WEIGHTED WEAK-TYPE $(1,1)$ ESTIMATES VIA RUBIO DE FRANCIA EXTRAPOLATION

MARÍA J. CARRO*, LOUKAS GRAFAKOS**, AND JAVIER SORIA*

Abstract. The classical Rubio de Francia extrapolation result asserts that if an operator $T : L^{p_0}(u) \to L^{p_0,\infty}(u)$ is bounded for some $p_0 > 1$ and every $u \in A_{p_0}$, then, for every $1 < p < \infty$ and every $u \in A_p$, $T : L^p(u) \to L^{p,\infty}(u)$ is bounded. However, there are examples showing that it is not possible to extrapolate to the end-point $p = 1$. In this paper we shall prove that there exists a class of weights, slightly larger than A_p, with the following property: If an operator $T : L^{p_0,1}(u) \to L^{p_0,\infty}(u)$ is bounded, for some $p_0 > 1$ and every u in this class then, for every $u \in A_1$,

(1) T is of restricted weak-type $(1,1)$;
(2) for every $\varepsilon > 0$,

$$T : L(\log L)^\varepsilon(u) \longrightarrow L^{1,\infty}_{\text{loc}}(u).$$

Moreover, for a big class of operators, including Calderón-Zygmund maximal operators, g-functions, the intrinsic square function, and the Haar shift operators, we obtain a weak-type $(1,1)$ estimate with respect to every $u \in A_1$.

1. Introduction

Let M be the Hardy-Littlewood maximal operator, defined by

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|\,dy,$$

where Q denotes a cube in $\mathbb{R}^n$. A positive locally integrable function w (called weight) is said to belong to the Muckenhoupt class A_r $(r > 1)$, if

$$\|w\|_{A_r} = \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-1/(r-1)}(x)\,dx\right)^{r-1} < \infty.$$

If $r = 1$, we say that $w \in A_1$, if $Mw(x) \le Cw(x)$, at almost every point $x \in \mathbb{R}^n$ and $\|w\|_{A_1}$ will be the least constant C satisfying such inequality.

2010 *Mathematics Subject Classification.* 42B35, 46E30.

Key words and phrases. Rubio de Francia Extrapolation; A_p weights; Hardy-Littlewood maximal function.

*Both authors have been partially supported by the grants MTM2013-40985-P (Spanish Government) and 2014SGR289 (Catalan Autonomous Government).

**Author supported in part by National Science Foundation under Grant DMS 0900946.

These classes of weights were introduced by B. Muckenhoupt [17], who proved that, if $p > 1$, then

$$M : L^p(w) \longrightarrow L^p(w)$$

is bounded if, and only if, $w \in A_p$. Also, for every $1 \leq p < \infty$, M is of weak-type (p,p) if, and only if, $w \in A_p$.

Concerning the behavior of the boundedness constant of M, depending on the weight w, we mention the work of Buckley [3], who proved that, for every $1 \leq p < \infty$,

$$\|M\|_{L^p(w) \to L^{p,\infty}(w)} \leq C \|w\|_{A_p}^{\frac{1}{p}},$$

and, if $p > 1$,

$$\|M\|_{L^p(w) \to L^p(w)} \leq C_p \|w\|_{A_p}^{\frac{1}{p-1}},$$

with C_p depending on p.

An important result, for our purpose, concerning A_p weights is the so-called extrapolation theorem of Rubio de Francia [20] (see also [10, 11, 12]), which says that if, for some $p_0 \geq 1$ and every $w \in A_{p_0}$,

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

is a bounded operator then, for every $p > 1$ and every $w \in A_p$,

$$T : L^p(w) \longrightarrow L^p(w)$$

is also bounded. We have to emphasize that it is not possible to extrapolate up to the endpoint $p = 1$; that is, there are examples of operators, for which the hypothesis of Rubio de Francia's theorem holds, which are not of weak-type $(1,1)$, as for example the operator $T = M \circ M$.

Since the above result was first proved, many other proofs and improvements have appeared in the literature. In particular, it was shown in [7, 6] that the operator T played no role and, in fact, the result could be obtained for a pair of functions (f,g) as follows: if, for some $1 \leq p_0 < \infty$, there exists an increasing function $\varphi_{p_0}(t)$, $t > 0$, such that, for every $w \in A_{p_0}$,

$$\|g\|_{L^{p_0}(w)} \leq \varphi_{p_0}(\|w\|_{A_{p_0}}) \|f\|_{L^{p_0}(w)},$$

then, for every $1 < p < \infty$, there exists an increasing function $\varphi_p(t)$, $t > 0$, such that, for every $w \in A_p$,

$$\|g\|_{L^p(w)} \leq \varphi_p(\|w\|_{A_p}) \|f\|_{L^p(w)}.$$

Moreover, an explicit construction of φ_p from φ_{p_0} was given in [8], where a version of the extrapolation theorem with sharp bounds was proved. Along these lines, we have to also mention the recent work [9] where a simple proof of the last result has been presented. As said in that paper, the three basic ingredients of any of the proofs of the extrapolation results are:

(i) Factorization.
(ii) Construction of A_1 weights via the Rubio de Francia algorithm.

(iii) Sharp bounds for the Hardy-Littlewood maximal function.

In fact, the main idea in [9] was to use a new factorization of A_{p_0} weights, while (ii) and (iii) were used in a standard way. We recall that the usual factorization result for A_{p_0} weights says that $w \in A_{p_0}$ if, and only if, there exist $u_0, u_1 \in A_1$ such that $w = u_0 u_1^{1-p_0}$, while the argument in [9] uses the fact that, if $w \in A_p$ and $u \in A_1$, then $wu^{p-p_0} \in A_{p_0}$, $1 \leq p \leq p_0 < \infty$, and

$$\|wu^{p-p_0}\|_{A_{p_0}} \leq \|w\|_{A_p} \|u\|_{A_1}^{p_0-p}.$$

We would like also to comment that, as explained in [9]: *The extrapolation results can be adapted to any situation in which factorization and the Rubio de Francia algorithm are available.*

One of the main results of this paper is to show that one can also extrapolate in cases where the Rubio de Francia algorithm is indeed not available and, what it is more important, these new extrapolation results allow us to obtain, in many cases, the weak-type $(1,1)$ boundedness.

In fact, we can prove that there exists a class of weights $\widehat{A}_{p_0}$ (see Definition 2.9), slightly bigger than A_{p_0}, for which, given a restricted weak-type (p_0, p_0) bounded operator $T : L^{p_0,1}(u) \to L^{p_0,\infty}(u)$, for some $p_0 > 1$ and every $u \in \widehat{A}_{p_0}$ then, for every $u \in A_1$, T is of restricted weak-type $(1,1)$ and, if T is sublinear (or even quasi-sublinear), we also obtain that, for every $\varepsilon > 0$,

$$T : L(\log L)^{\varepsilon}(u) \longrightarrow L^{1,\infty}_{\text{loc}}(u)$$

is bounded, where

$$L(\log L)^{\varepsilon}(u) = \left\{ f : \ \|f\|_{L(\log L)^{\varepsilon}(u)} = \int_0^{\infty} f_u^*(t) \Big(1 + \log^+ \frac{1}{t} \Big)^{\varepsilon} dt < \infty \right\},$$

and

$$L^1_{\text{loc}}(u) = \left\{ f : \|f\|_{L^1_{\text{loc}}(u)} = \sup_{0<t\leq 1} t f_u^*(t) < \infty \right\},$$

with f_u^* the decreasing rearrangement of f, with respect to the weight u, defined by

$$f_u^*(t) = \inf \left\{ y > 0 : \lambda_f^u(y) \leq t \right\},$$

and $\lambda_f^u(y) = u\big(\{x : |f(x)| > y\}\big)$ is the distribution function of f with respect to u (we shall write $\lambda_f(y)$ and f^*, if $u = 1$). We shall also use the standard notation $u(E) = \int_E u(x)\, dx$ (if $u = 1$ we simply write $|E|$).

Moreover, for a big class of operators, including Calderón-Zygmund maximal operators, g-functions, the intrinsic square function, and Haar shift operators, we obtain that, for every $u \in A_1$,

$$T : L^1(u) \longrightarrow L^{1,\infty}(u)$$

is bounded.

We state the version of the extrapolation result given in [9]:

Theorem 1.1. *Assume that, for some family of pairs of nonnegative functions, (f,g), for some $1 \le p_0 < \infty$, and for all $w \in A_{p_0}$, we have*

$$\left(\int_{\mathbb{R}^n} g^{p_0}(x)w(x)\,dx\right)^{1/p_0} \le N(\|w\|_{A_{p_0}})\left(\int_{\mathbb{R}^n} f^{p_0}(x)w(x)\,dx\right)^{1/p_0},$$

where $N(t)$, $t>0$, is an increasing function. Then, for all $1<p<\infty$ and all $w \in A_p$,

$$\left(\int_{\mathbb{R}^n} g^{p}(x)w(x)\,dx\right)^{1/p} \le K(w)\left(\int_{\mathbb{R}^n} f^{p}(x)w(x)\,dx\right)^{1/p},$$

where, if $p<p_0$,

$$K(w) = CN\Big(\|w\|_{A_p}(2\|M\|_{L^p(w)})^{p_0-p}\Big)$$

and, if $p>p_0$,

$$K(w) = CN\Big(\|w\|_{A_p}^{\frac{p_0-1}{p-1}}(2\|M\|_{L^{p'}(w^{1-p'})})^{\frac{p-p_0}{p-1}}\Big).$$

In particular,

$$K(w) \lesssim N\Big(C\|w\|_{A_p}^{\max\left(1,\frac{p_0-1}{p-1}\right)}\Big).$$

In this paper we are interested in the case $1<p<p_0$.

A constant C_p (independent of the weights) such that it remains bounded when p tends to 1, will be represented by C. As usual, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \le CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2. Main results: restricted weak-type extrapolation

Given a weight u, let $L^{p,1}(u)$ be the set of measurable functions such that

$$\|f\|_{L^{p,1}} = \int_0^\infty f_u^*(t)t^{\frac{1}{p}-1}\,dt = p\int_0^\infty \lambda_f^u(y)^{1/p}dy < \infty,$$

and let $L^{p,\infty}(u)$ be defined by the condition

$$\|f\|_{L^{p,\infty}(u)} = \sup_{t>0} t^{1/p} f_u^*(t) = \sup_{y>0} y\lambda_f^u(y)^{1/p} < \infty.$$

Definition 2.1. For every $1 \le p < \infty$, we define the restricted A_p class, $A_p^{\mathcal{R}}$, as those weights u for which the following quantity is finite:

$$\|u\|_{A_p^{\mathcal{R}}} = \sup_{E\subset Q} \frac{|E|}{|Q|}\left(\frac{u(Q)}{u(E)}\right)^{1/p},$$

where the supremum is taken over all cubes Q and all measurable sets $E \subset Q$.

We clearly have that

$$A_1^{\mathcal{R}} = A_1.$$

Then, the starting point of our theory is the following result in [14]:

Theorem 2.2. *For every $1 \le p < \infty$, it holds that*

$$M : L^{p,1}(u) \longrightarrow L^{p,\infty}(u)$$

is bounded if, and only if, $u \in A_p^{\mathcal{R}}$. Moreover, if $1 < p < \infty$,

$$(p-1)\|M\|_{L^{p,1}(u)\to L^{p,\infty}(u)} \lesssim \|u\|_{A_p^{\mathcal{R}}} \le \|M\|_{L^{p,1}(u)\to L^{p,\infty}(u)},$$

and, if $p = 1$,

$$\|M\|_{L^1(u)\to L^{1,\infty}(u)} \approx \|u\|_{A_1}.$$

Concerning the relation with A_p weights the following result holds:

Proposition 2.3. *For every $\varepsilon > 0$,*

$$A_p \subset A_p^{\mathcal{R}} \subset A_{p+\varepsilon}.$$

Moreover, for every $1 \le p < \infty$,

$$\|w\|_{A_p^{\mathcal{R}}} \le \|w\|_{A_p}^{1/p}. \tag{2.1}$$

Proof. The embeddings are clear. Now, for every $E \subset Q$,

$$|E| \le w(E)^{1/p}\Big(\int_Q w^{1-p'}(x)\,dx\Big)^{1/p'},$$

and hence

$$\frac{|E|}{|Q|}\left(\frac{w(Q)}{w(E)}\right)^{1/p} \le \frac{1}{|Q|}w(Q)^{1/p}\Big(\int_Q w^{1-p'}(x)\,dx\Big)^{1/p'} \le \|w\|_{A_p}^{1/p},$$

as we wanted to show. □

Our next goal is to study this class of weights and prove a factorization result for, at least, a sufficiently large subclass. Let us start with two lemmas:

Lemma 2.4. *If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies that $Mf(x) < \infty$, a.e. $x \in \mathbb{R}^n$ then, for every cube $Q \subset \mathbb{R}^n$ for which $f\chi_Q \not\equiv 0$, there exists $0 < s_{f,Q} < \infty$ such that*

$$Q \subset E_{f,Q} \subset 13Q,$$

where $E_{f,Q} = \big\{y \in \mathbb{R}^n : M(f\chi_Q)(y) > s_{f,Q}\big\}$.

Proof. For convenience, we will assume that Q is a closed cube. Let us first prove that, for every $x \notin 3Q$ and every $y \in Q$, we have that

$$M(f\chi_Q)(x) \le M(f\chi_Q)(y). \tag{2.2}$$

In fact, take any cube R containing x and such that $|R\cap Q| > 0$. Then, $|R| \ge |Q|$, and

$$\frac{1}{|R|}\int_{R\cap Q}|f(z)|\,dz \le \frac{1}{|Q|}\int_{R\cap Q}|f(z)|\,dz \le \frac{1}{|Q|}\int_Q |f(z)|\,dz \le M(f\chi_Q)(y).$$

Hence, taking the supremum in R, we obtain (2.2). In particular, if we define $s_{f,Q} = \frac{1}{2}\sup_{x\in\partial(3Q)} M(f\chi_Q)(x)$, then

$$0 < s_{f,Q} < \inf_{y\in Q} M(f\chi_Q)(y) < \infty. \tag{2.3}$$

Using (2.3) we observe that $Q \subset E_{f,Q}$. To finish, assume $x \notin 13Q$, and take any cube R containing x and such that $|R \cap Q| > 0$. Then, $|R| \geq 2^n|3Q|$, and

$$\frac{1}{|R|}\int_{R\cap Q} |f(z)|\,dz \leq \frac{1}{2^n}\frac{1}{|3Q|}\int_{3Q} |f(z)|\chi_Q(z)\,dz \leq \frac{2}{2^n}s_{f,Q} \leq s_{f,Q}.$$

Hence $M(f\chi_Q)(x) \leq s_{f,Q}$ and $E_{f,Q} \subset 13Q$. □

Lemma 2.5.

Let g be a positive measurable function such that, for every constant $a \geq 0$,

$$|\{x : g(x) = a\}| = 0. \tag{2.4}$$

Then, the following holds:

(i) *Let F be a measurable set with $|F| < \infty$ and let $\alpha \leq 0$. Then, for almost every $0 < t < |F|$,*

$$\left(g^\alpha\chi_F\right)^*(t) = \left((g\chi_F)^*(|F| - t)\right)^\alpha.$$

(ii) *For every $r > 0$,*

$$\left(g\chi_{\{g>r\}}\right)^*(t) = g^*(t)\chi_{\{g^*>r\}}(t).$$

(iii) *If $g > 0$, $t > 0$, $\alpha \leq 0$ and $|E| = t$, then*

$$\int_{\{x:g(x)>g^*(t)\}} g(x)^\alpha\,dx \leq \int_E g(x)^\alpha\,dx.$$

Proof. (i) and (ii) are immediate from the usual definitions. Let us see that (iii) follows from the previous estimates:

$$\begin{aligned}\int_E g(x)^\alpha\,dx &= \int_{\mathbb{R}^n} g(x)^\alpha\chi_E(x)\,dx \\ &= \int_0^t (g^\alpha\chi_E)^*(s)\,ds = \int_0^t \left((g\chi_E)^*(s)\right)^\alpha ds,\end{aligned} \tag{2.5}$$

where we have used (i) in the last equality. Similarly, using now (i) and (ii):

$$\begin{aligned}\int_{\{x:g(x)>g^*(t)\}} g(x)^\alpha\,dx &\leq \int_0^t (g^\alpha\chi_{\{x:g(x)>g^*(t)\}})^*(s)\,ds \\ &= \int_0^t \left((g\chi_{\{x:g(x)>g^*(t)\}})^*(s)\right)^\alpha ds = \int_0^t g^*(s)^\alpha\,ds.\end{aligned} \tag{2.6}$$

Thus, since $g^* \geq (g\chi_E)^*$ and $\alpha < 0$, (2.5) and (2.6) prove the result. □

Remark 2.6. Without loss of generality (see [5, Lemma 2.1]) we can assume that g satisfies condition (2.4) and, in this case, for every $r > 0$.

$$|\{x : g(x) > g^*(r)\}| = r.$$

The following result will be fundamental for our purposes.

Theorem 2.7. *For every positive and locally integrable function f and every $1 \le p < \infty$, the weight $u = (Mf)^{1-p} \in A_p^{\mathcal{R}}$, with constant independent of f.*

Proof. We have to prove that

$$\|u\|_{A_p^{\mathcal{R}}} = \sup_{E \subset Q} \frac{|E|}{|Q|} \left(\frac{\int_Q Mf(x)^{1-p}\,dx}{\int_E Mf(x)^{1-p}\,dx} \right)^{1/p} < \infty.$$

Let Q be an arbitrary cube and let us write, for every $x \in Q$,

$$Mf(x) \approx M(f\chi_{3Q})(x) + M(f\chi_{(3Q)^c})(x).$$

Now, it is easy to see that, for every $x, x' \in Q$,

$$M(f\chi_{(3Q)^c})(x) \approx M(f\chi_{(3Q)^c})(x');$$

that is, $M(f\chi_{(3Q)^c})$ is essentially constant $C_{Q,f}$ in Q. Hence, for every $x \in Q$,

$$Mf(x) \approx M(f\chi_{3Q})(x) + C_{Q,f}.$$

Hence,

$$\int_Q (Mf(x))^{1-p}\,dx \approx \int_Q (M(f\chi_{3Q})(x) + C_{Q,f})^{1-p}\,dx.$$

Let $E_{f,3Q}$ be as in Lemma 2.4, and let $g(x) = M(f\chi_{3Q})(x) + C_{Q,f}$. Then, by Lemmas 2.5 and 2.4, with $\alpha = 1 - p$, and Remark 2.6, we obtain that

$$\begin{aligned}
\int_Q (Mf(x))^{1-p}\,dx &\lesssim \int_0^{|Q|} \left(g^{1-p}\chi_{E_{f,3Q}}\right)^*(t)\,dt \\
&\le \int_0^{|E_{f,3Q}|} \left((g\chi_{E_{f,3Q}})^*(|E_{f,3Q}| - t)\right)^{1-p} dt \\
&= \int_0^{|E_{f,3Q}|} \left((g\chi_{E_{f,3Q}})^*(t)\right)^{1-p} dt = \int_0^{|E_{f,3Q}|} g^*(t)^{1-p}\,dt. \quad (2.7)
\end{aligned}$$

Using now that

$$\big(M(f\chi_{3Q})\big)^*(s) \approx \frac{1}{s}\int_0^s (f\chi_{3Q})^*(t)\,dt,$$

we deduce that

$$\begin{aligned}
\int_Q (Mf(x))^{1-p}\,dx &\lesssim \int_0^{|E_{f,3Q}|} \left(\frac{1}{s}\int_0^s (f\chi_{3Q})^*(t)\,dt + C_{Q,f}\right)^{1-p} ds \\
&\le \int_0^{|39Q|} s^{p-1} F(s)\,ds, \quad (2.8)
\end{aligned}$$

where

$$F(s) = \left(\int_0^s (f\chi_{3Q})^*(t)\,dt + C_{Q,f}\,s \right)^{1-p}$$

is a decreasing function.

Now, we want to prove that

$$B = \sup_{E\subset Q} \left(\frac{|E|}{|Q|}\right)^p \frac{\int_Q (M(f\chi_{3Q})(x) + C_{Q,f})^{1-p}\,dx}{\int_E (M(f\chi_{3Q})(x) + C_{Q,f})^{1-p}\,dx} < \infty.$$

Using (2.7) and (2.8), we see that

$$\int_Q (M(f\chi_{3Q})(x) + C_{Q,f})^{1-p}\,dx \lesssim \int_0^{|39Q|} s^{p-1}F(s)\,ds.$$

Similarly, if $E \subset Q$, with $|E| = t$, Lemma 2.5 gives:

$$\begin{aligned}
\int_E (M(f\chi_{3Q})(x) + C_{Q,f})^{1-p}\,dx &= \int_E (g(x))^{1-p}\,dx \geq \int_{\{x:g(x)>g^*(t)\}} (g(x))^{1-p}\,dx \\
&= \int_0^t (g^{1-p}\chi_{\{x:g(x)>g^*(t)\}})^*(s)\,ds = \int_0^t (g^{1-p})^*(s)\,ds \\
&= \int_0^t (g^*(s))^{1-p}\,ds \approx \int_0^t s^{p-1}F(s)\,ds.
\end{aligned}$$

Therefore,

$$B \lesssim \sup_{0<t<|Q|} \left(\frac{t}{|39Q|}\right)^p \frac{\int_0^{|39Q|} s^{p-1}F(s)\,ds}{\int_0^t s^{p-1}F(s)\,ds} = \sup_{0<t<|Q|} \frac{\int_0^1 s^{p-1}F(s|39Q|)\,ds}{\int_0^1 s^{p-1}F(ts)\,ds} \leq 1,$$

and the result follows. $\square$

Corollary 2.8. *For every $u \in A_1$, every positive and locally integrable function f and every $1 \leq p < \infty$, the weight $(Mf)^{1-p}u \in A_p^{\mathcal{R}}$ and*

$$\|(Mf)^{1-p}u\|_{A_p^{\mathcal{R}}} \lesssim \|u\|_{A_1}^{1/p}.$$

Proof. We have that

$$\int_Q (Mf)^{1-p}(x)u(x)\,dx \leq \frac{1}{\inf_{x\in Q} Mf(x)^{p-1}} \int_Q u(x)\,dx = \frac{u(Q)}{\inf_{x\in Q} Mf(x)^{p-1}}.$$

Let us take $0 < \alpha < 1$ such that $\alpha \leq p-1$. Then, since

$$\frac{|Q|}{\int_Q (Mf)^{\alpha}(x)\,dx} \leq \frac{\int_Q (Mf)^{-\alpha}(x)\,dx}{|Q|},$$

we have, using that $(Mf)^\alpha \in A_1$,

$$\begin{aligned}\frac{1}{\inf_{x\in Q} Mf(x)^{p-1}} &= \left(\frac{1}{\inf_{x\in Q} Mf(x)^{\alpha}}\right)^{\frac{p-1}{\alpha}} \lesssim \left(\frac{|Q|}{\int_Q (Mf)^\alpha(x)\,dx}\right)^{\frac{p-1}{\alpha}}\\ &\le \left(\frac{\int_Q (Mf)^{-\alpha}(x)\,dx}{|Q|}\right)^{\frac{p-1}{\alpha}} \le \frac{\int_Q (Mf)^{1-p}(x)\,dx}{|Q|}.\end{aligned}$$

By the previous theorem, we finally obtain that

$$\begin{aligned}\left(\frac{|E|}{|Q|}\right)^p \int_Q (Mf)^{1-p}(x)u(x)\,dx &\lesssim \frac{u(Q)}{|Q|}\int_E (Mf)^{1-p}(x)\,dx\\ &\le \|u\|_{A_1}\int_E (Mf)^{1-p}(x)u(x)\,dx,\end{aligned}$$

as we wanted to see. $\square$

The above corollary motivates the following definition:

Definition 2.9. We define

$$\widehat{A}_p = \left\{u : \exists\, f \in L^1_{\mathrm{loc}} \text{ and } \exists\, u_1 \in A_1 : u = (Mf)^{1-p}u_1\right\},$$

with

$$\|u\|_{\widehat{A}_p} = \inf \|u_1\|_{A_1}^{1/p}.$$

We have proved that, for every $1 \le p < \infty$,

$$\widehat{A}_p \subset A_p^{\mathcal{R}} \qquad \text{and} \qquad \|u\|_{A_p^{\mathcal{R}}} \lesssim \|u\|_{\widehat{A}_p}.$$

At this point, we should emphasize that the class of weights for which we are going to extend Rubio de Francia's extrapolation result is $\widehat{A}_p$. It is unknown to us whether $\widehat{A}_p = A_p^{\mathcal{R}}$.

We begin by proving the following important distribution inequality:

Proposition 2.10. *For every weight u, every pair of positive functions f and g, every $\gamma > 0$ and $1 \le p < p_0$,*

$$\lambda_g^u(y) \le \lambda_{Mf}^u(\gamma y) + \gamma^{p_0-p}\frac{y^{p_0}}{y^p}\int_{\{g>y\}} (Mf(x))^{p-p_0}u(x)\,dx. \tag{2.9}$$

Proof.

$$
\begin{aligned}
\lambda_g^u(y) &\le \lambda_{Mf}^u(\gamma y) + u(\{x : g(x) > y,\ Mf(x) \le \gamma y\}) \\
&= \lambda_{Mf}^u(\gamma y) + \int_{\{g>y, Mf\le\gamma y\}} u(x)\,dx \\
&\le \lambda_{Mf}^u(\gamma y) + \int_{\{g>y\}} \left(\frac{\gamma y}{Mf}\right)^{p_0-p} u(x)\,dx \\
&= \lambda_{Mf}^u(\gamma y) + \gamma^{p_0-p}\frac{y^{p_0}}{y^p}\int_{\{g>y\}} (Mf(x))^{p-p_0} u(x)\,dx.
\end{aligned}
$$

□

Now, we are ready to prove our first Rubio de Francia extrapolation result, based on the following facts:

(i) The $\widehat{A}_p$ class satisfies a factorization result.
(ii) We will not need to construct A_1 weights, so Rubio de Francia algorithm can be avoided.
(iii) Sharp bounds for the maximal operators are known.

Theorem 2.11. *Let T be an operator satisfying that, for some $p_0 > 1$ and every $v \in \widehat{A}_{p_0}$,*

$$T : L^{p_0,1}(v) \longrightarrow L^{p_0,\infty}(v)$$

is bounded, with constant less than or equal to $\varphi_{p_0}(\|v\|_{\widehat{A}_{p_0}})$, with φ_{p_0} an increasing function on $(0,\infty)$. Then, for every $v \in A_1$,

$$T : L^{1,\frac{1}{p_0}}(v) \longrightarrow L^{1,\infty}(v) \tag{2.10}$$

is bounded, with constant less than or equal to

$$\varphi_{p_0,1}(\|v\|_{A_1}) = C\|v\|_{A_1}^{1-\frac{1}{p_0}}\varphi_{p_0}(\|v\|_{A_1}^{\frac{1}{p_0}}).$$

In particular, T is of restricted weak-type $(1,1)$; that is, for every measurable set,

$$\|T\chi_E\|_{L^{1,\infty}(v)} \lesssim \varphi_{p_0,1}(\|v\|_{A_1})v(E). \tag{2.11}$$

Proof. Let $v \in A_1$ and let $w = (Mf)^{1-p_0}v$. Then, by (2.9) with $p = 1$ and the fact that $w \in \widehat{A}_{p_0}$ with $\|w\|_{\widehat{A}_{p_0}} \leq \|v\|_{A_1}^{1/p_0}$, we have that

$$\begin{aligned}
\lambda_{Tf}^v(y) &\leq \lambda_{Mf}^v(\gamma y) + \gamma^{p_0-1}\frac{y^{p_0}}{y}\int_{\{|Tf|>y\}}(Mf(x))^{1-p_0}v(x)\,dx \\
&\lesssim \lambda_{Mf}^v(\gamma y) + \gamma^{p_0-1}\frac{\varphi_{p_0}(\|w\|_{\widehat{A}_{p_0}})^{p_0}}{y}\left(\int_0^\infty\left(\int_{\{|f|>z\}}(Mf(x))^{1-p_0}v(x)\,dx\right)^{1/p_0}dz\right)^{p_0} \\
&\leq \lambda_{Mf}^v(\gamma y) + \gamma^{p_0-1}\frac{\varphi_{p_0}(\|w\|_{\widehat{A}_{p_0}})^{p_0}}{y}\left(\int_0^\infty z^{\frac{1}{p_0}-1}\left(\int_{\{|f|>z\}}v(x)\,dx\right)^{1/p_0}dz\right)^{p_0} \\
&\approx \lambda_{Mf}^v(\gamma y) + \gamma^{p_0-1}\frac{\varphi_{p_0}(\|w\|_{\widehat{A}_{p_0}})^{p_0}}{y}\|f\|_{L^{1,\frac{1}{p_0}}(v)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
y\lambda_{Tf}^v(y) &\lesssim y\lambda_{Mf}^v(\gamma y) + \gamma^{p_0-1}\varphi_{p_0}(\|v\|_{A_1}^{1/p_0})^{p_0}\|f\|_{L^{1,\frac{1}{p_0}}(v)} \\
&\lesssim \frac{1}{\gamma}\|v\|_{A_1}\|f\|_{L^1(v)} + \gamma^{p_0-1}\varphi_{p_0}(\|v\|_{A_1}^{1/p_0})^{p_0}\|f\|_{L^{1,\frac{1}{p_0}}(v)} \\
&\lesssim \left(\frac{1}{\gamma}\|v\|_{A_1} + \gamma^{p_0-1}\varphi_{p_0}(\|v\|_{A_1}^{1/p_0})^{p_0}\right)\|f\|_{L^{1,\frac{1}{p_0}}(v)},
\end{aligned}$$

from which the result follows taking the infimum in $\gamma > 0$. □

Remark 2.12. In general, it is not true that if T satisfies the hypotheses of Theorem 2.11, then T is of weak-type $(1, 1)$. To see this, we consider the following operator, which was introduced in [1],

$$Af(x) = \left\|\frac{f\chi_{(0,x)}}{x - \cdot}\right\|_{L^{1,\infty}(0,1)}.$$

This operator plays an important role in connection with Bourgain's return time theorems. Now, it is immediate to see that, for every measurable set E, $A\chi_E \leq M\chi_E$, and hence A satisfies the same restricted inequalities as M. However A is not of weak-type $(1, 1)$.

Theorem 2.13. *Let T be a sublinear operator such that, for some $p_0 > 1$ and every $v \in \widehat{A}_{p_0}$,*

$$\|Tf\|_{L^{p_0,\infty}(v)} \leq \varphi_{p_0}(\|v\|_{\widehat{A}_{p_0}})\|f\|_{L^{p_0,1}(v)},$$

with φ_{p_0} an increasing function on $(0, \infty)$. Then, for every $1 \leq p < p_0$ and every $v \in \widehat{A}_p$,

$$\|Tf\|_{L^{p,\infty}(v)} \leq \varphi_{p_0,p}(\|v\|_{\widehat{A}_p})\|f\|_{L^{p,\frac{p}{p_0}}(v)},$$

where

$$\varphi_{p_0,p}(t) \lesssim t^{1-\frac{p}{p_0}}\varphi_{p_0}(Ct^{\frac{p}{p_0}}). \tag{2.12}$$

Proof. The case $p=1$ was already solved in Theorem 2.11. Let $p>1$ and $v=(Mg)^{1-p}u\in\widehat{A}_p$, with $u\in A_1$ and $\|u\|_{A_1}^{1/p}\le 2\|v\|_{\widehat{A}_p}$. Then, we observe that

$$w=(Mf)^{p-p_0}v=\Big((Mf)^{1-p_0}u\Big)^{\frac{p_0-p}{p_0-1}}\Big((Mg)^{1-p_0}u\Big)^{\frac{p-1}{p_0-1}}=v_1^{\frac{p_0-p}{p_0-1}}v_2^{\frac{p-1}{p_0-1}}.$$

By hypothesis, for $j=1,2$, $T:L^{p_0,1}(v_j)\to L^{p_0,\infty}(v_j)$ is bounded, with constant less than or equal to $\varphi_{p_0}(\|v_j\|_{\widehat{A}_{p_0}})\le\varphi_{p_0}(\|u\|_{A_1}^{1/p_0})$ and hence, by interpolation

$$T:L^{p_0,1}(w)\to L^{p_0,\infty}(w) \tag{2.13}$$

is bounded, with constant less than or equal to $C\varphi_{p_0}(\|u\|_{A_1}^{1/p_0})$.

Using this fact and (2.9), we obtain that

$$\begin{aligned}
&\lambda^v_{Tf}(y)\le\lambda^v_{Mf}(\gamma y)+\gamma^{p_0-p}\frac{y^{p_0}}{y^p}\int_{\{|g|>y\}}(Mf(x))^{p-p_0}v(x)\,dx\\
\lesssim&\lambda^v_{Mf}(\gamma y)+\gamma^{p_0-p}\frac{\varphi_{p_0}(\|u\|_{A_1}^{1/p_0})^{p_0}}{y^p}\left(\int_0^\infty\left(\int_{\{|f|>z\}}(Mf(x))^{p-p_0}v(x)\,dx\right)^{1/p_0}dz\right)^{p_0}\\
\le&\lambda^v_{Mf}(\gamma y)+\gamma^{p_0-p}\frac{\varphi_{p_0}(C\|v\|_{\widehat{A}_p}^{p/p_0})^{p_0}}{y^p}\left(\int_0^\infty z^{\frac{p}{p_0}-1}\left(\int_{\{|f|>z\}}v(x)\,dx\right)^{1/p_0}dz\right)^{p_0}\\
\approx&\lambda^v_{Mf}(\gamma y)+\gamma^{p_0-p}\frac{\varphi_{p_0}(C\|v\|_{\widehat{A}_p}^{p/p_0})^{p_0}}{y^p}\|f\|^p_{L^{p,\frac{p}{p_0}}(v)},
\end{aligned}$$

and the result follows as in the proof of Theorem 2.11. □

Remark 2.14. The behavior of the extrapolation constant obtained in the previous theorem is sharp, since if we apply the result to the maximal operator we have that $\varphi_{p_0}(t)=t$ and the same holds for $\varphi_{p_0,p}$.

Since $\|\chi_E\|_{L^{p,\frac{p}{p_0}}(u)}=\|\chi_E\|_{L^{p,1}(u)}$, we obtain that, if $1<p<p_0$, we can extrapolate restricted weak-type inequalities as follows:

Corollary 2.15. *Let T be a sublinear operator satisfying that, for some $p_0>1$ and every $v\in\widehat{A}_{p_0}$,*

$$T:L^{p_0,1}(v)\longrightarrow L^{p_0,\infty}(v)$$

is bounded, with constant less than or equal to $\varphi_{p_0}(\|v\|_{\widehat{A}_{p_0}})$, with φ_{p_0} an increasing function on $(0,\infty)$. Then, with $\varphi_{p_0,p}$ as in (2.12), $1<p<p_0$ and every $v\in\widehat{A}_p$,

$$T:L^{p,1}(v)\longrightarrow L^{p,\infty}(v) \tag{2.14}$$

is bounded, with constant less than or equal to $\frac{C}{p-1}\varphi_{p_0,p}(\|v\|_{\widehat{A}_p})$.

Another consequence of our results is that, although we cannot, in general, obtain the weak-type $(1,1)$ boundedness for T, we can extrapolate up to a space quite near to $L^1(u)$:

Corollary 2.16. *Let T be a sublinear operator satisfying that, for some $p_0 > 1$ and every $v \in \widehat{A}_{p_0}$,*

$$T : L^{p_0,1}(v) \longrightarrow L^{p_0,\infty}(v)$$

is bounded, with constant less than or equal to $\varphi_{p_0}(\|v\|_{\widehat{A}_{p_0}})$, and φ_{p_0} an increasing function. Then, for every $\varepsilon > 0$ and every $u \in A_1$,

$$T : L(\log L)^{\varepsilon}(u) \longrightarrow L^{1,\infty}_{\mathrm{loc}}(u)$$

is bounded, with constant less than or equal to $\frac{C\|u\|_{A_1}^{1-\frac{1}{p_0}}}{\varepsilon}\varphi((\|u\|_{A_1}^{\frac{1}{p_0}})$.

Proof. Let $1 < p_1 < p_0$. Then, by Corollary 2.15 and (2.10), we get that, for every $u \in A_1$,

$$T : L^{1,\frac{1}{p_1}}(u) \longrightarrow L^{1,\infty}(u)$$

is bounded, with constant less than or equal to

$$\varphi_{p_1,1}(\|u\|_{A_1}) \lesssim \frac{1}{p_1 - 1}\|u\|_{A_1}^{1-\frac{1}{p_0}}\varphi_{p_0}(\|u\|_{A_1}^{\frac{1}{p_0}}).$$

Now, given $\varepsilon > 0$ and $u \in A_1$, let us take p_1 such that $\varepsilon(p_1' - 1) = 2$. Then, if we write $f = f_0 + f_1$, where $\|f_0\|_\infty \le f_u^*(1)$, we obtain that

$$\begin{aligned}
\|Tf_1\|_{L^{1,\infty}(u)} &\lesssim \frac{\|u\|_{A_1}^{1-\frac{1}{p_0}}}{p_1 - 1}\varphi_{p_0}(\|u\|_{A_1}^{\frac{1}{p_0}})\left(\int_0^1 f_u^*(t)^{1/p_1} t^{\frac{1}{p_1}-1}\, dt\right)^{p_1} \\
&\le \frac{\|u\|_{A_1}^{1-\frac{1}{p_0}}}{p_1 - 1}\varphi_{p_0}(\|u\|_{A_1}^{\frac{1}{p_0}})\left(\int_0^1 f_u^*(t)\left(1 + \log^+ \frac{1}{t}\right)^{\varepsilon} dt\right) \\
&\quad \times \left(\int_0^1 \left(1 + \log^+ \frac{1}{t}\right)^{-\varepsilon(p_1'-1)} \frac{dt}{t}\right)^{p_1/p_1'} \\
&\lesssim \frac{\|u\|_{A_1}^{1-\frac{1}{p_0}}}{\varepsilon}\varphi_{p_0}(\|u\|_{A_1}^{\frac{1}{p_0}})\|f\|_{L(\log L)^{\varepsilon}(u)}.
\end{aligned}$$

Finally, using interpolation, we have that T is bounded on $L^{\frac{p_0+p_1}{2}}(u)$ with a constant that can be also controlled by $\frac{\|u\|_{A_1}^{1-\frac{1}{p_0}}}{p_1-1}\varphi_{p_0}(\|u\|_{A_1}^{\frac{1}{p_0}})$, and hence

$$\begin{aligned}
\|Tf_0\|_{L^{1,\infty}_{\mathrm{loc}}(u)} &\lesssim \|Tf_0\|_{L^{\frac{p_0+p_1}{2}}(u)} \lesssim \frac{\|u\|_{A_1}^{1-\frac{1}{p_0}}}{p_1 - 1}\varphi_{p_0}(\|u\|_{A_1}^{\frac{1}{p_0}})\|f_0\|_{L^{\frac{p_0+p_1}{2}}(u)} \\
&\lesssim \frac{\|u\|_{A_1}^{1-\frac{1}{p_0}}}{\varepsilon}\varphi_{p_0}(\|u\|_{A_1}^{\frac{1}{p_0}})\|f\|_{L^1(u)},
\end{aligned}$$

and the result follows. $\square$

3. From restricted weak-type to weak-type

Taking into account the results in (2.11) and (2.14) we are now interested in studying when restricted weak-type implies weak-type. We shall consider two cases: $p=1$ and $p>1$.

3.1. The case $p = 1$.

In this case, we are interested in studying when the following implication holds, for every $u \in A_1$,

$$\|T\chi_E\|_{L^{1,\infty}(u)} \lesssim u(E), \forall E \Longrightarrow T : L^1(u) \longrightarrow L^{1,\infty}(u).$$

We know that this is not true in general. However, it was proved in [4] that for a quite big class of operators the above implication is true in the case $u=1$.

Our goal now, is to prove that the same holds in the weighted setting if $u \in A_1$.

Definition 3.1. Given $\delta > 0$, a function $a \in L^1(\mathbb{R}^n)$ is called a δ-atom if it satisfies the following properties:

(i) $\int_{\mathbb{R}^n} a(x)\, dx = 0$, and

(ii) there exists a cube Q such that $|Q| \leq \delta$ and supp $a \subset Q$.

Definition 3.2. (a) A sublinear operator T is (ε,δ)-atomic if, for every $\varepsilon > 0$, there exists $\delta > 0$ satisfying that

$$\|Ta\|_{L^1+L^\infty} \leq \varepsilon \|a\|_1, \tag{3.1}$$

for every δ-atom a.

(b) A sublinear operator T is (ε,δ)-atomic approximable if there exists a sequence $(T_n)_n$ of (ε,δ)-atomic operators such that, for every measurable set E, $|T_n\chi_E| \leq |T\chi_E|$ and, for every $f \in L^1$ such that $\|f\|_\infty \leq 1$, and for almost every x,

$$|Tf(x)| \leq \liminf_n |T_n f(x)| \cdot$$

Examples: If

$$T^* f(x) = \sup_{j\in\mathbb{N}} \left| \int_{\mathbb{R}^n} K_j(x,y) f(y)\, dy \right|,$$

with

$$\lim_{y\to x} \|K_j(\cdot,y) - K_j(\cdot,x)\|_{L^1+L^\infty} = 0,$$

then T^* is (ε,δ)-atomic approximable. In particular, standard maximal Calderón-Zygmund operators are of this type. In general, $T^* f(x) = \sup_n |T_n f(x)|$, where T_n is (ε,δ)-atomic, is (ε,δ)-atomic approximable and the same holds for $Tf(x) = \left(\sum_n |T_n f(x)|^q \right)^{1/q}$, with $q \geq 1$ (see [4] for other examples).

To formulate our main result, we first need the following definitions:

Definition 3.3. Given $\delta > 0$, we say that $\mathcal{F}_\delta$ is a δ-net if

$$\mathcal{F}_\delta = \{Q_j : |Q_j| = \delta, Q_j \text{ are pairwise disjoint}, \cup Q_j = \mathbb{R}^n\}.$$

Definition 3.4. A set E is said to be a δ-union of cubes, if $E = \cup_i \tilde{Q}_i$ where, for every i, there exists j such that $\tilde{Q}_i \subset Q_j$, with $Q_j \in \mathcal{F}_\delta$, $\mathcal{F}_\delta$ a δ-net, and every $Q_j \in \mathcal{F}_\delta$ contains at most one $\tilde{Q}_i$.

Finally, if E is a δ-union of cubes for some $\delta > 0$, we write $E \in \mathcal{F}$.

Theorem 3.5. *Let T be a sublinear operator (ε, δ)-atomic approximable and let $u \in A_1$. Then, if there exists $C_u > 0$ such that, for every measurable set E,*

$$\|T\chi_E\|_{L^{1,\infty}(u)} \leq C_u u(E),$$

we have that

$$T : L^1(u) \longrightarrow L^{1,\infty}(u)$$

is bounded, with constant $2^n C_u \|u\|_{A_1}$.

Proof. First of all let us assume that T is (ε, δ)-atomic. Let $N \in \mathbb{N}$ and let us consider $u_N = \min(u, N)$. Let $f \in L^1$ be a positive function such that $\|f\|_\infty \leq 1$ and, given $\varepsilon > 0$, let us consider a δ-net $\mathcal{F}_\delta$, where δ is the number associated to ε by the property that T is (ε, δ)-atomic.

Given $Q_i \in \mathcal{F}_\delta$, let $f_i = f\chi_{Q_i}$. Then,

$$\int_{\mathbb{R}^n} f_i(x)\, dx \leq |Q_i|.$$

For each i, we can find a finite collection of cubes $\{Q_{i,j}\}_j$ such that $Q_i = \cup_j Q_{i,j}$,

$$|Q_{i,j}| = \int_{\mathbb{R}^n} f_i(x)\, dx = \int_{Q_i} f(x)\, dx,$$

and $\sum_j \chi_{Q_{i,j}} \leq 2^n$. Now, let us take one $\tilde{Q}_i$, among these cubes, such that

$$\frac{u(\tilde{Q}_i)}{|\tilde{Q}_i|} = \min_j \frac{u(Q_{i,j})}{|Q_{i,j}|}.$$

Then, it is clear that the function $g_i = f_i - \chi_{\tilde{Q}_i}$ is a δ-atom and

$$\|g_i\|_1 \leq \int_{Q_i} |f(x)|\, dx + |\tilde{Q}_i| = 2\int_{Q_i} |f(x)|\, dx.$$

Now, $f = \sum_i f_i = \sum_i g_i + \chi_E$, where $E = \cup \tilde{Q}_i \in \mathcal{F}$. Then, by sublinearity, $|Tf| \leq \sum_i |Tg_i| + |T\chi_E|$ and therefore, for every $0 < \alpha < 1$,

$$\begin{aligned}(Tf)^*_{u_N}(t) &\leq \left(\sum_i |Tg_i|\right)^*_{u_N}(\alpha t) + (T\chi_E)^*_{u_n}((1-\alpha)t) \\ &\leq \left(\sum_i |Tg_i|\right)^*\left(\frac{\alpha t}{N}\right) + (T\chi_E)^*_u((1-\alpha)t).\end{aligned}$$

On the other hand, using the (ε,δ) property on each δ-atom g_i, we obtain that

$$\left(\sum_i |Tg_i|\right)^*\left(\frac{\alpha t}{N}\right) \le \frac{N}{\alpha t}\int_0^{\frac{\alpha t}{N}}\left(\sum_i |Tg_i|\right)^*(s)\,ds \le \frac{N}{\alpha t}\sum_i \int_0^{\frac{\alpha t}{N}} (Tg_i)^*(s)\,ds$$
$$\le \max\left(\frac{N}{\alpha t},1\right)\varepsilon\sum_i \|g_i\|_1 \le 2\max\left(\frac{N}{\alpha t},1\right)\varepsilon\|f\|_1.$$

Consequently,

$$(Tf)^*_{u_N}(t) \le 2\max\left(\frac{N}{\alpha t},1\right)\varepsilon\|f\|_1 + (T\chi_E)^*_u((1-\alpha)t)$$
$$\le 2\max\left(\frac{N}{\alpha t},1\right)\varepsilon\|f\|_1 + \frac{C_u}{(1-\alpha)t}u(E),$$

and, since,

$$u(E) = \sum_i u(\tilde{Q}_i) = \sum_i \frac{u(\tilde{Q}_i)}{|\tilde{Q}_i|}|\tilde{Q}_i| = \sum_i \frac{u(\tilde{Q}_i)}{|\tilde{Q}_i|}\int_{Q_i} f(x)\,dx$$
$$\le \sum_i \frac{u(\tilde{Q}_i)}{|\tilde{Q}_i|}\sum_j \int_{Q_{i,j}} f(x)\,dx \le \sum_{i,j}\frac{u(Q_{i,j})}{|Q_{i,j}|}\int_{Q_{i,j}} f(x)\,dx$$
$$\le \|u\|_{A_1}\sum_{i,j}\int_{Q_{i,j}} f(x)\; u(x)\,dx \le 2^n\|u\|_{A_1}\|f\|_{L^1(u)},$$

we obtain that

$$(Tf)^*_{u_N}(t) \le 2\max\left(\frac{N}{\alpha t},1\right)\varepsilon\|f\|_1 + \frac{2^n C_u\|u\|_{A_1}}{(1-\alpha)t}\|f\|_{L^1(u)}.$$

Letting first ε tend to zero, then $\alpha\to 0$ and finally $N\to\infty$, we get the result for the operator T.

To finish, if T is (ε,δ)-atomic approximable and $(T_m)_m$ is the corresponding sequence of (ε,δ)-atomic operators given in Definition 3.2, then

$$t(T_m\chi_E)^*_u(t) \le t(T\chi_E)^*_u(t) \le C_u u(E),$$

and hence, $t(T_m f)^*_u(t) \le 2^n C_u\|u\|_{A_1}\|f\|_{L^1(u)}$, for every positive function f such that $\|f\|_\infty \le 1$. Since $|Tf(x)| \le \lim_m \inf |T_m f(x)|$, we obtain the result. $\square$

Remark 3.6. Observe that we have proved that, if an arbitrary weight u satisfies $\|T\chi_E\|_{L^{1,\infty}(u)} \le C_u u(E)$, for every measurable set and T is (ε,δ)-atomic approximable, then

$$T : L^1(Mu) \longrightarrow L^{1,\infty}(u)$$

is bounded, with constant $2^n C_u$.

Corollary 3.7. *Let T be a sublinear (ε,δ)-atomic approximable operator satisfying that, for some $p_0>1$ and every $v\in\widehat{A}_{p_0}$,*

$$T : L^{p_0,1}(v) \longrightarrow L^{p_0,\infty}(v)$$

is bounded, with constant less than or equal to $\varphi_{p_0}(\|v\|_{\widehat{A}_{p_0}})$. *Then, for every* $v \in A_1$,

$$T : L^1(v) \longrightarrow L^{1,\infty}(v),$$

with constant less than or equal to $C\|v\|_{A_1}^{2-\frac{1}{p_0}} \varphi_{p_0}(\|v\|_{A_1}^{1/p_0})$.

3.2. **The case $p > 1$.** Analyzing the results obtained in the previous sections, it is natural to ask about the relation between the fact that an operator T satisfies that, for every $u \in \widehat{A}_p$,

$$T : L^{p,1}(u) \longrightarrow L^{p,\infty}(u) \tag{3.2}$$

is bounded and that, for every $u \in A_p$,

$$T : L^p(u) \longrightarrow L^{p,\infty}(u) \tag{3.3}$$

is bounded.

Clearly, both conditions are the same if $p = 1$. On the other hand, for $p > 1$, (3.3) cannot imply, in general, (3.2) since it is known that we cannot extrapolate up to $p = 1$ from the last condition, and we can do it from the first. Hence, it seems natural to think that (3.2) is a stronger condition than (3.3). This will be the content of our next theorem, following the next three lemmas.

Lemma 3.8. *Let* $0 < \theta < 1$. *If* $u = (Mh)^{\theta(1-p)} u_0 \in A_p$, *with* $u_0 \in A_1$ *then, for every positive locally integrable function* F,

$$v = u(MF)^{(1-\theta)(1-p)} \in A_p^{\mathcal{R}} \qquad \text{with} \qquad \|v\|_{A_p^{\mathcal{R}}} \lesssim \|u_0\|_{A_1}^{1/p},$$

and, for every measurable function g,

$$g_u^*(t) \le g_v^*\Bigg(\Big(\frac{qt}{\|F\|_{L^{p,1}(v)}^{p(1-q)} \|v\|_{A_p^{\mathcal{R}}}^{p(1-q)}}\Big)^{1/q}\Bigg),$$

with $q = 1 + \frac{(1-\theta)(1-p)}{p}$.

Proof. The first part of the lemma follows by interpolation as in (2.13). On the other hand,

$$\begin{aligned}
\lambda_g^u(y) &= \int_{\{g>y\}} u(x)\,dx = \int_{\{g>y\}} v(x)((MF)(x))^{(1-\theta)(p-1)}\,dx \\
&\le \int_0^{\lambda_g^v(y)} [t^{1/p}(MF)_v^*(t)]^{(1-\theta)(p-1)} t^{\frac{(1-\theta)(1-p)}{p}}\,dt \\
&\le \frac{1}{q}\|v\|_{A_p^{\mathcal{R}}}^{(1-\theta)(p-1)} \lambda_g^v(y)^{1+\frac{(1-\theta)(1-p)}{p}} \|F\|_{L^{p,1}(v)}^{(1-\theta)(p-1)} \\
&= \frac{1}{q}\|v\|_{A_p^{\mathcal{R}}}^{p(1-q)} \|F\|_{L^{p,1}(v)}^{p(1-q)} \lambda_g^v(y)^q,
\end{aligned}$$

and hence, the result follows. $\square$

Lemma 3.9. *Let $p>1$. If for every $v\in \widehat{A}_p$,*

$$\|Tf\|_{L^{p,\infty}(v)} \le \varphi(\|v\|_{\widehat{A}_p})\|f\|_{L^{p,1}(v)},$$

with φ an increasing function on $(0,\infty)$ then, for every $u=u_0u_1^{1-p}\in A_p$, $u_0,u_1\in A_1$, there exists $q\in[\frac{1}{p},1)$, such that

$$\|Tf\|_{L^{pq,\infty}(u)} \lesssim \|u_0\|_{A_1}^{\frac{1-q}{pq}}\varphi(C\|u_0\|_{A_1}^{1/p})\|f\|_{L^{pq,q}(u)}.$$

Moreover,

$$1-q\approx \frac{1}{p'\|u_1\|_{A_1}}. \tag{3.4}$$

Proof. Let $u=u_0u_1^{1-p}$, with $u_0,u_1\in A_1$. Then, there exists a locally integrable function h such that $u_1\approx (Mh)^{\theta(1-p)}$, with $0<\theta<1$ and $\|u_1\|_{A_1}\approx \frac{1}{1-\theta}$.

By the previous lemma, taking $q=1+\frac{(1-\theta)(1-p)}{p}$ and $F=f$, and using interpolation as in (2.13), we have that

$$\begin{aligned} t^{\frac{1}{pq}}(Tf)^*_u(t) &\lesssim \left(\|f\|_{L^{p,1}(v)}^{p(1-q)}\|v\|_{A_p^{\mathcal{R}}}^{p(1-q)}\right)^{\frac{1}{pq}}\varphi(C\|u_0\|_{A_1}^{1/p})\|f\|_{L^{p,1}(v)} \\ &\lesssim \|u_0\|_{A_1}^{\frac{1-q}{pq}}\varphi(C\|u_0\|_{A_1}^{1/p})\|f\|_{L^{p,1}(v)}^{\frac{1}{q}}. \end{aligned}$$

Now, since

$$\begin{aligned} \|f\|_{L^{p,1}(v)} &\approx \int_0^\infty \left(\int_{\{f>z\}} u(x)((Mf)(x))^{p(q-1)}\,dx\right)^{1/p} dz \\ &\le \int_0^\infty z^{q-1}\lambda_f^u(z)^{1/p}dz \approx \|f\|_{L^{pq,q}(u)}^q, \end{aligned}$$

we get the result. $\square$

Lemma 3.10. *Let $0<q_0,q_1\le 1<p_0<p_1<\infty$, and let T be a sublinear operator such that*

$$T: L^{p_j,q_j}(u)\longrightarrow L^{p_j,\infty}(u),$$

is bounded, with constant less than or equal to M_j, with $j=0,1$. Then, for every $0<\theta<1$ and $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$,

$$T: L^p(u)\longrightarrow L^{p,\infty}(u)$$

is bounded, with $\|T\|_{L^p(u)}\lesssim BM_0^{1-\theta}M_1^\theta$, where

$$B=\left(\frac{p_0(p-q_0)}{q_0(p-p_0)}\right)^{\frac{p-q_0}{pq_0}}+\left(\frac{p_1(p-q_1)}{q_1(p_1-p)}\right)^{\frac{p-q_1}{pq_1}}+\left(\frac{p_1}{q_1}\right)^{1/q_1}.$$

Proof. This is a classical interpolation result and the boundedness can be found in [2, Theorem 5.3.2] but, for us, the most important part is the behavior of the constant, which does not explicitly appear in classical books. Clearly, we have that

$$T : (L^{p_0,q_0}(u),\, L^{p_1,q_1}(u))_{\theta,\infty} \longrightarrow (L^{p_0,\infty}(u),\, L^{p_1,\infty}(u))_{\theta,\infty},$$

with constant less than or equal to $M_0^{1-\theta}M_1^{\theta}$. Hence, everything will follow if we prove:

(i) $\|f\|_{L^{p,\infty}(u)} \leq 2\|f\|_{(L^{p_0,\infty}(u),\, L^{p_1,\infty}(u))_{\theta,\infty}}$.

(ii) $\|f\|_{(L^{p_0,q_0}(u),\, L^{p_1,q_1}(u))_{\theta,\infty}} \leq CB\|f\|_{L^p(u)}$, with a uniform constant C.

The proof of (i) is easy and we omit it. To prove (ii), let $f \in L^p(u)$ and given $t > 0$, let $\gamma = \frac{p_0p_1}{p_1-p_0}$ and let us write $f = f_0 + f_1$, with $f_0 = f\chi_{\{|f|>f_u^*(t^\gamma)\}}$. Then,

$$K(t, f; L^{p_0,q_0}(u), L^{p_1,q_1}(u)) \leq \|f_0\|_{L^{p_0,q_0}(u)} + t\|f_1\|_{L^{p_1,q_1}(u)}.$$

Now, by Hölder's inequality,

$$\|f_0\|_{L^{p_0,q_0}(u)} \leq \left(\int_0^{t^\gamma} f_u^*(s)^{q_0} s^{\frac{q_0}{p_0}} \frac{ds}{s}\right)^{1/q_0} \leq \left[\frac{p-q_0}{q_0\left(\frac{p}{p_0}-1\right)}\right]^{\frac{p-q_0}{pq_0}} t^\theta \|f\|_{L^p(u)},$$

and

$$\begin{aligned}
\|f_1\|_{L^{p_1,q_1}(u)} &= \left(\int_0^\infty f_u^*(s+t^\gamma)^{q_1} s^{\frac{q_1}{p_1}} \frac{ds}{s}\right)^{1/q_1} \\
&\leq f_u^*(t^\gamma)\left(\int_0^{t^\gamma} s^{\frac{q_1}{p_1}-1}\, ds\right)^{1/q_1} + \left(\int_{t^\gamma}^\infty f_u^*(s)^{q_1} s^{\frac{q_1}{p_1}-1}\, ds\right)^{1/q_1} \\
&= I + II.
\end{aligned}$$

Now,

$$I \leq \left(\frac{p_1}{q_1}\right)^{1/q_1} f_u^*(t^\gamma) t^{\frac{\gamma}{p_1}} = \left(\frac{p_1}{q_1}\right)^{1/q_1} t^{\frac{\gamma}{p}} f_u^*(t^\gamma) t^{\gamma\left(\frac{1}{p_1}-\frac{1}{p}\right)} \leq \left(\frac{p_1}{q_1}\right)^{1/q_1} t^{\theta-1}\|f\|_{L^p(u)},$$

and to estimate II, we proceed as for f_0, and obtain

$$II \leq \left[\frac{p-q_1}{q_1\left(\frac{p}{p_1}-1\right)}\right]^{\frac{p-q_1}{pq_1}} t^{\theta-1}\|f\|_{L^p(u)},$$

from which the result follows. □

Theorem 3.11. *Let $1 < p < \infty$ and let T be a sublinear operator such that, for every $v \in \widehat{A}_p$,*

$$T : L^{p,1}(v) \longrightarrow L^{p,\infty}(v)$$

is bounded, with constant $\varphi(\|v\|_{\widehat{A}_p})$*, and* φ *an increasing function on* $(0,\infty)$*. Then, for every* $u \in A_p$*,*

$$T : L^p(u) \longrightarrow L^{p,\infty}(u)$$

is bounded, with constant less than or equal to

$$B(u) \lesssim \left(\frac{\varphi(C\|u\|_{A_p}^{2/p})}{p-1}\right) \inf_{u_0,u_1} \left(\|u_1\|_{A_1}^{\frac{p-1}{p}} \left(1+\log^+ A(u,u_0)\right)^{\frac{p-1}{p}}\right),$$

where the infimum is taken among all possible decompositions $u = u_0 u_1^{1-p}$, $u_0, u_1 \in A_1$ *and*

$$A(u;u_0) = \|u_0\|_{A_1} \left(\frac{\varphi(\|u_0\|_{A_1}^{1/p})}{\varphi(C\|u\|_{A_p}^{2/p})}\right)^{\frac{p}{p-1}}.$$

In particular,

$$B(u) \lesssim \left(\frac{1}{p-1}\right) \|u\|_{A_p}^{\frac{1}{p}} \left(1+\log^+ \|u\|_{A_p}\right)^{\frac{p-1}{p}} \varphi(C\|u\|_{A_p}^{2/p}). \tag{3.5}$$

We recall that the above estimates of $B(u)$ are for p near 1.

Proof. By Lemma 3.9 we have that, given $u = u_0 u_1^{1-p} \in A_p$, with $u_j \in A_1$, there exists $\frac{1}{p} \leq q < 1$ such that

$$T : L^{pq,q}(u) \longrightarrow L^{pq,\infty}(u)$$

is bounded, with constant less than or equal to $C\|u_0\|_{A_1}^{\frac{1-q}{pq}} \varphi(C\|u_0\|_{A_1}^{1/p})$.

Now, by hypothesis and using the fact that $\|u\|_{\widehat{A}_p} \lesssim \|u\|_{A_p}^{2/p}$ we have that, for every measurable set E,

$$\|T\chi_E\|_{L^{p,\infty}(u)} \leq \varphi(\|u\|_{A_p}^{2/p}) u(E)^{1/p},$$

and using Theorem 1.1 we obtain that, for every $p < r < \infty$,

$$T : L^{r,1}(u) \longrightarrow L^{r,\infty}(u),$$

with norm less than or equal to $\frac{C}{r-1}\varphi(C\|u\|_{A_p}^{2/p})$. Therefore, by Lemma 3.10,

$$T : L^p(u) \longrightarrow L^{p,\infty}(u)$$

is bounded, with norm less than or equal to

$$\begin{aligned} B(u) \lesssim & \left[\left(\frac{1}{1-q}\right)^{\frac{p-q}{pq}} + \left(\frac{r}{r-p}\right)^{\frac{p-1}{p}} + r\right](r-1)^{-\alpha} \\ & \times \|u_0\|_{A_1}^{\frac{1-q}{pq}(1-\alpha)} \varphi(C\|u_0\|_{A_1}^{1/p})^{1-\alpha} \varphi(C\|u\|_{A_p}^{2/p})^{\alpha}, \end{aligned} \tag{3.6}$$

where, $1-q < \alpha < 1$ and $r > p$ satisfy

$$\frac{1-\alpha}{pq} + \frac{\alpha}{r} = \frac{1}{p}.$$

In fact, we shall take $r < r_0$ for some fixed $r_0 < \infty$ big enough, and hence

$$\alpha\left(\frac{1}{pqr_0} - \frac{1}{pq}\right) < \alpha\left(\frac{1}{r} - \frac{1}{pq}\right) = \frac{1}{p} - \frac{1}{pq} = \frac{1}{p}\left(1 - \frac{1}{q}\right).$$

Thus, for some small enough constant c_0, we can take any α such that $\alpha \geq c_0(\frac{1}{q} - 1)$. On the other hand,

$$\frac{r}{r-p} = \frac{\alpha q}{(1-\alpha)(1-q)} \leq \frac{1}{(1-\alpha)(1-q)},$$

and

$$\left(\frac{1}{1-q}\right)^{\frac{p-q}{pq}} \lesssim \left(\frac{1}{1-q}\right)^{\frac{p-1}{p}}.$$

Now, by (3.4) and (3.6), we obtain that

$$B(u) \lesssim \inf_{c_0(p-1)<\alpha<1} \left(\frac{p'\|u_1\|_{A_1}}{1-\alpha}\right)^{\frac{p-1}{p}} (p')^\alpha \|u_0\|_{A_1}^{\frac{1-q}{pq}(1-\alpha)} \varphi(\|u_0\|_{A_1}^{1/p})^{1-\alpha} \varphi(C\|u\|_{A_p}^{2/p})^\alpha;$$

and since

$$\frac{1-q}{pq}(1-\alpha) \leq \frac{1-q}{pq} \leq 1 - \frac{1}{p},$$

we get that

$$B(u) \lesssim \left(\frac{1}{p-1}\right)^{2-\frac{1}{p}} \|u_1\|_{A_1}^{\frac{p-1}{p}} \varphi(C\|u\|_{A_p}^{2/p}) \left(\inf_{c_0(p-1)<\alpha<1} \frac{A(u,u_0)^{1-\alpha}}{1-\alpha}\right)^{\frac{p-1}{p}}.$$

Computing the infimum we obtain the result. Finally, (3.5) follows since we can always find a decomposition $u = u_0 u_1^{1-p}$ such that $\|u_0\|_{A_1} \leq \|u\|_{A_p}$ and $\|u_1\|_{A_1}^{p-1} \leq \|u\|_{A_p}$. □

Remark 3.12. If, in the above theorem, we assume that, for every $v \in A_p^{\mathcal{R}}$,

$$T : L^{p,1}(v) \longrightarrow L^{p,\infty}(v)$$

is bounded, with constant $\varphi(\|v\|_{A_p^{\mathcal{R}}})$ (as it happens in all the examples in Section 4), then using (2.1) and the same proof as before, one can see that the estimate in (3.5) can be improved as follows

$$B(u) \lesssim \left(\frac{1}{p-1}\right) \|u\|_{A_p}^{\frac{1}{p}} \left(1 + \log^+ \|u\|_{A_p}\right)^{\frac{p-1}{p}} \varphi(C\|u\|_{A_p}^{1/p}). \tag{3.7}$$

An estimate of this kind, where a logarithmic factor appears, has been obtained in [15] for the particular case of square functions.

Now, recently, the precise dependence on $\|u\|_{A_2}$ of the norm of the Calderón-Zygmund operators has attracted a lot of interest, and the following A_2-conjecture was formulated:

$$\|T\|_{L^2(u)} \lesssim \|u\|_{A_2}.$$

This conjecture has been solved in [13]. In fact, it was proved in [19] that

$$\|T\|_{L^2(u)} \lesssim \|u\|_{A_2} + \|T\|_{L^2(u)\to L^{2,\infty}(u)} + \|T^*\|_{L^2(u^{-1})\to L^{2,\infty}(u^{-1})},$$

but since $u^{-1} \in A_2$ and T^* is again a Calderón-Zygmund operator, one can conclude that to prove the A_2-conjecture it is enough to prove that

$$\|T\|_{L^2(u)\to L^{2,\infty}(u)} \lesssim \|u\|_{A_2}.$$

Therefore, if we could remove the logarithmic factor in (3.7), we would obtain that

$$\|T\chi_E\|_{L^{2,\infty}(u)} \le \|u\|_{A_2^{\mathcal{R}}} u(E)^{1/2} \Longrightarrow A_2 - \text{conjecture}.$$

Also, we observe that, by Theorem 2.11 and the fact that $\|u\|_{A_2^{\mathcal{R}}} \lesssim \|u\|_{\widehat{A}_2}$,

$$\|T\chi_E\|_{L^{2,\infty}(u)} \le \|u\|_{A_2^{\mathcal{R}}} u(E)^{1/2} \Longrightarrow \|T\chi_E\|_{L^{1,\infty}(u)} \le \|u\|_{A_1} u(E),$$

while it was recently proved that the behavior of the constant of

$$T : L^{1,\infty}(u) \longrightarrow L^1(u)$$

is not linear in $\|u\|_{A_1}$ [18]. So, it is an interesting open problem to study whether the following inequality is true, for every Calderón-Zygmund operator and every measurable set E:

$$\|T\chi_E\|_{L^{2,\infty}(u)} \lesssim \|u\|_{A_2^{\mathcal{R}}} u(E)^{1/2}.$$

4. Applications to weak-type $(1,1)$ estimates for classical operators in Harmonic Analysis

It is clear from the standard proof that, if an operator T satisfies a good-λ inequality

$$w(\{|Tf| > 3\lambda, Mf \le \gamma\lambda\}) \le C_w \gamma^\alpha w(\{|Tf| > \lambda\}),$$

then, for every $w \in \widehat{A}_2$,

$$T : L^{2,1}(w) \to L^{2,\infty}(w)$$

is bounded, and therefore any Calderón-Zygmund operator and any Calderón-Zygmund maximal operator satisfies the hypotheses of our theorems. However, we have to mention that the proof using the good-λ inequality does not give good estimates for the norm.

Square functions, g-function and the intrinsic square function: We refer to the papers [21] and [16] for this part. Let $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}^+$ and $\Gamma_\alpha(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < \alpha t\}$.

The classical square functions are defined as follows. If $u(x,t) = P_t * f(x)$ is the Poisson integral of f, the Lusin area integral is defined by

$$S_\alpha f(x) = \left(\int_{\Gamma_\alpha(x)} |\nabla u(y,t)|^2 \frac{dydt}{t^{n-1}} \right)^{1/2},$$

and the Littlewood-Paley g-function

$$g(f)(x) = \left(\int_0^\infty t |\nabla u(x,t)|^2 dt \right)^{1/2}.$$

Also, for a function $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $\int \varphi = 0$, the continuous square function is defined by

$$g_\varphi f(x) = \left(\int_0^\infty |(\varphi_t * f)(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

Finally, the intrinsic square function (introduced by M. Wilson in [21]) is defined by

$$G_\alpha f(x) = \left(\int_{\Gamma_\alpha(x)} |A_\alpha f(y,t)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

where

$$A_\alpha f(y,t) = \sup_{\varphi \in \mathcal{C}^\alpha} |(\varphi_t * f)(y)|,$$

with $\mathcal{C}^\alpha$ the family of functions φ supported in $B(0,1)$, such that $\int \varphi = 0$ and

$$|\varphi(x) - \varphi(x')| < |x - x'|^\alpha, \qquad \forall x, x'.$$

It was proved in [21] that, if S is any of the Littlewood operators defined above, it holds that

$$Sf(x) \lesssim G_\alpha f(x),$$

and hence it is enough to study the boundedness of G_α. In [16] it was proved that, for every $w \in A_3$,

$$\|G_1 f\|_{L^3(w)} \lesssim \|w\|_{A_3}^{1/2} \|f\|_{L^3(w)},$$

and using the extrapolation of Rubio de Francia, it was obtained that, for every $w \in A_p$, $p > 1$,

$$\|G_\alpha f\|_{L^p(w)} \lesssim \|w\|_{A_3}^{\max(\frac{1}{2}, \frac{1}{p-1})} \|f\|_{L^p(w)}.$$

Modifying slightly their proof, we obtain the corresponding result for $p = 1$. We shall follow the notation in [16] and we shall only present the modifications.

Theorem 4.1. *For every $w \in \widehat{A}_3$,*

$$\|G_\alpha f\|_{L^{3,\infty}(w)} \lesssim \|w\|_{\widehat{A}_3}^{5/2} \|f\|_{L^{3,1}(w)}.$$

Proof. Let $\mathbb{R}^n_i$ $(1 \le i \le 2n)$ be the n-dimensional quadrants in $\mathbb{R}^n$ and let Q_i^N be the dyadic cube adjacent to the origin of side length 2^N that is contained in $\mathbb{R}^n_i$. Then, by [16, (5.7) and (5.8)], we have that

$$\begin{aligned} \|G_\alpha f\|_{L^{3,\infty}(w;\mathbb{R}^n_i)} &\lesssim \lim_{N\to\infty} \|G_\alpha f\|_{L^{3,\infty}(w;Q_i^N)} \\ &\lesssim \|Mf\|_{L^{3,\infty}(w)} + \sum_{i=1}^{2^n} \lim_{N\to\infty} \|(A_{45}^{N,i} f)^{1/2}\|_{L^{3,\infty}(w;\mathbb{R}^n_i)}, \end{aligned}$$

where

$$A_{45}^{N,i} f(x) = \sum_{j,k} \left(\frac{1}{|45Q_{j,k}|} \int_{45Q_{j,k}} |f(y)| dy \right)^2 \chi_{Q_{j,k}}(x),$$

with $\{Q_{j,k}\}_{j,k}$ a particular family of cubes contained in Q_i^N satisfying the following fundamental property: there exist measurable sets $E_{j,k} \subset Q_{j,k}$ such that $\{E_{j,k}\}_{j,k}$ are pairwise disjoint and $|Q_{j,k}|/|E_{j,k}| \le 2$.

Since $\|Mf\|_{L^{3,\infty}(w)} \lesssim \|w\|_{\widehat{A}_3} \|f\|_{L^{3,1}(w)}$, it will be enough to prove that, for every N,

$$\|(A_{45}^{N,i} f)^{1/2}\|_{L^{3,\infty}(w;\, Q_i^N)} \lesssim \|w\|_{\widehat{A}_3}^{5/2} \|f\|_{L^{3,1}(w)}.$$

Let A_{45}^N be one of these $A_{45}^{N,i}$ and $Q^N = Q_i^N$. By duality, let us take $h \ge 0$ such that $\|h\|_{L^{3,1}(w)} = 1$. Then we have to prove (see [16, (5.4)]), that

$$\sum_{j,k} \left(\frac{1}{|45Q_{j,k}|} \int_{45Q_{j,k}} |f(y)| dy \right)^2 \int_{Q_{j,k}} h(x) w(x)\, dx \lesssim \|w\|_{\widehat{A}_3}^5 \|f\|_{L^{3,1}(w)}^2.$$

Let $c > 0$ such that, for every $x \in Q_{j,k}$, there exists another cube $\tilde{Q}_{j,k}$ centered at x such that $Q_{j,k} \subset \tilde{Q}_{j,k} \subset cQ_{j,k}$. Then, since $|Q_{j,k}|/|E_{j,k}| \le 2$, it holds that $|cQ_{j,k}|/|E_{j,k}| \le 2c^n$ and thus $w(cQ_{j,k})/w(E_{j,k}) \lesssim \|w\|_{\widehat{A}_3}^3$ and hence

$$\begin{aligned}
&\left(\frac{1}{|45Q_{j,k}|} \int_{45Q_{j,k}} |f(x)| \right)^2 \int_{Q_{j,k}} h(x) w(x)\, dx \\
\lesssim& \|w\|_{\widehat{A}_3}^3 \left(\frac{1}{|45Q_{j,k}|} \int_{45Q_{j,k}} |f(x)| \right)^2 \left(\frac{1}{w(cQ_{j,k})} \int_{Q_{j,k}} h(x) w(x)\, dx \right) w(E_{j,k}) \\
\lesssim& \|w\|_{\widehat{A}_3}^3 \left(\frac{1}{|45Q_{j,k}|} \int_{45Q_{j,k}} |f(x)| \right)^2 \left(\frac{1}{w(\tilde{Q}_{j,k})} \int_{\tilde{Q}_{j,k}} h(x) w(x)\, dx \right) w(E_{j,k}) \\
\lesssim& \|w\|_{\widehat{A}_3}^3 \int_{E_{j,k}} Mf(x)^2 (M_w^c h)(x) w(x)\, dx,
\end{aligned}$$

where

$$M_w^c(h)(x) = \sup_{Q_x} \frac{1}{w(Q_x)} \int_{Q_x} |h(x)| w(x)\, dx,$$

being Q_x cubes centered at x. Summing in j,k and using that $\{E_{j,k}\}_{j,k}$ are pairwise disjoint, we obtain that

$$\begin{aligned}
&\int_{Q^N} A_{45}^N f(x) h(x) w(x)\, dx \lesssim \|w\|_{\widehat{A}_3}^3 \int_{\mathbb{R}^n} Mf(x)^2 (M_w^c h)(x) w(x)\, dx \\
\lesssim& \|w\|_{\widehat{A}_3}^3 \|(Mf)^2\|_{L^{3/2,\infty}(w)} \|M_w^c h\|_{L^{3,1}(w)} \lesssim \|w\|_{\widehat{A}_3}^5 \|f\|_{L^{3,1}(w)}^2.
\end{aligned}$$

Consequently,

$$\|(A_{45}^N f)^{1/2}\|_{L^{3,\infty}(w;\, Q^N)} \lesssim \|w\|_{\widehat{A}_3}^{5/2} \|f\|_{L^{3,1}(w)},$$

as we wanted to prove. $\square$

As a consequence of our extrapolation results we can conclude that (see [21]):

Corollary 4.2. *For every $w \in A_1$,*

$$G_\alpha : L^1(w) \longrightarrow L^{1,\infty}(w)$$

is bounded.

Proof. The result will follow by Corollary 3.7 as soon as we prove that G_α is (ε, δ)-atomic approximable.

For every $M \in \mathbb{N}$, let

$$\Gamma_\alpha^M(x) = \Big\{(y,t) \in \mathbb{R}_+^{n+1} : \frac{1}{M} \le t \le M,\ |y - x| < \alpha t\Big\},$$

and set

$$G_\alpha^M f(x) = \bigg(\int_{\Gamma_\alpha^M(x)} |A_\alpha f(y,t)|^2 \frac{dydt}{t^{n+1}} \bigg)^{1/2}.$$

Then, G_α^M is (ε, δ)-atomic. To see this, let us take a δ-atom a and let us observe that, for every $\varphi \in \mathcal{C}^\alpha$,

$$|(\varphi_t * a)(y)| = \bigg| \int_Q \varphi_t(y-z) a(z) dz \bigg| = \bigg| \int_Q (\varphi_t(y-z) - \varphi_t(y - y_Q)) a(z) dz \bigg|,$$

where y_Q is the center of the cube Q. Therefore,

$$|(\varphi_t * a)(y)| \le \delta^\alpha \frac{1}{t^{n+\alpha}} \|a\|_1,$$

and hence

$$|A_\alpha f(y,t)| \lesssim \delta^\alpha \frac{1}{t^{n+\alpha}} \|a\|_1,$$

from which it follows that

$$\|G_\alpha^M f\|_{L^1 + L^\infty} \le \|G_\alpha^M f\|_{L^\infty} \lesssim \delta^\alpha \|a\|_1 C_M,$$

and hence, by choosing δ appropriately we obtain (3.1) and the result follows. □

References

[1] I. Assani, Z. Buczolich, and R.D. Mauldin, *An L^1 counting problem in ergodic theory*, J. Anal. Math. **95** (2005), 221–241.

[2] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Grundlehren der Mathematischen Wissenschaften, No. 223. Springer-Verlag, Berlin-New York, 1976.

[3] S. Buckley, *Estimates for operator norms on weighted spaces and reverse Jensen inequalities*, Trans. Amer. Math. Soc. **340** (1993), 253–272.

[4] M. J. Carro, *From restricted weak type to strong type estimates*, J. London Math. Soc. **70** (2004), no. 3, 750–762.

[5] M. J. Carro and J. Martín, *A useful estimate for the decreasing rearrangement of the sum of functions*, Quart. J. Math. **55** (2004), 41–45.

[6] D. Cruz-Uribe, J. M. Martell, and C. Pérez, *Extrapolation from A_∞ weights and applications*, J. Funct. Anal. **213** (2004), no. 2, 412–439.

[7] D. Cruz-Uribe and C. Pérez, *Two weight extrapolation via the maximal operator*, J. Funct. Anal. **174** (2000), 1–17.

[8] O. Dragičević, L. Grafakos, M. C. Pereyra, and S. Petermichl, *Extrapolation and sharp norm estimates for classical operators on weighted Lebesgue spaces*, Publ. Mat. **49** (2005), 73–91.

[9] J. Duoandikoetxea, *Extrapolation of weights revisited: new proofs and sharp bounds*, J. Funct. Anal. **260** (2011), no. 6, 1886–1901.

[10] J. García-Cuerva, *An extrapolation theorem in the theory of A_p weights*, Proc. Amer. Math. Soc. **87** (1983), 422–426.

[11] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Mathematics Studies, Vol. 116, Notas de Matemática [Mathematical Notes], 104. North-Holland Publishing Co., Amsterdam, 1985.

[12] L. Grafakos, *Modern Fourier Analysis*, Second edition. Graduate Texts in Mathematics, 250. Springer, New York, 2009.

[13] P.T. Hytönen, *The sharp weighted bound for general Calderón-Zygmund operators*, Ann. of Math. (2) **175** (2012), no. 3, 1473–1506.

[14] R. Kerman and A. Torchinsky, *Integral inequalities with weights for the Hardy maximal function*, Studia Math. **71** (1982), no. 2, 277–284.

[15] M.T. Lacey and J. Scurry, *Weighted weak type estimates for square functions*, `http://arxiv.org/pdf/1211.4219.pdf`.

[16] A. Lerner, *Sharp weighted norm inequalities for Littlewood-Paley operators and singular integrals*, Adv. Math. **226** (2011), no. 5, 3912–3926.

[17] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.

[18] F. Nazarov, A. Reznikov, V. Vasyunin, and A. Volberg, *A_1 conjecture: weak norm estimates of weighted singular operators and Bellman functions* (2010), available at `http://sashavolberg.files.wordpress.com/2010/11/a11_7loghilb11_21_2010.pdf`.

[19] C. Pérez, S. Treil, and A. Volberg, *On A_2 conjecture and corona decomposition of weights*, 2010, preprint.

[20] J. L. Rubio de Francia, *Factorization theory and A_p weights*, Amer. J. Math. **106** (1984), no. 3, 533–547.

[21] M. Wilson, *The intrinsic square function*, Rev. Mat. Iberoam. **3** (2007), no. 3, 771–791.

M. J. Carro and J. Soria, Department of Applied Mathematics and Analysis, University of Barcelona, 08007 Barcelona, Spain

E-mail address: `carro@ub.edu, soria@ub.edu`

L. Grafakos, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

E-mail address: `grafakosl@missouri.edu`