LONG-RANGE INSTABILITY OF LINEAR EVOLUTION PDE ON SEMI-BOUNDED DOMAINS VIA THE FOKAS METHOD

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ABSTRACT. We study the inhomogeneous Airy partial differential equation (also called Stokes or linearized Korteweg-de Vries equation with a negative sign) on the half-line with generic initial and boundary data in a classical smooth setting, via the formula provided by the Fokas unified transform method for linear evolution equations. We first present a suitable decomposition of that formula in the complex plane in order to appropriately interpret various terms appearing in it, thus securing convergence in a strict sense. Writing the solution in an Ehrenpreis-Palamodov form, our analysis allows for rigorous *a posteriori* verification of the full initial-boundary-value problem and a thorough investigation of the behavior of the solution near the boundaries of the spatiotemporal domain. We prove that the integrals in this representation converge uniformly to prescribed values and the solution admits a smooth extension up to the boundary only under certain data compatibility conditions (with implications for well-posedness, control theory and efficient numerical computations). Importantly, based on this analysis, we perform an effective asymptotic study of far-field dynamics. This yields new explicit asymptotic formulae which characterize the properties of the solution in terms of (in)compatibilities of the data at the 'corner' of the quadrant. In particular, the asymptotic behavior of the solution is sensitive to perturbations of the data at the origin. In all cases, even assuming the initial data to belong to the Schwartz class, the solution loses this property at soon as time becomes positive. Hereby, we report on the discovery of a novel type of a long-range instability phenomenon for linear dispersive differential equations. Our ideas are extendable to other Airy-like and more general problems for dispersive evolution equations.

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Contents

1. Introduction and preliminaries

The Unified Transform Method (henceforth UTM), also known as Fokas method, was introduced about 25 years ago [50] and was initially conceived as a method for solving initial-boundary value problems (IBVP) for completely integrable nonlinear partial differential equations (PDE). Typical examples of equations that can be studied in this way include the Korteweg-de Vries (KdV), the nonlinear Schrödinger (NLS) or, more generally, equations that can be formulated as evolutions in time of a linear differential operator $L(t)$ governed by the famous Lax pair equation $\frac{dL}{dt} = BL - LB$ where $B(L)$ is usually some auxiliary antisymmetric linear differential operator. While the investigation of an initial-value problem for KdV involves the study of the scattering transform for the associated linear Schrödinger operator L, the role of B being somewhat trivialized, the investigation of the IBVP involves the joint study of scattering data for both operators L and B; hence the term "Unified Transform". The interdependence of the two operators renders a highly nontrivial extension of the standard scattering method, e.g. [57], [90], [55]. Even though this method was initially proposed for nonlinear problems, it soon became evident that it was also applicable to linear problems e.g. [60], [51], [52], [35], [41], [54], [64] (in chronological order). While, prior to the new method, the existing tools for boundary value problems of linear PDE (such as the Laplace or the sine transform) were *explicitly* applicable only to very specific cases, the new method has been spectacularly successful in a much wider class of linear-PDE problem - of any order and in all sorts of domains - including elliptic ones and applications with variable coefficients or even with a moving boundary; it has also been implemented or extended to cases of mixed derivatives, nonlocal/nonseparable conditions, interface problems, systems of equations and fractional-order evolution equations too. Incidentally, this method has led to surprising results in spectral theory and control theory as well as to the development of a new effective approach (pioneered by Fokas and Himonas who conceived and implemented it) to establishing well-posedness for nonlinear evolution PDE. For all these areas of application of the Fokas methodology and extensions thereof by many mathematicians we refer to [59], [63], [38], [117], [4], [66], [37], [129], [88], [61], [126], [101], [106], [77], [40], [118], [58], [127], [46], [67], [112], [119], [32], [62], [78], [116], [85], [89], [39], [2], [107] (listed in chronological order) and references therein. In fact, the linear method offers insights into the nonlinear integrability theory by helping to realize that Lax pairs provide the generalization of the divergence formulation in the separable linear case. Not only explicit formulae for the solutions are provided, but such formulae have also been observed numerically to be very efficient, e.g. [48], [125], [65], [68], [73], [6], [96], [36] and have led to the development of novel hybrid numerical-analytical techniques. A crucial quality of these formulae appears to be that they are uniformly well-behaved near the boundary.

Recently, a new line of investigation and a rigorous analytical approach have been introduced in [18], applied in e.g. [26], and extended in [25], [27], [23], [19], [20], [21], [28], [22], in order to study miscellaneous properties (concerning e.g. regularity and qualitative theory, well-posedness and asymptotic analysis) of linear evolution PDE posed in the quarter-plane. This novel technique as a starting point utilized the integral formulae afforded by the method of Fokas. The current work expands this rigorously-minded program while achieving valuable new knowledge for a classical and important case of IBVP. At the same time, it inaugurates a new area of implementation of the UTM, that is the investigation and analysis of 'distant' sensitivity (or *long-range instabilities*) of solutions of dispersive PDE formulated on semi-(un)bounded domains.

In [25], the IBVP for the forced linearized KdV equation on the positive half-line was considered:

$$\begin{cases} U_t + U_{xxx} = f, & (x,t) \in Q := \mathbb{R}^+ \times \mathbb{R}^+ \\ U(x,0) = u_0(x), & x \in \mathbb{R}^+ \\ U(0,t) = g_0(t), & t \in \mathbb{R}^+, \end{cases}$$

where the data u_0, g_0 and f were given continuous functions defined in $\mathbb{R}^+ \cup \{0\}$. Further regularity assumptions on the data were explicitly stated in [25], each assumption tailor-made according to the different regularity properties one wants to secure for the solution. For a seemingly simple problem like this, there were no classical *spatial*-transform methods, i.e. in the x-variable, until the appearance of an explicit formula which is derived by a straightforward application of the new method. Importantly, therefore, the UTM provides the direct analogue of the Fourier transform in the case of IBVP.

At this stage, it should be noted that there is a vast body of literature on the KdV equation and its variants, so a review of that literature is outside the scope of our article. Nevertheless, within the plethora of such articles, we indicatively mention the related works (listed chronologically): [83], [87], [131], [9], [122], [1], [31], [94], [111], [56], [12], [99], [114], [124], [8], [10], [69], [29], [95], [134], [13], [42], [105], [14], [49], [102], [130], [121], [11], [34], [53], [71], [74], [108], [120], [33], [44], [133], [84], [93], [109], [103], [98], [100], [104], [115], [72], [110], [82], [113], [70], [86], [3], [15], [43], [16], [75], [123], [128], [45], [17], [97], [47], [132], [7], [76], [79], [80], and [81] which address either the whole-space Cauchy problem or IBVP (e.g. on half-line or circle) for KdV-type equations and offer fascinating and important results principally pertaining to long-time asymptotics and questions of well-posedness in low-regularity settings, which, in turn, or course, are based on appropriate estimates for the linear parts of the equations.

In this paper, we undertake a detailed study of IBVP for the forced Airy partial differential equation (or Stokes or linearized KdV equation with a negative sign) on the half-line. We begin with the homogeneous version, namely:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^3 u}{\partial x^3} = 0, & (x,t) \in Q := \mathbb{R}^+ \times \mathbb{R}^+ \\ \lim\limits_{t \to 0^+} u(x,t) = u_0(x), & x \in \mathbb{R}^+ \\ \lim\limits_{x \to 0^+} u(x,t) = g_0(t), & t \in \mathbb{R}^+ \\ \lim\limits_{x \to 0^+} \dfrac{\partial u(x,t)}{\partial x} = g_1(t), & t \in \mathbb{R}^+, \end{cases} \tag{1.1}$$

where $u = u(x,t)$. For clarity of exposition and increased readability, we will study the above problem under the following assumptions on the data:

$$u_0(x) \in S\big([0,\infty)\big) \quad \text{and} \quad g_0(t), g_1(t) \in C^\infty\big([0,\infty)\big).$$

Even though there is only a sign change in comparison to the linear KdV, the analysis of the present IBVP is significantly more involved. To begin with, an extra (Neumann) condition is required at the boundary $x = 0$. The most surprising, perhaps, of our findings is the sensitivity of the long-range behavior ($x \to +\infty$, for fixed times t) to perturbations of the data near the corner $(x,t) = (0,0)$. Moreover, as we continue the study of different linear equations, our intuition that the formulae provided by the UTM are numerically-friendly is being confirmed. This is another important outcome of the present work.

As in previous works [25], [27], we will not provide the initial construction of this formula by Fokas; we refer to [41] for a clear formal derivation. Here, we will rather completely and rigorously justify it a posteriori and prove that the function defined by this formula:

(1) is well-defined,
(2) satisfies the Airy equation for all positive x and t,

(3) converges uniformly to the given initial and boundary data as the variables approach the boundaries of our domain, i.e. the semi-axes $t = 0$ and $x = 0$,
(4) assuming appropriate conditions on the regularity of the data, derivatives of the solution also approach derivatives (of the corresponding order) of the data, as one approaches the boundary.
(5) depending on whether the data u_0, g_0, g_1, and their derivatives, satisfy certain compatibility conditions at $(x, t) = (0, 0)$, the solution and derivatives may tend to zero, as $x \to +\infty$. The rate of convergence also depends critically on the choice of such compatibility conditions!

The plan of this paper is as follows: In the next two sections, we present the formula provided by the UTM and show that the integrals in it are well-defined and provide a classical solution of the considered IBVP in the open quadrant $Q = \{x > 0, t > 0\}$ converging pointwise to the given data at the boundary, at least away from the point $(x = 0, t = 0)$. We prove that the convergence is actually uniform in appropriate compact sets. In the third section we prove the uniform convergence of the derivatives of the solution near the half-lines $\{x = 0, t > 0\}$, $\{t = 0, x > 0\}$. In the forth section of this paper, we investigate the behavior, near the point $(x = 0, t = 0)$, of both the solution and its derivatives, depending of course on the availability of appropriate compatibility conditions on the data u_0, g_0, g_1, and their derivatives at that point. In the two subsequent sections we provide asymptotic results for certain exponential integrals, which we then use to derive the long-space asymptotics of the solution and its derivatives. As pointed out, the rate of convergence, as $x \to \infty$, depends sensitively on the choice of the relevant compatibility conditions on the data u_0, g_0, g_1, and their derivatives at $(x = 0, t = 0)$! In the seventh section, we study the analogous questions for the inhomogeneous (forced) equation $\partial_t U - \partial_{xxx} U = f$, for an appropriate class of functions $f = f(x, t)$. As a byproduct, the formula in this case (i.e. with forcing term) has not appeared in the literature before, to the best of our knowledge, and might perhaps be used for investigation of wellposedness questions for nonlinear analogues via the Fokas-Himonas technique, as it has already been done for a variety of problems in e.g. [58], [116], [132], [76], [79], [80]; see also [81] for the recent extension of this approach by Himonas and Yan to *systems* of nonlinear pde. Finally, Section 8 summarizes and discusses our main results and their consequences while offering meaningful directions for future research.

Part of our findings has been announced in short during the 28$^{\text{th}}$ International Conference "Dynamical Systems and Complexity" (Crete, Greece, 18-26 July 2022) [24] which was dedicated to Prof. A. S. Fokas on the occasion of his 70$^{\text{th}}$ birthday.

The Fokas formula: The unified transform method yields the following solution to problem (1.1):

$$
\begin{aligned}
u(x,t) = \frac{1}{2\pi}\Bigg[\int_{\lambda=-\infty}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda - \int_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda \\
- \int_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)\,d\lambda\Bigg] \\
+ \frac{1}{2\pi}\Bigg[\int_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha^2)\lambda^2\tilde{g}_0(\omega(\lambda),t)\,d\lambda \\
+ \int_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha)\lambda^2\tilde{g}_0(\omega(\lambda),t)\,d\lambda\Bigg] \\
- \frac{i}{2\pi}\Bigg[\int_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha)\lambda\tilde{g}_1(\omega(\lambda),t)\,d\lambda \\
+ \int_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha^2)\lambda\tilde{g}_1(\omega(\lambda),t)\,d\lambda\Bigg],
\end{aligned}
\tag{1.2}
$$

for $x>0$ and $t>0$, where $\omega(\lambda)=i\lambda^3$, $\alpha=e^{2\pi i/3}$,

$$
\begin{aligned}
\Omega_1^- &= \{\lambda\in\mathbb{C} : \operatorname{Im}\lambda\ge 0,\ \operatorname{Re}\lambda\le 0 \text{ and } \operatorname{Re}\omega(\lambda)\le 0\}\\
&= \{\lambda\in\mathbb{C} : (2\pi/3)\le \arg\lambda\le\pi\},
\end{aligned}
$$

and

$$
\begin{aligned}
\Omega_2^- &= \{\lambda\in\mathbb{C} : \operatorname{Im}\lambda\ge 0,\ \operatorname{Re}\lambda\ge 0 \text{ and } \operatorname{Re}\omega(\lambda)\le 0\}\\
&= \{\lambda\in\mathbb{C} : 0\le \arg\lambda\le\pi/3\}
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{u}_0(\lambda) &= \int_{y=0}^{\infty} e^{-i\lambda y}u_0(y)dy,\\
\tilde{g}_0(\omega(\lambda),t) &= \int_{\tau=0}^{t} e^{\omega(\lambda)\tau}g_0(\tau)d\tau,\\
\tilde{g}_1(\omega(\lambda),t) &= \int_{\tau=0}^{t} e^{\omega(\lambda)\tau}g_1(\tau)d\tau.
\end{aligned}
$$

The regions Ω_1^- and Ω_2^- and the contours around them are shown in Figure 1.

Remarks: We recall the following facts:

(1) $\hat{u}_0(\lambda)$ is defined for $\lambda\in\mathbb{C}$ with $\operatorname{Im}\lambda\le 0$, C^∞ in $\{\lambda\in\mathbb{C} : \operatorname{Im}\lambda\le 0\}$ and is analytic in $\{\lambda\in\mathbb{C} : \operatorname{Im}\lambda<0\}$, while the functions $\tilde{g}_j(\omega(\lambda),t)$, $j=0,1$, are defined and are analytic for $\lambda\in\mathbb{C}$.

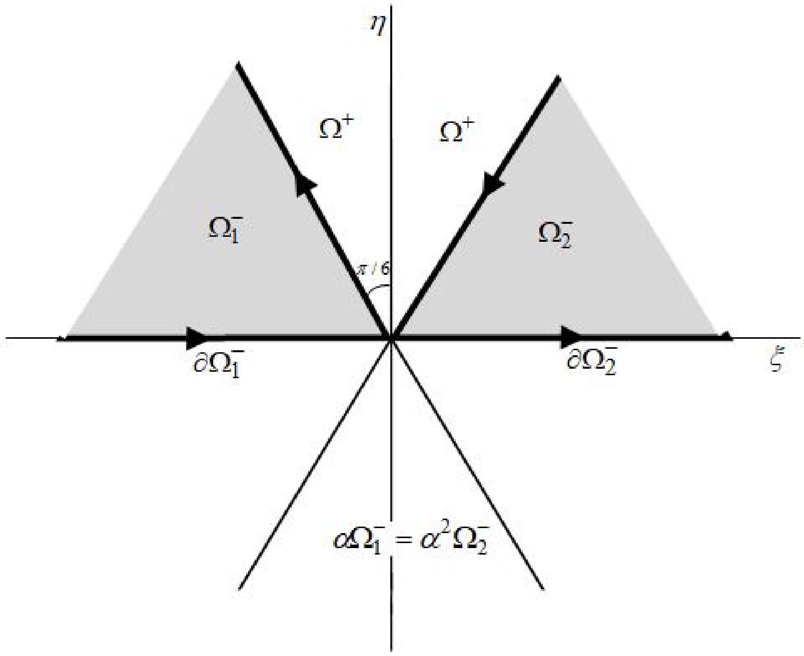

FIGURE 1. The contours used in the Fokas formula.

(2) Integration by parts shows

$$\hat{u}_0(\lambda) = \frac{u_0(0)}{i\lambda} + \int_{y=0}^{\infty} e^{-i\lambda y}\frac{du_0(y)}{dy}dy \quad (\lambda \neq 0,\ \operatorname{Im}\lambda \leq 0) \tag{1.3}$$

and for $\lambda \neq 0,\ j = 0, 1$

$$e^{-\omega(\lambda)t}\tilde{g}_j(\omega(\lambda), t) = \frac{g_j(t)}{\omega(\lambda)} - \frac{g_j(0)}{\omega(\lambda)}e^{-\omega(\lambda)t} - e^{-\omega(\lambda)t}\int_{\tau=0}^{t} e^{\omega(\lambda)\tau}\frac{dg_j(\tau)}{d\tau}d\tau. \tag{1.4}$$

In particular,

$$\hat{u}_0(\lambda) = O(1/\lambda), \quad \text{as } \mathbb{C} \ni \lambda \to \infty \text{ with } \operatorname{Im}\lambda \leq 0,$$

and

$$e^{-\omega(\lambda)t}\tilde{g}_j(\omega(\lambda), t) = O(1/\lambda^3), \quad \text{as } \mathbb{C} \ni \lambda \to \infty \text{ with } \operatorname{Re}\omega(\lambda) \geq 0.$$

(3) The integrals in the RHS of (1.2) taken over the contours $\partial\Omega_1^- \cap \{\arg\lambda = 2\pi/3\}$ and $\partial\Omega_2^- \cap \{\arg\lambda = \pi/3\}$ converge absolutely because of the presence of the factor $e^{i\lambda x}$ since, when $\arg\lambda = 2\pi/3$ or $\arg\lambda = \pi/3$,

$$\left|e^{i\lambda x - \omega(\lambda)t}\right| = \left|e^{i\lambda x}\right| = e^{-(\operatorname{Im}\lambda)x} = e^{-\frac{\sqrt{3}}{2}x|\lambda|}, \tag{1.5}$$

hence

$$\left|e^{i\lambda x}\right| \leq e^{-\frac{\sqrt{3}}{2}x_0|\lambda|} \quad \text{for} \quad x \geq x_0 > 0.$$

In this regard, let us notice also that

$$[\lambda \in \partial\Omega_1^- \Rightarrow \operatorname{Im}(\alpha\lambda) \leq 0] \quad \text{and} \quad [\lambda \in \partial\Omega_2^- \Rightarrow \operatorname{Im}(\alpha^2\lambda) \leq 0],$$

so that the integrals over $\partial\Omega_1^-$ and $\partial\Omega_2^-$, which involve the functions $\hat{u}_0(\alpha\lambda)$ and $\hat{u}_0(\alpha^2\lambda)$, are well defined.

Comment: The integrals in (1.2) are defined for $x > 0$ and $t > 0$. We will show later that the various parts of these integrals either converge absolutely or exist in the generalized sense, i.e., as limits of the form $\lim_{A\to\infty}\int_{|\lambda|\leq A}\cdots$, as long as $x > 0$ and $t > 0$. In particular, we will see that - in general - the parts of the integrals in (1.2), which are taken on the real line intervals $(-\infty, 0]$ or $[0, \infty)$ and which involve $\hat{u}_0$ or $\tilde{g}_0$, do not converge absolutely, and we will have to deform their contours in order to be able to handle them - even for $x > 0$ and $t > 0$. These integrals oscillate faster if we differentiate them under the integral sign - with respect to x or t - or if we let $x \to 0^+$ or $t \to 0^+$. As a matter of fact, all the integrals in (1.2), and their derivatives (i.e., including the ones taken on the contours $\partial\Omega_1^- \cap \{\arg \lambda = 2\pi/3\}$ and $\partial\Omega_2^- \cap \{\arg \lambda = \pi/3\}$) become oscillatory if we let $x \to 0^+$, as the factor $e^{i\lambda x}\big|_{x=0} = 1$ loses its rapid decrease, as $\lambda \to \infty$, even when $\arg\lambda = 2\pi/3$ or $\arg\lambda = \pi/3$. In this regard and in order to see how important it is to justify the interchange of limits and differentiations with the various integrals - whenever this holds; see for example the embedded Remark 2.3 in the proof of Theorem 1.1. Also in this respect, let us notice that if we write $\lambda = \xi + i\eta$, $\xi, \eta \in \mathbb{R}$, then

$$i\lambda x - \omega(\lambda)t = (-\eta x + 3\xi^2\eta t - \eta^3 t) + i(\xi x - \xi^3 t + 3\xi\eta^2 t)$$

and

$$\left|e^{i\lambda x-\omega(\lambda)t}\right|\Big|_{\eta=-\varepsilon} = e^{-\eta x+3\xi^2\eta t-\eta^3 t}\Big|_{\eta=-\varepsilon} = e^{-3\varepsilon\xi^2 t+\varepsilon^3 t-\varepsilon x} \approx e^{-3\varepsilon|\lambda|^2 t+\varepsilon^3 t+\varepsilon x} \tag{1.6}$$

as $\lambda = \xi + i\eta \to \infty$ with $\eta = \varepsilon$. We emphasize that the simple equations (1.3), (1.4), (1.5) and (1.6), combined with Cauchy's theorem and Jordan's lemma, play an important role in the deformation of the contours which are necessary to deal with the aforementioned limit and differentiation processes of the integrals, when Lebesgue's dominated convergence theorem cannot be applied immediately.

The integrals of the solution: As we have pointed out, some of the integrals in (1.2) do not converge absolutely and they deteriorate when we differentiate them under the integral sign with respect to x or t. Thus they have to be interpreted appropriately.

The first integral in (1.2) can be interpreted as follows:

$$\int_{\lambda=-\infty}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)\,d\lambda = \lim_{A\to\infty}\int_{\lambda=-A}^{A} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)\,d\lambda = \lim_{A\to\infty}\int_{\{\mathrm{Im}\lambda=-\varepsilon\}\cap\{-A\le\mathrm{Re}\lambda\le A\}} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)\,d\lambda, \tag{1.7}$$

where ε is a small positive constant. (We will see that the second limit in (1.7) is independent of ε). Indeed, applying Cauchy's theorem in the rectangle $\{\lambda\in\mathbb{C} : -A\le\mathrm{Re}\,\lambda\le A, -\varepsilon\le\mathrm{Im}\,\lambda\le 0\}$, we have

$$\int_{\lambda=-A}^{A} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda = \int_{\{\mathrm{Im}\lambda=-\varepsilon\}\cap\{-A\le\mathrm{Re}\lambda\le A\}} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda + \int_{\{0\le\mathrm{Im}\lambda\le-\varepsilon\}\cap\{\mathrm{Re}\lambda=\pm A\}} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda. \tag{1.8}$$

Now, since $\hat{u}_0(\lambda)=O(1/\lambda)$ for $\lambda\to\infty$ with $\mathrm{Im}\,\lambda\le 0$ and

$$\sup\{\left|e^{i\lambda x-\omega(\lambda)t}\right| : -1\le\mathrm{Im}\,\lambda\le 0\}<+\infty,$$

we have

$$\lim_{A\to\infty}\int_{\{0\le\mathrm{Im}\lambda\le-\varepsilon\}\cap\{\mathrm{Re}\lambda=\pm A\}} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda = 0. \tag{1.9}$$

Also, in view of (1.6), the integral

$$\int_{\{\mathrm{Im}\lambda=-\varepsilon\}\cap\{-\infty<\mathrm{Re}\lambda<+\infty\}} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda$$

converges absolutely, for fixed $x\ge 0$ and $t>0$, and is equal to the last limit in (1.7). Moreover,

$$\int_{\{\mathrm{Im}\lambda=-\varepsilon\}\cap\{-\infty<\mathrm{Re}\lambda<+\infty\}} \left|\lambda^N e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)\right||d\lambda|<+\infty \quad \text{for every} \quad N\in\mathbb{N} \tag{1.10}$$

This implies that the integral (1.7) defines a C^∞ function in $(x,t)\in Q$ and we may write

$$\frac{\partial^{n+k}}{\partial x^n\partial t^k}\left[\int_{\lambda=-\infty}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda\right] = \int_{\{\mathrm{Im}\lambda=-\varepsilon\}} (i\lambda)^n[-\omega(\lambda)]^k e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda.$$

The parts of all the integrals in (1.2) which are taken on the lines

$$\{\lambda \in \mathbb{C} : \arg \lambda = \pi/3\} \quad \text{or} \quad \{\lambda \in \mathbb{C} : \arg \lambda = 2\pi/3\}$$

converge absolutely and remain such after any number of differentiations with respect to x or t, provided that $x > 0$. This follows from the presence in these integrals of the factor $e^{i\lambda x}$ and (1.5). Let us notice also that $\hat{u}_0(\alpha\lambda) = O(1/\lambda)$ and $\hat{u}_0(\alpha^2\lambda) = O(1/\lambda)$, when $\lambda \to \infty$ with $\arg \lambda = \pi/3$ or $\arg \lambda = 2\pi/3$, and that

$$\left|e^{-\omega(\lambda)t}\tilde{g}_0(\omega(\lambda), t)\right| \leq \int_{\tau=0}^{t} |g_0(\tau)| d\tau \quad \text{and} \quad \left|e^{-\omega(\lambda)t}\tilde{g}_1(\omega(\lambda), t)\right| \leq \int_{\tau=0}^{t} |g_1(\tau)| d\tau,$$

for $\arg \lambda = \pi/3$ or $\arg \lambda = 2\pi/3$, since $\operatorname{Re} \omega(\lambda) = 0$ for these λ.

Next, working as with the integral (1.7), we obtain

$$\begin{aligned}
&\int_{\lambda=-\infty}^{0} e^{i\lambda x - \omega(\lambda)t} \hat{u}_0(\alpha\lambda) d\lambda \\
&= \int_{\lambda=-\infty}^{-1} e^{i\lambda x - \omega(\lambda)t} \hat{u}_0(\alpha\lambda) d\lambda + \int_{\lambda=-1}^{0} e^{i\lambda x - \omega(\lambda)t} \hat{u}_0(\alpha\lambda) d\lambda \\
&= \int_{\{\mathrm{Im}\lambda = -\varepsilon\} \cap \{\mathrm{Re}\lambda \leq -1\}} e^{i\lambda x - \omega(\lambda)t} \hat{u}_0(\alpha\lambda) d\lambda \\
&\qquad + \int_{\lambda \in [-1-\varepsilon i, -1]} e^{i\lambda x - \omega(\lambda)t} \hat{u}_0(\alpha\lambda) d\lambda + \int_{\lambda=-1}^{0} e^{i\lambda x - \omega(\lambda)t} \hat{u}_0(\alpha\lambda) d\lambda,
\end{aligned} \tag{1.11}$$

where the integrals taken over infinite parts of the half line $(-\infty, 0]$ have to be interpreted in the generalized sense. (Since $\hat{u}_0(\alpha\lambda)$ is defined provided that $\mathrm{Im}(\alpha\lambda) \leq 0$, we split $\int_{\lambda=-\infty}^{0}$ as the sum $\int_{\lambda=-\infty}^{-1} + \int_{\lambda=-1}^{0}$ so that the first two integrals in the RHS of (1.11) are well defined. Also, for this reason, we chose ε sufficiently small).

Similarly, we may write

$$\begin{aligned}
&\int_{\lambda=0}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda \\
&\quad= \int_{\lambda=0}^{1} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda + \int_{\lambda=1}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda \\
&\quad= \int_{\lambda=0}^{1} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda + \int_{\lambda\in[1,1-\varepsilon i]} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda \\
&\qquad\quad + \int_{\{\mathrm{Im}\lambda=-\varepsilon\}\cap\{\mathrm{Re}\lambda\geq 1\}} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda.
\end{aligned} \tag{1.12}$$

Now, it is easy to see that the integrals in (1.2) which involve the function $\hat{u}_0$ define C^∞ functions for $(x,t)\in Q$ which satisfy the equation $u_t = u_{xxx}$.

It remains to deal with the integrals of (1.2) which involve the functions $\tilde{g}_0$ and $\tilde{g}_1$ and are taken on the parts of the contours $\partial\Omega_1^-$ and $\partial\Omega_2^-$ which lie on the real line, i.e., $(\partial\Omega_1^-)\cap\mathbb{R}$ and $(\partial\Omega_2^-)\cap\mathbb{R}$.

First, we will show that, for $0<t<T$,

$$\begin{aligned}
\lim_{A\to\infty}\Bigg[&\int_{(\partial\Omega_1^-)\cap\{|\lambda|\leq A\}} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),t)\,d\lambda \\
&- \int_{(\partial\Omega_1^-)\cap\{|\lambda|\leq A\}} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),T)\,d\lambda\Bigg] = 0.
\end{aligned} \tag{1.13}$$

Indeed, by Cauchy's theorem, the difference of the two integrals in (1.13) is equal to

$$\begin{aligned}
&\int_{(\partial\Omega_1^-)\cap\{|\lambda|\leq A\}} e^{i\lambda x-\omega(\lambda)t}\lambda^2\left(\int_{\tau=t}^{T} e^{\omega(\lambda)\tau}g_0(\tau)d\tau\right)d\lambda \\
&\quad= \int_{\{|\lambda|=A\}\cap\{(2\pi/3)\leq\arg\lambda\leq\pi\}} e^{i\lambda x-\omega(\lambda)t}\lambda^2\left(\int_{\tau=t}^{T} e^{\omega(\lambda)\tau}g_0(\tau)d\tau\right)d\lambda.
\end{aligned} \tag{1.14}$$

But one has

$$e^{-\omega(\lambda)t}\lambda^2\left(\int_{\tau=t}^{T} e^{\omega(\lambda)\tau}g_0(\tau)d\tau\right) = O(1/\lambda) \text{ as } \lambda\to\infty \text{ with } \lambda\in\Omega_1^-. \tag{1.15}$$

Now, (1.15) and Jordan's lemma imply that the RHS of (1.14) tends to zero as $A \to \infty$, and (1.13) follows.

Similarly, we can prove that equations, analogous to (1.13), hold true for all the integrals in (1.2) which involve the functions $\tilde{g}_0$ or $\tilde{g}_1$. Thus, (1.2) is equivalent to

$$\begin{aligned} u(x,t) =& \frac{1}{2\pi}\Bigg[\int_{\lambda=-\infty}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)\,d\lambda - \int_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)\,d\lambda \\ &\qquad - \int_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)\,d\lambda\Bigg] \\ &+ \frac{1}{2\pi}\Bigg[\int_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha^2)\lambda^2\tilde{g}_0(\omega(\lambda),T)\,d\lambda \\ &\qquad + \int_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha)\lambda^2\tilde{g}_0(\omega(\lambda),T)\,d\lambda\Bigg] \\ &- \frac{i}{2\pi}\Bigg[\int_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha)\lambda\tilde{g}_1(\omega(\lambda),T)]\,d\lambda \\ &\qquad + \int_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha^2)\lambda\tilde{g}_1(\omega(\lambda),T)\,d\lambda\Bigg], \end{aligned} \tag{1.16}$$

for $x > 0$ and $0 < t < T$, in the sense that it suffices to deal with the parts of the four integrals in (1.16) which involve the functions $\tilde{g}_0$ and $\tilde{g}_1$ and are taken on $(\partial\Omega_1^-)\cap\mathbb{R}$ and $(\partial\Omega_2^-)\cap\mathbb{R}$. (The parts of these integrals in (1.16), which are taken on the lines $\{\lambda\in\mathbb{C} : \arg\lambda = \pi/3\}$ or $\{\lambda\in\mathbb{C} : \arg\lambda = 2\pi/3\}$, converge absolutely and remain such after any number of differentiations with respect to x or t, provided that $x > 0$, because of the presence of the factor $e^{i\lambda x}$, since, for λ with $\operatorname{Re}\omega(\lambda) = 0$,

$$\left|e^{-\omega(\lambda)t}\tilde{g}_0(\omega(\lambda),T)\right| \le \int_{\tau=0}^{T} |g_0(\tau)|d\tau \text{ and } \left|e^{-\omega(\lambda)t}\tilde{g}_1(\omega(\lambda),T)\right| \le \int_{\tau=0}^{T} |g_1(\tau)|d\tau.$$

To deal with the first of these integrals, we write

$$\int_{-A}^{0} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda = \int_{-A}^{-1}\cdots + \int_{-1}^{0}\cdots \tag{1.17}$$

and

$$\lim_{A\to\infty}\int_{-A}^{-1} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda$$
$$=\Bigg(\int_{\{\mathrm{Im}\lambda=\varepsilon\}\cap\{\mathrm{Re}\lambda\le -1\}}+\int_{\lambda\in[-1+\varepsilon i,-1]}\Bigg)e^{i\lambda x+\omega(\lambda)(T-t)}\lambda^2\sigma_N(T,\lambda)d\lambda$$
$$-\Bigg(\int_{\{\mathrm{Im}\lambda=-\varepsilon\}\cap\{\mathrm{Re}\lambda\le -1\}}+\int_{\lambda\in[-1-\varepsilon i,-1]}\Bigg)e^{i\lambda x-\omega(\lambda)t}\lambda^2\sigma_N(0,\lambda)d\lambda$$
$$-\int_{-\infty}^{-1}\lambda^2\Bigg(\frac{e^{i\lambda x-\omega(\lambda)t}}{[\omega(\lambda)]^N}\int_{\tau=0}^{T}e^{\omega(\lambda)\tau}g_0^{(N)}(\tau)d\tau\Bigg)d\lambda, \tag{1.18}$$

for $N\in\mathbb{N}$, $0<t<T$, and where we have set

$$\sigma_N(\tau,\lambda)=\frac{g_0(\tau)}{\omega(\lambda)}-\frac{g_0'(\tau)}{[\omega(\lambda)]^2}+\cdots+(-1)^{N-1}\frac{g_0^{(N-1)}(\tau)}{[\omega(\lambda)]^N}.$$

Equation (1.18) follows by integration by parts and Cauchy's theorem, observing also that

$$\lim_{A\to\infty}\int_{\{0\le\mathrm{Im}\lambda\le 1\}\cap\{\mathrm{Re}\lambda=-A\}} e^{i\lambda x+\omega(\lambda)(T-t)}\lambda^2\sigma_N(T,\lambda)\,d\lambda=0 \quad\text{and}$$
$$\lim_{A\to\infty}\int_{\{-1\le\mathrm{Im}\lambda\le 0\}\cap\{\mathrm{Re}\lambda=-A\}} e^{i\lambda x-\omega(\lambda)t}\lambda^2\sigma_N(0,\lambda)\,d\lambda=0.$$

Since

$\lambda^2\Big(\frac{e^{-\omega(\lambda)t}}{[\omega(\lambda)]^N}\int_{\tau=0}^{T}e^{\omega(\lambda)\tau}g_0^{(N)}(\tau)d\tau\Big)=O\Big(\frac{1}{\lambda^{3N+1}}\Big)$, as $\lambda\to-\infty$ $(\lambda\in\mathbb{R})$, it is easy to see that the LHS of (1.18) defines a C^∞ function for $(x,t)\in Q$ which satisfies the equation $u_t=u_{xxx}$. Now, in view of (1.17), we may interpret the integral

$$\int_{-\infty}^{0} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda$$

as

$$\Big(\lim_{A\to\infty}\int_{-A}^{-1}\cdots\Big)+\int_{-1}^{0}\cdots=[\text{RHS of (1.18)}]+\int_{-1}^{0}\cdots \tag{1.19}$$

and see that it defines a C^∞ function for $(x,t)\in Q$ which satisfies the equation $u_t=u_{xxx}$.

Dealing with the parts of the other three integrals in (1.16), which involve the functions $\tilde{g}_0$ and $\tilde{g}_1$ and are taken on $(\partial\Omega_1^-)\cap\mathbb{R}$ and $(\partial\Omega_2^-)\cap\mathbb{R}$, is similar.

Thus we have proved the 1st part of the following theorem, which is the main result of this article.

Theorem 1.1. *Given $u_0(x) \in S\big([0,\infty)\big)$ and $g_0(t), g_1(t) \in C^\infty\big([0,\infty)\big)$, the function $u(x,t)$ defined by (1.1) - equivalently by (1.11) - is C^∞ for $(x,t) \in Q$ and satisfies the following:*

1^{st} The differential equation $u_t = u_{xxx}$ for $x > 0$ and $t > 0$.

2^{nd} The limit condition $\lim\limits_{t\,\to\,0^+} u(x,t) = u_0(x)$ for each fixed $x > 0$.

3^{rd} The limit condition $\lim\limits_{x\,\to\,0^+} u(x,t) = g_0(t)$ for each fixed $t > 0$.

4^{th} The limit condition $\lim\limits_{x\,\to\,0^+} \dfrac{\partial u}{\partial x}(x,t) = g_1(t)$ for each fixed $t > 0$.

5^{th} The convergence in the 2^{nd} assertion is uniform for x on compact subsets of $(0,+\infty)$.

6^{th} The convergence in the 3^{rd} and 4^{th} assertion is uniform for t on compact subsets of $(0,+\infty)$.

Comment: Substituting (1.7), (1.11), (1.12), (1.19) and its analogue for g_1, in (1.16), we obtain an integral representation of the solution to problem (1.1) on a fixed strip $\{x > 0, 0 < t < T\}$ in an Ehrenpreis-Palamodov form, in the sense that the integrals, in the resulting representation, combine the exponential solutions $e^{i\lambda x-\omega(\lambda)t}$ $(\lambda \in \mathbb{C})$ of the equation (by means of appropriate measures), they converge absolutely and uniformly on every compact subset of $\{x > 0, 0 < t < T\}$ and remains so after any number of differentiations with respect to x and t, provided that N (in (1.19)) is chosen sufficiently large (depending on the number of differentiations). The measures that can be used in the integral representation of the solution of the equation which satisfies the given initial and boundary conditions are not unique.

2. Proof of Theorem 1.1

The proof consists of several steps.

Step 1: Evaluation of the solution for $t = 0$. Setting $t = 0$ in the integrals of (1.2) or (1.16), we find

$$u(x,t)|_{t=0} = \frac{1}{2\pi}\Bigg[\int\limits_{\lambda=-\infty}^{\infty} e^{i\lambda x}\hat{u}_0(\lambda)\,d\lambda - \int\limits_{\partial\Omega_1^-} e^{i\lambda x}\hat{u}_0(\alpha\lambda)\,d\lambda - \int\limits_{\partial\Omega_2^-} e^{i\lambda x}\hat{u}_0(\alpha^2\lambda)d\lambda\Bigg]. \tag{2.1}$$

(Let us observe that (1.13) holds also for $t = 0$, so that we may use either (1.2) or (1.16) to compute $u(x,t)|_{t=0}$). The above integrals need to be appropriately interpreted. For example

$$\int_{\lambda=-\infty}^{\infty} e^{i\lambda x - \omega(\lambda)t} \hat{u}_0(\lambda)\, d\lambda = \lim_{A\to\infty} \int_{\lambda=-A}^{A} e^{i\lambda x - \omega(\lambda)t} \hat{u}_0(\lambda)\, d\lambda$$

hence

$$\int_{\lambda=-\infty}^{\infty} e^{i\lambda x} \hat{u}_0(\lambda)\, d\lambda = \lim_{A\to\infty} \int_{\lambda=-A}^{A} e^{i\lambda x} \hat{u}_0(\lambda)\, d\lambda.$$

Thus, by Fourier's inversion formula, we have

$$\frac{1}{2\pi} \int_{\lambda=-\infty}^{\infty} e^{i\lambda x} \hat{u}_0(\lambda)\, d\lambda = u_0(x) \quad \text{for} \quad x > 0.$$

Also it follows from Cauchy's theorem and Jordan's lemma that

$$\int_{\partial\Omega_1^-} e^{i\lambda x} \hat{u}_0(\alpha\lambda)\, d\lambda = 0 \quad \text{and} \quad \int_{\partial\Omega_2^-} e^{i\lambda x} \hat{u}_0(\alpha^2\lambda)\, d\lambda = 0.$$

Therefore (2.1) can be written as follows: $u(x,t)|_{t=0} = u_0(x)$, $x > 0$.

Step 2: Evaluation of the solution for $x = 0$. First we have,

$$\Bigg[\int_{\lambda=-\infty}^{\infty} e^{i\lambda x - \omega(\lambda)t} \hat{u}_0(\lambda)\, d\lambda - \int_{\partial\Omega_1^-} e^{i\lambda x - \omega(\lambda)t} \hat{u}_0(\alpha\lambda)\, d\lambda - \int_{\partial\Omega_2^-} e^{i\lambda x - \omega(\lambda)t} \hat{u}_0(\alpha^2\lambda)\, d\lambda \Bigg]\Bigg|_{x=0}$$

$$= \Bigg\{ \lim_{A\to\infty} \Bigg[\int_{\lambda=-A}^{A} e^{i\lambda x - \omega(\lambda)t} \hat{u}_0(\lambda)\, d\lambda - \int_{(\partial\Omega_1^-)\cap\{|\lambda|\le A\}} e^{i\lambda x - \omega(\lambda)t} \hat{u}_0(\alpha\lambda)\, d\lambda - \int_{\partial\Omega_2^-\cap\{|\lambda|\le A\}} e^{i\lambda x - \omega(\lambda)t} \hat{u}_0(\alpha^2\lambda)\, d\lambda \Bigg]\Bigg\}\Bigg|_{x=0}$$

$$= \lim_{A\to\infty} \Bigg[\int_{\lambda=-A}^{A} e^{-\omega(\lambda)t} \hat{u}_0(\lambda) d\lambda - \int_{(\partial\Omega_1^-)\cap\{|\lambda|\le A\}} e^{-\omega(\lambda)t} \hat{u}_0(\alpha\lambda)\, d\lambda - \int_{\partial\Omega_2^-\cap\{|\lambda|\le A\}} e^{-\omega(\lambda)t} \hat{u}_0(\alpha^2\lambda)\, d\lambda \Bigg]. \tag{2.2}$$

We claim that the limit in (2.2) is equal to zero. Indeed, the quantity inside the last square brackets above is equal to the sum $B_1(A) + B_2(A)$, where

$$B_1(A) = \int_{\lambda=-A}^{0} e^{-\omega(\lambda)t}\hat{u}_0(\lambda)\,d\lambda - \alpha^2 \int_{\{\arg\lambda=2\pi/3\}\cap\{|\lambda|\le A\}} e^{-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)\,d(\alpha\lambda)$$
$$- \alpha \int_{\{\arg\lambda=0\}\cap[\{|\lambda|\le A\}} e^{-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)\,d(\alpha^2\lambda)$$

and

$$B_2(A) = \int_{\lambda=0}^{A} e^{-\omega(\lambda)t}\hat{u}_0(\lambda)\,d\lambda - \alpha^2 \int_{\{\arg\lambda=\pi\}\cap\{|\lambda|\le A\}} e^{-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)\,d(\alpha\lambda)$$
$$- \alpha \int_{\{\arg\lambda=\pi/3\}\cap[\{|\lambda|\le A\}} e^{-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)\,d(\alpha^2\lambda).$$

But

$$\int_{\{\arg\lambda=2\pi/3\}\cap\{|\lambda|\le A\}} e^{-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d(\alpha\lambda) = \int_{\arg\mu=4\pi/3\}\cap\{|\mu|\le A\}} e^{-\omega(\mu)t}\hat{u}_0(\mu)d\mu$$

and, with the appropriate orientation of the contours,

$$\lim_{A\to\infty}\left[\int_{\lambda=-A}^{0} e^{-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda + \int_{\{\arg\mu=4\pi/3\}\cap\{|\mu|\le A\}} e^{-\omega(\mu)t}\hat{u}_0(\mu)d\mu\right] = 0.$$

Writing similar equations also for the third integral in $B_1(A)$, we can see that $\lim_{A\to\infty} B_1(A) = 0$, taking into consideration also the fact that $1+\alpha+\alpha^2 = 0$. Similarly $\lim_{A\to\infty} B_2(A) = 0$, and these prove that the quantity (2.2) is equal to zero.

Next we will show that, for $0 < t < T$,

$$\frac{1}{2\pi}\left[\int_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha^2)\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda + \int_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha)\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda\right]\Bigg|_{x=0} = g_0(t). \tag{2.3}$$

The preceding integrals are interpreted as the limits

$$\lim_{A\to\infty}\int_{(\partial\Omega_1^-)\cap\{|\lambda|\le A\}} \cdots \quad\text{and}\quad \lim_{A\to\infty}\int_{(\partial\Omega_2^-)\cap\{|\lambda|\le A\}} \cdots,$$

keeping also in mind (1.18), and it is in these limits that we set $x=0$.

Indeed we have,

$$\left[\int_{\lambda\in\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda\right]\Big|_{x=0}$$

$$= \int_{\lambda\in\partial\Omega_1^-} e^{-\omega(\lambda)t}\lambda^2\left(\int_{\tau=0}^{T} e^{\omega(\lambda)\tau}g_0(\tau)d\tau\right)d\lambda$$

$$= \int_{\lambda=-\infty}^{0} e^{-\omega(\lambda)t}\lambda^2\left(\int_{\tau=0}^{T} e^{\omega(\lambda)\tau}g_0(\tau)d\tau\right)d\lambda$$

$$+ \int_{\partial\Omega_1^-\cap\{\arg\lambda=2\pi/3\}} e^{-\omega(\lambda)t}\lambda^2\left(\int_{\tau=0}^{T} e^{\omega(\lambda)\tau}g_0(\tau)d\tau\right)d\lambda$$

$$= \frac{1}{3}\int_{\mu=-\infty}^{0} e^{-\mu i t}\left(\int_{\tau=0}^{T} e^{\mu i\tau}g_0(\tau)d\tau\right)d\mu + \frac{1}{3}\int_{\mu=0}^{\infty} e^{-\mu i t}\left(\int_{\tau=0}^{T} e^{\mu i t}g_0(\tau)d\tau\right)d\mu$$

$$= \frac{1}{3}\int_{\mu=-\infty}^{\infty} e^{\mu i t}\left(\int_{\tau=0}^{T} e^{-\mu i\tau}g_0(\tau)d\tau\right)d\mu$$

$$= \frac{1}{3}2\pi g_0(t),$$

and similarly,

$$\left[\int_{\lambda\in\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda\right]\Big|_{x=0} = \frac{1}{3}2\pi g_0(t).$$

As far as the above equation is concerned, let us notice - alternatively - that the change of variables $\mu=\alpha^2\lambda$ gives

$$\int_{\lambda\in\partial\Omega_1^-} e^{[i(\alpha^2\lambda)x/\alpha^2]-\omega(\lambda)t}(\alpha^2\lambda)^2\tilde{g}_0(\omega(\lambda),T)d(\alpha^2\lambda)$$

$$= \int_{\mu\in\partial\Omega_2^-} e^{[i\mu x/\alpha^2]-\omega(\mu)t}\mu^2\tilde{g}_0(\omega(\mu),T)d\mu.$$

Therefore the LHS of (2.3) becomes

$$\frac{1-\alpha^2}{3}g_0(t)+\frac{1-\alpha}{3}g_0(t)=g_0(t),$$

since $1+\alpha+\alpha^2=0$.

Finally we have

$$\Bigg[\int_{\lambda\in\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}\lambda\tilde{g}_1(\omega(\lambda),T)\,d\lambda\Bigg]\Bigg|_{x=0} = \int_{\lambda\in\partial\Omega_1^-} e^{-\omega(\lambda)t}\lambda\tilde{g}_1(\omega(\lambda),T)\,d\lambda$$

$$= \frac{1}{\alpha^4}\int_{\lambda\in\partial\Omega_1^-} e^{-\omega(\lambda)t}(\alpha^2\lambda)\tilde{g}_1(\omega(\lambda),T)\,d(\alpha^2\lambda) = \frac{1}{\alpha^4}\int_{\mu\in\partial\Omega_2^-} e^{-\omega(\mu)t}\mu\tilde{g}_1(\omega(\mu),T)\,d\mu,$$

and therefore, in view of (1.16), we have

$$\Bigg[\int_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha)\lambda\tilde{g}_1(\omega(\lambda),T)]\,d\lambda + \int_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha^2)\lambda\tilde{g}_1(\omega(\lambda),T)]\,d\lambda\Bigg]\Bigg|_{x=0}$$

$$= \left[\frac{1}{\alpha^4}(1-\alpha)+(1-\alpha^2)\right]\int_{\mu\in\partial\Omega_2^-} e^{-\omega(\mu)t}\mu\tilde{g}_1(\omega(\mu),T)\,d\mu = 0.$$

Thus, we have proved that $u(x,t)|_{x=0} = g_0(t)$, $t>0$.

Step 3: Evaluation of the derivative $\partial u(x,t)/\partial x$ for $x=0$. Firstly,

$$\left[\frac{\partial}{\partial x}\left(\int_{\lambda=-\infty}^{0} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)\,d\lambda\right)\right]\Bigg|_{x=0}$$

$$= \int_{\{\mathrm{Im}\lambda=-\varepsilon\}\cap\{\mathrm{Re}\lambda\le 0\}} i\lambda e^{-\omega(\lambda)t}\hat{u}_0(\lambda)\,d\lambda + \int_{\lambda\in[-\varepsilon i,0]} i\lambda e^{-\omega(\lambda)t}\hat{u}_0(\lambda)\,d\lambda$$

$$= \int_{\{\arg\lambda=4\pi/3\}} i\lambda e^{-\omega(\lambda)t}\hat{u}_0(\lambda)\,d\lambda = \int_{\lambda=-\infty}^{0} i\lambda e^{-\omega(\lambda)t}\hat{u}_0(\lambda)\,d\lambda$$

and

$$\left[\frac{\partial}{\partial x}\left(\int_{\lambda=0}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)\,d\lambda\right)\right]\Bigg|_{x=0}$$

$$= \int_{\{\mathrm{Im}\lambda=-\varepsilon\}\cap\{\mathrm{Re}\lambda\ge 0\}} i\lambda e^{-\omega(\lambda)t}\hat{u}_0(\lambda)\,d\lambda + \int_{\lambda\in[0,-\varepsilon i]} i\lambda e^{-\omega(\lambda)t}\hat{u}_0(\lambda)\,d\lambda$$

$$= \int_{\{\arg\lambda=5\pi/3\}} i\lambda e^{-\omega(\lambda)t}\hat{u}_0(\lambda)\,d\lambda = \int_{\lambda=0}^{\infty} i\lambda e^{-\omega(\lambda)t}\hat{u}_0(\lambda)\,d\lambda.$$

(In the above calculations we used also Lemma 2.1 (Step 4 below), which is a Jordan's type lemma. It follows also that the integrals $\int_{-\infty}^{0}$ and $\int_{0}^{\infty}$ in the RHSs of the above equations exist in the generalized sense).

In analogy with the preceding identities, we write

$$\begin{aligned}
&\left[\frac{\partial}{\partial x}\left(\int_{\lambda=-\infty}^{0} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda\right)\right]\Bigg|_{x=0} \\
&= \int_{\{\mathrm{Im}\lambda=-\varepsilon\}\cap\{\mathrm{Re}\lambda\le -1\}} i\lambda e^{-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)\,d\lambda \\
&\qquad + \int_{\lambda\in[-1-\varepsilon i,-1]} i\lambda e^{-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)\,d\lambda + \int_{\lambda=-1}^{0} i\lambda e^{-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)\,d\lambda \\
&= \int_{\{\arg\lambda=4\pi/3\}} i\lambda e^{-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)\,d\lambda \\
&= \frac{1}{\alpha^2}\int_{\{\arg\lambda=4\pi/3\}} i\alpha\lambda e^{-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)\,d(\alpha\lambda) \\
&= -\frac{1}{\alpha^2}\int_{\mu=0}^{\infty} i\mu e^{-\omega(\mu)t}\hat{u}_0(\mu)\,d\mu
\end{aligned}$$

and

$$\begin{aligned}
\left[\frac{\partial}{\partial x}\left(\int_{\lambda=0}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)\,d\lambda\right)\right]\Bigg|_{x=0} &= \int_{\{\arg\lambda=5\pi/3\}} i\lambda e^{-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)\,d\lambda \\
&= -\frac{1}{\alpha^4}\int_{\mu=-\infty}^{0} i\mu e^{-\omega(\mu)t}\hat{u}_0(\mu)\,d\mu.
\end{aligned}$$

Moreover,

$$
\begin{aligned}
\left[\frac{\partial}{\partial x}\left(\int\limits_{\arg\lambda=2\pi/3} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda\right)\right]\Bigg|_{x=0}
&= \int\limits_{\arg\lambda=2\pi/3} i\lambda e^{-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)\,d\lambda \\
&= \frac{1}{\alpha^2}\int\limits_{\arg\lambda=2\pi/3} i\alpha\lambda e^{-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)\,d(\alpha\lambda) \\
&= \frac{1}{\alpha^2}\int\limits_{\arg\mu=4\pi/3} i\mu e^{-\omega(\mu)t}\hat{u}_0(\mu)\,d\mu \\
&= -\frac{1}{\alpha^2}\int\limits_{\mu=-\infty}^{0} i\mu e^{-\omega(\mu)t}\hat{u}_0(\mu)\,d\mu
\end{aligned}
$$

and

$$
\begin{aligned}
\left[\frac{\partial}{\partial x}\left(\int\limits_{\arg\lambda=\pi/3} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)\,d\lambda\right)\right]\Bigg|_{x=0}
&= -\int\limits_{\arg\lambda=\pi/3} i\lambda e^{-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)\,d\lambda \\
&= -\frac{1}{\alpha^4}\int\limits_{\mu=0}^{\infty} i\mu e^{-\omega(\mu)t}\hat{u}_0(\mu)\,d\mu.
\end{aligned}
$$

Now it follows from the above equations that

$$
\begin{aligned}
\Bigg[\frac{\partial}{\partial x}\Bigg(\int\limits_{\lambda=-\infty}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda - \int\limits_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)\,d\lambda \\
- \int\limits_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)\,d\lambda\Bigg)\Bigg]\Bigg|_{x=0} = 0.
\end{aligned}
\tag{2.4}
$$

(This computations is similar to the one which showed that the quantity (2.2) is equal to zero).

Next, using (1.18) and (1.19), we compute (with $N = 1$).

$$
\begin{aligned}
&\left[\frac{\partial}{\partial x}\left(\int_{-\infty}^{0} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda\right)\right]\Bigg|_{x=0} \\
&= \left(\int_{\{\mathrm{Im}\lambda=1\}\cap\{\mathrm{Re}\lambda\le -1\}} + \int_{\lambda\in[-1+i,-1]}\right) i\lambda e^{\omega(\lambda)(T-t)}\lambda^2\sigma_N(T,\lambda)\,d\lambda \\
&\qquad - \left(\int_{\{\mathrm{Im}\lambda=-1\}\cap\{\mathrm{Re}\lambda\le -1\}} + \int_{\lambda\in[-1-i,-1]}\right) i\lambda e^{-\omega(\lambda)t}\lambda^2\sigma_N(0,\lambda)\,d\lambda \\
&\qquad - \int_{-\infty}^{-1} i\lambda\lambda^2\left(\frac{e^{-\omega(\lambda)t}}{[\omega(\lambda)]^N}\int_{\tau=0}^{T} e^{\omega(\lambda)\tau}g_0^{(N)}(\tau)d\tau\right)d\lambda \\
&\qquad + \int_{-1}^{0} i\lambda e^{-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),T)\,d\lambda.
\end{aligned} \tag{2.5}
$$

Now we may write the RHS of (2.5) formally as the "integral"

$$
\int_{-\infty}^{0} i\lambda e^{-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda = i\int_{(-\infty,0]}^{*} e^{-\omega(\lambda)t}\lambda^3\tilde{g}_0(\omega(\lambda),T)d\lambda. \tag{2.6}
$$

(The symbol "$*$" is inserted to indicate that the corresponding integral must be appropriately interpreted).

Similar (formal) integrals may be written also for the other parts of the expression $[\partial u(x,t)/\partial x]|_{x=0}$ which contain the function $\tilde{g}_0(\omega(\lambda),T)$, namely,

$$
\begin{aligned}
&i\int_{\{\arg\lambda=2\pi/3\}}^{*} e^{-\omega(\lambda)t}\lambda^3\tilde{g}_0(\omega(\lambda),T)\,d\lambda, \\
&i\int_{\{\arg\lambda=\pi/3\}}^{*} e^{-\omega(\lambda)t}\lambda^3\tilde{g}_0(\omega(\lambda),T)\,d\lambda, \\
&i\int_{[0,\infty)}^{*} e^{-\omega(\lambda)t}\lambda^3\tilde{g}_0(\omega(\lambda),T)\,d\lambda.
\end{aligned} \tag{2.7}
$$

Using the interpretation (2.5) of the integral (2.6) and the analogous interpretations of the integrals (2.7), we see that

$$\int^{*}_{\lambda\in\partial\Omega_1^-} e^{-\omega(\lambda)t}\lambda^3\tilde{g}_0(\omega(\lambda),T)\,d\lambda = \frac{1}{\alpha^2}\int^{*}_{\lambda\in\partial\Omega_1^-} e^{-\omega(\lambda)t}(\alpha^2\lambda)^3\tilde{g}_0(\omega(\lambda),T)\,d(\alpha^2\lambda)$$
$$= \frac{1}{\alpha^2}\int^{*}_{\mu\in\partial\Omega_2^-} e^{-\omega(\mu)t}\mu^3\tilde{g}_0(\omega(\mu),T)\,d\mu,$$

having set $\mu = \alpha^2\lambda$.

It follows that

$$\Bigg[\frac{\partial}{\partial x}\Bigg(\int_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha^2)\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda$$
$$+\int_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha)\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda\Bigg)\Bigg]\Bigg|_{x=0} = 0.$$

Finally we have

$$\Bigg[\frac{\partial}{\partial x}\Bigg(\int_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}(-\alpha)\lambda\tilde{g}_1(\omega(\lambda),T)d\lambda$$
$$+\int_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha^2)\lambda\tilde{g}_1(\omega(\lambda),T)d\lambda\Bigg)\Bigg]\Bigg|_{x=0} = 2\pi i g_1(t).$$

(This computation is similar to the one providing (2.3)).

Thus $[\partial u(x,t)/\partial x]|_{x=0} = g_1(t)$, $t>0$, where we used (1.16) for $u(x,t)$, as in Step 2.

Step 4: Here we will make use of the following lemma.

Lemma 2.1. *We have*

$$\lim_{A\to\infty}\int_{\substack{|\lambda|=A\\ \pi\le arg\lambda\le 4\pi/3}} \lambda^2 e^{-i\lambda^3 t}\hat{u}_0(\lambda)d\lambda = 0$$

and

$$\lim_{A\to\infty}\int_{\substack{|\lambda|=A\\ (5\pi/3)\le arg\lambda\le 2\pi}} \lambda^2 e^{-i\lambda^3 t}\hat{u}_0(\lambda)d\lambda = 0.$$

Proof. Setting $\lambda = Ae^{i\theta}$, $\pi\le\theta\le 4\pi/3$, and writing

$$\int_{\substack{|\lambda|=A\\ \pi\le\arg\lambda\le 4\pi/3}} \lambda^2 e^{-i\lambda^3 t}\hat{u}_0(\lambda)d\lambda = \int_{\theta=\pi}^{4\pi/3} A^2e^{2i\theta}\exp[-iA^3e^{3i\theta}t]\hat{u}_0(Ae^{i\theta})Ae^{i\theta}id\theta,$$

we see that

$$\left| \int\limits_{\substack{|\lambda|=A \\ \pi\leq\arg\lambda\leq 4\pi/3}} \lambda^2 e^{-i\lambda^3 t}\hat{u}_0(\lambda)d\lambda \right| \preceq \int\limits_{\theta=\pi}^{4\pi/3} A^2 e^{tA^3\sin(3\theta)}d\theta$$
$$= \frac{1}{3}A^2 \int\limits_{\theta=3\pi}^{4\pi} e^{tA^3\sin\theta}d\theta$$
$$= \frac{1}{3}A^2 \int\limits_{\theta=0}^{\pi} e^{-tA^3\sin\theta}d\theta$$
$$\leq \frac{\pi}{3tA}.$$

Letting $A \to \infty$, we obtain the first limit. The proof of the second one is similar. □

Step 5: We claim that

$$\lim_{t\to 0^+}\Bigg[\int\limits_{\lambda=-\infty}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda - \int\limits_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda - \int\limits_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda\Bigg] = 2\pi u(x,t)|_{t=0}. \tag{2.8}$$

Indeed, one has

$$\lim_{t\to 0^+}\int\limits_{\lambda=-\infty}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda$$
$$= \lim_{t\to 0^+}\left(\lim_{A\to\infty}\int\limits_{\lambda=-A}^{A} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda\right)$$
$$= \lim_{t\to 0^+}\left[\int\limits_{-\infty}^{-1} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda + \int\limits_{1}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda\right] + \lim_{t\to 0^+}\int\limits_{[-1,1]} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda$$
$$= \lim_{t\to 0^+}\left[\int\limits_{-\infty}^{-1} e^{i\lambda x-\omega(\lambda)t}\frac{u_0(0)}{i\lambda}d\lambda + \int\limits_{1}^{\infty} e^{i\lambda x-\omega(\lambda)t}\frac{u_0(0)}{i\lambda}d\lambda\right]$$

$$+\lim_{t\,\to\,0^+}\Big[\int_{-\infty}^{-1} e^{i\lambda x-\omega(\lambda)t}\frac{1}{i\lambda}\hat{(u_0')}(\lambda)d\lambda+\int_{1}^{\infty} e^{i\lambda x-\omega(\lambda)t}\frac{1}{i\lambda}\hat{(u_0')}(\lambda)d\lambda\Big]$$
$$+\int_{[-1,1]} e^{i\lambda x}\hat{u}_0(\lambda)d\lambda$$
$$=\Big[\int_{-\infty}^{-1} e^{i\lambda x}\frac{u_0(0)}{i\lambda}d\lambda+\int_{1}^{\infty} e^{i\lambda x}\frac{u_0(0)}{i\lambda}d\lambda\Big]$$
$$+\Big[\int_{-\infty}^{-1} e^{i\lambda x}\frac{1}{i\lambda}\hat{(u_0')}(\lambda)d\lambda+\int_{1}^{\infty} e^{i\lambda x}\frac{1}{i\lambda}\hat{(u_0')}(\lambda)d\lambda\Big]+\int_{[-1,1]} e^{i\lambda x}\hat{u}_0(\lambda)d\lambda$$
$$=\int_{-\infty}^{-1} e^{i\lambda x}\hat{u}_0(\lambda)d\lambda+\int_{1}^{\infty} e^{i\lambda x}\hat{u}_0(\lambda)d\lambda+\int_{[-1,1]} e^{i\lambda x}\hat{u}_0(\lambda)d\lambda$$
$$=\int_{-\infty}^{\infty} e^{i\lambda x}\hat{u}_0(\lambda)d\lambda.$$

[In the preceding calculations we made use of Lemma 2.2 (Step 6, below).]

Similar computations show that

$$\lim_{t\,\to\,0^+}\int_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)\,d\lambda=\int_{\partial\Omega_1^-} e^{i\lambda x}\hat{u}_0(\alpha\lambda)\,d\lambda$$

and

$$\lim_{t\,\to\,0^+}\int_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)\,d\lambda=\int_{\partial\Omega_2^-} d^{i\lambda x}\hat{u}_0(\alpha^2\lambda)\,d\lambda.$$

Therefore (2.8) follows from (2.1).

Step 6:

Lemma 2.2. *We have*

$$\lim_{t\,\to\,0^+}\int_{1}^{\infty} e^{i\lambda}[1-e^{-i\lambda^3t}]\frac{d\lambda}{\lambda}=0 \quad \textit{and} \quad \lim_{t\,\to\,0^+}\int_{-\infty}^{-1} e^{i\lambda}[1-e^{-i\lambda^3t}]\frac{d\lambda}{\lambda}=0.$$

Also, for $x>0$ we have

$$\lim_{t\,\to\,0^+}\int_{1}^{\infty} e^{i\lambda x-\omega(\lambda)t}\frac{1}{i\lambda}d\lambda=\int_{1}^{\infty} e^{i\lambda x}\frac{1}{i\lambda}\,d\lambda$$

and

$$\lim_{t\to 0^+}\int_{-\infty}^{-1} e^{i\lambda x-\omega(\lambda)t}\frac{1}{i\lambda}d\lambda=\int_{-\infty}^{-1} e^{i\lambda x}\frac{1}{i\lambda}\,d\lambda.$$

Moreover the convergence in the above limits is uniform for x in compact subsets of $(0,+\infty)$*.*

Proof. Let us write

$$\int_1^\infty e^{i\lambda}[1-e^{-i\lambda^3 t}]\frac{d\lambda}{\lambda}=I_1(t)+I_2(t)+I_3(t),$$

where

$$I_1(t)=\int_{1/\sqrt{6t}}^{1/\sqrt{t}} e^{i\lambda}[1-e^{-i\lambda^3 t}]\frac{d\lambda}{\lambda}$$

$$I_2(t)=\int_{1}^{1/\sqrt{6t}} e^{i\lambda}[1-e^{-i\lambda^3 t}]\frac{d\lambda}{\lambda},$$

$$I_3(t)=\int_{1/\sqrt{t}}^{\infty} e^{i\lambda}[1-e^{-i\lambda^3 t}]\frac{d\lambda}{\lambda}.$$

Setting $\lambda=\mu/\sqrt{t}$ and $A=1/\sqrt{t}$ (with $A\to\infty$ as $t\to 0^+$) we have

$$\begin{aligned}
I_1(t)&=\int_{\mu=1/\sqrt{6}}^{1} e^{i\mu/\sqrt{t}}[1-e^{-i\mu^3/\sqrt{t}}]\frac{d\mu}{\mu}\\
&=\int_{\mu=1/\sqrt{6}}^{1} e^{iA\mu}[1-e^{-iA\mu^3}]\frac{d\mu}{\mu}\\
&=\int_{\mu=1/\sqrt{6}}^{1} e^{iA\mu}\frac{d\mu}{\mu}-\int_{\mu=1/\sqrt{6}}^{1} e^{i(\mu-\mu^3)A}\frac{d\mu}{\mu}.
\end{aligned}$$

By the Riemann-Lebesgue lemma, we obtain

$$\lim_{A\to\infty}\int_{\mu=1/\sqrt{6}}^{1} e^{iA\mu}\frac{d\mu}{\mu}=0.$$

On the other hand, integrating by parts, we write

$$\int_{\mu=1/\sqrt{6}}^{1} e^{i(\mu-\mu^3)A}\frac{d\mu}{\mu} = \int_{y=1/\sqrt{6}}^{1} e^{i(y-y^3)A}dy + \int_{\mu=1/\sqrt{6}}^{1}\left[\int_{y=1/\sqrt{6}}^{\mu} e^{i(y-y^3)A}dy\right]\frac{d\mu}{\mu^2}.$$

Since the second derivative of $y - y^3$ is $-6y$ and satisfies $|-6y| \geq \sqrt{6}$ for $y \in [1/\sqrt{6}, 1]$, by the Van der Corput's lemma,

$$\left|\int_{y=1/\sqrt{6}}^{1} e^{i(y-y^3)A}dy\right| \leq \frac{\text{constant}}{\sqrt{A}}$$

and

$$\left|\int_{y=1/\sqrt{6}}^{\mu} e^{i(y-y^3)A}dy\right| \leq \frac{\text{constant}}{\sqrt{A}} \quad \text{for } \mu \in [1/\sqrt{6}, 1].$$

Therefore, $\lim\limits_{t\to 0^+} I(t) = 0$.

Next, setting $\delta_2 = \delta_2(t) = [1, 1/\sqrt{6t}]$ and $\delta_3 = \delta_3(t) = [1/\sqrt{t}, \infty)$, we have

$$\begin{aligned} I_2(t) + I_3(t) &= \int_{\delta_2\cup\delta_3} [e^{i\lambda} - e^{i(\lambda-\lambda^3 t)}]\frac{d\lambda}{\lambda} = \int_{\delta_2\cup\delta_3}\left[\frac{(e^{i\lambda})'}{i} - \frac{[e^{i(\lambda-\lambda^3 t)}]'}{i(1-3\lambda^2 t)}\right]\frac{d\lambda}{\lambda} \\ &= \left[\frac{e^{i\lambda}}{i\lambda} - \frac{e^{i(\lambda-\lambda^3 t)}}{i(1-3\lambda^2 t)\lambda}\right]_{\lambda=1}^{\lambda=1/\sqrt{6t}} + \left[\frac{e^{i\lambda}}{i\lambda} - \frac{e^{i(\lambda-\lambda^3 t)}}{i(1-3\lambda^2 t)\lambda}\right]\Bigg|_{\lambda=1/\sqrt{t}}^{\lambda=\infty} \\ &\quad + \int_{\delta_2\cup\delta_3}\left\{\frac{e^{i\lambda}}{i\lambda^2} - e^{i(\lambda-\lambda^3 t)}\left(\frac{1}{i(1-3\lambda^2 t)\lambda^2} - \frac{6t}{i(1-3\lambda^2 t)^2}\right)d\lambda\right\}. \end{aligned}$$

Now the expressions inside the square brackets are bounded by a constant $\times\sqrt{t}$.

For $\lambda \in \delta_3 \Rightarrow 3\lambda^2 t - 1 \geq 2\lambda^2 t \geq 2$, the integrand over δ_3 is bounded by a constant $/\lambda^2$ and the integral over δ_3 is bounded by a constant $\times\sqrt{t}$.

For $\lambda \in \delta_2 \Rightarrow (1/2) \leq 1 - 3\lambda^2 t \leq 1$, the part of the integral over δ_2 which contains the term $6t$ is bounded by a constant $\times\sqrt{t}$. The integrand of the other part is

$$\frac{e^{i\lambda}}{i\lambda^2} - e^{i(\lambda-\lambda^3 t)}\frac{1}{i(1-3\lambda^2 t)\lambda^2},$$

and is bounded by a constant divided by λ^2. Also

$$\lim_{t\to 0^+}\left\{\frac{e^{i\lambda}}{i\lambda^2} - e^{i(\lambda-\lambda^3 t)}\frac{1}{i(1-3\lambda^2 t)\lambda^2}\right\}\chi_{\delta_2(t)}(\lambda) = 0 \quad \text{for every} \quad \lambda \in [1, \infty).$$

Thus, by the Lebesgue dominated convergence theorem, we obtain

$$\lim_{t\to 0^+}\int_{\delta_2(t)}\left\{\frac{e^{i\lambda}}{i\lambda^2}-e^{i(\lambda-\lambda^3 t)}\frac{1}{i(1-3\lambda^2 t)\lambda^2}\right\}d\lambda=0.$$

Now, the first assertion of the lemma follows from the above results. The second assertion follows similarly. The next two assertions follow immediately by rescaling, i.e., changing the variables in the integrals by setting $\mu=\lambda x$.

Finally, to prove that this convergence is uniform for x in compact sets, let us restrict x between two positive constants: $\beta_1\le x\le\beta_2$. If $\beta_2\le 1$, let us write

$$\int_1^\infty[e^{i\lambda x-\omega(\lambda)t}-e^{i\lambda x}]\frac{1}{i\lambda}d\lambda=\int_x^\infty[e^{i\mu-\omega(\mu)(t/x^3)}-e^{i\mu}]\frac{1}{i\mu}d\mu$$
$$=\int_1^\infty[e^{i\mu-\omega(\mu)(t/x^3)}-e^{i\mu}]\frac{1}{i\mu}d\mu+\int_x^1[e^{i\mu-\omega(\mu)(t/x^3)}-e^{i\mu}]\frac{1}{i\mu}d\mu.$$

We claim that

$$\int_1^\infty[e^{i\mu-\omega(\mu)(t/x^3)}-e^{i\mu}]\frac{1}{i\mu}d\mu\ \to\ 0\ \text{ as }\ t\ \to\ 0^+,\ \text{ uniformly for }\ x\in[\beta_1,\beta_2].$$

To prove this, let us consider $\varepsilon>0$. Then there is $\delta=\delta(\varepsilon)>0$ so that

$$\left|\int_1^\infty[e^{i\mu-\omega(\mu)t}-e^{i\mu}]\frac{1}{i\mu}d\mu\right|<\varepsilon\ \text{ for }\ 0<t<\delta.$$

Therefore, if $0<t<\beta_1^3\delta$ and $x\in[\beta_1,\beta_2]$ then $0<(t/x^3)<\delta$, and, therefore,

$$\left|\int_1^\infty[e^{i\mu-\omega(\mu)(t/x^3)}-e^{i\mu}]\frac{1}{i\mu}d\mu\right|<\varepsilon.$$

On the other hand, for $x\in[\beta_1,\beta_2]$,

$$\left|\int_x^1[e^{i\mu-\omega(\mu)(t/x^3)}-e^{i\mu}]\frac{1}{i\mu}d\mu\right|=\left|\int_1^{1/x}[e^{i\lambda x-\omega(\lambda)t}-e^{i\lambda x}]\frac{1}{i\lambda}d\lambda\right|$$
$$\le\int_1^{1/\beta_1}|e^{-\omega(\lambda)t}-1|\frac{1}{\lambda}d\lambda,$$

and, therefore,

$$\int_x^1 [e^{i\mu-\omega(\mu)(t/x^3)} - e^{i\mu}]\frac{1}{i\mu}d\mu \;\to\; 0, \quad \text{as} \quad t \;\to\; 0^+, \quad \text{uniformly in } x \in [\beta_1, \beta_2].$$

The above computations show that the convergence

$$\lim_{t\,\to\,0^+} \int_1^\infty e^{i\lambda x-\omega(\lambda)t}\frac{1}{i\lambda}d\lambda = \int_1^\infty e^{i\lambda x}\frac{1}{i\lambda}d\lambda$$

is uniform for $\beta_1 \leq x \leq \beta_2$, if $\beta_2 \leq 1$. The case $\beta_1 \geq 1$ is similar.

Analogously one obtains the proof that the convergence

$$\lim_{t\,\to\,0^+} \int_{-\infty}^{-1} e^{i\lambda x-\omega(\lambda)t}\frac{1}{i\lambda}d\lambda = \int_{-\infty}^{-1} e^{i\lambda x}\frac{1}{i\lambda}d\lambda$$

is uniform for x in compact subsets of $(0, +\infty)$. This completes the proof of the lemma. $\square$

Step 7: We claim that

$$\lim_{t\,\to\,0^+} \int_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda), t)d\lambda = 0, \quad \text{for} \quad x > 0. \tag{2.9}$$

It suffices to show that

$$\lim_{t\,\to\,0^+} \int_{-\infty}^{-1} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda), t)d\lambda = 0, \tag{2.10}$$

since the other parts of the integral in (2.9) clearly tend to zero as $t \;\to\; 0^+$. Now, writing

$$\begin{aligned} e^{-\omega(\lambda)t}\tilde{g}_0(\omega(\lambda), t) &= e^{-\omega(\lambda)t}\int_{\tau=0}^{t} e^{\omega(\lambda)\tau}g_0(\tau)d\tau \\ &= \frac{g_0(t)}{\omega(\lambda)} - \frac{g_0(0)}{\omega(\lambda)}e^{-\omega(\lambda)t} - \frac{e^{-\omega(\lambda)t}}{\omega(\lambda)}\int_{\tau=0}^{t} e^{\omega(\lambda)\tau}\frac{dg_0(\tau)}{d\tau}d\tau, \ \lambda \neq 0, \end{aligned}$$

and taking into consideration that

$$\lim_{t\,\to\,0^+} \left\{[g_0(t) - g_0(0)]\int_{-\infty}^{-1} e^{i\lambda x}\frac{d\lambda}{\lambda}\right\} = 0$$

and that

$$\frac{e^{-\omega(\lambda)t}}{\omega(\lambda)}\int_{\tau=0}^{t} e^{\omega(\lambda)\tau}\frac{dg_0(\tau)}{d\tau}d\tau = O(1/\lambda^6),$$

we see that the proof of (2.10) is reduced to

$$\lim_{t\to 0^+}\int_{-\infty}^{-1} e^{i\lambda x}[1-e^{-\omega(\lambda)t}]\frac{d\lambda}{\lambda}=0 \iff \lim_{t\to 0^+}\int_{-\infty}^{-1} e^{i\lambda}[1-e^{-i\lambda^3 t}]\frac{d\lambda}{\lambda}=0,$$

a fact proved in Step 5. This completes the proof of (2.9).

In analogy with (2.9),

$$\lim_{t\to 0^+}\int_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),t)d\lambda=0, \tag{2.11}$$

and its proof is reduced this time to

$$\lim_{t\to 0^+}\int_{1}^{\infty} e^{i\lambda x}[1-e^{-\omega(\lambda)t}]\frac{d\lambda}{\lambda}=0 \iff \lim_{t\to 0^+}\int_{1}^{\infty} e^{i\lambda}[1-e^{-i\lambda^3 t}]\frac{d\lambda}{\lambda}=0.$$

The last equation is proved again in Lemma 2.2 of Step 6.

Also,

$$\begin{aligned}\lim_{t\to 0^+}\int_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}\lambda\tilde{g}_1(\omega(\lambda),t)d\lambda=0,\\ \lim_{t\to 0^+}\int_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}\lambda\tilde{g}_1(\omega(\lambda),t)d\lambda=0.\end{aligned} \tag{2.12}$$

The preceding equations are straightforward consequences of the fact that

$$e^{i\lambda x-\omega(\lambda)t}\lambda\tilde{g}_1(\omega(\lambda),t)=O(1/\lambda^2)\quad\text{for}\quad\lambda\to\infty\quad(\text{with }\lambda\in\mathbb{R}).$$

Finally, combining (2.8), (2.9), (2.11), (2.12) and the equation $u(x,t)|_{t=0}=u_0(x)$ (proved in Step 1), we obtain $\lim_{t\to 0^+}u(x,t)=u_0(x)$ for each fixed $x>0$. This completes the proof of the 2^{nd} part of Theorem 1.1.

Step 8: Since (1.2), (1.3) and (1.4) hold also for $x=0$, it follows that

$$\lim_{x\to 0^+}\int_{\lambda=-\infty}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda=\int_{\lambda=-\infty}^{\infty} e^{-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda. \tag{2.13}$$

Similarly, letting $x\to 0^+$ in (1.7) and (1.8), we obtain

$$\lim_{x\to 0^+}\int_{(\partial\Omega_1^-)\cap\mathbb{R}} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda=\int_{(\partial\Omega_1^-)\cap\mathbb{R}} e^{-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda \tag{2.14}$$

and

$$\lim_{x\to 0^+}\int_{(\partial\Omega_2^-)\cap\mathbb{R}} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda=\int_{(\partial\Omega_2^-)\cap\mathbb{R}} e^{-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda. \tag{2.15}$$

On the other hand

$$\int_{\{\arg\lambda=2\pi/3\}} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda = \frac{1}{\alpha^2}\int_{\{\arg\mu=0\}} e^{i(\mu/\alpha^2)x-\omega(\mu)t}\hat{u}_0(\mu/\alpha)d\mu,$$

hence

$$\begin{aligned}\lim_{x\,\to\,0^+}\int_{\{\arg\lambda=2\pi/3\}} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda &= \frac{1}{\alpha^2}\int_{\{\arg\mu=0\}} e^{-\omega(\mu)t}\hat{u}_0(\mu/\alpha)d\mu \\ &= \int_{\{\arg\lambda=2\pi/3\}} e^{-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda.\end{aligned} \tag{2.16}$$

Similarly one has

$$\lim_{x\,\to\,0^+}\int_{\{\arg\lambda=\pi/3\}} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda = \int_{\{\arg\lambda=\pi/3\}} e^{-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda. \tag{2.17}$$

It follows from (2.13) - (2.17) that

$$\begin{aligned}\lim_{x\,\to\,0^+}\Bigg[\int_{\lambda=-\infty}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda - \int_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda \\ - \int_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda\Bigg] = 0,\end{aligned} \tag{2.18}$$

where we used also the fact that the quantity (2.2) is equal to zero, as we showed in Step 2.

Step 9: We claim that

$$\begin{aligned}\lim_{x\,\to\,0^+}\Bigg[\int_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha^2)\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda \\ + \int_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha)\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda\Bigg] = 2\pi g_0(t).\end{aligned} \tag{2.19}$$

First, if we let $x \to 0^+$ in (1.13) and we use the fact that (1.13) holds also for $x=0$, we obtain

$$\begin{aligned}\lim_{x\,\to\,0^+}\int_{(\partial\Omega_1^-)\cap\mathbb{R}} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda \\ = \int_{(\partial\Omega_1^-)\cap\mathbb{R}} e^{-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda.\end{aligned} \tag{2.20}$$

Similarly,

$$\lim_{x \to 0^+} \int_{(\partial\Omega_2^-)\cap\mathbb{R}} e^{i\lambda x - \omega(\lambda)t} \lambda^2 \tilde{g}_0(\omega(\lambda), T) d\lambda = \int_{(\partial\Omega_2^-)\cap\mathbb{R}} e^{-\omega(\lambda)t} \lambda^2 \tilde{g}_0(\omega(\lambda), T) d\lambda. \tag{2.21}$$

On the other hand, setting $\mu = \alpha^2\lambda$, we have

$$\int_{\{\arg\lambda = 2\pi/3\}} e^{i\lambda x - \omega(\lambda)t} \lambda^2 \tilde{g}_0(\omega(\lambda), T) d\lambda = \int_{\{\arg\mu = 0\}} e^{i(\mu/\alpha^2)x - \omega(\mu)t} \mu^2 \tilde{g}_0(\omega(\mu), T) d\mu,$$

whereas

$$\begin{aligned} \lim_{x \to 0^+} \int_{\{\arg\lambda = 2\pi/3\}} & e^{i\lambda x - \omega(\lambda)t} \lambda^2 \tilde{g}_0(\omega(\lambda), T) d\lambda \\ &= \int_{\{\arg\mu = 0\}} e^{-\omega(\mu)t} \mu^2 \tilde{g}_0(\omega(\mu), T) d\mu \\ &= \int_{\{\arg\lambda = 2\pi/3\}} e^{-\omega(\lambda)t} \lambda^2 \tilde{g}_0(\omega(\lambda), T) d\lambda. \end{aligned} \tag{2.22}$$

Similarly

$$\lim_{x \to 0^+} \int_{\{\arg\lambda = \pi/3\}} e^{i\lambda x - \omega(\lambda)t} \lambda^2 \tilde{g}_0(\omega(\lambda), T) d\lambda = \int_{\{\arg\lambda = \pi/3\}} e^{-\omega(\lambda)t} \lambda^2 \tilde{g}_0(\omega(\lambda), T) d\lambda. \tag{2.23}$$

Now (2.19) follows from (2.20)-(2.23) and the corresponding calculation made in Step 2 (which yields (2.3)).

Proof of the 3$^{\text{rd}}$ part. Since $e^{-\omega(\lambda)t}\lambda\tilde{g}_1(\omega(\lambda), T) = O(1/\lambda^2)$, when $\lambda \to \infty$ with $\operatorname{Re}\lambda = 0$, we see that the integrals in (1.1) - equivalently (1.11) - which contain the function $\tilde{g}_1$ tend to zero as $x \to 0^+$. (We also recall the last calculation of Step 2, which completed the proof of the equation $u(x,t)|_{x=0} = g_0(t)$). Therefore the 3$^{\text{rd}}$ part of the theorem follows from (2.18) and (2.19).

Remark 2.3. A computation similar to the one in Step 2, leading to (2.3), yields

$$\int_{\partial\Omega_1^-} \lim_{x \to 0^+} [e^{i\lambda x - \omega(\lambda)t} \lambda^2 \tilde{g}_0(\omega(\lambda), t)] d\lambda = \frac{\pi g_0(t)}{3}$$

and

$$\lim_{x\to 0^+} \int\limits_{\partial\Omega_2^-} [e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),t)]d\lambda = \frac{\pi g_0(t)}{3}$$

while, according to the proof of (2.19).

$$\lim_{x\to 0^+} \int\limits_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),t)d\lambda = \frac{2\pi g_0(t)}{3}$$

and

$$\lim_{x\to 0^+} \int\limits_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),t)d\lambda = \frac{2\pi g_0(t)}{3}.$$

Thus the order of limit and integration, in the above equations, cannot be interchanged in general.

Step 10: We claim that

$$\lim_{x\to 0^+}\left\{\frac{\partial}{\partial x}\left[\int\limits_{\lambda=-\infty}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda \right.\right.$$
$$\left.\left. -\int\limits_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda - \int\limits_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda\right]\right\} = 0. \tag{2.24}$$

To verify this assertion, writing (1.6) with $n = 1$ and $k = 0$, and letting $x \to 0^+$, we obtain

$$\begin{aligned}
&\lim_{x\to 0^+}\left\{\frac{\partial}{\partial x}\left[\int\limits_{\lambda=-\infty}^{0} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda\right]\right\} \\
&= \int\limits_{\{\mathrm{Im}\lambda=-\varepsilon\}\cap\{\mathrm{Re}\lambda\le 0\}} i\lambda e^{-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda + \int\limits_{[-\varepsilon i,0]} i\lambda e^{-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda \\
&= -\int\limits_{\{\arg\lambda=4\pi/3\}}^{*} i\lambda e^{-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda.
\end{aligned} \tag{2.25}$$

(The symbol "$*$" indicates that the corresponding integral must be appropriately interpreted).

Similarly, using (1.8), we obtain

$$
\begin{aligned}
&\lim_{x\,\to\,0^+}\left\{\frac{\partial}{\partial x}\left[\int\limits_{\lambda=0}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda\right]\right\}\\
&\qquad=\lim_{x\,\to\,0^+}\frac{\partial}{\partial x}\left[\int\limits_{\arg\lambda=0}^{*} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda\right]\\
&\qquad=\lim_{x\,\to\,0^+}\int\limits_{\arg\lambda=0}^{*} i\lambda e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda \qquad (2.26)\\
&\qquad=\int\limits_{\arg\lambda=0}^{*} i\lambda e^{-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda\\
&\qquad=\frac{1}{\alpha}\int\limits_{\{\arg\mu=4\pi/3\}}^{*} i\mu e^{-\omega(\mu)t}\hat{u}_0(\mu)d\mu.
\end{aligned}
$$

Also

$$
\begin{aligned}
&\lim_{x\,\to\,0^+}\left\{\frac{\partial}{\partial x}\left[\int\limits_{\{\arg\lambda=2\pi/3\}} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda\right]\right\}\\
&\qquad=\frac{1}{\alpha^2}\int\limits_{\{\arg\mu=4\pi/3\}}^{*} i\mu e^{-\omega(\mu)t}\hat{u}_0(\mu)d\mu.
\end{aligned}
\qquad (2.27)
$$

As $1+\alpha+\alpha^2=0$, it follows from (2.25) - (2.27) that

$$
\begin{aligned}
&\lim_{x\,\to\,0^+}\left\{\frac{\partial}{\partial x}\left[\int\limits_{\lambda=-\infty}^{0} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda-\int\limits_{\{\arg\lambda=2\pi/3\}} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda\right.\right.\\
&\qquad\left.\left.-\int\limits_{\lambda=0}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda\right]\right\}=0.
\end{aligned}
\qquad (2.28)
$$

Similarly, we have

$$\lim_{x\to 0^+}\left\{\frac{\partial}{\partial x}\left[\int_{\lambda=0}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda - \int_{\{\arg\lambda=\pi/3\}} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda \right.\right.$$
$$\left.\left. - \int_{\lambda=-\infty}^{0} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda\right]\right\} = 0. \tag{2.29}$$

Finally, (2.24) is a consequence of (2.28) and of (2.29).

Step 11: Using (2.5) and (2.6), and letting $x \to 0^+$, we obtain

$$\lim_{x\to 0^+}\frac{\partial}{\partial x}\left[\int_{-\infty}^{0} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda\right] = \int_{(-\infty,0]}^{*} e^{-\omega(\lambda)t}i\lambda^3\tilde{g}_0(\omega(\lambda),T)d\lambda. \tag{2.30}$$

Also, setting $\lambda = \mu\alpha^2$, we have

$$\int_{\{\arg\lambda=2\pi/3\}} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda = \int_{[0,\infty)} e^{i(\mu/\alpha^2)x-\omega(\mu)t}\mu^2\tilde{g}_0(\omega(\mu),T)d\mu.$$

Now, interpreting the integral in the RHS of the above equation as the (1.18) and (1.19), we obtain

$$\lim_{x\to 0^+}\frac{\partial}{\partial x}\left[\int_{\{\arg\lambda=2\pi/3\}} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda\right]$$
$$= \frac{1}{\alpha^2}\int_{[0,\infty)}^{*} e^{-\omega(\mu)t}i\mu^3\tilde{g}_0(\omega(\mu),T)d\mu,$$

where the integral $\int_{[0,\infty)}^{*}$ is defined analogously to (2.6).

Thus, we may write

$$\int_{[0,\infty)}^{*} e^{-\omega(\mu)t}i\mu^3\tilde{g}_0(\omega(\mu),T)d\mu = \alpha^2\int_{\{\arg\lambda=2\pi/3\}}^{*} e^{-\omega(\lambda)t}i\lambda^3\tilde{g}_0(\omega(\lambda),T)d\lambda$$

or

$$\lim_{x\to 0^+}\frac{\partial}{\partial x}\left[\int\limits_{\{\arg\lambda=2\pi/3\}} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda\right]$$
$$=\int\limits_{\{\arg\lambda=2\pi/3\}}^{*} e^{-\omega(\lambda)t}i\lambda^3\tilde{g}_0(\omega(\lambda),T)d\lambda.$$

The preceding calculations lead to

$$\lim_{x\to 0^+}\frac{\partial}{\partial x}\left[\int\limits_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda\right]$$
$$=\int\limits_{\partial\Omega_1^-}^{*} e^{-\omega(\lambda)t}i\lambda^3\tilde{g}_0(\omega(\lambda),T)d\lambda.$$

Similarly,

$$\lim_{x\to 0^+}\frac{\partial}{\partial x}\left[\int\limits_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda\right]=\int\limits_{\partial\Omega_2^-}^{*} e^{-\omega(\lambda)t}i\lambda^3\tilde{g}_0(\omega(\lambda),T)d\lambda.$$

Therefore, setting $\mu=\alpha\lambda$ in the integral $\int_{\partial\Omega_2^-}^{*}\cdots d\lambda$, we obtain

$$\begin{aligned}\lim_{x\to 0^+}\frac{\partial}{\partial x}\Bigg[&\int\limits_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha^2)\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda\\ &+\int\limits_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha)\lambda^2\tilde{g}_0(\omega(\lambda),T)d\lambda\Bigg]\\ =\left[(1-\alpha^2)+\frac{1}{\alpha}(1-\alpha)\right]&\int\limits_{\partial\Omega_1^-}^{*} e^{-\omega(\lambda)t}i\lambda^3\tilde{g}_0(\omega(\lambda),T)d\lambda=0.\end{aligned}\tag{2.31}$$

Proof of 4^{th} part. We have

$$\begin{aligned}
&\lim_{x\to 0^+}\frac{\partial}{\partial x}\Bigg[\int\limits_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha)\lambda\tilde{g}_1(\omega(\lambda),T)]d\lambda \\
&\qquad\qquad + \int\limits_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha^2)\lambda\tilde{g}_1(\omega(\lambda),T)]d\lambda\Bigg] \\
&\quad = \int\limits_{\partial\Omega_1^-} e^{-\omega(\lambda)t}(1-\alpha)i\lambda^2\tilde{g}_1(\omega(\lambda),T)]d\lambda \\
&\qquad\qquad + \int\limits_{\partial\Omega_2^-} e^{-\omega(\lambda)t}(1-\alpha^2)i\lambda^2\tilde{g}_1(\omega(\lambda),T)]d\lambda \\
&\quad = 2\pi i g_1(t).
\end{aligned} \tag{2.32}$$

The last equation is (2.19) with g_1 in place of g_0. Now, the 4^{th} part follows from (2.24), (2.31) and (2.32).

Proof of 5^{th} part. It follows from Lemma 2.2 of Step 6 that the convergence in (2.8), (2.9) and (2.11) is uniform for x in compact subsets of $(0,+\infty)$. It is easy to see that this holds also for the limit in (2.12).

Proof of 6^{th} part. Examining the proofs in Steps 7, 8, 9, 10, and the proof of (2.32), it is easy to see that the limits are uniform for t in compact subsets of $(0,+\infty)$. This completes the proof of the theorem.

3. Behavior of the derivatives $\partial^n u/\partial x^n$ as $x \to 0^+$ or as $t \to 0^+$

Under the assumptions of Theorem 1.1, the 3^{rd} and 6^{th} conclusions of this theorem imply that the function $u(x,t)$, defined by (2.1), satisfies $\lim\limits_{Q\ni(x,t)\to(0,t_0)} u(x,t) = g_0(t_0)$, for every $t_0>0$, i.e. $u(x,t)$ extends to a continuous function along the semi-axis $\{x=0, t>0\}$ by setting $u(0,t)=g_0(t)$ for $t>0$. Now, we will extend this result also to the derivatives of the function $u(x,t)$.

Theorem 3.1. *Under the assumptions of Theorem 1.1 the following assertions hold:*

1^{st} For each fixed $t>0$, the function $u(x,t)$ belongs to the space $C^\infty([0,\infty))$ with respect to x, i.e., the limits

$$g_n(t) := \lim_{x\to 0^+}\frac{\partial^n u(x,t)}{\partial x^n}, \quad n=0,1,2,\dots,$$

exist. Moreover the functions $g_n(t)$ are C^∞ for $t\in(0,\infty)$ and the above limits are uniform for t in compact subsets of $(0,+\infty)$.

2^{nd} The function $u(x,t)$, originally defined for $(x,t)\in Q$, extends to a C^∞ function on $Q\cup\{(0,t): t>0\}$, by setting $u(0,t)=g_0(t)$ for $t>0$.

3^{rd} For $k = 0, 1, 2, \dots$, and $t > 0$,

$$\lim_{x \to 0^+} \frac{\partial^{3k} u(x,t)}{\partial x^{3k}} = \frac{d^k g_0(t)}{dt^k} \quad \textit{and} \quad \lim_{x \to 0^+} \frac{\partial^{3k+1} u(x,t)}{\partial x^{3k+1}} = \frac{d^k g_1(t)}{dt^k}.$$

Proof. ***Step 1***. The cases $n = 0$ and $n = 1$ of the 1^{st} assertion are contained in Theorem 1.1. Now we prove the existence of these limits in the general case. First, it follows from (1.7) that the limit

$$\lim_{x \to 0^+} \frac{\partial^n}{\partial x^n} \left[\int\limits_{\lambda=-\infty}^{\infty} e^{i\lambda x - \omega(\lambda)t} \hat{u}_0(\lambda) d\lambda \right]$$

exists and defines a C^∞ functionb of $t \in (0, \infty)$. It is easy to see that a similar result holds for all the parts of the integrals in (1.16) which are taken over the half-lines

$$\{\arg \lambda = \pi\} = (-\infty, 0) \quad \text{or} \quad \{\arg \lambda = 0\} = [0, +\infty).$$

Now writing the equation

$$\int\limits_{\{\arg\lambda = 2\pi/3\}} e^{i\lambda x - \omega(\lambda)t} \hat{u}_0(\alpha\lambda) d\lambda = \frac{1}{\alpha^2} \int\limits_{\{\arg\mu = 0\}} e^{i(\mu/\alpha^2)x - \omega(\mu)t} \hat{u}_0(\mu/\alpha) d\mu,$$

we see, as previously, that the limit

$$\lim_{x \to 0^+} \frac{\partial^n}{\partial x^n} \left[\int\limits_{\{\arg\lambda = 2\pi/3\}} e^{i\lambda x - \omega(\lambda)t} \hat{u}(\alpha\lambda) d\lambda \right]$$

exists and defines a C^∞ function of $t \in (0, \infty)$.
Similar is also the treatment of the parts of all the other integrals in (1.16) which are taken on the lines

$$\{\arg \lambda = 2\pi/3\} \quad \text{or} \quad \{\arg \lambda = \pi/3\}.$$

This completes the proof of the 1^{st} assertion.

Step 2. It follows from the 1^{st} conclusion that

$$\lim_{Q \ni (x,t) \to (0,t_0)} \frac{\partial^n u(x,t)}{\partial x^n} = g_n(t_0) \quad \text{for} \quad t_0 \in (0, +\infty).$$

Since for every nonnegative integers n and m,

$$\frac{\partial^{n+m} u(x,t)}{\partial x^n \partial t^m} = \frac{\partial^{n+3m} u(x,t)}{\partial x^{n+3m}} \quad \text{for} \quad (x,t) \in Q,$$

it follows that

$$\lim_{Q \ni (x,t) \to (0,t_0)} \frac{\partial^{n+m} u(x,t)}{\partial x^n \partial t^m} = g_{n+3m}(t_0) \quad \text{for} \quad t_0 \in (0, +\infty).$$

Therefore, the function $u(x,t)$, originally defined for $(x,t) \in Q$, extends to a C^∞ functions on $Q \cup \{(0,t) : t > 0\}$. This proves the 2^{nd} assertion of the theorem.

Step 3. We now prove the 3^{rd} assertion. The case $k = 0$ is contained in Theorem 1.1. Let $u^*(x,t)$ denote a smooth extension of $u(x,t)$, from Q to an open neighborhood of semi-axis $\{x = 0, t > 0\}$, which exists from the 2^{nd} conclusion. Then

$$\lim_{x \to 0^+} \frac{\partial^3 u(x,t)}{\partial x^3} = \lim_{x \to 0^+} \frac{\partial u(x,t)}{\partial t} = \left.\frac{\partial u^*(x,t)}{\partial t}\right|_{x=0} = \frac{dg_0(t)}{dt}.$$

Similarly,

$$\lim_{x \to 0^+} \frac{\partial^4 u(x,t)}{\partial x^4} = \lim_{x \to 0^+} \frac{\partial}{\partial t}\left(\frac{\partial u(x,t)}{\partial x}\right) = \left.\frac{\partial}{\partial t}\left(\frac{\partial u^*(x,t)}{\partial x}\right)\right|_{x=0} = \frac{dg_1(t)}{dt}.$$

We now proceed inductively to complete the proof of the 3^{rd} assertion. □

Next we study the limits of the derivatives as $t \to 0^+$. Firstly the 2^{nd} and the 5^{th} conclusions of Theorem 1.1 imply that the function $u(x,t)$, defined by (1.2), satisfies

$$\lim_{Q \ni (x,t) \to (x_0,0)} u(x,t) = u_0(x_0)$$

for every $x_0 > 0$, i.e., $u(x,t)$ extends to a continuous function along the semi-axis $\{x > 0, t = 0\}$ by setting $u(x,0) = u_0(x)$ for $x > 0$. As we will see, extending this result to derivatives of the function $u(x,t)$ requires certain compatibility conditions for the data $u_0(x)$, $g_0(t)$ and $g_1(t)$, at the point $(x,t) = (0,0)$. A first result in this direction is the following theorem.

Theorem 3.2. *With the assumptions as in Theorem 1.1, we have:*

1^{st} If $u_0(0) = g_0(0)$ *then* $\displaystyle\lim_{t \to 0^+} \frac{\partial u(x,t)}{\partial x} = \frac{du_0(x)}{dx}$ *for* $x > 0$.

2^{nd} If $u_0(0) = g_0(0)$ *and* $u_0'(0) = g_1(0)$ *then*

$$\lim_{t \to 0^+} \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{d^2 u_0(x)}{dx^2} \quad \textit{and} \quad \lim_{t \to 0^+} \frac{\partial^3 u(x,t)}{\partial x^3} = \frac{d^3 u_0(x)}{dx^3} \quad \textit{for} \quad x > 0.$$

3^{rd} All the above limits are uniform for x in compact subsets of $(0, +\infty)$.

Proof. ***Step 1***. Computations analogous to the ones leading to (1.18) and (1.19) (with g_1 in place of g_0) imply that

$$\begin{aligned}
&\frac{\partial}{\partial x}\Big[\int\limits_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha)\lambda\tilde{g}_1(\omega(\lambda),t)]d\lambda \\
&\qquad\qquad + \int\limits_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha^2)\lambda\tilde{g}_1(\omega(\lambda),t]d\lambda\Big] \\
&= i\int\limits_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha)\lambda^2\tilde{g}_1(\omega(\lambda),t)]d\lambda \\
&\qquad\qquad + i\int\limits_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha^2)\lambda^2\tilde{g}_1(\omega(\lambda),t)]d\lambda,
\end{aligned} \tag{3.1}$$

with the appropriate interpretation of the preceding integrals.

Now it follows from (2.9) and (2.11) (with g_1 in place of g_0) that the limit as $t \to 0^+$ of the expression in (3.1) is zero.

Step 2. For $x > 0$ and $t > 0$, let us consider the quantity

$$\begin{aligned}
E(x,t) = &\int\limits_{\lambda=-\infty}^{0} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda - \int\limits_{\lambda=-\infty}^{0} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda \\
&\qquad - \int\limits_{\arg\lambda=2\pi/3} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda \\
&\qquad + (1-\alpha^2)\int\limits_{\lambda=-\infty}^{0} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),t)d\lambda,
\end{aligned}$$

which is part of the RHS of (2.1).

Considering the contours

$$\Gamma_1 = (-\infty - i, -1 - i] \cup [-1 - i, -1]$$

and

$$\gamma_1 = -[\{|\lambda| = 1,\ \pi \le \arg\lambda \le 2\pi/3\} \cup \{|\lambda| \ge 1, \arg\lambda = 2\pi/3\}]$$

and keeping in mind the interpretation of integrals given by (1.11) and (1.18), we write

$$E(x,t) = \int\limits_{\lambda=-\infty}^{-1} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda + \int\limits_{\lambda=-1}^{0} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda$$

$$
\begin{aligned}
&- \int_{\lambda=-\infty}^{-1} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda \\
&- \int_{\lambda=-1}^{0} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda - \int_{\arg\lambda=2\pi/3} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda \\
&+ (1-\alpha^2)\int_{\lambda=-\infty}^{-1} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),t)d\lambda \\
&+ (1-\alpha^2)\int_{\lambda=-1}^{0} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),t)d\lambda
\end{aligned}
$$

and we express the preceding sum as

$$
\begin{aligned}
E(x,t) = u_0(0)&\int_{\Gamma_1} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{i\lambda} \\
&+ \int_{\Gamma_1} e^{i\lambda x-\omega(\lambda)t}\widehat{(u_0')}(\lambda)\frac{d\lambda}{i\lambda} + \int_{\lambda=-1}^{0} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda \\
&- u_0(0)\int_{\Gamma} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{i\alpha\lambda} - \int_{\Gamma} e^{i\lambda x-\omega(\lambda)t}\widehat{(u_0')}(\alpha\lambda)\frac{d\lambda}{i\alpha\lambda} \\
&- \int_{\lambda=-1}^{0} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda - \int_{\arg\lambda=2\pi/3} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda \\
&- (1-\alpha^2)g_0(0)\int_{\Gamma_1} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{i\lambda} + (1-\alpha^2)g_0(t)\int_{\gamma_1} e^{i\lambda x}\frac{d\lambda}{i\lambda} \\
&- (1-\alpha^2)\int_{\lambda=-\infty}^{-1} e^{i\lambda x-\omega(\lambda)t}\widetilde{(g_0')}(\omega(\lambda),t)\frac{d\lambda}{i\lambda} \\
&+ (1-\alpha^2)\int_{\lambda=-1}^{0} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),t)d\lambda.
\end{aligned}
\tag{3.2}
$$

Now taking into consideration the assumption "$u_0(0) = g_0(0)$" and differentiating with respect to x we obtain

$$\begin{aligned}\frac{\partial E(x,t)}{\partial x} &= \int_{\Gamma_1} e^{i\lambda x-\omega(\lambda)t}(\hat{u_0'})(\lambda)d\lambda + \int_{\lambda=-1}^{0} e^{i\lambda x-\omega(\lambda)t} i\lambda\hat{u}_0(\lambda)d\lambda \\ &\quad - \alpha^2 \int_{\Gamma_1} e^{i\lambda x-\omega(\lambda)t}(\hat{u_0'})(\alpha\lambda)d\lambda - \int_{\lambda=-1}^{0} e^{i\lambda x-\omega(\lambda)t} i\lambda\hat{u}_0(\alpha\lambda)d\lambda \\ &\quad - \int_{\arg\lambda=2\pi/3} e^{i\lambda x-\omega(\lambda)t} i\lambda\hat{u}_0(\alpha\lambda)d\lambda \\ &\quad + (1-\alpha^2)g_0(t)\int_{\gamma_1} e^{i\lambda x}d\lambda - (1-\alpha^2)\int_{\lambda-\infty}^{-1} e^{i\lambda x-\omega(\lambda)t}(\tilde{g_0'})(\omega(\lambda),t)d\lambda \\ &\quad + (1-\alpha^2)\int_{\lambda=-1}^{0} e^{i\lambda x-\omega(\lambda)t}\omega(\lambda)\tilde{g}_0(\omega(\lambda),t)d\lambda\end{aligned}$$

from which we deduce that

$$\begin{aligned}\frac{\partial E(x,t)}{\partial x} &= \int_{\Gamma} e^{i\lambda x-\omega(\lambda)t}(\hat{u_0'})(\lambda)d\lambda + \int_{\lambda=-1}^{0} e^{i\lambda x-\omega(\lambda)t}(\hat{u_0'})(\lambda)d\lambda \\ &\quad + u_0(0)\int_{\lambda=-1}^{0} e^{i\lambda x-\omega(\lambda)t}d\lambda - \alpha^2\int_{\Gamma} e^{i\lambda x-\omega(\lambda)t}(\hat{u_0'})(\alpha\lambda)d\lambda \\ &\quad - \alpha^2\int_{\lambda=-1}^{0} e^{i\lambda x-\omega(\lambda)t}(\hat{u_0'})(\alpha\lambda)d\lambda - \alpha^2 u_0(0)\int_{\lambda=-1}^{0} e^{i\lambda x-\omega(\lambda)t}d\lambda \\ &\quad - \alpha^2\int_{\arg\lambda=2\pi/3} e^{i\lambda x-\omega(\lambda)t}(\hat{u_0'})(\alpha\lambda)d\lambda - \alpha^2 u_0(0)\int_{\arg\lambda=2\pi/3} e^{i\lambda x-\omega(\lambda)t}d\lambda \\ &\quad + (1-\alpha^2)g_0(t)\int_{\gamma_1} e^{i\lambda x}d\lambda + Y(x,t),\end{aligned} \tag{3.3}$$

where the term $Y(x,t)$ has the property $\lim_{t\to 0^+} Y(x,t) = 0$. Thus, letting $t \to 0^+$ in (3.3), we conclude that

$$\lim_{t\to 0^+}\frac{\partial E(x,t)}{\partial x} = \lim_{t\to 0^+}\int_{\lambda=-\infty}^{0} e^{i\lambda x-\omega(\lambda)t}(\hat{u_0'})(\lambda)d\lambda$$

$$-\alpha^2 \lim_{t\to 0^+} \int_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}\hat{(u_0')}(\alpha\lambda)d\lambda$$

$$+u_0(0)\Big[\int_{\lambda=-1}^{0} e^{i\lambda x}d\lambda - \alpha^2 \int_{\lambda=-1}^{0} e^{i\lambda x}d\lambda - \alpha^2 \int_{\arg\lambda=2\pi/3} e^{i\lambda x}d\lambda + (1-\alpha^2)\int_{\gamma_1} e^{i\lambda x}d\lambda\Big],$$

from which we deduce that

$$\begin{aligned}\lim_{t\to 0^+}\frac{\partial E(x,t)}{\partial x} &= \lim_{t\to 0^+}\int_{\lambda=-\infty}^{0} e^{i\lambda x-\omega(\lambda)t}\hat{(u_0')}(\lambda)d\lambda \\ &\quad -\alpha^2 \lim_{t\to 0^+}\int_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}\hat{(u_0')}(\alpha\lambda)d\lambda + u_0(0)F(0),\end{aligned} \tag{3.4}$$

having set $F(\lambda) = e^{i\lambda x}/ix$.

Step 3. Again for $x > 0$ and $t > 0$, we consider the quantity

$$\begin{aligned}Z(x,t) = &\int_{\lambda=0}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda - \int_{\lambda=0}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda \\ &- \int_{\arg\lambda=\pi/3} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda + (1-\alpha)\int_{\lambda=0}^{\infty} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),t)d\lambda,\end{aligned}$$

which is also part of the RHS of (1.2). Working as in Step 2, we prove that

$$\begin{aligned}\lim_{t\to 0^+}\frac{\partial Z(x,t)}{\partial x} &= \lim_{t\to 0^+}\int_{\lambda=0}^{\infty} e^{i\lambda x-\omega(\lambda)t}(u_0')(\lambda)d\lambda \\ &\quad -\alpha \lim_{t\to 0^+}\int_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}\hat{(u_0')}(\alpha^2\lambda)d\lambda - u_0(0)F(0).\end{aligned} \tag{3.5}$$

Step 4. It is straightforward that

$$\begin{aligned}\lim_{t\to 0^+}\frac{\partial}{\partial x}\Big[&\int_{\arg\lambda=2\pi/3} e^{i\lambda x-\omega(\lambda)t}(1-\alpha^2)\lambda^2\tilde{g}_0(\omega(\lambda),t)d\lambda \\ &+\int_{\arg\lambda=\pi/3} e^{i\lambda x-\omega(\lambda)t}(1-\alpha)\lambda^2\tilde{g}_0(\omega(\lambda),t)d\lambda\Big] = 0.\end{aligned} \tag{3.6}$$

Proof of the 1st conclusion. It follows from (1.2), the result of Step 1, (3.4) - (3.6), and applying (2.8)(and its proof) to u_0' (in place of u_0).

Proof of the 2nd conclusion. It is similar to the previous one.

Proof of the 3rd conclusion. It follows easily examining the above proofs. □

Remark 3.3. In view of the asymptotic behavior of the involved integrals given in Lemma 5.4 below, we see that the existence of the limit $\lim\limits_{t \to 0^+} \dfrac{\partial u(x,t)}{\partial x}$ implies that $u_0(0) = g_0(0)$. Similarly, the existence of the limits $\lim\limits_{t \to 0^+} \dfrac{\partial^2 u(x,t)}{\partial x^2}$ and $\lim\limits_{t \to 0^+} \dfrac{\partial^3 u(x,t)}{\partial x^3}$ implies that $u_0(0) = g_0(0)$ and $u_0'(0) = g_1(0)$.

4. The limit of the solution and its derivatives as $(x,t) \to (0,0)$

It follows from Theorem 1.1 that the function $u(x,t)$ is continuous for $(x,t) \in \overline{Q} - \{(0,0)\}$. It is clear that if the function $u(x,t)$ extends continuously also to the point $(0,0)$ the necessarily $u_0(0) = g_0(0)$. In the following theorem we show that this condition is also sufficient.

Theorem 4.1. *Under the assumptions as in Theorem 1.1 and if* $u_0(0) = g_0(0)$ *then*

$$\lim_{\overline{Q} \ni (x,t) \to (0,0)} u(x,t) = u_0(0).$$

Proof. We introduce the following notation: For two functions $V(x,t)$ and $W(x,t)$, defined for $x > 0$ and $t > 0$, we will write $V(x,t) \approx W(x,t)$ if and only if all the limits

$$\lim_{Q \ni (x,t) \to (0,0)} [V(x,t) - W(x,t)],$$

$$\lim_{x \to 0^+} \left(\lim_{\substack{t \to 0^+ \\ x > 0}} [V(x,t) - W(x,t)] \right),$$

$$\lim_{t \to 0^+} \left(\lim_{\substack{x \to 0^+ \\ t > 0}} [V(x,t) - W(x,t)] \right)$$

exist and are equal.

As $u_0(0) = g_0(0)$, it follows from (3.2) that

$$E(x,t) \approx - \int\limits_{\arg\lambda = 2\pi/3} e^{i\lambda x - \omega(\lambda)t} \hat{u}_0(\alpha\lambda) d\lambda + (1 - \alpha^2) g_0(t) \int\limits_{\gamma_1} e^{i\lambda x} \frac{d\lambda}{i\lambda},$$

whereas

$$\begin{aligned}
E(x,t) &\approx - \int_{\{\arg\lambda=2\pi/3, |\lambda|\geq 1\}} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda \\
&\qquad + (1-\alpha^2)g_0(t) \int_{\{\arg\lambda=2\pi/3, |\lambda|\geq 1\}} e^{i\lambda x}\frac{d\lambda}{i\lambda} \\
&\approx -u_0(0) \int_{\{\arg\lambda=2\pi/3, |\lambda|\geq 1\}} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{i\alpha\lambda} \\
&\qquad + (1-\alpha^2)g_0(t) \int_{\{\arg\lambda=2\pi/3, |\lambda|\geq 1\}} e^{i\lambda}\frac{d\lambda}{i\lambda}.
\end{aligned} \tag{4.1}$$

Similarly,

$$\begin{aligned}
Z(x,t) \approx -u_0(0) &\int_{\{\arg\lambda=\pi/3, |\lambda|\geq 1\}} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{i\alpha^2\lambda} \\
&+ (1-\alpha)g_0(t) \int_{\{\arg\lambda=\pi/3, |\lambda|\geq 1\}} e^{i\lambda x}\frac{d\lambda}{i\lambda}.
\end{aligned} \tag{4.2}$$

Let us keep in mind that the orientation of $\{\arg\lambda = 2\pi/3, |\lambda| \geq 1\}$ in (4.1) is the one that this line inherits from $\partial\Omega_1^-$, while the orientation of $\{\arg\lambda = \pi/3, |\lambda| \geq 1\}$ (in (7.2)) is that of $\partial\Omega_2^-$.

Furthermore

$$\begin{aligned}
&\int_{\arg\lambda=2\pi/3} e^{i\lambda x-\omega(\lambda)t}(1-\alpha^2)\lambda^2\tilde{g}_0(\omega(\lambda),t)d\lambda \\
&\qquad \approx -(1-\alpha^2)g_0(0) \int_{\{\arg\lambda=2\pi/3, |\lambda|\geq 1\}} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{i\lambda} \\
&\qquad\qquad + (1-\alpha^2)g_0(t) \int_{\{\arg\lambda=2\pi/3, |\lambda|\geq 1\}} e^{i\lambda x}\frac{d\lambda}{i\lambda}
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
&\int_{\arg\lambda=\pi/3} e^{i\lambda x-\omega(\lambda)t}(1-\alpha)\lambda^2\tilde{g}_0(\omega(\lambda),t)d\lambda \\
&\qquad \approx -(1-\alpha)g_0(0) \int_{\{\arg\lambda=\pi/3, |\lambda|\geq 1\}} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{i\lambda} \\
&\qquad + (1-\alpha)g_0(t) \int_{\{\arg\lambda=\pi/3, |\lambda|\geq 1\}} e^{i\lambda x}\frac{d\lambda}{i\lambda}.
\end{aligned} \tag{4.4}$$

Let us also notice that

$$\int_{\{\arg\lambda=2\pi/3,|\lambda|\geq 1\}} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{i\lambda}+\int_{\{\arg\lambda=\pi/3,|\lambda|\geq 1\}} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{i\lambda}\approx 0. \tag{4.5}$$

Finally, one has

$$\begin{aligned}\int_{\partial\Omega_1^-} & e^{i\lambda x-\omega(\lambda)t}(1-\alpha)\lambda\tilde{g}_1(\omega(\lambda),t)]d\lambda \\ & +\int_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha^2)\lambda\tilde{g}_1(\omega(\lambda),t)]d\lambda\approx 0.\end{aligned} \tag{4.6}$$

Now the conclusion of the theorem follows from the definitions of $E(x,t)$ and $Z(x,t)$, (1.2) and the relations (4.1) - 4.6).

It follows from Theorem 1.1, 3.1 and 3.2, that if $u_0(0)=g_0(0)$ then the derivative $\partial u/\partial x$ has continuous extension for $(x,t)\in\overline{Q}-\{(0,0)\}$. If this derivative extends continuously to the point $(0,0)$ then we must also have $u_0'(0)=g_1(0)$. The following theorem shows that these conditions are also sufficient for the continuous extendability of $\partial u/\partial x$ to $\overline{Q}$. □

Theorem 4.2. *Under the assumptions of Theorem 1.1 and if* $u_0(0)=g_0(0)$ *and* $u_0'(0)=g_1(0)$ *we have*

$$\lim_{\overline{Q}\ni(x,t)\,\to\,(0,0)}\frac{\partial u(x,t)}{\partial x}=u_0'(0).$$

Proof. It follows from the first part of (3.3) that

$$\begin{aligned}\frac{\partial E(x,t)}{\partial x}\approx ((1-\alpha)u_0'(0)\int_{\Gamma_1} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{i\lambda}-\alpha^2u_0(0)\int_{\arg\lambda=2\pi/3} e^{i\lambda x-\omega(\lambda)t}d\lambda \\ -\alpha u_0'(0)\int_{\{\arg\lambda=2\pi/3,|\lambda|\geq 1\}} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{i\lambda}+(1-\alpha^2)g_0(t)\int_{\gamma_1} e^{i\lambda x}d\lambda.\end{aligned} \tag{4.7}$$

Similarly,

$$\begin{aligned}\frac{\partial Z(x,t)}{\partial x}\approx (1-\alpha^2)u_0'(0)\int_{\Gamma_2} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{i\lambda}-\alpha u_0(0)\int_{\arg\lambda=\pi/3} e^{i\lambda x-\omega(\lambda)t}d\lambda \\ -\alpha^2u_0'(0)\int_{\{\arg\lambda=\pi/3,|\lambda|\geq 1\}} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{i\lambda}+(1-\alpha)g_0(t)\int_{\gamma_2} e^{i\lambda x}d\lambda,\end{aligned} \tag{4.8}$$

where

$$\Gamma_2=[1,1-i]\cup[1-i,+\infty-i)$$

and

$$\gamma_2 = \{|\lambda| = 1,\ \ 0 \le \arg\lambda \le \pi/3\} \cup \{|\lambda| \ge 1,\ \arg\lambda = \pi/3\}.$$

Also we have

$$\begin{aligned}
\frac{\partial}{\partial x}\Bigg[& \int\limits_{\arg\lambda=2\pi/3} e^{i\lambda x-\omega(\lambda)t}(1-\alpha^2)\lambda^2\tilde{g}_0(\omega(\lambda),t)d\lambda \\
& + \int\limits_{\arg\lambda=\pi/3} e^{i\lambda x-\omega(\lambda)t}(1-\alpha)\lambda^2\tilde{g}_0(\omega(\lambda),t)d\lambda\Bigg] \\
& \approx -(1-\alpha^2)g_0(0)\int\limits_{\arg\lambda=2\pi/3} e^{i\lambda x-\omega(\lambda)t}d\lambda + (1-\alpha^2)g_0(t)\int\limits_{\arg\lambda=2\pi/3} e^{i\lambda x}d\lambda \\
& \quad -(1-\alpha)g_0(0)\int\limits_{\arg\lambda=\pi/3} e^{i\lambda x-\omega(\lambda)t}d\lambda + (1-\alpha)g_0(t)\int\limits_{\arg\lambda=\pi/3} e^{i\lambda x}d\lambda.
\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}
-i\frac{\partial}{\partial x}\Bigg[& \int\limits_{\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha)\lambda\tilde{g}_1(\omega(\lambda),t)d\lambda \\
& + \int\limits_{\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}(1-\alpha^2)\lambda\tilde{g}_1(\omega(\lambda),t)d\lambda\Bigg] \\
& \approx -(1-\alpha)g_1(0)\int\limits_{\Gamma_1} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{i\lambda} + (1-\alpha)g_1(t)\int\limits_{\gamma_1} e^{i\lambda x}\frac{d\lambda}{i\lambda} \\
& \quad -(1-\alpha)g_1(0)\int\limits_{\{\arg\lambda=2\pi/3,|\lambda|\ge 1\}} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{i\lambda} \\
& \quad +(1-\alpha)g_1(t)\int\limits_{\{\arg\lambda=2\pi/3,|\lambda|\ge 1\}} e^{i\lambda x}\frac{d\lambda}{i\lambda} \\
& \quad -(1-\alpha^2)g_1(0)\int\limits_{\Gamma_2} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{i\lambda} + (1-\alpha^2)g_1(t)\int\limits_{\gamma_2} e^{i\lambda x}\frac{d\lambda}{i\lambda} \\
& \quad -(1-\alpha^2)g_1(0)\int\limits_{\{\arg\lambda=\pi/3,|\lambda|\ge 1\}} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{i\lambda} \\
& \quad +(1-\alpha^2)g_1(t)\int\limits_{\{\arg\lambda=\pi/3,|\lambda|\ge 1\}} e^{i\lambda x}\frac{d\lambda}{i\lambda}.
\end{aligned} \tag{4.10}$$

Now the conclusion of the theorem is a consequence of (4.7) - (4.10). □

Remark 4.3. Further computations show that the continuous extension of the derivatives $\partial^n u/\partial x^n$ to $\overline{Q}$ need higher order compatibility conditions for the data.

5. The asymptotic behavior of certain integrals

Lemma 5.1. *For fixed $\varepsilon > 0$ we have*

$$\int\limits_{\substack{Im\lambda=-\varepsilon\\ Re\lambda\geq 0}} e^{i\lambda}e^{-i\lambda^3 t}d\lambda = 2\frac{\sqrt{\pi}}{\sqrt[4]{3}}\frac{1}{\sqrt[4]{t}}\exp\left[i\left(\frac{2}{3\sqrt{3}}\frac{1}{\sqrt{t}}-\frac{\pi}{4}\right)\right]+o\big(1/\sqrt[4]{t}\big), \quad as \quad t\to 0^+.$$

In particular, the limit

$$\lim_{t\to 0^+}\int\limits_{\substack{Im\lambda=-\varepsilon\\ Re\lambda\geq 0}} e^{i\lambda x}e^{-i\lambda^3 t}d\lambda$$

does not exist for any $x > 0$.

Proof. Since for $\varepsilon_2 > \varepsilon_1 > 0$ the function defined by the integral

$$\int\limits_{\lambda\in[-\varepsilon_1 i,-\varepsilon_2 i]} e^{i\lambda}e^{-i\lambda^3 t}d\lambda$$

is bounded for t close to 0, the assertion of the lemma is independent of ε. Thus, we may assume that $\varepsilon = 1$.

Setting $\lambda = \xi - \varepsilon i = \xi - i$, with $0 \leq \xi < +\infty$, we have

$$\int\limits_{\substack{\mathrm{Im}\lambda=-\varepsilon\\ \mathrm{Re}\lambda\geq 0}} e^{i\lambda}e^{-i\lambda^3 t}d\lambda = \int\limits_{\xi=0}^{\infty} e^{i(\xi-i)}e^{-i(\xi-i)^3 t}d\xi = e^{1+t}\int\limits_{\xi=0}^{\infty} e^{-3\xi^2 t}e^{-i(\xi^3 t-\xi-3t\xi)}d\xi. \tag{5.1}$$

Changing the variable from ξ to y, setting $\xi^2 t = y$, we see that the asymptotic behavior of the above integral, as $t\to 0^+$, is reduced to that of the integral

$$\frac{1}{\sqrt{t}}\int\limits_{y=0}^{\infty}\frac{e^{-3y}}{\sqrt{y}}e^{-i\frac{1}{\sqrt{t}}\sqrt{y}[(y-1)-3t]}dy.$$

Since $|e^{iw_1}-e^{iw_2}| \leq |w_1-w_2|$, for $w_1, w_2 \in \mathbb{R}$, we have

$$\left|\frac{1}{\sqrt{t}}\int\limits_{y=0}^{\infty}\frac{e^{-3y}}{\sqrt{y}}e^{-i\frac{1}{\sqrt{t}}\sqrt{y}[(y-1)-3t]}dy-\frac{1}{\sqrt{t}}\int\limits_{y=0}^{\infty}\frac{e^{-3y}}{\sqrt{y}}e^{-i\frac{1}{\sqrt{t}}\sqrt{y}(y-1)}dy\right| \leq 3\int\limits_{y=0}^{\infty}e^{-3y}dy = 1.$$

Thus it suffices to study the integral

$$\frac{1}{\sqrt{t}}\int\limits_{y=0}^{\infty}\frac{e^{-3y}}{\sqrt{y}}e^{-i\frac{1}{\sqrt{t}}\sqrt{y}(y-1)}dy = \frac{1}{\sqrt{t}}\left[\int\limits_{y=0}^{1/6}+\int\limits_{y=1/6}^{4/3}+\int\limits_{y=4/3}^{\infty}\right]. \tag{5.2}$$

Now

$$\begin{aligned}\frac{1}{\sqrt{t}}\int\limits_{y=4/3}^{\infty}\frac{e^{-3y}}{\sqrt{y}}e^{-i\frac{1}{\sqrt{t}}\sqrt{y}(y-1)}dy &= \frac{1}{\sqrt{t}}\int\limits_{y=4/3}^{\infty}\frac{e^{-3y}}{\sqrt{y}}\frac{d}{dy}\Bigg[\int\limits_{z=4/3}^{y}e^{-i\frac{1}{\sqrt{t}}\sqrt{z}(z-1)}dz\Bigg]dy\\ &= -\frac{1}{\sqrt{t}}\int\limits_{y=4/3}^{\infty}\frac{d}{dy}\Bigg[\frac{e^{-3y}}{\sqrt{y}}\Bigg]\Bigg[\int\limits_{z=4/3}^{y}e^{-i\frac{1}{\sqrt{t}}\sqrt{z}(z-1)}dz\Bigg]dy\end{aligned}\tag{5.3}$$

and, by van der Corput's lemma,

$$\left|\int\limits_{z=4/3}^{y}e^{-i\frac{1}{\sqrt{t}}\sqrt{z}(z-1)}dz\right| \leq (\text{constant})\,\sqrt{t}.$$

It follows that the integral in the RHS of (5.3) is bounded for $t>0$.

Also

$$\begin{aligned}&\frac{1}{\sqrt{t}}\int\limits_{y=0}^{1/6}\frac{e^{-3y}}{\sqrt{y}}e^{-i\frac{1}{\sqrt{t}}\sqrt{y}(y-1)}dy\\ &= \frac{2i}{3}\int\limits_{y=0}^{1/6}\frac{e^{-3y}}{y-(1/3)}\frac{d}{dy}\Big[e^{-i\frac{1}{\sqrt{t}}\sqrt{y}(y-1)}\Big]dy\\ &= \frac{2i}{3}\Bigg[\frac{e^{-3y}}{y-(1/3)}e^{-i\frac{1}{\sqrt{t}}\sqrt{y}(y-1)}\Bigg]\Bigg|_{y=0}^{y=1/6}\\ &\qquad -\frac{2i}{3}\int\limits_{y=0}^{1/6}\frac{d}{dy}\Bigg[\frac{e^{-3y}}{y-(1/3)}\Bigg]e^{-i\frac{1}{\sqrt{t}}\sqrt{y}(y-1)}dy.\end{aligned}\tag{5.4}$$

It follows that the integral in the RHS of (5.4) is also bounded for $t>0$.

Finally, by the stationary phase asymptotic relation, we obtain

$$\begin{aligned}&\frac{1}{\sqrt{t}}\int\limits_{y=1/6}^{4/3}\frac{e^{-3y}}{\sqrt{y}}e^{-i\frac{1}{\sqrt{t}}\sqrt{y}(y-1)}dy\\ &\approx \frac{2}{e}\sqrt{\frac{\pi}{\sqrt{3}}}\frac{1}{\sqrt[4]{t}}\exp\Bigg[i\Bigg(\frac{2}{3\sqrt{3}}\frac{1}{\sqrt{t}}-\frac{\pi}{4}\Bigg)\Bigg],\quad \text{as}\quad t\to 0^{+},\end{aligned}\tag{5.5}$$

in the sense that the difference of the two quantities is $o\big(1/\sqrt[4]{t}\big)$.

Thus the lemma follows from (5.1) - (5.5). $\square$

Lemma 5.2. *Let $\varepsilon > 0$, $x > 0$ and $n \in \mathbb{N} \cup \{0\}$. Then*

$$\int\limits_{\substack{Im\lambda=-\varepsilon \\ Re\lambda\geq 0}} (i\lambda)^n e^{i\lambda x} e^{-i\lambda^3 t} d\lambda = \frac{2i^n\sqrt{\pi}}{3^{(2n+1)/4}} x^{(2n-1)/4} \frac{1}{t^{n/2}} \frac{1}{\sqrt[4]{t}} \exp\left[i\left(\frac{2}{3\sqrt{3}} \frac{x^{3/2}}{\sqrt{t}} - \frac{\pi}{4}\right)\right]$$

$$+ o\big(1/(t^{n/2}\sqrt[4]{t})\big), \quad as \ t \ \to \ 0^+$$

and the limit of the preceding integral does not exist as $t \ \to \ 0^+$.

Proof. We consider first the case $x = 1$. Writing $\lambda = \xi - \varepsilon i = \xi - i$, $0 \leq \xi < +\infty$, as in the previous proof, and substituting

$$\lambda^n = (\xi - i)^n = \xi^n - ni\xi^{n-1} + \cdots$$

in the integral, we see that we have to deal with the integrals

$$\int\limits_{\xi=0}^{\infty} \xi^k e^{i(\xi-i)} e^{-i(\xi-i)^3 t} d\xi = e^{1+t} \int\limits_{\xi=0}^{\infty} \xi^k e^{-3\xi^2 t} e^{-i(\xi^3 t - \xi - 3t\xi)} d\xi, \quad k = n, n-1, \ldots, 0.$$

Working with the above integrals as in the Lemma 5.1 and combining the results, we obtain the formula in the case $x = 1$. In fact it suffices to notice that the "dominant" term of the integral is the one with $k = n$.

For the general case in which $x > 0$ (but fixed), we set $\mu = \lambda x$ and we have

$$\int\limits_{\substack{\mathrm{Im}\lambda=-\varepsilon \\ \mathrm{Re}\lambda\geq 0}} (i\lambda)^n e^{i\lambda x} e^{-i\lambda^3 t} d\lambda = \frac{1}{x^{n+1}} \int\limits_{\substack{\mathrm{Im}\mu=-\varepsilon x \\ \mathrm{Re}\mu\geq 0}} (i\mu)^n e^{i\mu} e^{-i\mu^3 (t/x^3)} d\mu,$$

and the formula follows from the previous case applied with t/x^3 in place of t. ☐

Lemma 5.3. *Let $\varepsilon > 0$, $x > 0$ and $n \in \mathbb{N} \cup \{0\}$. Then, as $t \ \to \ 0^+$.*

$$\int\limits_{\substack{Im\lambda=-\varepsilon \\ Re\lambda\leq 0}} (i\lambda)^n e^{i\lambda x} e^{-i\lambda^3 t} d\lambda = \frac{2(-i)^n\sqrt{\pi}}{3^{(2n+1)/4}} x^{(2n-1)/4} \frac{1}{t^{n/2}} \frac{1}{\sqrt[4]{t}} \exp\left[-i\left(\frac{2}{3\sqrt{3}} \frac{x^{3/2}}{\sqrt{t}} - \frac{\pi}{4}\right)\right]$$

$$+ o\big(1/(t^{n/2}\sqrt[4]{t})\big).$$

Proof. Setting $\lambda = \xi - \varepsilon i$ with $-\infty < \xi \leq 0$, we obtain that

$$\int\limits_{\substack{\mathrm{Im}\lambda=-\varepsilon \\ \mathrm{Re}\lambda\leq 0}} (i\lambda)^n e^{i\lambda} e^{-i\lambda^3 t} d\lambda = i^n \int\limits_{\xi=-\infty}^{0} (\xi - \varepsilon i)^n e^{i(\xi-\varepsilon i)} e^{-i(\xi-\varepsilon i)^3 t} d\xi$$

$$= i^n \int\limits_{\xi=0}^{\infty} (-\xi - \varepsilon i)^n e^{i(-\xi-\varepsilon i)} e^{-i(-\xi-\varepsilon i)^3 t} d\xi.$$

Therefore

$$\int\limits_{\substack{\mathrm{Im}\lambda=-\varepsilon\\ \mathrm{Re}\lambda\le 0}} (i\lambda)^n e^{i\lambda} e^{-i\lambda^3 t} d\lambda = \overline{\int\limits_{\substack{\mathrm{Im}\lambda=-\varepsilon\\ \mathrm{Re}\lambda\ge 0}} (i\lambda)^n e^{i\lambda} e^{-i\lambda^3 t} d\lambda},$$

and the formula follows from Lemma 5.2. ☐

Lemma 5.4. *Let $\varepsilon > 0$, $x > 0$ and $n \in \mathbb{N} \cup \{0\}$. Then, as $t \to 0^+$,*

$$\begin{aligned}(1-\alpha^2) \int\limits_{\substack{Im\lambda=-\varepsilon\\ Re\lambda\le 0}} (i\lambda)^n e^{i\lambda x - i\lambda^3 t} d\lambda + (1-\alpha) \int\limits_{\substack{Im\lambda=-\varepsilon\\ Re\lambda\ge 0}} (i\lambda)^n e^{i\lambda x - i\lambda^3 t} d\lambda \\ = \frac{4\sqrt{\pi}}{3^{(2n-1)/4}} x^{(2n-1)/4} \frac{1}{t^{n/2}} \frac{1}{\sqrt[4]{t}} Re\Big\{ i^n \exp\Big[i\Big(\frac{2}{3\sqrt{3}} \frac{x^{3/2}}{\sqrt{t}} - \frac{5\pi}{12} \Big) \Big] \Big\} \\ + o\big(1/(t^{n/2} \sqrt[4]{t})\big).\end{aligned}$$

In particular the limit

$$\lim_{t\to 0^+} \Big[(1-\alpha^2) \int\limits_{\substack{Im\lambda=-\varepsilon\\ Re\lambda\le 0}} (i\lambda)^n e^{i\lambda x - i\lambda^3 t} d\lambda + (1-\alpha) \int\limits_{\substack{Im\lambda=-\varepsilon\\ Re\lambda\ge 0}} (i\lambda)^n e^{i\lambda x - i\lambda^3 t} d\lambda \Big]$$

does not exist.

Proof. Since $1-\alpha^2 = \sqrt{3}e^{i\pi/6}$ and $1-\alpha = \sqrt{3}e^{-i\pi/6}$, the required formula follows immediately from the formulas of Lemma 5.2 and 5.3. ☐

Lemma 5.5. *Let $t > 0$ and $n \in \mathbb{N} \cup \{0\}$. Then, as $x \to +\infty$*

$$\begin{aligned}(1-\alpha^2) \int\limits_{\substack{Im\lambda=-1/x\\ Re\lambda\le 0}} (i\lambda)^n e^{i\lambda x - i\lambda^3 t} d\lambda + (1-\alpha) \int\limits_{\substack{Im\lambda=-1/x\\ Re\lambda\ge 0}} (i\lambda)^n e^{i\lambda x - i\lambda^3 t} d\lambda \\ = \frac{4\sqrt{\pi}}{3^{(2n-1)/4}} \frac{1}{t^{n/2}} \frac{1}{\sqrt[4]{t}} x^{(2n-1)/4} Re\Big\{ i^n \exp\Big[i\Big(\frac{2}{3\sqrt{3}} \frac{x^{3/2}}{\sqrt{t}} - \frac{5\pi}{12} \Big) \Big] \Big\} \\ + o\big(x^{(2n-1)/4}\big).\end{aligned}$$

Moreover the above relation is uniform in t on compact subsets of $(0, +\infty)$.

Proof. Setting $\mu = \lambda x$ in the integrals, the LHS of the equation becomes

$$\frac{1}{x^{n+1}} \Big[(1-\alpha^2) \int\limits_{\substack{\mathrm{Im}\mu=-1\\ \mathrm{Re}\mu\le 0}} (i\mu)^n e^{i\mu - i\mu^3 (t/x^3)} d\mu + (1-\alpha) \int\limits_{\substack{\mathrm{Im}\mu=-1\\ \mathrm{Re}\mu\ge 0}} (i\mu)^n e^{i\mu - i\mu^3 (t/x^3)} d\mu \Big],$$

and the formula of the lemma follows from Lemma 5.4. ☐

Lemma 5.6. *As $x \to +\infty$ and uniformly for t in compact subsets of* $(0, +\infty)$,

$$\int_{\lambda=1}^{\infty} e^{i\lambda x - i\lambda^3 t} \frac{d\lambda}{i\lambda} = -i\sqrt{\pi}\sqrt[4]{3} \frac{t^{1/4}}{x^{3/4}} \exp\Big[i\Big(\frac{2}{3\sqrt{3}} \frac{x^{3/2}}{\sqrt{t}} - \frac{\pi}{4} \Big) \Big] + O(1/x).$$

and, therefore,

$$(1-\alpha^2)\int\limits_{\lambda=-\infty}^{-1} e^{i\lambda x - i\lambda^3 t}\frac{d\lambda}{i\lambda} + (1-\alpha)\int\limits_{\lambda=1}^{\infty} e^{i\lambda x - i\lambda^3 t}\frac{d\lambda}{i\lambda}$$
$$= 2\sqrt{\pi}\sqrt{3}\sqrt[4]{3}\frac{t^{1/4}}{x^{3/4}}\sin\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}} - \frac{5\pi}{12}\right) + O(1/x).$$

Proof. First we consider the case $t = 1$. Setting $\lambda = \mu\sqrt{x}$ we find that

$$\int\limits_{\lambda=1}^{\infty} e^{i\lambda x - i\lambda^3}\frac{d\lambda}{i\lambda} = \int\limits_{\mu=1/\sqrt{x}}^{\infty} e^{ix^{3/2}(\mu-\mu^3)}\frac{d\mu}{i\mu} = \int\limits_{\gamma_x} e^{ix^{3/2}(\mu-\mu^3)}\frac{d\mu}{i\mu} + \int\limits_{\Gamma} e^{ix^{3/2}(\mu-\mu^3)}\frac{d\mu}{i\mu},$$

where γ_x is a smooth curve from the point $1/\sqrt{x}$ to $i\infty$ and Γ is a smooth curce from $i\infty$, through $1/\sqrt{3}$, to $e^{-i\pi/6}\infty$, as in Figure 2 below. (The above equation follows from Cauchy's theorem and Jordan's lemma, provided that the curves γ_x and Γ are appropriately chosen). For instance, we choose

$$\gamma_x : [0,1) \;\to\; \mathbb{C}, \gamma_x(s) := \frac{1-s}{\sqrt{x}} + i\left(\frac{1}{1-s} - 1\right), \quad 0 \le s < 1.$$

The curve Γ is chosen so that $\operatorname{Im}[i(\mu-\mu^3)] = 2/(3\sqrt{3})$ for μ on Γ, close to the point $1/\sqrt{3}$, and $\operatorname{Re}[i(\mu-\mu^3)] < 0$ for every μ on Γ with $\mu \neq 1/\sqrt{3}$. Also the part of Γ in the half plane $\{\operatorname{Im}\mu < 0\}$ should lie in the set $\{-\pi/6 < \arg\mu < 0\}$ and be asymptotic to the half line $\{\arg\mu = -\pi/6\}$, while the part of Γ in the half plane $\{\operatorname{Im}\mu > 0\}$ should be asymptotic to the half line $\{\arg\mu = \pi/2\}$.

Now

$$\int\limits_{\gamma_x} e^{ix^{3/2}(\mu-\mu^3)}\frac{d\mu}{i\mu} = \int\limits_{\tilde\gamma_x} e^{i\lambda x - i\lambda^3}\frac{d\lambda}{i\lambda}$$

where

$$\tilde\gamma_x : [0,1) \;\to\; \mathbb{C},\ \tilde\gamma_x(s) := \sqrt{x}\gamma_x(s) = (1-s) + i\left(\frac{1}{1-s} - 1\right)\sqrt{x}, \quad 0 \le s < 1.$$

Then $\tilde\gamma_x$ is a curve from 1 to $i\infty$ (see Figure 2) and integration by parts shows that

$$\int\limits_{\tilde\gamma_x} e^{i\lambda x - i\lambda^3}\frac{d\lambda}{i\lambda} \approx e^{(x-1)i}\left(\frac{1}{x} + \frac{3-i}{x^2} + \frac{7}{x^3} + \cdots\right), \quad \text{as } x \;\to\; +\infty.$$

On the other hand, by the steepest descent method,

$$\int\limits_{\Gamma} e^{ix^{3/2}(\mu-\mu^3)}\frac{d\mu}{i\mu} = -i\sqrt{\pi}\sqrt[4]{3}\frac{1}{x^{3/4}}\exp\left[i\left(\frac{2x^{3/2}}{3\sqrt{3}} - \frac{\pi}{4}\right)\right] + O(1/x^{3/2}), \text{ as } x \;\to\; +\infty.$$

This proves the estimate of the lemma in the case $t = 1$.

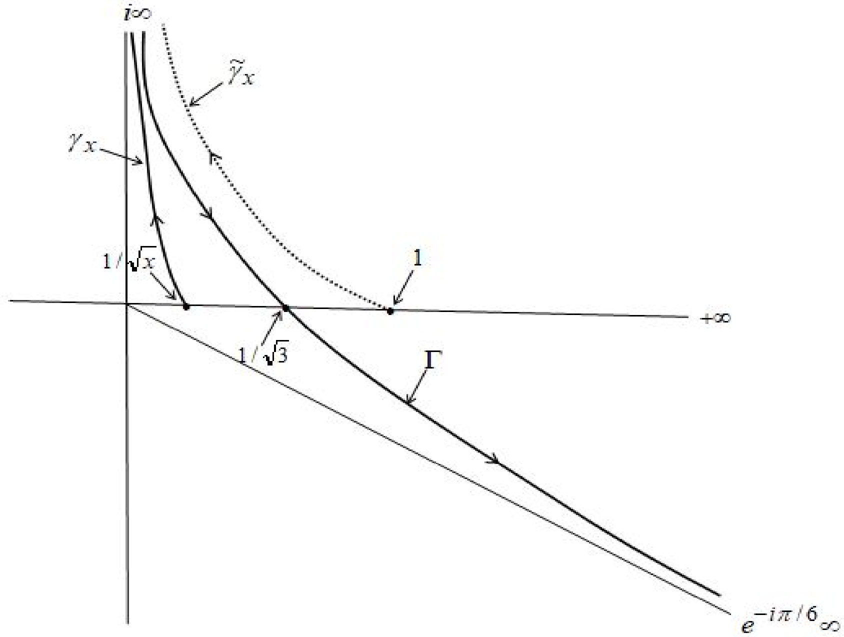

FIGURE 2. The contours γ_x, $\tilde{\gamma}_x$, and Γ used in the proofs of Lemmas 5.6 and 5.7

For the general case, i.e., when t ranges within a compact subset of the interval $(0, +\infty)$, the estimate follows by rescaling. Indeed, setting $\mu = \lambda t^{1/3}$, we have

$$\int_{\lambda=1}^{\infty} e^{i\lambda x - i\lambda^3 t} \frac{d\lambda}{i\lambda} = \int_{\mu=t^{1/3}}^{\infty} e^{i\mu(x/t^{1/3}) - i\mu^3} \frac{d\mu}{i\mu}$$
$$= \int_{\mu=t^{1/3}}^{1} e^{i\mu(x/t^{1/3}) - i\mu^3} \frac{d\mu}{i\mu} + \int_{\mu=1}^{\infty} e^{i\mu(x/t^{1/3}) - i\mu^3} \frac{d\mu}{i\mu}$$

and

$$\int_{\mu=t^{1/3}}^{1} e^{i\mu(x/t^{1/3}) - i\mu^3} \frac{d\mu}{i\mu} = \int_{\lambda=1}^{1/t^{1/3}} e^{i\lambda x - i\lambda^3 t} \frac{d\lambda}{i\lambda} = \frac{1}{ix} \int_{\lambda=1}^{1/t^{1/3}} (e^{i\lambda x})' e^{-i\lambda^3 t} \frac{d\lambda}{i\lambda} = O(1/x).$$

This completes the proof of the lemma. $\square$

Lemma 5.7. *For $n \geq 2$, we have*

$$\int_{\lambda=1}^{\infty} e^{i\lambda x - i\lambda^3 t} \frac{d\lambda}{\lambda^n} = O(1/x).$$

as $x \to +\infty$*, uniformly for* t *in compact subsets of* $(0, +\infty)$*.*

Proof. We consider first the case $n = 2$. As in the proof of Lemma 5.6, setting $\lambda = \mu\sqrt{x}$ we find that

$$\int_{\lambda=1}^{\infty} e^{i\lambda x - i\lambda^3} \frac{d\lambda}{\lambda^2} = \frac{1}{\sqrt{x}} \int_{\mu=1/\sqrt{x}}^{\infty} e^{ix^{3/2}(\mu-\mu^3)} \frac{d\mu}{\mu}$$
$$= \frac{1}{\sqrt{x}} \int_{\gamma_x} e^{ix^{3/2}(\mu-\mu^3)} \frac{d\mu}{\mu^2} + \frac{1}{\sqrt{x}} \int_{\Gamma} e^{ix^{3/2}(\mu-\mu^3)} \frac{d\mu}{\mu^2}$$

and

$$\frac{1}{\sqrt{x}} \int_{\gamma_x} e^{ix^{3/2}(\mu-\mu^3)} \frac{d\mu}{\mu^2} = \int_{\tilde{\gamma}_x} e^{i\lambda x - i\lambda^3} \frac{d\lambda}{\lambda^2} = O(1/x).$$

Also, by the steepest descent method,

$$\frac{1}{\sqrt{x}} \int_{\Gamma} e^{ix^{3/2}(\mu-\mu^3)} \frac{d\mu}{\mu^2} = \sqrt{\pi}\sqrt[4]{3}\sqrt{3} \frac{1}{\sqrt{x}} \frac{1}{x^{3/4}} \exp\left[i\left(\frac{2x^{3/2}}{3\sqrt{3}} - \frac{\pi}{4} \right) \right] + O(1/x^2)$$
$$= O(1/x^{5/4}) \quad \text{as} \quad x \to +\infty,$$

and the required estimate - in the case $n = 2$ - follows when $t = 1$. When t ranges within a compact subset of the interval $(0, +\infty)$, the uniformity of the estimate follows by rescaling.

The proof in the case $n = 3$ is similar. If $n \geq 4$ the estimate follows by integration by parts:

$$\int_{\lambda=1}^{\infty} e^{i\lambda x - i\lambda^3 t} \frac{d\lambda}{\lambda^n} = \frac{1}{ix} \int_{\lambda=1}^{\infty} (e^{i\lambda x})' e^{-i\lambda^3 t} \frac{d\lambda}{\lambda^n}$$
$$= \frac{1}{ix} \left[e^{i\lambda x} e^{-i\lambda^3 t} \frac{1}{\lambda^n} \right] \Bigg|_{\lambda=1}^{\lambda=\infty} - \frac{1}{ix} \int_{\lambda=1}^{\infty} e^{i\lambda x} \frac{d}{d\lambda} \left[e^{-i\lambda^3 t} \frac{1}{\lambda^n} \right] d\lambda.$$

See also the proof of Lemma 6.3 that follows. □

6. Behavior of the solution and its derivatives as $x \to +\infty$

Theorem 6.1. *Let* $t_1 > t_0 > 0$ *be fixed. Under the assumptions of Theorem 1.1 the solution* $u(x, t)$ *given ?? (1.1) satisfies as* $x \to +\infty$*, uniformly in* $t_0 \leq t \leq t_1$*:*

1st

$$u(x, t) = [u_0(0) - g_0(0)] \frac{\sqrt{3}\sqrt[4]{3}}{\sqrt{\pi}} \frac{t^{1/4}}{x^{3/4}} \sin\left(\frac{2}{3\sqrt{3}} \frac{x^{3/2}}{\sqrt{t}} - \frac{5\pi}{12} \right) + O(1/x),$$

2^{nd}

$$\begin{aligned}\frac{\partial u(x,t)}{\partial x} =&[u_0(0)-g_0(0)]\frac{2\sqrt{3}\sqrt[4]{3}}{\sqrt{\pi}}\frac{1}{t^{1/4}x^{1/4}}\cos\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}}-\frac{5\pi}{12}\right)\\ &+[u_0'(0)-g_1(0)]\frac{\sqrt{3}\sqrt[4]{3}}{\sqrt{\pi}}\frac{t^{1/4}}{x^{3/4}}\sin\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}}-\frac{\pi}{12}\right)+O(1/x)\end{aligned}$$

3^{rd}

$$\begin{aligned}\frac{\partial^2 u(x,t)}{\partial x^2} =& -[u_0(0)-g_0(0)]\frac{2\sqrt[4]{3}}{\sqrt{\pi}}\frac{x^{1/4}}{t^{3/4}}\sin\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}}-\frac{5\pi}{12}\right)\Bigg)\\ &+[u_0'(0)-g_1(0)]\frac{2\sqrt{3}\sqrt[4]{3}}{\sqrt{\pi}}\frac{1}{t^{1/4}x^{1/4}}\cos\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}}-\frac{\pi}{12}\right)+O(1/x),\end{aligned}$$

4^{th}

$$\begin{aligned}\frac{\partial^3 u(x,t)}{\partial x^3} =& -[u_0(0)-g_0(0)]\frac{2}{\sqrt[4]{3}\sqrt{\pi}}\frac{x^{3/4}}{t^{5/4}}\cos\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}}-\frac{5\pi}{12}\right)\\ &+[u_0'(0)-g_1(0)]\frac{2\sqrt{3}\sqrt[4]{3}}{\sqrt{\pi}}\frac{x^{1/4}}{t^{3/4}}\sin\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}}-\frac{\pi}{12}\right)\\ &+[u_0'''(0)-g_0'(0)]\frac{\sqrt{3}\sqrt[4]{3}}{\sqrt{\pi}}\frac{t^{1/4}}{x^{3/4}}\sin\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}}-\frac{5\pi}{12}\right)+O(1/x).\end{aligned}$$

Lemma 6.2. *If $a,b\in\mathbb{R}$ and the function $\varphi(\lambda)$ is C^1 in $\{\lambda\in\mathbb{C}: Im\,\lambda<0\}\cup[a,b]$ and analytic in $\{\lambda\in\mathbb{C}: Im\,\lambda<0\}$, then, uniformly for $t_0\le t\le t_1$,*

$$\int\limits_{\lambda\in[a,b]} e^{i\lambda x-i\lambda^3 t}\varphi(\lambda)d\lambda = O(1/x) \quad \textit{and} \quad \int\limits_{\lambda\in[\alpha,\alpha-i/x]} e^{i\lambda x-i\lambda^3 t}\varphi(\lambda)d\lambda = O(1/x),$$

as $x\to+\infty$.

Proof. For the first relation it suffices to integrate by parts

$$\begin{aligned}\int\limits_{\lambda=\alpha}^{b} e^{i\lambda x-\omega(\lambda)t}\varphi(\lambda)d\lambda &= \frac{1}{ix}\int\limits_{\lambda=\alpha}^{b}(e^{i\lambda x})'e^{-\omega(\lambda)t}\varphi(\lambda)d\lambda\\ &=\frac{1}{ix}\left\{[e^{i\lambda x-\omega(\lambda)t}\varphi(\lambda)]\Big|_{\lambda=\alpha}^{\lambda=b}-\int\limits_{\lambda=\alpha}^{b}e^{i\lambda x}[e^{-\omega(\lambda)t}\varphi(\lambda)]'d\lambda\right\}\\ &=O(1/x).\end{aligned}$$

For the second one, we see that setting $\lambda=\alpha-si/x$, $0\le s\le 1$, we have

$$\int\limits_{\lambda\in[\alpha,\alpha-i/x]} e^{i\lambda x-i\lambda^3 t}\varphi(\lambda)d\lambda = -\frac{ie^{i\alpha}}{x}\int\limits_{s=0}^{1} e^{s}e^{-i(\alpha-si/x)^3 t}\varphi(\alpha-si/x)ds.$$

and the conclusion follows. □

Lemma 6.3. *We have*

$$\int_{\lambda=1}^{\infty} e^{i\lambda x - i\lambda^3 t}\frac{(u_0''')\hat{}(\lambda)}{(i\lambda)^3}d\lambda = O(1/x) \quad \textit{and} \quad \int_{\lambda=1}^{\infty} e^{i\lambda x - i\lambda^3 t}\frac{(u_0''')\hat{}(\alpha^2\lambda)}{(i\lambda)^3}d\lambda = O(1/x)$$

as $x \to +\infty$*, uniformly for* $t_0 \leq t \leq t_1$*.*

Proof. It suffices to notice that

$$\begin{aligned}\frac{1}{ix}\int_{\lambda=1}^{\infty} &(e^{i\lambda x})' e^{-i\lambda^3 t}\frac{(u_0''')\hat{}(\lambda)}{(i\lambda)^3}d\lambda \\ &= \frac{1}{ix}\left[e^{i\lambda x}e^{-i\lambda^3 t}\frac{(u_0''')\hat{}(\lambda)}{(i\lambda)^3}\right]\Bigg|_{\lambda=1}^{\lambda=\infty} - \frac{1}{ix}\int_{\lambda=1}^{\infty} e^{i\lambda x}\frac{d}{d\lambda}\left[e^{-i\lambda^3 t}\frac{(u_0''')\hat{}(\lambda)}{(i\lambda)^3}\right]d\lambda,\end{aligned}$$

since

$$\left|\frac{d}{d\lambda}\left[e^{-i\lambda^3 t}\frac{(u_0''')\hat{}(\lambda)}{(i\lambda)^3}\right]\right| \preceq \frac{1}{\lambda^2}.$$

The last estimate follows from the fact that

$$(u_0''')\hat{}(\lambda) = O(1/\lambda) \quad \text{and} \quad \frac{d}{d\lambda}\left[(u_0''')\hat{}(\lambda)\right] = O(1/\lambda).$$

This concludes the proof. □

Lemma 6.4. *We have*

$$\int_{\lambda=1}^{\infty} e^{i\lambda x - \omega(\lambda)t}(g_0')\tilde{}(\omega(\lambda), t)\frac{d\lambda}{i\lambda} = O(1/x)$$

and

$$\int_{\lambda=1}^{\infty} e^{i\lambda x - \omega(\lambda)t}(g_1')\tilde{}(\omega(\lambda), t)\frac{d\lambda}{\lambda^2} = O(1/x)$$

as $x \to +\infty$*, uniformly for* $t_0 \leq t \leq t_1$*.*

Proof. Once we notice that

$$\left|\frac{d}{d\lambda}[e^{-\omega(\lambda)t}(g_0')\tilde{}(\omega(\lambda), t]\right| \preceq \frac{1}{\lambda}.$$

the proof proceeds like that of the previous lemma. □

Lemma 6.5. *For* $n \in \mathbb{N} \cup \{0\}$ *and as* $x \to +\infty$*,*

$$\frac{\partial^n}{\partial x^n}\left(\int_{\lambda=1}^{\infty} e^{i\lambda x}\frac{d\lambda}{\lambda^m}d\lambda\right) = O(1/x), \quad m = 1, 2, 3, \dots .$$

Proof. It follows by differentiating the equation

$$\int_{\lambda=1}^{\infty} e^{i\lambda x}\frac{d\lambda}{\lambda^m} = \int_{0\le \arg\lambda\le\pi/2, |\lambda|=1} e^{i\lambda x}\frac{d\lambda}{\lambda^m} + \int_{\arg\lambda=\pi/2, |\lambda|\ge 1} e^{i\lambda x}\frac{d\lambda}{\lambda^m},$$

writing $e^{i\lambda x} = (e^{i\lambda x})'/ix$ and integrating by parts. □

Lemma 6.6. *If $\nu(x,t)$ is the function defined by any of the following integrals:*

$$\int_{\{arg\lambda=2\pi/3\}} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha\lambda)d\lambda,$$

$$\int_{\{arg\lambda=\pi/3\}} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\alpha^2\lambda)d\lambda,$$

$$\int_{\{arg\lambda=2\pi/3\}} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),t)d\lambda,$$

$$\int_{\{arg\lambda=\pi/3\}} e^{i\lambda x-\omega(\lambda)t}\lambda^2\tilde{g}_0(\omega(\lambda),t)d\lambda,$$

$$\int_{\{arg\lambda=2\pi/3\}} e^{i\lambda x-\omega(\lambda)t}\lambda\tilde{g}_1(\omega(\lambda),t)d\lambda,$$

$$\int_{\{arg\lambda=\pi/3\}} e^{i\lambda x-\omega(\lambda)t}\lambda\tilde{g}_1(\omega(\lambda),t)d\lambda,$$

then

$$\lim_{x\to+\infty}\left[x^{\ell}\frac{\partial^k\nu(x,t)}{\partial x^k}\right] = 0,$$

for nonnegative integers k and ℓ, uniformly for $t_0\le t\le t_1$.

Proof. The conclusion follows by repeated use of integration by parts and the fact that

$$\left|e^{i\lambda x}\right| = e^{-x\sqrt{3}|\lambda|/2} \quad \text{and} \quad \operatorname{Re}\omega(\lambda) = 0$$

when $\arg\lambda = \pi/3$ or $\arg\lambda = 2\pi/3$. □

6.1. Proof of the 1st part ot Theorem 6.1.

Consider the function

$$I^+(x,t)$$

$$:= \int_0^{\infty} e^{i\lambda x-\omega(\lambda)t}[\hat{u}_0(\lambda) - \hat{u}_0(\alpha^2\lambda) + (1-\alpha)\lambda^2\tilde{g}_0(\omega(\lambda),t) - i(1-\alpha^2)\lambda\tilde{g}_1(\omega(\lambda),t)]d\lambda$$

and write it as $I^+ = I_0^+ + I_1^+$ where

$$I_0^+ := \int_{\lambda=0}^{1} \cdots \quad \text{and} \quad I_1^+ := \int_{\lambda=1}^{\infty} \cdots .$$

By Lemma 6.2 we have

$$I_0^+(x,t) = O(1/x) \quad \text{as} \quad x \to +\infty, \quad \text{uniformly for} \quad t_1 \le t \le t_2.$$

Thinking of the integrals which are parts of I_1^+ as limits of the form $\lim_{A \to +\infty} \int_{\lambda=1}^{A}$ and using the equations

$$\hat{u}_0(\lambda) = \frac{u_0(0)}{i\lambda} + \frac{u_0'(0)}{(i\lambda)^2} + \frac{u_0''(0)}{(i\lambda)^3} + \frac{u_0^{(3)}(0)}{(i\lambda)^4} + \frac{u_0^{(4)}(0)}{(i\lambda)^5} + \frac{(u_0^{(5)})\hat{}(\lambda)}{(i\lambda)^5},$$

$$\hat{u}_0(\alpha^2\lambda) = \alpha\frac{u_0(0)}{i\lambda} + \alpha^2\frac{u_0'(0)}{(i\lambda)^2} + \frac{u_0''(0)}{(i\lambda)^3} + \alpha\frac{u_0^{(3)}(0)}{(i\lambda)^4} + \alpha^2\frac{u_0^{(4)}(0)}{(i\lambda)^5} + \frac{(u_0^{(5)})\hat{}(\alpha^2\lambda)}{(i\lambda)^5},$$

$$\tilde{g}_0(\omega(\lambda),t) = \frac{g_0(t)}{\omega(\lambda)}e^{\omega(\lambda)t} - \frac{g_0(0)}{\omega(\lambda)} - \frac{g_0'(t)}{[\omega(\lambda)]^2}e^{\omega(\lambda)t} + \frac{g_0'(0)}{[\omega(\lambda)]^2} + \frac{1}{[\omega(\lambda)]^2}(g_0'')\tilde{}(\omega(\lambda),t),$$

$$\tilde{g}_1(\omega(\lambda),t) = \frac{g_1(t)}{\omega(\lambda)}e^{\omega(\lambda)t} - \frac{g_1(0)}{\omega(\lambda)} - \frac{g_1'(t)}{[\omega(\lambda)]^2}e^{\omega(\lambda)t} + \frac{g_1'(0)}{[\omega(\lambda)]^2} + \frac{1}{[\omega(\lambda)]^2}(g_1'')\tilde{}(\omega(\lambda),t)$$

for $\lambda \neq 0$ we find that

$$\begin{aligned} I_1^+ = (1-\alpha)[u_0(0) - g_0(0)] \int_{\lambda=1}^{\infty} e^{i\lambda x - i\lambda^3 t}\frac{d\lambda}{i\lambda} \\ + (1-\alpha^2)[u_0'(0) - g_1(0)] \int_{\lambda=1}^{\infty} e^{i\lambda x - i\lambda^3 t}\frac{d\lambda}{(i\lambda)^2} \\ + \int_{\lambda=1}^{\infty} e^{i\lambda x - i\lambda^3 t}\frac{(u_0''')\hat{}(\lambda)}{(i\lambda)^3}d\lambda - \int_{\lambda=1}^{\infty} e^{i\lambda x - i\lambda^3 t}\frac{(u_0''')\hat{}(\alpha^2\lambda)}{(i\lambda)^3}d\lambda \\ + (1-\alpha)g_0(t)\int_{\lambda=1}^{\infty} e^{i\lambda x}\frac{d\lambda}{i\lambda} - (1-\alpha)\int_{\lambda=1}^{\infty} e^{i\lambda x - \omega(\lambda)t}(g_0')\tilde{}(\omega(\lambda),t)\frac{d\lambda}{i\lambda} \\ - (1-\alpha^2)g_1(t)\int_{\lambda=1}^{\infty} e^{i\lambda x}\frac{d\lambda}{\lambda^2} + (1-\alpha^2)\int_{\lambda=1}^{\infty} e^{i\lambda x - \omega(\lambda)t}(g_1')\tilde{}(\omega(\lambda),t)\frac{d\lambda}{\lambda^2}. \end{aligned} \tag{6.1}$$

Applying Lemma 5.7 to the second term in the RHS of (6.1) and Lemmas 6.3, 6.4 and 6.5 to the other terms of (6.1) we obtain

$$I^{+}(x,t) = (1-\alpha)[u_0(0) - g_0(0)] \int_{\lambda=1}^{\infty} e^{i\lambda x - i\lambda^3 t} \frac{d\lambda}{i\lambda} + O(1/x). \tag{6.2}$$

Similarly, setting

$$I^{-}(x,t) := \int_{\lambda=-\infty}^{0} e^{i\lambda x - \omega(\lambda)t} \Big[\hat{u}_0(\lambda) - \hat{u}_0(\alpha\lambda) + (1-\alpha^2)\lambda^2 \tilde{g}_0(\omega(\lambda),t) - i(1-\alpha)\lambda \tilde{g}_1(\omega(\lambda),t)\Big]\, d\lambda$$

we find that

$$I^{-}(x,t) = (1-\alpha^2)[u_0(0) - g_0(0)] \int_{\lambda=-\infty}^{-1} e^{i\lambda x - i\lambda^3 t} \frac{d\lambda}{i\lambda} + O(1/x). \tag{6.3}$$

Now adding (6.2) and (6.3), noticing that

$$(1-\alpha^2) \int_{\lambda=-\infty}^{-1} e^{i\lambda x - i\lambda^3 t} \frac{d\lambda}{i\lambda} = \overline{(1-\alpha) \int_{\lambda=1}^{\infty} e^{i\lambda x - i\lambda^3 t} \frac{d\lambda}{i\lambda}}$$

and using the results of Lemma 5.6 and 6.6, we obtain the 1st part of the theorem.

Lemma 6.7. *We have*

$$\frac{\partial}{\partial x}\Big[\int_{\lambda=1}^{\infty} e^{i\lambda x - i\lambda^3 t} \frac{\widehat{(u_0''')}(\lambda)}{(i\lambda)^3} d\lambda\Big] = O(1/x)$$

and

$$\frac{\partial}{\partial x}\Big[\int_{\lambda=1}^{\infty} e^{i\lambda x - i\lambda^3 t} \frac{\widehat{(u_0''')}(\alpha^2\lambda)}{(i\lambda)^3} d\lambda\Big] = O(1/x)$$

as $x \to +\infty$, uniformly for $t_0 \le t \le t_1$.

Proof. We have

$$\frac{\partial}{\partial x}\Big[\int_{\lambda=1}^{\infty} e^{i\lambda x - i\lambda^3 t} \frac{\widehat{(u_0''')}(\lambda)}{(i\lambda)^3} d\lambda\Big] = \int_{\lambda=1}^{\infty} e^{i\lambda x - i\lambda^3 t} \frac{\widehat{(u_0''')}(\lambda)}{(i\lambda)^2} d\lambda.$$

Also

$$\frac{\widehat{(u_0''')}(\lambda)}{(i\lambda)^2} = \frac{\widehat{(u_0''')}(0)}{(i\lambda)^3} + \frac{\widehat{(u_0'''')}(\lambda)}{(i\lambda)^3} \quad (\lambda \neq 0).$$

Thus the first estimate of the lemma follows from Lemmas 5.7 and 6.3. The proof of the second estimate is similar. ∎

Lemma 6.8. *As $x \to +\infty$ and uniformly for $t_0 \leq t \leq t_1$, we have*

$$\frac{\partial}{\partial x}\left[\int_{\lambda=1}^{\infty} e^{i\lambda x-\omega(\lambda)t}(g_0')\tilde{}(\omega(\lambda),t)\frac{d\lambda}{i\lambda}\right] = O(1/x)$$

and

$$\frac{\partial}{\partial x}\left[\int_{\lambda=1}^{\infty} e^{i\lambda x-\omega(\lambda)t}(g_1')\tilde{}(\omega(\lambda),t)\frac{d\lambda}{\lambda^2}\right] = O(1/x).$$

Proof. We have

$$\frac{\partial}{\partial x}\left[\int_{\lambda=1}^{\infty} e^{i\lambda x-\omega(\lambda)t}(g_0')\tilde{}(\omega(\lambda),t)\frac{d\lambda}{i\lambda}\right] = \int_{\lambda=1}^{\infty} e^{i\lambda x-\omega(\lambda)t}(g_0')\tilde{}(\omega(\lambda),t)d\lambda$$

and

$$e^{i\lambda x-\omega(\lambda)t}(g_0')\tilde{}(\omega(\lambda),t) = e^{i\lambda x}\frac{g_0'(t)}{\omega(\lambda)} - e^{i\lambda x-\omega(\lambda)t}\frac{g_0'(0)}{\omega(\lambda)} - e^{i\lambda x-\omega(\lambda)t}\frac{1}{\omega(\lambda)}(g_0'')\tilde{}(\omega(\lambda),t).$$

Thus, the first estimate of the lemma follows from Lemmas 5.7, 6.5 and the proof of Lemma 6.4. The proof of the second estimate is easier. □

6.2. **Proof of the 2$^{\text{nd}}$ part of Theorem 6.1.** Deforming the contours in the first two terms in the RHS of (6.1) we may write the function I_1^+ in the following way: for $\varepsilon > 0$,

$$\begin{aligned} I_1^+ = {}& (1-\alpha)[u_0(0)-g_0(0)]\left[\int_{[1,1-\varepsilon i]} e^{i\lambda x - i\lambda^3 t}\frac{d\lambda}{i\lambda} + \int_{[1-\varepsilon i,+\infty-\varepsilon i]} e^{i\lambda x - i\lambda^3 t}\frac{d\lambda}{i\lambda}\right] \\ & + (1-\alpha^2)[u_0'(0)-g_1(0)]\left[\int_{[1,1-\varepsilon i]} e^{i\lambda x - i\lambda^3 t}\frac{d\lambda}{(i\lambda)^2} + \int_{[1-\varepsilon i,+\infty-\varepsilon i]} e^{i\lambda x - i\lambda^3 t}\frac{d\lambda}{(i\lambda)^2}\right] \\ & + \tilde{I}_1^+ \end{aligned}$$

where $\tilde{I}_1^+$ is the part of I_1^+ consisting of the remaining terms. Now, keeping ε fixed, we may differentiate with respect to x. The result is the following:

$$\begin{aligned} \frac{\partial I_1^+}{\partial x} ={}& (1-\alpha)[u_0(0)-g_0(0)\left[\int_{[1,1-\varepsilon i]} e^{i\lambda x - i\lambda^3 t}d\lambda + \int_{[1-\varepsilon i,+\infty-\varepsilon i]} e^{i\lambda x - i\lambda^3 t}d\lambda\right] \\ & + (1-\alpha^2)[u_0'(0)-g_1(0)]\left[\int_{[1,1-\varepsilon i]} e^{i\lambda x - i\lambda^3 t}\frac{d\lambda}{i\lambda} + \int_{[1-\varepsilon i,+\infty-\varepsilon i]} e^{i\lambda x - i\lambda^3 t}\frac{d\lambda}{i\lambda}\right] \\ & + \frac{\partial \tilde{I}_1^+}{\partial x}. \end{aligned} \tag{6.4}$$

By Lemmas 6.5, 6.7 and 6.8, we have

$$\frac{\partial \tilde{I}_1^+}{\partial x} = O(1/x).$$

Thus setting $\varepsilon = 1/x$ in (6.4) and using Lemma 6.2 we obtain

$$\begin{aligned}\frac{\partial I_1^+}{\partial x} = &(1-\alpha)[u_0(0)-g_0(0)] \int\limits_{\substack{\mathrm{Im}\lambda=-1/x \\ \mathrm{Re}\lambda\geq 1}} e^{i\lambda x - i\lambda^3 t} d\lambda \\ &+ (1-\alpha^2)[u_0'(0)-g_1(0)] \int\limits_{\substack{\mathrm{Im}\lambda=-1/x \\ \mathrm{Re}\lambda\geq 1}} e^{i\lambda x - i\lambda^3 t} \frac{d\lambda}{i\lambda} + O(1/x).\end{aligned}$$

In view of Lemma 6.2 again, the above equation can be written as follows:

$$\begin{aligned}\frac{\partial I^+}{\partial x} = &(1-\alpha)[u_0(0)-g_0(0)] \int\limits_{\substack{\mathrm{Im}\lambda=-1/x \\ \mathrm{Re}\lambda\geq 0}} e^{i\lambda x - i\lambda^3 t} d\lambda \\ &+ (1-\alpha^2)[u_0'(0)-g_1(0)] \int\limits_{\lambda=1}^{\infty} e^{i\lambda x - i\lambda^3 t} \frac{d\lambda}{i\lambda} + O(1/x).\end{aligned} \tag{6.5}$$

Likewise we obtain

$$\begin{aligned}\frac{\partial I^-}{\partial x} = &(1-\alpha^2)[u_0(0)-g_0(0)] \int\limits_{\substack{\mathrm{Im}\lambda=-1/x \\ \mathrm{Re}\lambda\leq 0}} e^{i\lambda x - i\lambda^3 t} d\lambda \\ &+ (1-\alpha)[u_0'(0)-g_1(0)] \int\limits_{\lambda=-\infty}^{-1} e^{i\lambda x - i\lambda^3 t} \frac{d\lambda}{i\lambda} + O(1/x).\end{aligned} \tag{6.6}$$

Adding (6.5) and (6.6), and taking into consideration Lemmas 5.5, 5.6 and 6.6, we obtain the formula of the 2^{nd} part of the theorem.
The proof of the 3^{rd} part is similar.

Remark 6.9. With the notation and in the sense of Theorem 6.1, we have

(1) If $u_0(0) = g_0(0)$ and $u_0'(0) = g_1(0)$ then

$$\frac{\partial^3 u(x,t)}{\partial x^3} = [u_0'''(0) - g_0'(0)] \frac{\sqrt{3}\sqrt[4]{3}}{\sqrt{\pi}} \frac{t^{1/4}}{x^{3/4}} \sin\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}} - \frac{5\pi}{12}\right) + O(1/x).$$

(2) If $\lim_{x\to\infty} \frac{\partial^2 u(x,t)}{\partial x^2}$ exists for some $t>0$, then $u_0(0) = g_0(0)$. Conversely, if $u_0(0) = g_0(0)$ then

$$\frac{\partial^2 u(x,t)}{\partial x^2} = [u_0'(0) - g_1(0)] \frac{2\sqrt{3}\sqrt[4]{3}}{\sqrt{\pi}} \frac{1}{t^{1/4}x^{1/4}} \cos\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}} - \frac{\pi}{12}\right) + O(1/x).$$

uniformly for t in compact subsets of $(0, +\infty)$.

(3) For $n \geq 4$,

$$\frac{\partial^n u(x,t)}{\partial x^n} = [u_0(0) - g_0(0)]\frac{2}{3^{(2n-3)/4}\sqrt{\pi}}\frac{x^{(2n-3)/4}}{t^{(2n-1)/4}}\mathrm{Re}\left\{ i^{n-1} \exp\left[i\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}} - \frac{5\pi}{12} \right) \right] \right\} + O(x^{(2n-5)/4}).$$

(4) Let $k \in \mathbb{N}$. If the limit $\lim\limits_{x \to \infty} \dfrac{\partial^{4k-1} u(x,t)}{\partial x^{4k-1}}$ exists for some $t > 0$ then

$$u_0^{(3\ell-3)}(0) = g_0^{(\ell-1)}(0) \quad \text{and} \quad u_0^{(3\ell-2)}(0) = g_1^{(\ell-1)}(0), \quad \text{for} \quad \ell = 1, 2, \ldots, k. \tag{6.7}$$

Conversely, (6.7) implies that

$$\lim_{x \to \infty} \frac{\partial^n u(x,t)}{\partial x^n} = 0$$

uniformly for t in compact subsets of $(0, +\infty)$, for $n = 0, 1, 2, \ldots, 4k$.

7. The equation $U_t - U_{xxx} = f$

In the section we study the analogous questions for the inhomogeneous equation, i.e., the solution of the following:

Problem. Solve

$$\begin{cases} \dfrac{\partial U}{\partial t} - \dfrac{\partial^3 U}{\partial x^3} = f, & (x,t) \in Q := \mathbb{R}^+ \times \mathbb{R}^+ \\ \lim\limits_{t \to 0^+} U(x,t) = u_0(x), & x \in \mathbb{R}^+ \\ \lim\limits_{x \to 0^+} U(x,t) = g_0(t), & t \in \mathbb{R}^+ \\ \lim\limits_{x \to 0^+} \dfrac{\partial U(x,t)}{\partial x}, & t \in \mathbb{R}^+, \end{cases} \tag{7.1}$$

for $U = U(x,t)$, where $u_0(x) \in S\big([0,\infty)\big)$, $g_0(t), g_1(t) \in C^\infty\big([0,\infty)\big)$ and $f = f(x,t) \in C^\infty(\overline{Q})$ such that $\frac{\partial^m f}{\partial t^m}(\cdot, t) \in S\big([0,\infty)\big)$ with respect to x, uniformly for t in compact subsets of $[0, +\infty)$ for all m.

The Fokas solution. Defining

$$\hat{f}(\lambda, t) = \int_{y=0}^{\infty} e^{-i\lambda y} f(y,t)dy \quad \text{and} \quad \tilde{\hat{f}}(\lambda, \omega(\lambda), t) = \int_{\tau=0}^{t} e^{\omega(\lambda)\tau} \hat{f}(\lambda, \tau)d\tau,$$

for $\lambda \in \mathbb{C}$ with $\mathrm{Im}\,\lambda \leq 0$, and setting

$$\Phi_{\mathbb{R}}(x,t) = \int_{\lambda=-\infty}^{\infty} e^{i\lambda x - \omega(\lambda)t}\, \tilde{\hat{f}}(\lambda, \omega(\lambda), t)d\lambda,$$

$$\Phi_{\partial\Omega_1^-}(x,t) = \int\limits_{\lambda\in\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}\,\tilde{\hat{f}}(\alpha\lambda,\omega(\lambda),t)$$

and

$$\Phi_{\partial\Omega_2^-}(x,t) = \int\limits_{\lambda\in\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}\,\tilde{\hat{f}}(\alpha^2\lambda,\omega(\lambda),t),$$

for $x>0$ and $t>0$, we will show that the solution $U(x,t)$ of (7.1) has the following integral representation:

$$U(x,t) = u(x,t) + \frac{1}{2\pi}\Phi_{\mathbb{R}}(x,t) - \frac{1}{2\pi}\Phi_{\partial\Omega_1^-}(x,t) - \frac{1}{2\pi}\Phi_{\partial\Omega_2^-}(x,t), \tag{7.2}$$

where $u(x,t)$ is the function given by (1.2).

Let us notice also that, for $0<t<T$,

$$\begin{aligned}\Phi_{\partial\Omega_1^-}(x,t) &= \int\limits_{\lambda\in\partial\Omega_1^-} e^{i\lambda x-\omega(\lambda)t}\,\tilde{\hat{f}}(\alpha\lambda,\omega(\lambda),T),\\ \Phi_{\partial\Omega_2^-}(x,t) &= \int\limits_{\lambda\in\partial\Omega_2^-} e^{i\lambda x-\omega(\lambda)t}\,\tilde{\hat{f}}(\alpha^2\lambda,\omega(\lambda),T).\end{aligned} \tag{7.3}$$

(These are proved as (1.13)).

Comment. The condition $\frac{\partial^m f}{\partial t^m}(\cdot,t)\in S\big([0,\infty)\big)$, that we impose on $f(x,t)$, means the following: For every $N,n,m\in\mathbb{N}\cup\{0\}$ and $t_0>0$,

$$\sup\left\{x^N\left|\frac{\partial^{n+m}f(x,t)}{\partial x^n\partial t^m}\right| : x\geq 0, 0\leq t\leq t_0\right\} < +\infty. \tag{7.4}$$

Some computations

(1). Integrating by parts we obtain

$$\hat{f}(\lambda,t) = \int\limits_{y=0}^{\infty} e^{-i\lambda y}f(y,t)dy = \frac{1}{i\lambda}f(0,t) + \frac{1}{i\lambda}\int\limits_{y=0}^{\infty} e^{-i\lambda x}\frac{\partial f(y,t)}{\partial y}dy \tag{7.5}$$

when $\lambda \in \mathbb{C} - \{0\}$, $\operatorname{Im}\lambda \leq 0, t \geq 0$ and

$$\begin{aligned}
&\tilde{\hat{f}}(\lambda,\omega(\lambda),t)\\
&\quad= \int\limits_{\tau=0}^{t} e^{\omega(\lambda)\tau}\hat{f}(\lambda,\tau)d\tau\\
&\quad= \frac{1}{i\lambda}\int\limits_{\tau=0}^{t} \frac{\partial}{\partial\tau}\left(\frac{e^{\omega(\lambda)\tau}}{\omega(\lambda)}\right) f(0,\tau)d\tau\\
&\qquad\qquad + \frac{1}{i\lambda}\int\limits_{\tau=0}^{t} \frac{\partial}{\partial\tau}\left(\frac{e^{\omega(\lambda)t}}{\omega(\lambda)}\right)\left(\int\limits_{y=0}^{\infty} e^{-i\lambda y}\frac{\partial f(y,\tau)}{\partial y}dy\right)d\tau\\
&\quad= \frac{1}{i\lambda}\frac{e^{\omega(\lambda)t}}{\omega(\lambda)}f(0,t) - \frac{1}{i\lambda}\frac{1}{\omega(\lambda)}f(0,0) - \frac{1}{i\lambda}\frac{1}{\omega(\lambda)}\int\limits_{\tau=0}^{t} e^{\omega(\lambda)\tau}\frac{\partial}{\partial\tau}[f(0,\tau)]d\tau\\
&\qquad\qquad + \frac{1}{i\lambda}\frac{e^{\omega(\lambda)t}}{\omega(\lambda)}\left(\int\limits_{y=0}^{\infty} e^{-i\lambda y}\frac{\partial f(y,t)}{\partial y}dy\right)\\
&\qquad\qquad - \frac{1}{i\lambda}\frac{1}{\omega(\lambda)}\left(\int\limits_{y=0}^{\infty} e^{-i\lambda y}\frac{\partial f(y,0)}{\partial y}dy\right)\\
&\qquad\qquad - \frac{1}{i\lambda}\frac{1}{\omega(\lambda)}\int\limits_{\tau=0}^{t} e^{\omega(\lambda)\tau}\left(\int\limits_{y=0}^{\infty} e^{-i\lambda y}\frac{\partial^2 f(y,t)}{\partial\tau\partial y}dy\right)d\tau.
\end{aligned} \tag{7.6}$$

(2) In view of (7.4), it follows from (7.6) that

$$e^{-\omega(\lambda)t}\tilde{\hat{f}}(\lambda,\omega(\lambda),t) = O(1/\lambda^4) \quad \text{as} \quad \lambda \to \infty, \tag{7.7}$$

with $\operatorname{Im}\lambda \leq 0$ and $\operatorname{Re}(\omega(\lambda) \geq 0$ in the sense that

$$\sup\left\{\left|\lambda^4 \cdot e^{-\omega(\lambda)t}\tilde{\hat{f}}(\lambda,\omega(\lambda),t)\right| : |\lambda| \geq 1,\ \lambda \in \Omega^+,\ 0 \leq t \leq t_0\right\} < +\infty.$$

(3) Generalizing (7.5), in analogy with the equation (1.3), integration by parts gives

$$\hat{f}(\lambda,t) = h_N(\lambda,t) + \frac{1}{(i\lambda)^N}\int\limits_{y=0}^{\infty} e^{-i\lambda y}\frac{\partial^N f(y,t)}{\partial y^N}dy$$

where

$$h_N(\lambda,t) := \sum_{m=1}^{N}\frac{1}{(i\lambda)^m}\left.\frac{\partial^{m-1}f(y,t)}{\partial y^{m-1}}\right|_{y=0}.$$

Therefore,

$$\delta_N(\lambda,t) := \tilde{f}(\lambda,t) - h_N(\lambda,t) = O(1/\lambda^{N+1}), \quad \text{as } \lambda \to \infty, \quad \text{with } \operatorname{Im}\lambda \le 0. \tag{7.8}$$

Now we have

$$\begin{aligned}\tilde{\hat{f}}(\lambda,\omega(\lambda),t) &= \int\limits_{\tau=0}^{t} e^{\omega(\lambda)\tau}\hat{f}(\lambda,\tau)d\tau \\ &= \int\limits_{\tau=0}^{t} e^{\omega(\lambda)\tau}\delta_N(\lambda,\tau)d\tau + \int\limits_{\tau=0}^{t} e^{\omega(\lambda)\tau}h_N(\lambda,\tau)d\tau \\ &= \tilde{\delta}_N(\lambda,\omega(\lambda),t) + \tilde{h}_N(\lambda,\omega(\lambda),t).\end{aligned} \tag{7.9}$$

Let us notice that, although (7.9) holds for $\lambda \in \mathbb{C}\setminus\{0\}$ with $\operatorname{Im}\lambda \le 0$, the last integral in (7.9), i.e. $\tilde{h}_N(\lambda,\omega(\lambda),t)$, is defined for every $\lambda \in \mathbb{C}\setminus\{0\}$.

Additional integrations by parts give

$$\begin{aligned}\tilde{h}_N(\lambda,\omega(\lambda),t) &= e^{\omega(\lambda)t}\mu_{N,M}(\lambda,t) - \mu_{N,M}(\lambda,0) \\ &\qquad - \frac{1}{[\omega(\lambda)]^M}\int\limits_{\tau=0}^{t} e^{\omega(\lambda)\tau}h_N^{(M)}(\lambda,\tau)d\tau,\end{aligned} \tag{7.10}$$

where

$$\mu_{N,M}(\lambda,t) := \frac{h_N(\lambda,t)}{\omega(\lambda)} - \frac{h_N'(\lambda,t)}{[\omega(\lambda)]^2} + \cdots + (-1)^{M-1}\frac{h_N^{(M-1)}(\lambda,t)}{[\omega(\lambda)]^M} \quad (\lambda \in \mathbb{C}\setminus\{0\}).$$

(The derivatives of $h_N(\lambda,t)$, in the above quantity, are taken with respect to t).

(4) In analogy to (7.7), we have

$$e^{-\omega(\lambda)t}\tilde{h}_N(\lambda,\omega(\lambda),t) = O(1/\lambda^4) \quad \text{for } \lambda \to \infty, \quad \text{with } \operatorname{Re}\omega(\lambda) \ge 0, \tag{7.11}$$

and

$$e^{-\omega(\lambda)t}\tilde{\delta}_N(\lambda,\omega(\lambda),t) = O(1/\lambda^{N+4}) \quad \text{as } \lambda \to \infty, \tag{7.12}$$

with $\operatorname{Im}\lambda \le 0$ and $\operatorname{Re}\omega(\lambda) \ge 0$.

Also

$$\frac{1}{[\omega(\lambda)]^M}e^{-\omega(\lambda)t}\int\limits_{\tau=0}^{t} e^{\omega(\lambda)\tau}h_N^{(M)}(\lambda,\tau)d\tau = O(1/\lambda^{3M+4}) \quad \text{as } \lambda \to \infty, \tag{7.13}$$

with $\operatorname{Re}\omega(\lambda) \ge 0$.

Theorem 7.1. *Under the assumptions in (7.1), the function $U(x,t)$, defined by (7.2), is C^∞ for $(x,t) \in Q$ and satisfies the following:*

1st The differential equation $U_t - U_{xxx} = f$ for $x > 0$ and $t > 0$.

2^{nd} The limit condition $\lim\limits_{t \to 0^+} U(x,t) = u_0(x)$, *uniformly in* x *on compact subsets of* $(0,+\infty)$.

3^{rd} The limit condition $\lim\limits_{x \to 0^+} U(x,t) = g_0(t)$, *uniformly in* t *on compact subsets of* $(0,+\infty)$.

4^{th} The limit condition $\lim\limits_{x \to 0^+} \dfrac{\partial U(x,t)}{\partial x} = g_1(t)$, *uniformly in* t *on compact subsets of* $(0,+\infty)$.

Proof. In view of Theorem 1.1, it suffices to deal with the integrals

$$\Phi_{\mathbb{R}}(x,t), \quad \Phi_{\partial\Omega_1^-}(x,t) \text{ and } \Phi_{\partial\Omega_2^-}(x,t).$$

Step 1. Substituting (7.9) and (7.10) in the integral defining $\Phi_{\mathbb{R}}(x,t)$, we obtain

$$\begin{aligned}\Phi_{\mathbb{R}}(x,t) = &\int\limits_{\lambda=-1}^{1} e^{i\lambda x - \omega(\lambda)t}\tilde{\tilde{f}}(\lambda,\omega(\lambda),t)d\lambda \\ &+ \int\limits_{\lambda\in\mathbb{R},|\lambda|\geq 1} e^{i\lambda x-\omega(\lambda)t}\tilde{\delta}_N(\lambda,\omega(\lambda),t)d\lambda \\ &+ \int\limits_{\lambda\in\mathbb{R},|\lambda|\geq 1} e^{i\lambda x}\mu_{N,M}(\lambda,t)d\lambda \\ &- \int\limits_{\lambda\in\mathbb{R},|\lambda|\geq 1} e^{i\lambda x-\omega(\lambda)t}\mu_{N,M}(\lambda,0)d\lambda \\ &- \int\limits_{\lambda\in\mathbb{R},|\lambda|\geq 1} \left[\frac{e^{i\lambda x-\omega(\lambda)t}}{[\omega(\lambda)]^M}\int\limits_{\tau=0}^{t} e^{\omega(\lambda)\tau}h_N^{(M)}(\lambda,\tau)d\tau\right]d\lambda.\end{aligned} \tag{7.14}$$

(We point out that (7.14) holds for any positive integers N and M.)
Also we have

$$\int\limits_{\lambda\in\mathbb{R},|\lambda|\geq 1} e^{i\lambda x}\mu_{N,M}(\lambda,t)d\lambda = \int\limits_{|\lambda|=1,0\leq\arg\lambda\leq\pi} e^{i\lambda x}\mu_{N,M}(\lambda,t)d\lambda \tag{7.15}$$

and

$$\int\limits_{\lambda\in\mathbb{R},|\lambda|\geq 1} e^{i\lambda x-\omega(\lambda)t}\mu_{N,M}(\lambda,0)d\lambda$$
$$= \int\limits_{\mathrm{Im}\lambda=-\varepsilon,|\mathrm{Re}\lambda|\geq 1} e^{i\lambda x-\omega(\lambda)t}\mu_{N,M}(\lambda,0)d\lambda \tag{7.16}$$
$$+ \int\limits_{[-1,-1-\varepsilon i]\cup[1-\varepsilon i,1]} e^{i\lambda x-\omega(\lambda)t}\mu_{N,M}(\lambda,0)d\lambda.$$

Now substituting (7.15) and (7.16) in (7.14), and taking into consideration (7.12) and (7.13), we see that $\Phi_{\mathbb{R}}(x,t)$ is C^∞ for $(x,t)\in Q$. In order to deal with $\Phi_{\partial\Omega_1^-}(x,t)$ we write it as $\int_{\lambda=-\infty}^0 + \int\limits_{\arg\lambda=2\pi/3}$, and we work with the part $\int_{\lambda=-\infty}^0$ as we did previously with $\Phi_{\mathbb{R}}(x,t)$ (i.e. writing equations analogous to (7.14) - (7.16)), proving that this defines a C^∞ function of $(x,t)\in Q$. $\big($Alternatively, this follows also from (7.25), below$\big)$. Dealing with $\int\limits_{\arg\lambda=2\pi/3}$ is easier because of (1.5). Similar computations can be made also for the integral $\Phi_{\partial\Omega_2^-}(x,t)$. The result is that the function $U(x,t)$, defined by (7.2), is indeed C^∞ for $(x,t)\in Q$.

Step 2. Substituting (7.15) and (7.16) in (7.14), applying the resulting formula with $N=1$ and $M=1$, and differentiating (keeping in mind (7.11) - (7.13)), we obtain

$$\frac{\partial^3}{\partial x^3}\Big[\int\limits_{\lambda=-\infty}^{\infty} e^{i\lambda x-\omega(\lambda)t}\tilde{\hat{f}}(\lambda,\omega(\lambda),t)d\lambda\Big]$$
$$= \int\limits_{\lambda=-\infty}^{\infty} (i\lambda)^3 e^{i\lambda x-\omega(\lambda)t}\tilde{\hat{f}}(\lambda,\omega(\lambda),t)d\lambda, \tag{7.17}$$

where the integral in the RHS is interpreted in the generalized sense. $\big($For this computation we apply the operator $\partial^3/\partial x^3$ firstly to the integrals in the RHS of (7.14), taking into consideration (7.12) and (7.13), we interchange the order of differentiations and integrations, and then we write the resulting integrals as generalized integrals. We point out that (7.7) does not guarantee the interchange of the order of differentiation and integration in the LHS of (7.17) immediately, and that is why we involved (7.14), (7.15) and (7.16)$\big)$.

Similarly,

$$\begin{aligned}\frac{\partial}{\partial t}\Big[\int\limits_{\lambda=-\infty}^{\infty} e^{i\lambda x-\omega(\lambda)t}\tilde{\hat{f}}(\lambda,\omega(\lambda),t)d\lambda\Big]\\ = \int\limits_{\lambda=-\infty}^{\infty}(-i\lambda^3)e^{i\lambda x-\omega(\lambda)t}\tilde{\hat{f}}(\lambda,\omega(\lambda),t)d\lambda + \int\limits_{\lambda=-\infty}^{\infty} e^{i\lambda x}\hat{f}(\lambda,t)d\lambda.\end{aligned} \tag{7.18}$$

(Both integrals in the RHS are interpreted in the generalized sense).

Proof of the 1st assertion. It follows from (7.17), (7.18) and Fourier's inversion formula that

$$\partial_t\Phi_{\mathbb{R}}(x,t) - \partial_{xxx}\Phi_{\mathbb{R}}(x,t) = f(x,t) \ \ (x>0, t>0). \tag{7.19}$$

Next writing again $\Phi_{\partial\Omega_1^-}(x,t) = \int\limits_{\lambda=-\infty}^{0} + \int\limits_{\arg\lambda=2\pi/3}$, we can derive equations analogous to (7.17) and (7.18) for both of these integrals and show that

$$\partial_t\Phi_{\partial\Omega_1^-}(x,t) - \partial_{xxx}\Phi_{\partial\Omega_1^-}(x,t) = 0 \ \ (x>0, t>0). \tag{7.20}$$

For the integral $\int\limits_{\lambda=-\infty}^{0}$ we work as in Step 2, while dealing with the integral $\int\limits_{\arg\lambda=2\pi/3}$ is easier because of (1.5) .

Similarly we prove that

$$\partial_t\Phi_{\partial\Omega_2^-}(x,t) - \partial_{xxx}\Phi_{\partial\Omega_2^-}(x,t) = 0. \tag{7.21}$$

Thus, the equation $U_t - U_{xxx} = f$ follows from (7.19), (7.20) and (7.21), and the equation $u_t - u_{xxx} = 0$ of Theorem 1.1. □

Proof of the 2nd, 3rd and 4th assertions. In view of (7.11).

$$\lim_{t\to 0^+}\Phi_{\mathbb{R}}(x,t) = 0, \quad \lim_{t\to 0^+}\Phi_{\partial\Omega_1^-}(x,t) = 0 \ \text{ and } \ \lim_{t\to 0^+}\Phi_{\partial\Omega_2^-}(x,t) = 0,$$

and the 2nd conclusion follows from the corresponding conclusion of Theorem 1.1.

Now, a computation similar to the one made in Step 2 of the proof of Theorem 1.1 shows that

$$\lim_{x\to 0^+}[\Phi_{\mathbb{R}}(x,t) - \Phi_{\partial\Omega_1^-}(x,t) - \Phi_{\partial\Omega_2^-}(x,t)] = 0$$

Indeed, interchanging the order of limit and the integrals in the above equation is immediate in view of (7.11). Let us notice also that the above limit is "similar" to the quantity (2.2). Thus the 3rd conclusion follows again from the corresponding conclusion of Theorem 1.1.

The proof of the 4th assertion is similar.

Theorem 7.2. *1st For the function $U(x,t)$ as in Theorem 7.1 and for each fixed $t>0$, the limits*

$$G_n(t) := \lim_{x \to 0^+} \frac{\partial^n U(x,t)}{\partial x^n}, \quad n = 0,1,2,\dots,$$

exist. Moreover the functions $G_n(t)$ are C^∞ for $t \in (0,\infty)$ and the above convergence is uniform for t in compact subsets of $(0,\infty)$.

2nd The function $U(x,t)$, originally defined for $(x,t) \in Q$, extends to a C^∞ function on $Q \cup \{(0,t) : t > 0\}$, by setting $U(0,t) = g_0(t)$ for $t > 0$.

3rd For $k = 0,1,2,\dots$ and $t > 0$,

$$\lim_{x \to 0^+} \frac{\partial^k U(x,t)}{\partial t^k} = \frac{d^k g_0(t)}{dt^k} \quad \textit{and} \quad \lim_{x \to 0^+} \frac{\partial^k}{\partial t^k}\left(\frac{\partial U(x,t)}{\partial x}\right) = \frac{d^k g_1(t)}{dt^k}.$$

Proof. Firstly, substituting (7.15) and (7.16) in (7.14), and applying the operator $\partial^n/\partial x^n$, we see that the limit $\lim_{x \to 0^+} \frac{\partial^n \Phi_{\mathbb{R}}(x,t)}{\partial x^n}$ exists and defines a C^∞ function of $t \in (0,\infty)$.

Next, using (7.3), we write

$$\begin{aligned} \Phi_{\partial\Omega_1^-}(x,T) = &\int_{\lambda=-\infty}^{0} e^{i\lambda x - \omega(\lambda)t} \tilde{\hat{f}}(\alpha\lambda, \omega(\lambda), T) d\lambda \\ &+ \int_{\lambda=0}^{\infty} e^{i\alpha\lambda x - \omega(\lambda)t} \tilde{\hat{f}}(\alpha^2\lambda, \omega(\lambda), T) d\lambda. \end{aligned} \tag{7.22}$$

Also writing (7.9) and (7.10) in the form

$$\begin{aligned} \tilde{\hat{f}}(\alpha\lambda, \omega(\lambda), T) &= \int_{\tau=0}^{T} e^{\omega(\lambda)\tau} \hat{f}(\alpha\lambda, \tau) d\tau \\ &= \int_{\tau=0}^{T} e^{\omega(\lambda)\tau} \delta_N(\alpha\lambda, \tau) d\tau + \int_{\tau=0}^{T} e^{\omega(\lambda)\tau} h_N(\alpha\lambda, \tau) d\tau \\ &= \tilde{\delta}_N(\alpha\lambda, \omega(\lambda), T) + \tilde{h}_N(\alpha\lambda, \omega(\lambda), T) \end{aligned} \tag{7.23}$$

and

$$\begin{aligned} \tilde{h}_N(\alpha\lambda, \omega(\lambda), T) = &e^{\omega(\lambda)T} \mu_{N,M}(\alpha\lambda, T) - \mu_{N,M}(\alpha\lambda, 0) \\ &- \frac{1}{[\omega(\lambda)]^M} \int_{\tau=0}^{T} e^{\omega(\lambda)\tau} h_N^{(M)}(\alpha\lambda, \tau) d\tau, \end{aligned} \tag{7.24}$$

we obtain, for $0 < t < T$,

$$\begin{aligned}\int\limits_{\lambda=-\infty}^{0} & e^{i\lambda x-\omega(\lambda)t}\tilde{\hat{f}}(\alpha\lambda,\omega(\lambda),T)d\lambda \\ &= \int\limits_{\lambda=-1}^{0} e^{i\lambda x-\omega(\lambda)t}\tilde{\hat{f}}(\alpha\lambda,\omega(\lambda),T)d\lambda \\ &\quad + \int\limits_{\lambda=-\infty}^{-1} e^{i\lambda x-\omega(\lambda)t}\tilde{\delta}_N(\alpha\lambda,\omega(\lambda),T)d\lambda \\ &\quad + \int\limits_{\substack{\mathrm{Im}\lambda=\varepsilon\\ \mathrm{Re}\lambda\le -1}} e^{i\lambda x+\omega(\lambda)(T-t)}\mu_{N,M}(\alpha\lambda,T)d\lambda \\ &\quad + \int\limits_{[-1+\varepsilon i,-1]} e^{i\lambda x-\omega(\lambda)(T-t)}\mu_{N,M}(\alpha\lambda,T)d\lambda \\ &\quad - \int\limits_{\substack{\mathrm{Im}\lambda=-\varepsilon\\ \mathrm{Re}\lambda\le -1}} e^{i\lambda x-\omega(\lambda)t}\mu_{N,M}(\alpha\lambda,T)d\lambda \\ &\quad - \int\limits_{[-1-\varepsilon i,-1]} e^{i\lambda x-\omega(\lambda)t}\mu_{N,M}(\alpha\lambda,T)d\lambda \\ &\quad - \int\limits_{\lambda=-\infty}^{-1} \frac{e^{i\lambda x-\omega(\lambda)t}}{[\omega(\lambda)]^M}\int\limits_{\tau=0}^{T} e^{\omega(\lambda)\tau}h_N^{(M)}(\alpha\lambda,\tau)d\tau.\end{aligned} \tag{7.25}$$

Applying the operator $\partial^n/\partial x^n$ to (7.25) and letting $x \to 0^+$ we see that the limit

$$\lim_{x\to 0^+}\frac{\partial^n}{\partial x^n}\Bigg[\int\limits_{\lambda=-\infty}^{0} e^{i\lambda x-\omega(\lambda)t}\tilde{\hat{f}}(\alpha\lambda,\omega(\lambda),T)d\lambda\Bigg]$$

exists and defines a C^∞ function of $t \in (0,\infty)$. Similarly, deriving an equation also for the second integral in the RHS of (7.22), analogous to (7.25), we can show that the limit

$$\lim_{x\to 0^+}\frac{\partial^n}{\partial x^n}\Bigg[\int\limits_{\lambda=0}^{\infty} e^{i\alpha\lambda x-\omega(\lambda)t}\tilde{\hat{f}}(\alpha^2\lambda,\omega(\lambda),T)d\lambda\Bigg]$$

also exists and defines a C^∞ function of $t \in (0,\infty)$.

Thus, in view of (7.22), the limit $\lim\limits_{x\to 0^+}[\partial^n\Phi_{\partial\Omega_1^-}(x,T)/\partial x^n]$ exists and defines a C^∞ function of $t \in (0,\infty)$.

Analogously, writing

$$\Phi_{\partial\Omega_2^-}(x,T) = \int_{\lambda=0}^{\infty} e^{i\lambda x-\omega(\lambda)t}\tilde{\tilde{f}}(\alpha^2\lambda,\omega(\lambda),T)d\lambda$$
$$+\int_{\lambda=-\infty}^{0} e^{i\alpha^2\lambda x-\omega(\lambda)t}\tilde{\tilde{f}}(\alpha\lambda,\omega(\lambda),T)d\lambda,$$

and working analogously, we conclude that also the limit $\lim\limits_{x\to 0^+}[\partial^n\Phi_{\partial\Omega_2^-}(x,T)/\partial x^n]$ exists and defines a C^∞ function of $t\in(0,\infty)$. Now it is easy to complete the proof of the 1st assertion.

The proof of the 2nd and 3rd assertions can be completed as in Theorem 3.1. □

Theorem 7.3. *Under the assumptions ofTheorem 7.1 we have:*

1st *If* $u_0(0)=g_0(0)$ *then* $\lim\limits_{t\to 0^+}\dfrac{\partial U(x,t)}{\partial x}=\dfrac{du_0(x)}{dx}$ *for* $x>0$.

2nd *If* $u_0(0)=g_0(0)$ *and* $u_0'(0)=g_1(0)$ *then*

$$\lim_{t\to 0^+}\frac{\partial^2 U(x,t)}{\partial x^2}=\frac{d^2u_0(x)}{dx^2}\quad and\quad \lim_{t\to 0^+}\frac{\partial^3 U(x,t)}{\partial x^3}=\frac{d^3u_0(x)}{dx^3}\quad for\ \ x>0.$$

3rd *All the preceding limits are uniform in* x *on compact subsets of* $(0,+\infty)$.

Proof. In view of Theorem 3.2, it suffices to show that

$$\begin{aligned}\lim_{t\to 0^+}\frac{\partial^k\Phi_{\mathbb{R}}(x,t)}{\partial x^k}&=0,\\ \lim_{t\to 0^+}\frac{\partial^k\Phi_{\partial\Omega_1^-}(x,t)}{\partial x^k}&=0,\\ \lim_{t\to 0^+}\frac{\partial^k\Phi_{\partial\Omega_2^-}(x,t)}{\partial x^k}&=0,\end{aligned}\tag{7.26}$$

for $k=1,2,3$. The cases $k=1$ and $k=2$ follow immediately from (7.7). In the case $k=3$, however, (7.7) does not guarantee the interchange of limit and integration in the equations (7.26).

Thus, for this case, writing (7.14) with $N = 1$ and $M = 1$, and differentiating, we obtain

$$\begin{aligned}\frac{\partial^3 \Phi_{\mathbb{R}}(x,t)}{\partial x^3} = &\int\limits_{\lambda=-1}^{1} (i\lambda)^3 e^{i\lambda x - \omega(\lambda)t} \tilde{\hat{f}}(\lambda, \omega(\lambda), t) d\lambda \\ &+ \int\limits_{\lambda\in\mathbb{R}, |\lambda|\geq 1} (i\lambda)^3 e^{i\lambda x - \omega(\lambda)t} \tilde{\delta}_1(\lambda, \omega(\lambda), t) d\lambda \\ &+ \int\limits_{\lambda\in\mathbb{R}, |\lambda|\geq 1} (i\lambda)^3 e^{i\lambda x} \mu_{1,1}(\lambda, t) d\lambda \\ &- \int\limits_{\lambda\in\mathbb{R}, |\lambda|\geq 1} (i\lambda)^3 e^{i\lambda x - \omega(\lambda)t} \mu_{1,1}(\lambda, 0) d\lambda \\ &- \int\limits_{\lambda\in\mathbb{R}, |\lambda|\geq 1} (i\lambda)^3 \left[\frac{e^{i\lambda x - \omega(\lambda)t}}{\omega(\lambda)} \int\limits_{\tau=0}^{t} e^{\omega(\lambda)\tau} h_1^{(1)}(\lambda, \tau) d\tau \right] d\lambda.\end{aligned} \tag{7.27}$$

This is essentially a rewriting of (7.17). We claim that

$$\begin{aligned}\lim_{t\to 0^+} \Bigg[&\int\limits_{\lambda\in\mathbb{R}, |\lambda|\geq 1} (i\lambda)^3 e^{i\lambda x} \mu_{1,1}(\lambda, t) d\lambda \\ &- \int\limits_{\lambda\in\mathbb{R}, |\lambda|\geq 1} (i\lambda)^3 e^{i\lambda x - \omega(\lambda)t} \mu_{1,1}(\lambda, 0) d\lambda \Bigg] = 0.\end{aligned} \tag{7.28}$$

(The integrals in (7.28) are understood in the generalized sense.)

Indeed, since

$$\mu_{1,1}(\lambda, t) = \frac{h_1(\lambda, t)}{\omega(\lambda)} = \frac{f(0,t)}{(i\lambda)(i\lambda^3)},$$

(7.28) follows Lemma 2.2.

Now it follows from (7.27), (7.28), (7.12) and (7.13) that

$$\lim_{t\to 0^+} [\partial^3 \Phi_{\mathbb{R}}(x,t)/\partial x^3] = 0.$$

Similarly, we show that

$$\lim_{t\to 0^+} [\partial^3 \Phi_{\partial\Omega_1^-}(x,t)/\partial x^3] = 0, \quad \lim_{t\to 0^+} [\partial^3 \Phi_{\partial\Omega_2^-}(x,t)/\partial x^3] = 0.$$

This completes the proof of (7.26). $\square$

Theorem 7.4. *With the assumptions as in Theorem 7.1, we have*

1st *If* $u_0(0) = g_0(0)$ *then* $\lim\limits_{\overline{Q}\ni(x,t)\to(0,0)} U(x,t) = u_0(0).$

2^{nd} *If* $u_0(0) = g_0(0)$ *and* $u_0'(0) = g_1(0)$ *then* $\displaystyle\lim_{\overline{Q}\ni(x,t)\,\to\,(0,0)} \frac{\partial U(x,t)}{\partial x} = u_0'(0)$.

Proof. It follows immediately from Theorem 4.1, 4.2 and the fact that for $k = 0$ and $k = 1$ all the limits

$$\lim_{\overline{Q}\ni(x,t)\,\to\,(0,0)} \frac{\partial^k \Phi_{\mathbb{R}}(x,t)}{\partial x^k}, \lim_{\overline{Q}\ni(x,t)\,\to\,(o,o)} \frac{\partial^k \Phi_{\partial\Omega_1^-}(x,t)}{\partial x^k}, \lim_{\overline{Q}\ni(x,t)\,\to\,(0,0)} \frac{\partial^k \Phi_{\partial\Omega_2^-}(x,t)}{\partial x^k}$$

are equal to zero. $\square$

7.5 Asymptotic behavior of the solution $U(x,t)$ as $x \to +\infty$. Keeping in mind the computations made in the proof of Theorem 6.1, let us consider the quantity

$$\Phi^+(x,t) := \int_{\lambda=1}^{\infty} e^{i\lambda x-\omega(\lambda)t}\tilde{\hat{f}}(\lambda,\omega(\lambda),t)d\lambda - \int_{\lambda=1}^{\infty} e^{i\lambda x-\omega(\lambda)t}\tilde{\hat{f}}(\alpha^2\lambda,\omega(\lambda),t)d\lambda.$$

Writing $e^{i\lambda x} = (e^{i\lambda x})'/(ix)$ in the above integrals, integrating by parts and using (7.7), we see that $\Phi^+(x,t) = O(1/x)$.

Similarly, for the quantity

$$\Phi^-(x,t) := \int_{\lambda=-\infty}^{-1} e^{i\lambda x-\omega(\lambda)t}\tilde{\hat{f}}(\lambda,\omega(\lambda),t)d\lambda - \int_{\lambda=-\infty}^{-1} e^{i\lambda x-\omega(\lambda)t}\tilde{\hat{f}}(\alpha\lambda,\omega(\lambda),t)d\lambda,$$

we obtain $\Phi^-(x,t) = O(1/x)$.

Next, we have

$$\frac{\partial\Phi^+(x,t)}{\partial x} = \int_{\lambda=1}^{\infty} (i\lambda)e^{i\lambda x-\omega(\lambda)t}\tilde{\hat{f}}(\lambda,\omega(\lambda),t)d\lambda - \int_{\lambda=1}^{\infty} (i\lambda)e^{i\lambda x-\omega(\lambda)t}\tilde{\hat{f}}(\alpha^2\lambda,\omega(\lambda),t)d\lambda, \tag{7.29}$$

from which we obtain

$$\begin{aligned}\frac{\partial\Phi^+(x,t)}{\partial x} &= f(0,0)\int_{\lambda=1}^{\infty} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{i\lambda^3} - f(0,0)\int_{\lambda=1}^{\infty} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{\alpha^2 i(\alpha^2\lambda)^3} + O(1/x)\\ &= (1-\alpha)f(0,0)\int_{\lambda=1}^{\infty} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{i\lambda^3} + O(1/x) = O(1/x). \qquad (7.30)\end{aligned}$$

Differentiating (7.29) (and keeping in mind (7.30)) we see that

$$\frac{\partial^2\Phi^+(x,t)}{\partial x^2} = (1-\alpha)f(0,0)\int_{\lambda=1}^{\infty} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{\lambda^2} + O(1/x) = O(1/x).$$

Further differentiation gives

$$\frac{\partial^3 \Phi^+(x,t)}{\partial x^3} = -(1-\alpha)f(0,0)\int_{\lambda=1}^{\infty} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{i\lambda} + O(1/x).$$

Similarly

$$\frac{\partial^3 \Phi^-(x,t)}{\partial x^3} = -(1-\alpha^2)f(0,0)\int_{\lambda=-\infty}^{-1} e^{i\lambda x-\omega(\lambda)t}\frac{d\lambda}{i\lambda} + O(1/x).$$

It follows from the above computations that, as $x \to \infty$,

$$\frac{\partial^k}{\partial x^k}[\Phi_{\mathbb{R}}(x,t) - \Phi_{\partial\Omega_1^-}(x,t) - \Phi_{\partial\Omega_2^-}(x,t)] = O(1/x), \quad \text{for} \quad k=0,1,2,$$

while

$$\begin{aligned}&\frac{1}{2\pi}\frac{\partial^3}{\partial x^3}[\Phi_{\mathbb{R}}(x,t) - \Phi_{\partial\Omega_1^-}(x,t) - \Phi_{\partial\Omega_2^-}(x,t)]\\ &\qquad = -f(0,0)\frac{\sqrt{3}\sqrt[4]{3}}{\sqrt{\pi}}\frac{t^{1/4}}{x^{3/4}}\sin\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}} - \frac{5\pi}{12}\right) + O(1/x).\end{aligned}$$

Thus, we are led to the following:

Proposition. The solution $U(x,t)$ of (7.1) satisfies the 1st, 2nd and 3rd estimates of Theorem 6.1 (i.e., these estimates hold with $U(x,t)$ in place of $u(x,t)$), while

$$\begin{aligned}\frac{\partial^3 U(x,t)}{\partial x^3} &= -[u_0(0)-g_0(0)]\frac{2}{\sqrt[4]{3}\sqrt{\pi}}\frac{x^{3/4}}{t^{5/4}}\cos\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}} - \frac{5\pi}{12}\right)\\ &\quad + [u_0'(0)-g_1(0)]\frac{2\sqrt{3}\sqrt[4]{3}}{\sqrt{\pi}}\frac{x^{1/4}}{t^{3/4}}\sin\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}} - \frac{\pi}{12}\right)\\ &\quad + [u_0'''(0)-g_0'(0)-f(0,0)]\frac{\sqrt{3}\sqrt[4]{3}}{\sqrt{\pi}}\frac{t^{1/4}}{x^{3/4}}\sin\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}} - \frac{5\pi}{12}\right) + O(1/x).\end{aligned}$$

Asymptotic expressions for higher-order derivatives can be found by continuing these calculations in an obvious way, while time-derivatives can clearly be associated to space-derivatives via the use of the PDE at hand itself.

8. Conclusions, Discussion and Outlook

Linear differential equations have traditionally been of paramount importance, both in their own right (e.g. as practical models of physical and engineering processes) and in the study of nonlinear counterparts (in view of e.g. *linearizability* of certain classes of PDE [30], linear estimates necessary for answering well-posedness questions, and so on).

In the article, the inhomogeneous Airy PDE (which is also referred to as the Stokes equation and corresponds to the linearized KdV with a negative sign) posed on the half-line, with arbitrary initial and boundary data, has been studied in a classical, smooth setting via the formula provided by the Fokas *unified method* for linear dispersive PDE. We employed IBVP for the Airy equation as a concrete example of implementation of a novel approach to studying existence, regularity and asymptotic properties of dispersive equations which, most importantly, has been instrumental for the discovery of a new instability effect.

More specifically, we first addressed the issue of giving accurate meaning to the UTM integral representation formula, by suitable decomposition of the kernels of the integrals and deformation of their contours, pointing out that some terms in that formula are oscillatory, thus securing convergence in a precisely defined sense. Our analysis, then, allowed for the necessary rigorous *a posteriori* verification of the full IBVP within the quarter-plane, thus properly establishing, for the first time, the validity of the solution formula (from which a 'constructive' *existence* result is forthwith established too for the IBVP at hand). This was followed by a detailed investigation of the behavior of the solution (and its mixed derivatives, of any order) near the boundaries - i.e., the axes - of the spatiotemporal domain. In this regard, we proved that the Fokas representation can be written in an equivalent form consistent with the Ehrnpreis-Palamodov *fundamental principle* and converges *uniformly* to the prescribed (depending on the given data) values as well as that it admits a smooth extension up to - and beyond - the boundary (facts that have significant implications for efficient numerical computations and other topics).

Based on the expression offered for the solution, most importantly, we are able to perform an effective asymptotic analysis of far-field dynamics. This yielded explicit asymptotic formulae, apparently not known before, which characterize, both quantitatively and qualitatively, the properties of the solution and its derivatives at infinity, in terms of (in)compatibilities of the data at the 'corner' of the quadrant. In particular, unless certain *compatibility conditions* are imposed on the given functions (at the point) which would guarantee *conditional* decay (but not rapid - its rate also depending on the 'degree' of prescribed compatibility) for large values of the spatial variable, the solution or derivatives thereof may become unbounded. Stated otherwise, the asymptotic behavior of the solution depends in an extremely sensitive way on minute perturbations of the data values at the very point $(0, 0)$, where compatibilities are, for all practical purposes, virtually always violated. In all cases, even if the initial data are assumed to belong to the Schwartz class, the solution loses this property as soon as time becomes positive! The 'singular' behavior exhibited here is in stark contrast to the heat and the linearized KdV equations (on the positive half-line) whose solutions have already been studied in a similar high-regularity context and proven to inherit the exponential decay of the data. The present classically-minded work has ultimately led us to the discovery of a novel type of a *long-range instability* phenomenon for linear dispersive differential equations, which, to the best of our knowledge, has not been reported before. Our ideas and techniques are extendable to other *Airy-like* and more general dispersive equations of evolution. Nevertheless, a unified and algorithmic methodology to obtain such results for *all* evolution PDE which admit a solution formula via the unified transform, remains an interesting open problem. It is believed that our analytical approach opens up a new avenue of investigation into the dynamics of linear evolution differential equations formulated on partially (un)bounded domains, and many analogous studies are expected to follow.

To conclude, by virtue of the Fokas UTM, explicit formulae can be provided for the solution of IBVP for a great variety of linear PDE. It turns out that these formulae are more efficient than the traditional formulae derived, in very special cases, by classical transforms. More specifically, the convergence to the given data as one approaches the boundary is uniform; even the convergence of the derivatives is uniform, if there is sufficient regularity of the data. In this paper, we provide a rigorous proof of this fact in the case of the Airy PDE. Moreover, given the power of computing nowadays, the role of PDE theory is partly relegated to the qualitative study of solutions, with particular attention to instabilities. An interesting new consequence of the spectacularly successful Fokas theory for the solution of IBVP for linear PDE is the observation and analysis of instabilities. In effect, the importance of this work is not limited within the theorems stated here but it also lies in the facts that these findings essentially offer insight into a new type of instability - which, albeit simple, has not been observed before and very likely is characteristic to a wide class of certain evolution PDE - and that our approach, ideas and techniques provide the way forward to further explorations.

It is intriguing to study which kind of linear equations exhibit such instabilities and which do not. Also, it is interesting to study whether there is a similar effect on long-time asymptotics. So far, this seems an extremely difficult task to achieve in a unified fashion. Proofs analogous to ours - which serve as an illustrative paradigm - for *all* the IBVP that admit analysis through the UTM. Evidently, the Fokas method has thus rejuvenated the study of *linear* PDE and instigated the generation of new results of both pure and practical interest. Finally, attempting to extend these ideas and obtain analogous results for nonlinear "completely" integrable PDE is an interesting direction for future research too. In this case, of course, rather than analyzing integral formulae, we need to analyze much less explicit formulae related to Riermann-Hilbert factorization problems on a hyperelliptic curve [90], [91]. A reasonable starting point is provided by the moving boundary problem for the focusing NLS [63] as it offers a good basis for the rigorous study of the related Dirichlet-to-Neumann problem (initiated in [5]) which, in turn, is necessary for the rigorous justification of the uniform validity of the Riemann-Hilbert formulation.

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