Instabilities of Linear Evolution PDEs via the Fokas Method

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Dedicated to Professor A. S. Fokas on the occasion of his 70th birthday

Abstract. In this short article, we use the formula provided by the Fokas method for initial-boundary-value problems (ibvp) for the linearised KdV equation on the half-line for positive time. Depending on the sign of the dispersive term, the long range asymptotics can depend in a very sensitive way on the behavior of the data at the point $(0, 0)$. Such instabilities have apparently not been noticed before and they are expected to appear for a large set of equations. As to which equations are unstable and which are not, this is an open question worthy of further investigation.

Keywords: Fokas formula; IBVPs for evolution PDE; half-line linearized KdV equation; long-space asymptotics; conditional decay; instability effects

1 INTRODUCTION

When the Fokas method was introduced by Fokas about 25 years ago [5] (see also [6], [7] and [8]), it was initially conceived as a method for solving initial-boundary value problems for completely integrable nonlinear equations like KdV, NLS, or more generally equations that can be formulated as evolutions in time of a linear differential operator $L(t)$ governed by the famous Lax pair equation $dL/dt = BL - LB$, where $B(L)$ is usually some auxiliary anti-symmetric linear differential operator.

While the study of an initial-value problem for KdV involves the study of the scattering transform for the associated linear Schrödinger operator L, the role of B being somewhat trivialised, the study of the initial – boundary value problem involves the joint study of scattering data for both operators L, B; thus the term "Unified Transform". The interdependence of the two operators renders this new method a highly nontrivial extension of the standard scattering method.

Even though this method was initially proposed for nonlinear problems, it soon became evident that it was also applicable to linear problems. While, before the new method, the existing tools for boundary value problems of linear PDEs (like the Laplace or the sine transform) were explicitly applicable to very specific equations, the new method has been spectacularly successful in a much wider class of problems, of any order, even elliptic [1], even with non-constant coefficients and in all sorts of domains in the (x, t)-plane; see, for instance, [2], [3], [4] and references cited therein. In fact, the linear method even offered some insights to the nonlinear integrability theory by helping to realize that Lax pairs provide the generalization of the divergence formulation from a separable linear to an integrable nonlinear PDE [9]. Not only very explicit formulae for the solutions are provided, but such formulae are very efficient numerically. It is fair to say that the Fokas method has thus rejuvenated the study of linear equations.

In this paper, we focus on one simple consequence of the Fokas theory. We report on the discovery of an instability phenomenon, apparently not noticed before. Let us, for example, consider the very specific initial-boundary value problem for the two linear KdV equations:

$$\begin{cases} \partial_t u + \partial_{xxx} u = 0, & (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ u(x,0) = u_0(x), & x \in \mathbb{R}^+ \\ u(0,t) = g_0(t), & t \in \mathbb{R}^+, \end{cases} \tag{1}$$

and

$$\begin{cases} \partial_t u - \partial_{xxx} u = 0, & (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ u(x,0) = u_0(x), & x \in \mathbb{R}^+ \\ u(0,t) = g_0(t), & t \in \mathbb{R}^+ \\ u_x(0,t) = g_1(t), & t \in \mathbb{R}^+, \end{cases} \tag{2}$$

where the initial and boundary data u_0, g_0 and g_1 are functions defined in $\mathbb{R}^+$ and satisfy appropriate conditions (see Theorem 1 and Section 3).

2 THE EQUATION $\partial_t u + \partial_{xxx} u = 0$

The Fokas formula for the solution of (1) is

$$\begin{aligned} u(x,t) = \frac{1}{2\pi} \int\limits_{\lambda=-\infty}^{\infty} & e^{i\lambda x - \omega(\lambda)t} \hat{u}_0(\lambda) d\lambda \\ & + \frac{1}{2\pi} \int\limits_{\lambda \in \Gamma} e^{i\lambda x - \omega(\lambda)t} [\alpha \hat{u}_0(\alpha\lambda) + \alpha^2 \hat{u}_0(\alpha^2\lambda)] d\lambda \\ & - \frac{1}{2\pi} \int\limits_{\lambda \in \Gamma} e^{i\lambda x - \omega(\lambda)t} 3\lambda^2 \tilde{g}_0(\omega(\lambda), t) d\lambda, \end{aligned} \tag{3}$$

where $\hat{u}_0(\lambda) = \int\limits_{y=0}^{\infty} e^{-i\lambda y} u_0(y) dy$ (defined for $\lambda \in \mathbb{C}$ with $\Im \lambda \leq 0$), $\tilde{g}_0(\omega(\lambda), t) = \int\limits_{\tau=0}^{t} e^{\omega(\lambda)\tau} g_0(\tau) d\tau$ with $\omega(\lambda) = -i\lambda^3$, $\alpha = e^{2\pi i/3}$, and $\Gamma = \partial\Omega^-$ with $\Omega^- = \{\lambda \in \mathbb{C} : \operatorname{Im} \lambda \geq 0 \text{ and } \operatorname{Re} \omega(\lambda) \leq 0\}$.

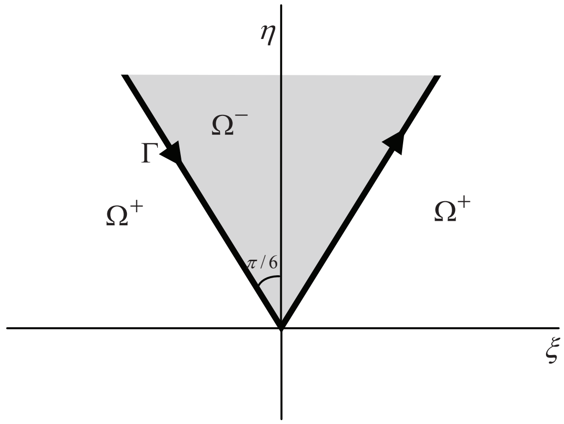

Fig. 1. The contour Γ is the boundary of Ω^-

Theorem 1. (see [3]) *If* $u_0(x) \in \mathcal{S}\big([0,\infty)\big)$ *and* $g_0(t) \in C^\infty\big([0,\infty)\big)$ *then the function* $u(x,t)$*, defined by (3), satisfies the following relation*

$$\lim_{x \to +\infty} \frac{\partial^k u(x,t)}{\partial x^k} = 0 \tag{4}$$

for every nonnegative integer k*, uniformly for* t *in compact subsets of* $(0,\infty)$.

Proof. Firstly, let us fix a $t > 0$. By appropriate deformation of the contours, we have that

$$\begin{aligned}
\frac{1}{i^k}\frac{\partial^k}{\partial x^k}\Bigg[\int_{\lambda=-\infty}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda\Bigg] = \\
\int_{\lambda=-1}^{1} \lambda^k e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda \\
+\Bigg(\int_{\lambda=-\infty}^{-1} + \int_{\lambda=1}^{\infty}\Bigg)\lambda^k e^{i\lambda x-\omega(\lambda)t}[\hat{u}_0(\lambda)-\sigma_N(\lambda)]d\lambda \\
+ \int_{\substack{\operatorname{Im}\lambda=1 \\ -\infty<\operatorname{Re}\lambda\le -1 \text{ or } 1\le \operatorname{Re}\lambda<\infty}} \lambda^k e^{i\lambda x-\omega(\lambda)t}\sigma_N(\lambda)d\lambda \\
+ \int_{\lambda\in[-1+i,-1]\cup[1,1+i]} \lambda^k e^{i\lambda x-\omega(\lambda)t}\sigma_N(\lambda)d\lambda
\end{aligned} \tag{5}$$

provided that $N > k$.

We claim that

$$\lim_{x\to\infty}\frac{\partial^k}{\partial x^k}\left[\int_{\lambda=-\infty}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda\right]=0. \tag{6}$$

For its proof, it suffices to show the following:

$$\lim_{x\to\infty}\int_{\lambda=-1}^{1} \lambda^k e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda=0 \tag{7}$$

$$\lim_{x\to\infty}\left(\int_{\lambda=-\infty}^{-1}+\int_{\lambda=1}^{\infty}\right)\left(\lambda^k e^{i\lambda x-\omega(\lambda)t}[\hat{u}_0(\lambda)-\sigma_N(\lambda)]d\lambda\right)=0 \tag{8}$$

$$\lim_{x\to\infty}\int_{\substack{\operatorname{Im}\lambda=1\\ -\infty<\operatorname{Re}\lambda\le -1 \text{ or } 1\le \operatorname{Re}\lambda<\infty}} \lambda^k e^{i\lambda x-\omega(\lambda)t}\sigma_N(\lambda)d\lambda=0 \tag{9}$$

$$\lim_{x\to\infty}\int_{\lambda\in[-1+i,-1]\cup[1,1+i]} \lambda^k e^{i\lambda x-\omega(\lambda)t}\sigma_N(\lambda)d\lambda=0. \tag{10}$$

Applying the Riemann-Lebesgue lemma to the function

$$\varphi(\lambda)=\begin{cases}\lambda^k e^{-\omega(\lambda)t}\hat{u}_0(\lambda) & \text{for } \quad -1\le\lambda\le 1\\ 0 & \text{for } \lambda\in\mathbb{R}-[-1,1]\end{cases}$$

which is clearly L^1 in $\mathbb{R}$, we obtain (7).

Similarly, (8) follows from the Riemann-Lebesgue lemma applied to the function

$$\Phi(\lambda)=\begin{cases}\lambda^k e^{-\omega(\lambda)t}[\hat{u}_0(\lambda)-\sigma_N(\lambda)] & \text{for } \lambda\in\mathbb{R}-[-1,1]\\ 0 & \text{for } \quad -1\le\lambda\le 1\end{cases}$$

which is also L^1 in $\mathbb{R}$, since $N>k$.

Now, for $\lambda=\xi+i\eta$ with $\eta=1$, $\left|e^{i\lambda x}\right|=e^{-x}$. Therefore, the absolute value of the integral in (9) is

$$\le e^{-x}\int_{\substack{\operatorname{Im}\lambda=1\\ -\infty<\operatorname{Re}\lambda\le -1 \text{ or } 1\le \operatorname{Re}\lambda<\infty}} \left|\lambda^k e^{-\omega(\lambda)t}\sigma_N(\lambda)\right|d|\lambda|,$$

and (9) follows.

Finally, for $\lambda = \xi + i\eta$, $\left|e^{i\lambda x}\right| = e^{-\eta x}$. It follows that if $\lambda = \xi + i\eta \in [-1+i, -1] \cup [1, 1+i]$ and $\lambda \neq \pm 1$ then $\eta > 0$, whence $\lim\limits_{x \to \infty} [\lambda^k e^{i\lambda x - \omega(\lambda)t} \sigma_N(\lambda)] = 0$. Therefore, (10) follows from Lebesgue's dominated convergence theorem.

Also,

$$\lim_{x \to \infty} \int_\Gamma \lambda^k e^{i\lambda x - \omega(\lambda)t} [\alpha \hat{u}_0(\alpha\lambda) + \alpha^2 \hat{u}_0(\alpha^2\lambda)] d\lambda = 0, \tag{11}$$

since the factor $e^{i\lambda x}$ in the above integral has absolute value $e^{-\sqrt{3}x|\lambda|/2}$ when $\lambda \in \Gamma$, and, for $x \geq 1$, the integrand is dominated by $|\lambda|^{k-1} e^{-\sqrt{3}|\lambda|/2}$ – up to a constant – and $\int\limits_\Gamma |\lambda|^{k-1} e^{-\frac{\sqrt{3}}{2}|\lambda|} d|\lambda| < +\infty$.
Similarly,

$$\lim_{x \to \infty} \int_\Gamma \lambda^k e^{i\lambda x - \omega(\lambda)t} 3\lambda^2 \widetilde{g}_0(\omega(\lambda), t) d\lambda = 0. \tag{12}$$

Now, (4) follows from (3), (6), (11) and (12).
Finally, it is easy to see that all the above limits are uniform for t in compact subsets of $(0, \infty)$.

Theorem 2. ([3]) *With the assumptions as in Theorem 1, the function $u(x,t)$, defined by (3), satisfies the following equation*

$$\lim_{x \to +\infty} [x u(x,t)] = 0 \tag{13}$$

uniformly for t in compact subsets of $(0, \infty)$.

Proof. With N sufficiently large, integration by parts gives

$$
\begin{aligned}
& ix \int_{\lambda=-\infty}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda \\
&\quad = \int_{\lambda=-1}^{1} \frac{d}{d\lambda}(e^{i\lambda x})e^{-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda \\
&\qquad + \left(\int_{\lambda=-\infty}^{-1} + \int_{\lambda=1}^{\infty}\right)\left(\frac{d}{d\lambda}(e^{i\lambda x})e^{-\omega(\lambda)t}[\hat{u}_0(\lambda)-\sigma_N(\lambda)]d\lambda\right) \\
&\qquad + \int_{\substack{\mathrm{Im}\,\lambda=1 \\ -\infty<\mathrm{Re}\lambda\le -1 \text{ or } 1\le \mathrm{Re}\,\lambda<\infty}} \frac{d}{d\lambda}(e^{i\lambda x})e^{-\omega(\lambda)t}\sigma_N(\lambda)d\lambda \\
&\qquad + \int_{\lambda\in[-1+i,-1]\cup[1,1+i]} \frac{d}{d\lambda}(e^{i\lambda x})e^{-\omega(\lambda)t}\sigma_N(\lambda)d\lambda \\
&\quad = -\int_{\lambda=-1}^{1} e^{i\lambda x}\frac{d}{d\lambda}[e^{-\omega(\lambda)t}\hat{u}_0(\lambda)]d\lambda \\
&\qquad - \left(\int_{\lambda=-\infty}^{-1} + \int_{\lambda=1}^{\infty}\right)\left(e^{i\lambda x}\frac{d}{d\lambda}\{e^{-\omega(\lambda)t}[\hat{u}_0(\lambda)-\sigma_N(\lambda)]\}d\lambda\right) \\
&\qquad + \int_{\substack{\mathrm{Im}\lambda=1 \\ -\infty<\mathrm{Re}\,\lambda\le -1 \text{ or } 1\le \mathrm{Re}\,\lambda<\infty}} e^{i\lambda x}\frac{d}{d\lambda}[e^{-\omega(\lambda)t}\sigma_N(\lambda)]d\lambda \\
&\qquad - \int_{\lambda\in[-1+i,-1]\cup[1,1+i]} e^{i\lambda x}\frac{d}{d\lambda}[e^{-\omega(\lambda)t}\sigma_N(\lambda)d\lambda],
\end{aligned}
\tag{14}
$$

since the “intermediate evaluations” cancel each other.
Now, the integrals in RHS of (14) tend to zero, as $x \to +\infty$, by the Riemann-Lebesgue lemma, as in the proof of Theorem 3. Therefore, (14) implies that

$$
\lim_{x\to\infty}\left(x\int_{\lambda=-\infty}^{\infty} e^{i\lambda x-\omega(\lambda)t}\hat{u}_0(\lambda)d\lambda\right) = 0. \tag{15}
$$

Next, by the presence of the factor $e^{i\lambda x}$ and the fact that integration is taken on $\Gamma \cap \{|\lambda| \geq 1\}$,

$$\lim_{x \to \infty} \left[x \int_{\Gamma \cap \{|\lambda| \geq 1\}} e^{i\lambda x - \omega(\lambda) t} [\alpha \hat{u}_0(\alpha\lambda) + \alpha^2 \hat{u}_0(\alpha^2 \lambda)] d\lambda \right] = 0,$$

$$\lim_{x \to \infty} \left[x \int_{\Gamma \cap \{|\lambda| \geq 1\}} e^{i\lambda x - \omega(\lambda) t} 3\lambda^2 \widetilde{g}_0(\omega(\lambda), t) d\lambda \right] = 0. \tag{16}$$

On the other hand, writing $x e^{i\lambda x} = d(e^{i\lambda x})/i d\lambda$ and integrating by parts, we obtain

$$\lim_{x \to \infty} \left[x \int_{\Gamma \cap \{|\lambda| \leq 1\}} e^{i\lambda x - \omega(\lambda) t} [\alpha \hat{u}_0(\alpha\lambda) + \alpha^2 \hat{u}_0(\alpha^2 \lambda)] d\lambda \right] = 0,$$

$$\lim_{x \to \infty} \left[x \int_{\Gamma \cap \{|\lambda| \leq 1\}} e^{i\lambda x - \omega(\lambda) t} 3\lambda^2 \widetilde{g}_0(\omega(\lambda), t) d\lambda \right] = 0, \tag{17}$$

Now, (13) follows from (15), (16) and (17).

Examining the proofs of the previous two theorems, we easily see that we can prove the following more general theorem.

Theorem 3. ([3]) *With the assumptions as in Theorem 1, the function $u(x,t)$, defined by (3), satisfies the following equation:*

$$\lim_{x \to +\infty} \left(x^{\ell} \frac{\partial^k u(x,t)}{\partial x^k} \right) = 0$$

for nonnegative integers k and ℓ, uniformly for t in compact subsets of $(0, \infty)$.

3 THE EQUATION $\partial_t u - \partial_{xxx} u = 0$

Assuming that $u_0(x) \in \mathcal{S}\big([0,\infty)\big)$ and $g_0(t), g_1(t) \in C^\infty\big([0,\infty)\big)$, the Fokas solution for (2) is

$$\begin{aligned} u(x,t) =& \frac{1}{2\pi} \Bigg[\int_{\lambda=-\infty}^{\infty} e^{i\lambda x - \omega(\lambda) t} \hat{u}_0(\lambda) d\lambda - \int_{\partial\Omega_1^-} e^{i\lambda x - \omega(\lambda) t} \hat{u}_0(\alpha\lambda) d\lambda - \int_{\partial\Omega_2^-} e^{i\lambda x - \omega(\lambda) t} \hat{u}_0(\alpha^2\lambda) d\lambda \Bigg] \\ &+ \frac{1}{2\pi} \Bigg[\int_{\partial\Omega_1^-} e^{i\lambda x - \omega(\lambda) t} (1-\alpha^2) \lambda^2 \widetilde{g}_0(\omega(\lambda), t) d\lambda + \int_{\partial\Omega_2^-} e^{i\lambda x - \omega(\lambda) t} (1-\alpha) \lambda^2 \widetilde{g}_0(\omega(\lambda), t) d\lambda \Bigg] \\ &- \frac{i}{2\pi} \Bigg[\int_{\partial\Omega_1^-} e^{i\lambda x - \omega(\lambda) t} (1-\alpha) \lambda \widetilde{g}_1(\omega(\lambda), t)] d\lambda + \int_{\partial\Omega_2^-} e^{i\lambda x - \omega(\lambda) t} (1-\alpha^2) \lambda \widetilde{g}_1(\omega(\lambda), t)] d\lambda \Bigg], \end{aligned} \tag{18}$$

for $x > 0$ and $t > 0$, where $\omega(\lambda) = i\lambda^3$, $\alpha = e^{2\pi i/3}$,

$$\begin{aligned}\Omega_1^- &= \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda \geq 0,\ \operatorname{Re}\lambda \leq 0, \text{and } \operatorname{Re}\omega(\lambda) \leq 0\} \\ &= \{\lambda \in \mathbb{C} : (2\pi/3) \leq \arg\lambda \leq \pi\},\end{aligned}$$

and

$$\begin{aligned}\Omega_2^- &= \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda \geq 0,\ \operatorname{Re}\lambda \geq 0, \text{and } \operatorname{Re}\omega(\lambda) \leq 0\} \\ &= \{\lambda \in \mathbb{C} : 0 \leq \arg\lambda \leq \pi/3\}.\end{aligned}$$

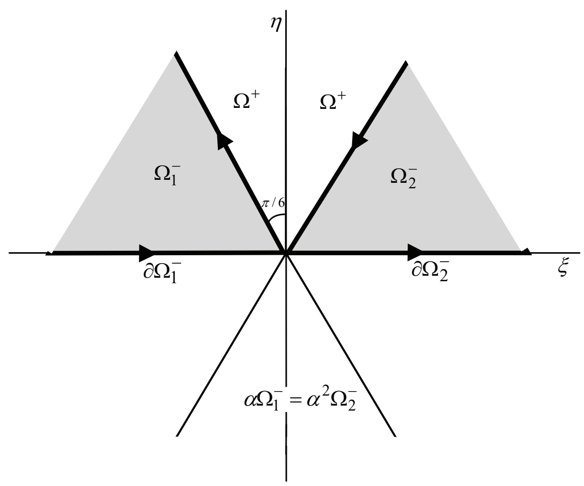

Fig. 2. The sets Ω_1^- and Ω_2^-, and their boundaries.

For the function $u(x,t)$, defined by (18), the following theorems hold. (Detailed proofs will appear in [2].)

Theorem 4. ([2]) *With fixed $t_1 > t_0 > 0$, the solution $u(x,t)$, given by (18), satisfies the following:*

As $x \longrightarrow +\infty$ and uniformly for $t_0 \leq t \leq t_1$,

1st $u(x,t) = [u_0(0) - g_0(0)]\dfrac{\sqrt{3}\sqrt[4]{3}}{\sqrt{\pi}}\dfrac{t^{1/4}}{x^{3/4}}\sin\left(\dfrac{2}{3\sqrt{3}}\dfrac{x^{3/2}}{\sqrt{t}} - \dfrac{5\pi}{12}\right) + O(1/x)$,

2nd $\dfrac{\partial u(x,t)}{\partial x} = [u_0(0) - g_0(0)]\dfrac{2\sqrt{3}\sqrt[4]{3}}{\sqrt{\pi}}\dfrac{1}{t^{1/4}x^{1/4}}\cos\left(\dfrac{2}{3\sqrt{3}}\dfrac{x^{3/2}}{\sqrt{t}} - \dfrac{5\pi}{12}\right)$

$$+[u_0'(0)-g_1(0)]\frac{\sqrt{3}\sqrt[4]{3}}{\sqrt{\pi}}\frac{t^{1/4}}{x^{3/4}}\sin\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}}-\frac{\pi}{12}\right)+O(1/x),$$

3rd $$\frac{\partial^2 u(x,t)}{\partial x^2}=-[u_0(0)-g_0(0)]\frac{2\sqrt[4]{3}}{\sqrt{\pi}}\frac{x^{1/4}}{t^{3/4}}\sin\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}}-\frac{5\pi}{12}\right)$$

$$+[u_0'(0)-g_1(0)]\frac{2\sqrt{3}\sqrt[4]{3}}{\sqrt{\pi}}\frac{1}{t^{1/4}x^{1/4}}\cos\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}}-\frac{\pi}{12}\right)+O(1/x),$$

4th $$\frac{\partial^3 u(x,t)}{\partial x^3}=-[u_0(0)-g_0(0)]\frac{2}{\sqrt[4]{3}\sqrt{\pi}}\frac{x^{3/4}}{t^{5/4}}\cos\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}}-\frac{5\pi}{12}\right)$$

$$+[u_0'(0)-g_1(0)]\frac{2\sqrt{3}\sqrt[4]{3}}{\sqrt{\pi}}\frac{x^{1/4}}{t^{3/4}}\sin\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}}-\frac{\pi}{12}\right)$$

$$+[u_0'''(0)-g_0'(0)]\frac{\sqrt{3}\sqrt[4]{3}}{\sqrt{\pi}}\frac{t^{1/4}}{x^{3/4}}\sin\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}}-\frac{5\pi}{12}\right)+O(1/x).$$

Theorem 5. ([2]) *With the notation and in the sense of Theorem 4, we have:*
1st If $u_0(0)=g_0(0)$ *and* $u_0'(0)=g_1(0)$, *then*

$$\frac{\partial^3 u(x,t)}{\partial x^3}=[u_0'''(0)-g_0'(0)]\frac{\sqrt{3}\sqrt[4]{3}}{\sqrt{\pi}}\frac{t^{1/4}}{x^{3/4}}\sin\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}}-\frac{5\pi}{12}\right)+O(1/x).$$

2nd If $\lim_{x\to\infty}\frac{\partial^2 u(x,t)}{\partial x^2}$ *exists for some* $t>0$, *then* $u_0(0)=g_0(0)$. *Conversely, if* $u_0(0)=g_0(0)$ *then*

$$\frac{\partial^2 u(x,t)}{\partial x^2}=[u_0'(0)-g_1(0)]\frac{2\sqrt{3}\sqrt[4]{3}}{\sqrt{\pi}}\frac{1}{t^{1/4}x^{1/4}}\cos\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}}-\frac{\pi}{12}\right)+O(1/x),$$

uniformly for t in compact subsets of $(0,+\infty)$.
3rd For $n\geq 4$,

$$\frac{\partial^n u(x,t)}{\partial x^n}=[u_0(0)-g_0(0)]\frac{2}{3^{(2n-3)/4}\sqrt{\pi}}\frac{x^{(2n-3)/4}}{t^{(2n-1)/4}}\cdot$$
$$\cdot\,\mathrm{Re}\left\{i^{n-1}\exp\left[i\left(\frac{2}{3\sqrt{3}}\frac{x^{3/2}}{\sqrt{t}}-\frac{5\pi}{12}\right)\right]\right\}+O(x^{(2n-5)/4}).$$

4th Let $k\in\mathbb{N}$. *If the limit* $\lim_{x\to\infty}\frac{\partial^{4k-1}u(x,t)}{\partial x^{4k-1}}$ *exists for some* $t>0$, *then*

$$u_0^{(3\ell-3)}(0)=g_0^{(\ell-1)}(0)\ \text{ and }\ u_0^{(3\ell-2)}(0)=g_1^{(\ell-1)}(0),\ \text{ for }\ \ell=1,2,\ldots,k. \quad (19)$$

Conversely, (19) implies that

$$\lim_{x\to\infty}\frac{\partial^n u(x,t)}{\partial x^n}=0,$$

uniformly for t in compact subsets of $(0,+\infty)$, *for* $n=0,1,2,\ldots,4k$.

Comment: The above theorems show that the behavior of the solution, for large x, depends in a very sensitive way on the given data at the point $(x, t) = (0, 0)$.

4 CONCLUSION

Given the enormous power of today's computers, the role of PDE theory is partly relegated to the qualitative study of solutions, with particular attention to instabilities. An interesting consequence of the spectacularly successful Fokas theory for the solution of initial-boundary value problems for linear PDEs is the observation of instabilities. For some (not all) equations, the behavior of the solution, for large x, depends in a very sensitive way on the compatibility conditions at the point $(x, t) = (0, 0)$. Apparently this is a phenomenon not observed before. In the nonlinear case, the stable/unstable dichotomy is apparent mostly in the zero dispersion (semiclassical) limit and is related to the self-adjoint/non-self-adjoint dichotomy for the associated (spatial) Lax operator ([10]). It would be interesting to study what kind of linear evolution equations exhibit long-range instabilities and which equations do not. Also, it will be interesting to study whether there is a similar effect on long-time asymptotics. Such investigations for large values of spatial and temporal variables, for a variety of dispersive equations, are in progress and results will appear in subsequent publications. For the particular case of the linearized NLS with t-periodic boundary data, we refer to [11] where the large-t behavior of the solution is considered.

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