\int_{|\xi|^2}^1 \frac{1}{(1+\ln u^{-1})^{\frac{\gamma}{2}}}\ \frac{du}{u}$$

and then by performing another change of variables with $w = 1+\ln u^{-1}$, it would be

$$\begin{aligned}\int_1^{1+2\ln|\xi|^{-1}} \frac{1}{w^{\frac{\gamma}{2}}}\ dw &\sim_\gamma \begin{cases}\ln\left(1+2\ln|\xi|^{-1}\right) & \text{if }\ \gamma=2\\ (1+2\ln|\xi|^{-1})^{1-\frac{\gamma}{2}} & \text{if }\ 0<\gamma<2\end{cases}\\ &= \mathfrak{T}^{(n,\gamma)}(\xi),\end{aligned}$$

as desired.

□

Acknowledgment: We would like to thank Danqing He for pointing out to us an error in Lemma 4.1.

References

[1] L. Grafakos, *Modern Fourier Analysis*, 3rd edition, Graduate Texts in Mathematics 250, Springer, NY 2014.

L. Grafakos, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
Email address: grafakosl@missouri.edu

B. Park, Department of Mathematics, Sungkyunkwan University, Suwon 16419, Republic of Korea
Email address: bpark43@skku.edu