MAXIMAL FUNCTIONS ASSOCIATED WITH FOURIER MULTIPLIERS OF MIKHLIN-HÖRMANDER TYPE

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Abstract. We show that maximal operators formed by dilations of Mikhlin-Hörmander multipliers are typically not bounded on $L^p(\mathbb{R}^d)$. We also give rather weak conditions in terms of the decay of such multipliers under which L^p boundedness of the maximal operators holds.

1. Introduction

For a bounded Fourier multiplier m on $\mathbb{R}^d$ and a Schwartz function f in $\mathcal{S}(\mathbb{R}^d)$ define the maximal function associated with m by

$$\mathcal{M}_m f(x) = \sup_{t>0} \big|\mathcal{F}^{-1}[m(t\cdot)\widehat{f}\,](x)\big|.$$

We are interested in the class of multipliers that satisfy the estimates of the standard Mikhlin-Hörmander multiplier theorem

$$|\partial^\alpha m(\xi)| \le C_\alpha |\xi|^{-\alpha} \tag{1.1}$$

for all (or sufficiently large) multiindices α. More precisely, let L^r_γ be the standard Bessel-potential (or Sobolev) space with norm

$$\|f\|_{L^r_\gamma} = \|(I-\Delta)^{\gamma/2} f\|_r;$$

here we include the case $r=1$. Let ϕ be a smooth function supported in $\{\xi : 1/2 < |\xi| < 2\}$ which is nonvanishing on $\{\xi : 1/\sqrt{2} \le |\xi| \le \sqrt{2}\}$. Then one imposes conditions on m of the form

$$\sup_{k\in\mathbb{Z}} \|\phi m(2^k\cdot)\|_{L^r_\gamma} < \infty. \tag{1.2}$$

The function m is a Fourier multiplier on all L^p, $1<p<\infty$ if (1.2) holds for $\gamma > d/r$, with $1 \le r \le 2$ and the condition for $r=2$ is the least restrictive one (see [7]). Concerning the maximal operator Dappa and Trebels [4] showed using Calderón-Zygmund theory that if $\mathcal{M}_m$ is a priori bounded on some L^q, $q>1$ and if (1.2) holds for $r=1$, $\gamma>d$, then $\mathcal{M}_m$ is of weak type $(1,1)$ and thus bounded on L^p for $1<p<q$. Using square function estimates, the L^2 boundedness of $\mathcal{M}_m$ has been shown in [2], [4]

Date: March 7, 2004.

Christ, Grafakos and Seeger were supported in part by NSF grants. Honzík was supported by 201/03/0931 Grant Agency of the Czech Republic.

under certain additional decay assumptions (*cf.* also [9]). For instance, it follows from [4] that

$$\|\mathcal{M}_m f\|_p \le C_p \Big(\sum_{k\in\mathbb{Z}} \|\phi m(2^k\cdot)\|_X^2 \Big)^{1/2} \|f\|_p, \tag{1.3}$$

with $X = L^p_{d/p+\epsilon}$ for $1 < p \le 2$, and with $X = L^2_{d/2+\epsilon}$, for $2 \le p < \infty$. Further results in terms of weaker differentiability assumptions are in [2], [4], especially for classes of radial multipliers. Moreover, if m is homogeneous of degree 0 then trivially $|\mathcal{M}_m f| = |\mathcal{F}^{-1}[m\widehat{f}\,]|$; this observation can be used to build more general classes of symbols without decay assumptions for which $\mathcal{M}_m$ is L^p bounded.

A problem left open in [4] is whether the Mikhlin-Hörmander type assumption in (1.1) or (1.2) alone is sufficient to prove boundedness of the maximal operator $\mathcal{M}_m$. We show here that some additional assumption is needed; indeed this applies already to the dyadic maximal function associated with m, defined by

$$M_m f = \sup_{k\in\mathbb{Z}} \big|\mathcal{F}^{-1}[m(2^k\cdot)\widehat{f}\,]\big|, \tag{1.4}$$

which of course is dominated by $\mathcal{M}_m f$.

Example. *Let $\{v(l)\}_{l=0}^\infty$ be a positive increasing and unbounded sequence. Then there is a Fourier multiplier m satisfying*

$$\sup_\xi \big|\partial_\xi^\alpha\big(\phi(\xi)m(2^k\xi)\big)\big| \le C_\alpha \frac{v(|k|)}{\sqrt{\log(|k|+2)}}, \quad k \in \mathbb{Z}, \tag{1.5}$$

with $C_\alpha < \infty$ for all multiindices α, so that the associated dyadic maximal operator M_m is unbounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$.

This counterexample will be explicitly constructed in §2. Taking $v(l) = \sqrt{\log(l+2)}$ we see that there exists m satisfying (1.1), so that M_m, and hence $\mathcal{M}_m$, are unbounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$. In view of these examples it is not unexpected that unboundedness of M_m holds in fact for the *typical* multiplier satisfying (1.1), *i.e.* on a residual set in the sense of Baire category. In order to formulate a result let $\mathfrak{S}$ be the space of functions $m \in C^\infty(\mathbb{R}^d \setminus \{0\})$ satisfying (1.1) with $C_\alpha < \infty$ for all multiindices α. It is easy to see that $\mathfrak{S}$ is a Fréchet-space with the topology given by the countable family of norms

$$\|m\|_{(j)} = \sup_{|\alpha|\le j} \sup_{\xi\in\mathbb{R}^d} |\xi|^{|\alpha|} |\partial_\xi^\alpha m(\xi)|. \tag{1.6}$$

Let $\mathcal{S}_0$ denote the space of Schwartz functions whose Fourier transform have compact support in $\mathbb{R}^d \setminus \{0\}$ and let $\mathfrak{S}^M$ be the space of all $m \in \mathfrak{S}$ for which

$$\sup\{\|M_m f\|_p : f \in \mathcal{S}_0, \|f\|_p \le 1\}$$

is finite for some $p \in (1,\infty)$. Thus $m \in \mathfrak{S}^M$ if and only if the linear operator $f \mapsto \{\mathcal{F}^{-1}[m(t\cdot)\widehat{f}\,]\}_{t>0}$ extends to a bounded operator from L^p to $L^p(L^\infty)$

for some $p \in (1,\infty)$; in other words $m \in \mathfrak{S}^M$ if and only if M_m extends to a bounded operator on $L^p(\mathbb{R}^d)$, for some $p \in (1,\infty)$.

Theorem 1.1. *$\mathfrak{S}^M$ is of first category in $\mathfrak{S}$, in the sense of Baire.*

In terms of positive results we note that there is a significant gap between the known conditions in (1.3) and the weak decay (1.5). Assuming $\|\phi m(2^k\cdot)\|_{L^1_{d+\epsilon}} = O(|k|^{-\alpha})$, then (1.3) yields L^p boundedness for $1 < p < \infty$ only when $\alpha > 1/2$. We shall see that this result remains in fact valid under the weaker assumption

$$\|\phi m(2^k\cdot)\|_{L^1_{d+\epsilon}} \lesssim (\log(|k|+2))^{-1-\epsilon}. \tag{1.7}$$

In what follows we shall mainly aim for minimal decay but will also try to formulate reasonable smoothness assumptions.

To formulate a general result we recall the definition of the nonincreasing rearrangement of a sequence ω, defined for $t \ge 0$ by

$$\omega^*(t) = \sup\big\{\lambda > 0 : \mathrm{card}\big(\{k : |\omega(k)| > \lambda\}\big) > t\big\};$$

note that $\omega^*(0) = \sup_k |\omega(k)|$ and ω^* is constant on the intervals $[n, n+1)$, $n = 0, 1, 2, \dots$.

Theorem 1.2. *Let $1 < p < \infty$, $1/p + 1/p' = 1$ and let $\omega : \mathbb{Z} \to [0,\infty)$ satisfy*

$$\omega^*(0) + \sum_{l=1}^{\infty} \frac{\omega^*(l)}{l} < \infty. \tag{1.8}$$

(i) Suppose that for some $\alpha > d/p$ we have

$$\Big(\int_{\mathbb{R}^d} \big|\mathcal{F}^{-1}[\phi m(2^k\cdot)]\big|^{p'} (1+|x|)^{\alpha p'} dx\Big)^{1/p'} \le \omega(k), \quad k \in \mathbb{Z}, \tag{1.9}$$

then $\mathcal{M}_m$ is bounded on $L^p(\mathbb{R}^d)$.

(ii) If (1.9) holds for $p' = 1$ then $\mathcal{M}_m$ maps L^∞ to BMO.

(iii) If for some $\varepsilon > 0$

$$\sup_x (1+|x|)^{d+\epsilon} \big|\mathcal{F}^{-1}[\phi m(2^k\cdot)](x)\big| \le \omega(k), \quad k \in \mathbb{Z}. \tag{1.10}$$

Then $\mathcal{M}_m$ is of weak type $(1,1)$, and $\mathcal{M}_m$ maps H^1 to L^1.

By the Hausdorff-Young inequality for $p \le 2$ one deduces

Corollary 1.3. *Suppose $1 < p < \infty$, $r = \min\{p, 2\}$, and $\alpha > d/r$. Suppose that*

$$\|\phi m(2^k\cdot)\|_{L^r_\alpha} \le \omega(k), \quad k \in \mathbb{Z}, \tag{1.11}$$

where ω satisfies (1.8). Then $\mathcal{M}_m$ is bounded on $L^p(\mathbb{R}^d)$.

In particular we conclude that the condition

$$\Big(\sum_{k\in\mathbb{Z}} \|\phi m(2^k\cdot)\|^q_{L^r_\alpha}\Big)^{1/q} < \infty \tag{1.12}$$

with α, r as in the corollary, implies L^p boundedness. Indeed (1.12) implies that $\omega^*(l) = O(l^{-1/q})$ as $l \to \infty$. Of course L^p boundedness also holds if $\omega^*(l) \lesssim (\log(2+l))^{-1-\varepsilon}$ etc. which covers condition (1.7).

Finally we state a more elementary but closely related result about maximal functions for a finite number of Hörmander-Mikhlin type multipliers m_ν, with no decay assumptions and not necessarily generated by dilating a single multiplier.

Theorem 1.4. *Let $1 < p < \infty$, $1/p + 1/p' = 1$ and let $\{m_\nu\}_{\nu\ge 1}$ be a sequence of multipliers and define a maximal operator by*

$$\mathfrak{M}_n f(x) = \sup_{1\le\nu\le n} \big|\mathcal{F}^{-1}[m_\nu \widehat{f}](x)\big|.$$

Suppose that

$$\sup_\nu \sup_{k\in\mathbb{Z}} \Big(\int_{\mathbb{R}^d} \big|\mathcal{F}^{-1}[\phi m_\nu(2^k\cdot)]\big|^{p'} (1+|x|)^{\alpha p'} dx\Big)^{1/p'} \le A \tag{1.13}$$

for some $\alpha > d/p$. *Then for $f \in L^p(\mathbb{R}^d)$*

$$\|\mathfrak{M}_n f\|_p \le C_p A \log(n+1) \|f\|_p. \tag{1.14}$$

Again if the above assumptions hold for $p = 1$ then a weak type (1,1) inequality and an $H^1 \to L^1$ inequality hold, and if $p = \infty$ we have an $L^\infty \to BMO$ inequality, all with constant $O(\log(n+1))$.

Structure of the paper. In §2 we shall provide the above mentioned examples for unboundedness and prove Theorem 1.1. A tiling lemma for finite sets of integers and other preliminaries needed in the proof of Theorem 1.2 are provided in §3. §4 contains the main relevant estimates for multipliers supported in a finite union of annuli. In §5 we conclude the proof of Theorem 1.2 and in §6 we give the proof of Theorem 1.4. Finally we state some extensions and open problems.

2. Unboundedness of the maximal operator

We shall explicitly construct an example satisfying (1.5) and then use our example to prove Theorem 1.1.

Define $S = \{1, -1, i, -i\}$ and let S^N be the set of sequences of length N on S. Enumerate the 4^N elements in S^N by $\{s_\kappa\}_{\kappa=1}^{4^N}$. Let Φ be a smooth function supported in $3/4 \le |\xi| \le 5/4$, so that $\Phi(\xi) = 1$ whenever $7/8 \le |\xi| \le 9/8$. We let

$$m_N(\xi) := \sum_{\kappa=1}^{4^N} \sum_{j=1}^{N} s_\kappa(j) \Phi(2^{-N\kappa - j}\xi)$$

which is supported in $\{\xi : 1/2 \le |\xi| \le 2^{N4^N+N+1}\}$, and define m by

$$m(\xi) = \sum_{N=1}^{\infty} N^{-1/2} v(4^N) m_N(2^{-N8^N}\xi). \tag{2.1}$$

One observes that the terms in this sum have disjoint supports and that m satisfies condition (1.5).

Fix $1 < p < \infty$. We will test the maximal operator M_m on functions $f_{N,p}$ defined as follows. Pick a Schwartz function Ψ such that $\|\Psi\|_p = 1$ and so that supp $\widehat{\Psi}$ is contained in the ball $|\xi| \le 1/8$. For $1 \le j \le N$ define

$$g_N(x) = \sum_{j=1}^{N} e^{2\pi i 2^j x_1} \Psi(x),$$

and set

$$f_{N,p}(x) = N^{-1/2} 2^{dN8^N/p} g_N(2^{N8^N} x). \tag{2.2}$$

Then $\widehat{g_N}(\xi) = \sum_{j=1}^{N} \widehat{\Psi}(\xi - 2^j e_1)$ and, by Littlewood-Paley theory,

$$\|g_N\|_p \le c_p N^{1/2}, \quad 1 < p < \infty.$$

Thus

$$\|f_{N,p}\|_p \le c_p < \infty, \quad 1 < p < \infty,$$

uniformly in N.

The main observation is

$$\Big\| \sup_{1 \le k \le N4^N} \Big| \mathcal{F}^{-1}[m_N(2^k \cdot)\widehat{g_N}] \Big| \Big\|_p \ge CN. \tag{2.3}$$

Given (2.3) we quickly derive the asserted unboundedness of M_m. Namely, by the support properties of the m_n it follows that

$$m_n(2^{k-n8^n}\xi)\widehat{g_N}(2^{-N8^N}\xi) = 0 \quad \text{ if } N \neq n, \quad 1 \le k \le N4^N.$$

Thus, setting $a_n = n^{-1/2} v(4^n)$, we obtain

$$\begin{aligned} M_m f_{N,p}(x) &\ge \sup_{1 \le k \le N4^N} \Big| \sum_{n=1}^{\infty} a_n \mathcal{F}^{-1}[m_n(2^{k-n8^n} \cdot)\widehat{f_{N,p}}](x) \Big| \\ &\ge \sup_{1 \le k \le N4^N} a_N N^{-\frac{1}{2}} \big| \mathcal{F}^{-1}[m_N(2^{k-N8^N} \cdot) 2^{-\frac{dN8^N}{p'}} \widehat{g_N}(2^{-N8^N} \cdot)](x) \big| \\ &= a_N N^{-\frac{1}{2}} 2^{\frac{dN8^N}{p}} \sup_{1 \le k \le N4^N} \big| \mathcal{F}^{-1}[m_N(2^k \cdot)\widehat{g_N}](2^{N8^N} x) \big|. \end{aligned}$$

Taking $L^p(\mathbb{R}^d)$ norms and using (2.3) we conclude that

$$\|M_m f_{N,p}\|_p \ge C a_N N^{1/2} = C v(4^N). \tag{2.4}$$

By the assumed unboundedness of the increasing sequence v it follows that M_m is not bounded on L^p.

Proof of (2.3). For any complex number z the quantity $\sup_{c \in S} \operatorname{Re}(cz)$ is at least $|z|/\sqrt{2}$. Thus for $x \in \mathbb{R}^d$ and $1 \le j \le N$ we may pick $c_j(x) \in S$ such that

$$\operatorname{Re}\big(c_j(x) e^{2\pi i 2^j x_1} \Psi(x)\big) \ge |\Psi(x)|/\sqrt{2}. \tag{2.5}$$

We can find κ_x in $\{1,\dots,4^N\}$ such that

$$c_j(x) = s_{\kappa_x}(j), \quad j = 1,\dots,N.$$

Taking $k = \kappa_x N$ we obtain

$$\sup_{1\le k\le N4^N} \big|\mathcal{F}^{-1}[m_N(2^k\cdot)\widehat{g_N}](x)\big|$$
$$\ge \operatorname{Re}\int \sum_{l=1}^{4^N}\sum_{\nu=1}^{N} s_l(\nu)\Phi(2^{-Nl-\nu}2^{N\kappa_x}\xi)\sum_{j=1}^{N}\widehat{\Psi}(\xi-2^je_1)e^{2\pi i\langle x,\xi\rangle}d\xi.$$

Since $1\le j\le N$, the supports of $\Phi(2^{-Nl-\nu}2^{N\kappa_x}\xi)$ and $\widehat{\Psi}(\xi-2^je_1)$ intersect only when $l=\kappa_x$ and $j=\nu$. In this case $\Phi(2^{-Nl-\nu}2^{N\kappa_x}\xi)=\Phi(2^{-j}\xi)$ is equal to 1 on the support of $\widehat{\Psi}(\xi-2^j)$. Therefore we obtain from (2.5) the pointwise estimate

$$\sup_{1\le k\le N4^N} \big|\mathcal{F}^{-1}[m_N(2^k\cdot)\widehat{g_N}](x)\big|$$
$$\ge \sum_{j=1}^{N}\operatorname{Re}\Big(s_{\kappa_x}(j)\mathcal{F}^{-1}[\widehat{\Psi}(\cdot-2^j)](x)\Big) \ge N|\Psi(x)|/\sqrt{2}\,.$$

Taking L^p norms yields (2.3). ☐

Proof of Theorem 1.1. The space $\mathfrak{S}$ is a complete metric space and the metric is given by

$$d(m_1,m_2)=\sum_{j=0}^{\infty}2^{-j}\frac{\|m_1-m_2\|_{(j)}}{1+\|m_1-m_2\|_{(j)}}$$

where $\|\cdot\|_{(j)}$ is defined in (1.6).

Let $f_{N,r'}$ be as in (2.2) (with $p=r'$) and for integers r,n,N, all ≥ 2, consider the set

$$\mathfrak{S}(r,n,N)=\{m\in\mathfrak{S}: \|M_mf_{N,r'}\|_{r'}\le n\},$$

here $r'=r/(r-1)$, and the set

$$\mathfrak{S}(r,n)=\bigcap_{N=2}^{\infty}\mathfrak{S}(r,n,N).$$

We shall show that $\mathfrak{S}(r,n)$ is closed in $\mathfrak{S}$, and nowhere dense. We also observe that

$$\mathfrak{S}^M\subset\bigcup_{r=2}^{\infty}\bigcup_{n=2}^{\infty}\mathfrak{S}(r,n); \tag{2.6}$$

thus $\mathfrak{S}^M$ is of first category. To see (2.6) assume that M_m is bounded on L^{p_0}, for some $p_0>1$. By the theorem by Dappa and Trebels mentioned before (*cf.* Proposition 3.2 below) it follows that M_m is bounded on L^p for $1<p<p_0$, in particular bounded on $L^{r'}$ for some integer $r\ge 2$. We

note that $f_{N,r'} \in \mathcal{S}_0$ is such that $\|f_{N,r'}\|_{r'} \le C_r$, independently of N. Thus $m \in \mathfrak{S}(r,n)$ for sufficiently large n.

Next, in order to show that the sets $\mathfrak{S}(r,n)$ are closed it suffices to show that the sets $\mathfrak{S}(r,n,N)$ are closed for all $N \ge 2$. For integers $l_1 \le l_2$ denote by $\mathcal{S}(l_1,l_2)$ the class of Schwartz functions whose Fourier transform is supported in the annulus $\{\xi : 2^{l_1-1} \le |\xi| \le 2^{l_2+1}\}$. We observe the following inequality

$$\|M_m f\|_p \le C(p)\|m\|_{(d+1)}(1+|l_2-l_1|)\|f\|_p, \quad \text{if } f \in \mathcal{S}(l_1,l_2),$$

which (in view of the dependence on l_1, l_2) can be obtained by standard techniques, see e.g. [4] or [9]. Note that every $f_{N,r'}$ is in some class $\mathcal{S}(l_1,l_2)$ with $l_2 - l_1 \le N$. Now, if $m_\nu \in \mathfrak{S}(r,n,N)$ and $\lim_{\nu\to\infty} d(m_\nu,m) = 0$ then

$$\|M_m f_{N,r'}\|_{r'} \le n + \|M_{m-m_\nu} f_{N,r'}\|_{r'} \le n + C(r')\|m-m_\nu\|_{(d+1)}\|f_{N,r'}\|_{r'}$$

and since $\|m-m_\nu\|_{(d+1)} \to 0$ we see that $m \in \mathfrak{S}(r,n,N)$. Thus $\mathfrak{S}(r,n,N)$ is closed.

Finally we need to show that $\mathfrak{S}(r,n)$ is nowhere dense in $\mathfrak{S}$; since this set is closed we need to show that it does not contain any open balls. Now if $g \in \mathfrak{S}(r,n)$ then consider the sequence $g_\nu = g + 2^{-\nu}m$ where m is as in (2.1). Clearly $d(g_\nu, g) \to 0$. However by (2.4) we have that $g_\nu \notin \mathfrak{S}(r,n,N)$ for sufficiently large N and thus $g_\nu \notin \mathfrak{S}(r,n)$ for any r, n. Thus $\mathfrak{S}(r,n)$ is nowhere dense. □

3. Preliminaries.

A tiling lemma. In §5 below we shall decompose the multiplier into pieces with compact but large support. In order to effectively estimate the maximal function associated to these pieces we shall use the following "tiling" lemma for integers.

Lemma 3.1. *Let $N > 0$ and let E be a set of integers of cardinality $\le 2^N$. Then we can find a set $B = \{b_i\}_{i\in\mathbb{Z}}$ of integers, such that*
(i) the sets $b_i + E$ are pairwise disjoint,
(ii) $b_i \in [i4^{N+1}, (i+1)4^{N+1})$, and
(iii) $\mathbb{Z} = \cup_{n=-4^{N+1}}^{4^{N+1}} (n+B)$.

Proof: Clearly (iii) is an immediate consequence of (ii). We enumerate the set $E = \{e_\nu\}_{\nu=1}^{2^N}$.

We set $b_0 = 0$, and construct b_j, b_{-j} for $j > 0$ by induction. Assume that $b_i \in [i4^{N+1}, (i+1)4^{N+1})$ has been constructed for $-j < i < j$ so that the sets $b_i + E$ are pairwise disjoint.

For $\nu = 1, \dots, 2^N$ we denote by C_ν^j the subset of all integers c in $[j4^{N+1}, (j+1)4^{N+1})$ with the property that $e_\nu + c \in \cup_{i=1-j}^{j-1}(b_i + E)$.

We shall verify

$$\text{card}\big(\cup_{\nu=1}^{2^N} C_\nu^j\big) \le 2^{2N+1} < 4^{N+1}. \tag{3.1}$$

Given (3.1) we may simply take

$$b_j \in [j4^{N+1}, (j+1)4^{N+1}) \setminus \big(\cup_{\nu=1}^{2^N} C_\nu^j \big)$$

and by construction the sets $b_{1-j}+E, \dots, b_j+E$ are disjoint.

In order to verify (3.1) observe that $e_\nu + c \in [j4^{N+1}+e_\nu, (j+1)4^{N+1}+e_\nu)$ if $c \in C_\nu^j$. Thus

$$\mathrm{card}(C_\nu^j) = \mathrm{card}\big([j4^{N+1}+e_\nu, (j+1)4^{N+1}+e_\nu\big) \cap \cup_{i=1-j}^{j-1}(b_i+E)\big).$$

Since by the induction assumption $b_{i+2}-b_i > 4^{N+1}$, if $i \ge 1-j$, $i+2 \le j-1$, this gives

$$\mathrm{card}\big([j4^{N+1}+e_\nu, (j+1)4^{N+1}+e_\nu) \cap \cup_{i=1-j}^{j-1}(b_i+\{e_\nu\})\big) \le 2$$

for all ν. This means $\mathrm{card}(C_\nu^j) \le 2\mathrm{card}(E) \le 2^{N+1}$ and thus the cardinality of $\cup_{\nu=1}^{2^N} C_\nu^j$ is bounded by $2^N 2^{N+1} < 4^{N+1}$, as claimed.

To finish the induction step we repeat this argument to construct b_{-j}. For $\nu = 1, \dots, 2^N$ we denote by C_ν^{-j} the subset of all integers c in $[-j4^{N+1}, (1-j)4^{N+1})$ with the property that $e_\nu + c \in \cup_{i=1-j}^{j}(b_i+E)$. Again we verify (by repeating the argument above) that the cardinality of $\cup_{\nu=1}^{2^N} C_\nu^{-j}$ is $< 4^{N+1}$ and then we may choose $b_{-j} \in [-j4^{N+1}, (1-j)4^{N+1})$ so that b_{-j} does not belong to $\cup_{\nu=1}^{2^N} C_\nu^{-j}$. Then by construction the sets $b_{-j}+E, \dots, b_j+E$ are disjoint. □

Weak type (1,1) and Hardy space estimates. For a countable set of multipliers $\{m_\nu\}_{\nu\in\mathcal{I}}$ consider the maximal function given by

$$\mathfrak{M}f(x) = \sup_{\nu\in\mathcal{I}} |\mathcal{F}^{-1}[m_\nu \widehat{f}\,](x)|.$$

We shall apply the following result on maximal functions which is based on Calderón-Zygmund theory and essentially proved in [4]. In what follows H^1 denotes the standard Hardy space.

Proposition 3.2. *Suppose that for some positive* $\epsilon \le 1$

$$\sup_{\nu\in\mathcal{I}} \sup_{k\in\mathbb{Z}} \sup_{x\in\mathbb{R}^d} (1+|x|)^{d+\epsilon} \big|\mathcal{F}^{-1}[\phi m_\nu(2^k\cdot)](x)\big| \le A_0$$

and suppose that $\mathfrak{M}$ *is bounded on* L^q *(for some* $q > 1$*) with operator norm* A_1*. Then* $\mathfrak{M}$ *is bounded from* H^1 *to* L^1 *with operator norm at most* $C_d(A_0\varepsilon^{-1}+A_1)$*; moreover* $\mathfrak{M}$ *is of weak type (1,1) with the estimate*

$$\sup_{\alpha>0} \alpha\, \mathrm{meas}\big(\{x : |\mathfrak{M}f(x)| > \alpha\}\big) \le C_d(A_0\varepsilon^{-1}+A_1)\|f\|_1.$$

Proof. We prove the weak-type $(1,1)$ bound. Fix $\alpha > 0$. We use the standard Calderón-Zygmund decomposition (see [11]) at level $\beta = (2^{d+1}A_1)^{-1}\alpha$. Thus we decompose $f = g_\beta + b_\beta$ where $|g_\beta| \le 2^d\beta$ and $b_\beta = \sum b_{\beta,Q}$, where $b_{\beta,Q}$ is supported on Q and has mean value 0. Moreover, if Q^* denotes the

$2\sqrt{d}$-dilate of Q with same center, then the dilated cubes Q^* have bounded overlap and

$$\sum_Q \text{meas}(Q^*) \le C(d)\beta^{-1}\|f\|_1 \le 2^{d+1}C(d)A_1\alpha^{-1}\|f\|_1.$$

Let $K_{\nu,j} = \mathcal{F}^{-1}[\phi(2^{-j}\cdot)m_\nu]$. We argue similarly as in Lemma 1 of [4] to verify the following vector-valued Hörmander condition for maximal operators (see [13]):

$$\int_{|x|\ge 2|y|} \sup_\nu \sum_j \Big|K_{\nu,j}(x-y) - K_{\nu,j}(x)\Big| dx \le C_d \epsilon^{-1} A_0. \tag{3.2}$$

Let $\widetilde{K}_{\nu,j} = \mathcal{F}^{-1}[\phi m_\nu(2^j\cdot)]$ so that $K_{\nu,j}(x) = 2^{jd}\widetilde{K}_{\nu,j}(2^j x)$. By assumption we have the pointwise estimate

$$|\widetilde{K}_{\nu,j}(x)| + |\nabla \widetilde{K}_{\nu,j}(x)| \lesssim A_0(1+|x|)^{-d-\varepsilon},$$

uniformly in ν and j. This quickly yields

$$\int_{|x|\ge 2|y|} \sup_\nu \Big|K_{\nu,j}(x-y) - K_{\nu,j}(x)\Big| dx \lesssim A_0 \min\{(2^j|y|)^{-\varepsilon}, 2^j|y|\};$$

thus after summing in j we obtain (3.2). This inequality implies in the usual way

$$\text{meas}\{x \notin \cup Q^* : \mathfrak{M}b_\beta(x) > \alpha/2\} \lesssim A_0\varepsilon^{-1}\alpha^{-1}\|f\|_1.$$

For the contribution of the "good" function g_β we obtain

$$\begin{aligned} &\text{meas}\{x : \mathfrak{M}g_\beta(x) > \alpha/2\} \le 2^q\alpha^{-q}\|\mathfrak{M}g_\beta\|_q^q \\ &\le 2^q\alpha^{-q}A_1^q\|g_\beta\|_q^q \le 2^q\alpha^{-q}A_1^q(2^d\beta)^{q-1}\|f\|_1 \\ &\le 2A_1\alpha^{-1}\|f\|_1. \end{aligned} \tag{3.3}$$

A combination of these estimates yields the weak-type $(1,1)$ estimate.

Finally for the $H^1 - L^1$ bound we use the atomic decomposition of H^1 and it suffices to prove the estimate

$$\|\mathfrak{M}f_Q\|_1 \lesssim (A_0\varepsilon^{-1} + A_1)$$

for functions f_Q supported on a cube Q satisfying $\int f_Q dx = 0$ and $\|f_Q\|_\infty \le |Q|^{-1}$. If Q^* denotes the expanded cube, then we get

$$\|\mathfrak{M}f_Q\|_{L^1(Q^*)} \lesssim |Q|^{\frac{1}{q'}}\|\mathfrak{M}f_Q\|_q \lesssim |Q|^{\frac{1}{q'}}A_1\|f_Q\|_q \lesssim A_1.$$

Using the cancellation of the atom we see that (3.2) implies

$$\|\mathfrak{M}f_Q\|_{L^1(\mathbb{R}^d\setminus Q^*)} \lesssim A_0\varepsilon^{-1}$$

and combining the two estimates we get the asserted $H^1 \to L^1$ estimate. □

Note that the hypothesis in the proposition is implied by

$$\sup_{\nu\in\mathcal{I}} \sup_{k\in\mathbb{Z}} \|\phi m_\nu(2^k\cdot)\|_{L^1_{d+\epsilon}} \le A_0.$$

The result of Dappa and Trebels [4] mentioned in the introduction corresponds to the special case where $m_\nu = m(t_\nu \cdot)$ and $\{t_\nu\}$ is an enumeration of the positive rational numbers.

4. Results on L^p boundedness

In this section E will be a set of integers satisfying

$$\text{card}(E) \le 2^N; \tag{4.1}$$

here N is a nonnegative integer.

Let $\psi \in C_c^\infty$ be supported in $\{\xi : 1/4 < |\xi| < 4\}$ and set $\Psi = \mathcal{F}^{-1}[\psi]$; later we shall work with a specific ψ satisfying (5.1).

Let $\chi \in C_c^\infty(\mathbb{R}^d)$ so that χ is radial and supported where $R_0 \le |x| \le R_1$ with $1/2 < R_0 < R_1 < 2$; moreover assume that χ is positive for $R_0 < |x| < R_1$ and that $\sum_{l=-\infty}^{\infty} \chi(2^{-l}x) = 1$ for $x \ne 0$. Now set $\chi_0(x) = \big(1 - \sum_{l>0} \chi(2^{-l}x)\big)$ so that χ_0 is supported where $|x| \le 2$. Let $\chi_l(x) = \chi(2^{-l}x)$ for $l > 0$; then $\sum_{l=0}^{\infty} \chi_l(x) \equiv 1$.

For a function g define by $\delta_t g$ the L^1 dilate; *i.e.*

$$\delta_t g(x) = t^{-d} g(t^{-1}x).$$

For a sequence $H = \{h_k\}_{k\in\mathbb{Z}}$ of locally integrable functons we then consider the operator

$$\mathcal{T}_t^{E,l}[H,f] = \sum_{k\in E} \delta_{2^k t}\big[\Psi * (\chi_l h_k)\big] * f \tag{4.2}$$

and the maximal function

$$\mathcal{M}^{E,l}[H,f] = \sup_{t>0} \big|\mathcal{T}_t^{E,l}[H,f]\big|.$$

In §5 we shall decompose $\mathcal{F}^{-1}[m(t\cdot)\widehat{f}]$ in terms of operators of the form (4.2).

The following L^q bound is favorable when $q \ge N + l$.

Proposition 4.1. *Assuming* (4.1) *we have for* $q \ge 2$

$$\|\mathcal{M}^{E,l}[H,f]\|_q \le C\, q\, 4^{N/q}\, (1+l) 2^{l/q}\, \|H\|_{\ell^\infty(L^1)} \|f\|_q.$$

Proof. In what follows we shall use the notation $A \lesssim B$ to indicate an inequality $A \le CB$ where C may only depend on d (and not on q or other parameters).

Define

$$g_{k,l}(\xi) = \psi(\xi)\widehat{h_k\chi_l}(\xi)$$

and

$$m^l(\xi) \equiv m^{l,E}(\xi) = \sum_{k\in E} g_{k,l}(2^{-k}\xi); \tag{4.3}$$

then

$$\mathcal{T}_t^{E,l}[H,f] = \mathcal{F}^{-1}[m^l(t\cdot)\widehat{f}].$$

Also note that

$$\partial_s\big[m^l(s\xi)\big] = \sum_{k\in E} \widetilde{g}_{k,l}(s2^{-k}\xi) \tag{4.4}$$

where

$$\widetilde{g}_{k,l}(\xi) = \psi(\xi)\langle\xi,\nabla\rangle\widehat{\chi_l h_k}(\xi) + \widehat{\chi_l h_k}(\xi)\langle\xi,\nabla\rangle\psi(\xi). \tag{4.5}$$

Now apply Lemma 3.1 for the set E, and let b_j be as in Lemma 3.1 (ii). By (iii) of Lemma 3.1 we may write

$$\sup_{t>0}|\mathcal{F}^{-1}[m^l(t\cdot)\widehat{f}\,]| = \sup_{|n|\le 4^{N+1}}\sup_{j\in\mathbb{Z}}\sup_{1\le s\le 2}\big|\mathcal{F}^{-1}[m^l(2^{-b_j+n}s\cdot)\widehat{f}\,]\big|.$$

Now one replaces the supremum in n and j by ℓ^q norms, takes the L^q norms, then interchanges the order of summation and integration. This yields for $\mathcal{M}^{E,l}[H,f] \equiv \mathcal{M}_{m^l}f$ the estimate

$$\big\|\mathcal{M}_{m^l}f\big\|_q \le 4^{\frac{(N+2)}{q}} \sup_{|n|\le 4^{N+1}} \Big(\sum_j \Big\|\sup_{1\le s\le 2}\big|\mathcal{F}^{-1}[m^l(2^{-b_j+n}s\cdot)\widehat{f}\,]\big|\Big\|_q^q\Big)^{\frac1q}. \tag{4.6}$$

Thus it remains to verify that for $|n|\le 4^{N+1}$

$$\Big(\sum_j \Big\|\sup_{1\le s\le 2}\big|\mathcal{F}^{-1}[m^l(2^{-b_j+n}s\cdot)\widehat{f}\,]\big|\Big\|_q^q\Big)^{\frac1q} \lesssim q(1+l)2^{\frac{l}{q}}\|H\|_{\ell^\infty(L^1)}\|f\|_q. \tag{4.7}$$

In what follows we may assume that $n=0$ since the general case follows by scaling.

To estimate the supremum in s it is standard to use the elementary inequality

$$|F(s)|^q \le |F(1)|^q + q\Big(\int_1^s |F(\sigma)|^q d\sigma\Big)^{\frac{q-1}{q}}\Big(\int_1^s |F'(\sigma)|^q d\sigma\Big)^{\frac1q}$$

which is obtained by applying the fundamental theorem of calculus to $|F|^q$ and Hölder's inequality.

Taking L^q norms and applying Hölder's inequality twice yields

$$\begin{aligned}\sum_j \big\|\sup_{1\le s\le 2}\big|\mathcal{F}^{-1}[m^l(2^{-b_j}s\cdot)\widehat{f}\,]\big|\big\|_q^q &\lesssim \sum_{j\in\mathbb{Z}}\big\|III_j^l\big\|_q^q \\ &+ q\Big(\int_1^2\sum_{j\in\mathbb{Z}}\big\|I_j^l(s)\big\|_q^q ds\Big)^{\frac{q-1}{q}}\Big(\int_1^2\sum_{j\in\mathbb{Z}}\big\|II_j^l(s)\big\|_q^q ds\Big)^{\frac1q},\end{aligned} \tag{4.8}$$

where

$$\begin{aligned} I_j^l(s) &= \mathcal{F}^{-1}\big[m^l(2^{-b_j}s\cdot)\widehat{f}\,\big]\\ II_j^l(s) &= \mathcal{F}^{-1}\big[\partial_s\big(m^l(2^{-b_j}\cdot)\big)\widehat{f}\,\big]\\ III_j^l &= \mathcal{F}^{-1}\big[m^l(2^{-b_j}\cdot)\widehat{f}\,\big].\end{aligned}$$

Next, we interchange the j-summations and integrations in (4.8) and use the imbedding of ℓ^2 into ℓ^q. This yields

(4.9)
$$\Big(\sum_j \big\| \sup_{1\le s\le 2} |\mathcal{F}^{-1}[m^l(2^{-b_j}s\cdot)\widehat{f}\,]|\big\|_q^q\Big)^{1/q}$$
$$\lesssim \Big(\int_1^2 \Big\|\Big(\sum_{j\in\mathbb{Z}} |I_j^l(s)|^2\Big)^{1/2}\Big\|_q^q ds\Big)^{\frac{q-1}{q^2}} \Big(\int_1^2 \Big\|\Big(\sum_{j\in\mathbb{Z}} |II_j^l(s)|^2\Big)^{1/2}\Big\|_q^q ds\Big)^{\frac{1}{q^2}}$$
$$+ \Big\|\Big(\sum_{j\in\mathbb{Z}} |III_j^l|^2\Big)^{1/2}\Big\|_q.$$

In order to estimate these terms we need the following estimates for vector valued singular integrals.

Sublemma. *For $2 \le q < \infty$ we have*

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big|\sum_{k\in E}\mathcal{F}^{-1}[g_{k,l}(2^{-b_j-k}\cdot)\widehat{f}\,]\Big|^2\Big)^{1/2}\Big\|_q \lesssim q(1+l)\|H\|_{\ell^\infty(L^1)}\|f\|_q \tag{4.10}$$

and

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big|\sum_{k\in E}\mathcal{F}^{-1}[\widetilde{g}_{k,l}(2^{-b_j-k}\cdot)\widehat{f}\,]\Big|^2\Big)^{1/2}\Big\|_q \lesssim q(1+l)2^l\|H\|_{\ell^\infty(L^1)}\|f\|_q. \tag{4.11}$$

Sketch of Proof. By duality (4.10) for $2 \le q < \infty$ is equivalent to

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big|\sum_{k\in E}\mathcal{F}^{-1}[g_{k,l}(2^{-b_j-k}\cdot)\widehat{f_j}]\Big|^2\Big)^{1/2}\Big\|_p \lesssim p'(1+l)^{-1+2/p}\|H\|_{\ell^\infty(L^1)}\Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_p \tag{4.12}$$

for $1 < p \le 2$, $p' = p/(p-1)$. For $p = 2$ this (and in fact a slightly better) bound follows from the essential disjointness of the supports of $g_{k,l}$ and the estimate
$$\|g_{k,l}\|_\infty \le C\|\widehat{h_k}\|_\infty \le C\|H\|_{\ell^\infty(L^1)}.$$
For $1 < p \le 2$ the inequality (4.12) follows from the weak type bound

$$\text{meas}\Big(\Big\{x : \Big(\sum_{j\in\mathbb{Z}}\Big|\sum_{k\in E}\mathcal{F}^{-1}[g_{k,l}(2^{-b_j-k}\cdot)\widehat{f_j}]\Big|^2\Big)^{1/2} > \lambda\Big\}\Big) \lesssim C\lambda^{-1}(1+l)\|H\|_{\ell^\infty(L^1)}\Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_1 \tag{4.13}$$

where in the interpolation we have to take into account the behavior of the constants in the Marcinkiewicz interpolation theorem (see e.g. [6], p. 33).

The weak type estimate follows by standard arguments in Calderón-Zygmund theory from the inequality

$$\int\limits_{|x|>2|y|} \Big(\sum_j \Big|\sum_{k\in E} \Big(\mathcal{F}^{-1}[g_{k,l}(2^{-b_j-k}\cdot)](x-y) \\ - \mathcal{F}^{-1}[g_{k,l}(2^{-b_j-k}\cdot)](x)\Big)\Big|^2\Big)^{1/2} dx \lesssim (1+l)\|H\|_{\ell^\infty(L_1)}$$

which, since the sets $\{b_j+E\}_{j\in\mathbb{Z}}$ are disjoint, is quickly derived from the inequalities

$$\int_{|x|>2|y|} \big|\mathcal{F}^{-1}[g_{k,l}(2^{-M}\cdot)](x-y) - \mathcal{F}^{-1}[g_{k,l}(2^{-M}\cdot)](x)\big| dx \\ \lesssim \|h_k\|_1 \times \begin{cases} 2^M|y| & \text{if } 2^M|y|\le 1 \\ 1 & \text{if } 1\le 2^M|y|\le 2^{2l} \\ (2^M|y|)^{-1} & \text{if } 2^M|y|\ge 2^{2l} \end{cases}. \tag{4.14}$$

Summing in M yields a blowup of order $O(1+l)$. The bound (4.14) is straightforward given the localization of χ_l and the decay of the Schwartz-function Ψ. This finishes the proof of (4.10).

In order to verify (4.11) we note from (4.5) that

$$\mathcal{F}^{-1}[\widetilde{g}_{k,l}](x) = c_1\sum_{i=1}^d \int \partial_{x_i}\Psi(x-y)y_i\chi_l(y)h_k(y)dy \\ + c_2\sum_{i=1}^d \int x_i\partial_{x_i}\Psi(x-y)\chi_l(y)h_k(y)dy.$$

Here the second term has the same quantitative properties as $\mathcal{F}^{-1}[g_{k,l}]$ while the first has similar estimates as $2^l\mathcal{F}^{-1}[g_{k,l}]$. Thus the above arguments show (4.11) as well. □

Proof of Proposition 4.1, cont. For fixed s we may perform the scaling $\xi\to s^{-1}\xi$ in (4.9); this puts us in the position to apply the sublemma. We then see that the right hand side of (4.9) can be estimated by a constant times

$$q(1+l)\|H\|_{\ell^\infty(L^1)}(1+2^{l/q})\|f\|_q$$

which implies the desired bound (4.7). □

Corollary 4.2. *Assuming* (4.1) *we have*

$$\|\mathcal{M}^{E,l}[H,f]\|_{BMO} \le C(N+l)(1+l)\|H\|_{\ell^\infty(L^1)}\|f\|_{L^\infty}.$$

Proof. Let Q be a cube in $\mathbb{R}^d$ with center x_Q and let Q^* be the $2\sqrt{d}$-dilate with same center. By the definition of BMO and Hölder's inequality the

assertion follows from
(4.15)
$$\Big(\frac{1}{|Q|}\int_Q |\mathcal{M}^{E,l}[H,f\chi_{Q^*}]|^{N+l}dx\Big)^{\frac{1}{N+l}} \lesssim (N+l)(1+l)\|H\|_{\ell^\infty(L^1)}\|f\|_{L^\infty}$$

and, with m^l as in (4.3),

$$\sup_{x\in Q}\sup_{t>0}\int_{\mathbb{R}^d\setminus Q^*}\big|\mathcal{F}^{-1}[m^l(t\cdot)](x-y)-\mathcal{F}^{-1}[m^l(t\cdot)](x_Q-y)\big|dy \lesssim (1+l)\|H\|_{\ell^\infty(L^1)}\|f\|_{L^\infty}. \tag{4.16}$$

The left hand side of (4.15) is bounded by the L^{N+l} operator norm of $\mathcal{M}^{E,l}(H,\cdot)$ times $|Q|^{-1/(N+l)}\|f\chi_{Q^*}\|_{N+l}$ and (4.15) follows from Proposition 4.1. Inequality (4.16) is deduced from the estimates in the Sublemma. □

Proposition 4.3. *Suppose that* (4.1) *holds. Then for* $1<p<\infty$, $1/p+1/p'=1$,

$$\|\mathcal{M}^{E,l}[H,f]\|_p \le C_p(N+l)(1+l)2^{ld/p}\|H\|_{\ell^\infty(L^{p'})}\|f\|_p.$$

Moreover the operator $f\mapsto \mathcal{M}^{E,l}[H,f]$ *is bounded from* H^1 *to* L^1*, and of weak type* $(1,1)$ *with operator bound* $CN(1+l)2^{ld}\|H\|_{\ell^\infty(L^\infty)}$.

Proof. Let $\tilde\chi_l$ be the characteristic function of the annulus $\{x: 2^{l-3}\le |x|\le 2^{l+3}\}$, for $l\ge 1$, and let $\tilde\chi_0$ be the characteristic function of the ball $\{x: |x|\le 8\}$. Let $h_k^l=h_k\tilde\chi_l$ and $H^l=\{h_k^l\}_{k\in\mathbb{Z}}$. Then observe that by the support property of χ_l (in the definition of $\mathcal{T}^{E,l}$) we have $\mathcal{T}^{E,l}[H,f]=\mathcal{T}^{E,l}[H^l,f]$. Proposition 4.1 yields the estimate

$$\|\mathcal{M}^{E,l}[H,f]\|_q = \|\mathcal{M}^{E,l}[H^l,f]\|_q \le Cq(1+l)\|H^l\|_{\ell^\infty(L^1)}, \quad q\ge N+l$$

This implies the assertion for $N+l\le p<\infty$.

To prove the Hardy space estimate we apply Proposition 3.2 in conjunction with Proposition 4.1 (for H^l and $q=N+l$) and we obtain

$$\|\mathcal{M}^{E,l}[H,f]\|_1 \lesssim \big((N+l)(1+l)\|H^l\|_{\ell^\infty(L^1)}+(1+l)\varepsilon^{-1}2^{l\epsilon}2^{ld}\|H^l\|_{\ell^\infty(L^\infty)}\big)\|f\|_{H^1}.$$

The asserted bound follows if we choose $\varepsilon=(1+l)^{-1}$. The weak type $(1,1)$ bound follows similarly.

We may now use the complex method for bilinear operators (which is a variant of Stein's theorem for analytic families, see [1], §4.4) together with the interpolation formula $[H_1,L^{p_1}]_\vartheta=L^p$, for $(1-\vartheta)+\vartheta/p_1=1/p$, see [5]. Now define $s=p'/p_1'$ so that $\vartheta=(1-\vartheta)/\infty+\vartheta/1=1/s$. We then obtain the estimate

$$\|\mathcal{M}^{E,l}[H,f]\|_p \le C_p(1+l)(N+l)2^{ld/s'}\|H\|_{\ell^\infty(L^s)}\|f\|_p$$

but since $\mathcal{M}^{E,l}[H,f] = \mathcal{M}^{E,l}[H^l,f]$ we may replace H with H^l on the right hand side of this inequality. Note that $s < p'$ and thus

$$2^{ld/s'}\|H^l\|_{\ell^\infty(L^s)} \lesssim 2^{ld/p}\|H^l\|_{\ell^\infty(L^{p'})},$$

by Hölder's inequality. This yields the asserted bound. □

Remark. One could also analytically interpolate the $H^1 \to L^1$ estimate with the $L^\infty \to BMO$ estimate of Corollary 4.2 using the formula $[H_1, BMO]_\theta = L^p$, $\theta = 1/p'$. This formula follows from the result in [5] for $[H_1, L^{p_1}]_{\vartheta_1}$ and $[L^{p_0}, BMO]_{\vartheta_2}$, $1 < p_0 < p_1 < \infty$, by Wolff's four space reiteration theorem, see [12].

5. Conclusion: Proof of Theorem 1.2

We only have to prove the L^p estimates for $p > 1$ since the asserted weak type $(1,1)$ bound is then a consequence of Proposition 3.2.

We need to decompose m in terms of the rearrangement function ω^*. Let

$$E_0 = \{k \in \mathbb{Z} : \omega^*(2) < |\omega(k)| \le \omega^*(0)\}$$

and for $j = 1, 2, \dots$ let

$$E_j = \{k \in \mathbb{Z} : \omega^*(2^{2^j}) < |\omega(k)| \le \omega^*(2^{2^{j-1}})\}.$$

As in the introduction let $\phi \in C_c^\infty$ be supported in $\{\xi : 1/2 < |\xi| < 2\}$ so that $\phi(\xi) \ne 0$ for $2^{-1/2} \le |\xi| \le 2^{1/2}$. Set $\psi(\xi) = \overline{\phi(\xi)}\big(\sum_{j\in\mathbb{Z}} |\phi(2^{-j}\xi)|^2\big)^{-1}$, then ψ is smooth and we have

$$\sum_{k\in\mathbb{Z}} \psi(2^{-k}\xi)\phi(2^{-k}\xi) = 1, \quad \xi \ne 0. \tag{5.1}$$

Let

$$m_j(\xi) = \sum_{k\in E_j} \psi(2^{-k}\xi)\phi(2^{-k}\xi)m(\xi)$$

then $m = \sum_{j=0}^\infty m_j$; here we use that $\omega^*(k) \to 0$ as $k \to \infty$.

We now decompose $\mathcal{F}^{-1}[\phi m(2^k\cdot)]$ using the dyadic cutoff functions χ_l. Define $h_k^{j,0}(x) = \mathcal{F}^{-1}[\phi m(2^k\cdot)](x)$ if $|x| \le 4$ and $k \in E_j$ and $h_k^{j,0}(x) = 0$, if $|x| > 4$ or $k \notin E_j$. Moreover for $l > 0$ let $h_k^{j,l}(x) = \mathcal{F}^{-1}[\phi m(2^k\cdot)](x)$ if $2^{l-4} \le |x| \le 2^{l+4}$ and $k \in E_j$ and $h_k^{j,l}(x) = 0$, if $|x| \notin [2^{l-4}, 2^{l+4}]$ or $k \notin E_j$. Then

$$\mathcal{F}^{-1}[\phi m(2^k\cdot)] = \sum_{j=0}^\infty \sum_{l=0}^\infty h_k^{j,l}\chi_l$$

and with $H^{j,l} = \{h_k^{j,l}\}_{k\in\mathbb{Z}}$, and $\Psi = \mathcal{F}^{-1}\psi$, we may write

$$\mathcal{F}^{-1}[m_j(t\cdot)\widehat{f}] = \sum_{k\in E_j} \delta_{2^k t}\big(\Psi * \sum_{l=0}^{\infty}(h_k^{j,l}\chi_l)\big)$$
$$= \sum_{l=0}^{\infty} \mathcal{T}_t^{E_j,l}[H^{j,l}, f].$$

The assumption (1.9) implies that

$$\|H^{j,l}\|_{\ell^\infty(L^{p'})} \lesssim 2^{-l\alpha}\omega^*(2^{2^{j-1}})$$

for $j \geq 1$ (and a similar estimate with $\omega^*(0)$ for $j = 0$).

Note from the definition of the rearrangement function that

$$\mathrm{card}(E_j) \leq 2^{2^j}.$$

Thus we obtain by Proposition 4.3

$$\big\|\mathcal{M}^{E_j,l}[H^{j,l}, f]\big\|_p \lesssim (1+l)^2 2^{-l(\alpha-d/p')} 2^j \omega^*(2^{2^{j-1}}), \quad j \geq 1,$$

and a similar estimate with $2^j\omega^*(2^{2^{j-1}})$ replaced by $\omega^*(0)$ when $j = 0$. By our assumption $\alpha > d/p'$ we therefore get

$$\|\mathcal{M}_m f\|_p \leq \sum_{j=0}^{\infty} \|\mathcal{M}_{m_j} f\|_p \leq \sum_{l=0}^{\infty}\sum_{j=0}^{\infty} \|\mathcal{M}^{E_j,l}[H^{j,l}, f]\|_p$$
$$\leq C_p \sum_{l=0}^{\infty}(1+l)^2 2^{-l(\alpha-d/p')}\Big[\omega^*(0) + \sum_{j=1}^{\infty} 2^j\omega^*(2^{2^{j-1}})\Big]\|f\|_p$$
$$\lesssim C_p'\Big[\omega^*(0) + \sum_{n=2}^{\infty}\frac{\omega^*(n)}{n}\Big]\|f\|_p.$$

This concludes the proof of Theorem 1.2. □

6. A sketch of the proof of Theorem 1.4.

The proof is simpler than the proof of Theorem 1.2 but relies on the same idea. Write $T_\nu f = \mathcal{F}^{-1}[m_\nu \widehat{f}]$. Assume that (1.13) holds with $p' = 1$, $\alpha > 0$. Then for $10\alpha^{-1} < q < \infty$ the operator T_ν is bounded on L^q with operator norm $O(q)$, uniformly in ν. We replace ℓ^∞ norms by ℓ^q norms and estimate for those q

$$\|\mathfrak{M}_n f\|_q \leq \Big\|\big(\sum_{\nu=1}^{n} |T_\nu f|^q\big)^{1/q}\Big\|_q = \Big(\sum_{\nu=1}^{n} \|T_\nu f\|_q^q\Big)^{1/q} \leq C_\alpha n^{1/q} q \|f\|_p.$$

This yields the desired result for $q \geq \log(n+1) \geq 10\alpha^{-1}$ since $n^{1/\log(n+1)}$ is bounded as $n \to \infty$. Arguing as in the proof of Corollary 4.2 one also gets an $L^\infty \to BMO$ estimate with bound $O(\log(n+1))$. Finally, under the analogue of (1.13) for $p = 1$ we derive an $H^1 \to L^1$ estimate (using

Proposition 3.2) and by an interpolation argument as used in Proposition 4.3 we may derive the asserted result for L^p boundedness. □

7. Remarks and open problems

7.1. The main open problem is to completely close the gap in terms of the power of logarithms in (1.5) and (1.7). In particular it should be interesting to know assuming (1.11) for which $s > 1$ the condition

$$\omega^*(0) + \Big(\sum_{l=1}^{\infty} \frac{[\omega^*(l)]^s}{l}\Big)^{1/s} < \infty$$

implies L^p boundedness of $\mathcal{M}_m$. We have shown that $s = 1$ is sufficient, but $s > 2$ is not.

Similarly, in (1.14) it would be interesting to investigate whether the bound $O(\log n)$ can be replaced by $O(\log^{1/s} n)$ for suitable $s \le 2$.

7.2. If $d = 1$ the assumptions (1.13) on the kernels (which are differentiability assumptions on the multiplier) are essentially sharp; this is seen by examining the multipliers $e^{i\xi}|\xi|^{-\alpha} \sum_{k=1}^{N} \phi(2^{-k}\xi)$ for suitable ϕ.

7.3. If m is radial, $m(\xi) = g(|\xi|)$, then the space $L^1_{d+\varepsilon}$ may be replaced by $L^1_{(d+1)/2+\varepsilon}$ in the weak type (1,1) estimate (see [4]), of Corollary 1.3. By analytic interpolation one obtain L^p boundedness for $1 \le p \le 2$ under the conditions (1.11), (1.8) with with $r = \min\{p, 2\}$, $\alpha > d/2 + (1/r - 1/2)$.

7.4. If (1.8) is replaced by a stronger decay assumption then much weaker smoothness assumptions suffice, as demonstrated in [2], [4] under the assumption $\omega \in \ell^2$. Various intermediate estimates can be derived by analytic interpolation. It should be interesting to obtain in higher dimensions the minimal smoothness assumption requiring only the decay in (1.8). The same question can be formulated for the dyadic maximal operators.

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