BILINEAR MAXIMAL FUNCTIONS ASSOCIATED WITH SURFACES

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ABSTRACT. We obtain $L^2 \times L^2 \to L^1$ boundedness for bilinear maximal functions associated with general compact hypersurfaces. Our method is based on the strategy introduced in [2] and a new multiplier result established in [7].

The spherical maximal function

$$\mathcal{M}(f)(x) := \sup_{t>0} \int_{\mathbb{S}^{n-1}} f(x-ty)d\sigma(y)$$

was introduced by Stein [13], who proved the L^p-boundedness of this operator in the sharp range of exponents when $n \geq 3$. One decade later, Bourgain [1] extended this result to the case when $n=2$ using geometric techniques involving intersection of circles. Much work has focused on this operator and its generalization to more general surfaces; see, for instance, [14], [8], [4], [12].

The (sub)bilinear analogue of the spherical maximal operator was introduced in [3], and the study of its $L^2 \times L^2 \to L^1$ boundedness was initiated in [2], where it was shown for $n \geq 8$. The dimension restriction was lowered to $n \geq 4$ in [6], as an application of a more general result for bilinear multipliers with limited decay. In this note we extend the aforementioned results to more general surfaces by making use of the strategy employed in [2] and of a new sharp criterion for the $(2,2,1)$ boundedness of bilinear multipliers obtained in [7].

Let S be a $(2n-1)$-dimensional compact smooth surface in $\mathbb{R}^{2n}$ without boundary such that k of the $2n-1$ principal curvatures of S are different from zero, and let μ be a smooth measure supported on it. For instance μ could be the canonical normalized surface measure on S. The bilinear maximal function

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associated with the surface S and the measure μ is defined by

$$\mathcal{M}_S(f,g)(x) = \sup_{t>0} \Big| \int_S f(x-ty)g(x-tz)d\mu(y,z) \Big|.$$

An important property of μ that will be needed is as follows.

Lemma 1 ([11]). *Let S and μ be as described above, then all derivatives of the Fourier transform of μ satisfy the estimate*

$$|\partial^\alpha(\widehat{\mu})(\xi)| \le C_\alpha(1+|\xi|)^{-k/2}.$$

The main result of this note is as follows:

Theorem 2. *The bilinear maximal operator $\mathcal{M}_S$ associated with S is bounded from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ whenever $k > n+2$.*

As the decay of the Fourier transform of $(2n-1)$-dimensional surface measure is $(1+|\xi|)^{-\frac{2n-1}{2}}$ we deduce as a corollary that the bilinear spherical maximal function is bounded from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ whenever $n \ge 4$. Recently it was shown in [10] that the bilinear spherical maximal function is bounded in the sharp range of exponents for all $n \ge 2$; the key idea of the proof is a change of variables which exploits the symmetry of the sphere (see [9, p. 136] and [2, Lemma 9]); unfortunately this idea is not applicable in the general case studied here. So the main contribution of this note is the extension of this result to general surfaces where symmetry does not appear. We work with surfaces with k nonzero principal curvatures and our approach requires the restriction $k > n+2$.

Our main result is Theorem 2 which covers the situation where k principal curvatures of the surface are nonzero. Such examples of surfaces include the products $\mathbb{S}^k \times [0,1]^{2n-k-1}$.

1. Some preliminaries

We recall some notation and the strategy in [2]. Let φ be a nonnegative smooth function supported in the unit annulus such that $\sum_{j\in\mathbb{Z}} \varphi(2^{-j}(\xi,\eta)) = 1$, and define $m_j(\xi,\eta) = \widehat{d\mu}(\xi,\eta)\varphi(2^{-j}(\xi,\eta))$. Associated with m_j, we define

$$\mathcal{M}_j(f,g)(x) = \sup_{t>0} \Big| \int_{\mathbb{R}^{2n}} \widehat{f}(\xi)\widehat{g}(\eta)m_j(t\xi,t\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta \Big|.$$

Obviously $\mathcal{M} \le \sum_j \mathcal{M}_j$.

Concerning the boundedness of $\mathcal{M}_j$ when S is the unit sphere $\mathbb{S}^{2n-1}$ and μ is the normalized measure on $\mathbb{S}^{2n-1}$, we have the following result proved in [2].

Proposition 3 ([2, Proposition 4]). *Let $n \geq 8$. Then there exists a positive constant C such that for all $j \geq 1$ and all functions $f, g \in L^2(\mathbb{R}^n)$ we have*

$$\|\mathcal{M}_j(f,g)\|_{L^1} \leq C\, j 2^{-\delta_n j} \|f\|_{L^2} \|g\|_{L^2}, \tag{1}$$

where $\delta_n = \frac{n}{5} - \frac{3}{2}$.

In particular, $\mathcal{M}$ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ when $n \geq 8$. To improve this result, we need to improve the exponent δ_n in (1).

In the proof of Proposition 3 in [2], we decompose m_j into diagonal and off-diagonal parts. To describe this decomposition, we take $\rho \in \mathcal{S}(\mathbb{R})$ satisfying $\chi_{[\varepsilon-1,1-\varepsilon]} \leq \rho \leq \chi_{[-1,1]}$ and define $m_j^1(\xi,\eta) = m_j(\xi,\eta)\rho(\frac{1}{j}(\log_2 \frac{|\xi|}{|\eta|}))$, then we obtain the decomposition $m_j = m_j^1 + m_j^2$, where m_j^1 is called the diagonal part, and m_j^2 is called the off-diagonal part. The off-diagonal part is handled by a standard square function argument, and the estimate of the diagonal part is obtained as a consequence of [5, Corollary 8].

In [7] we improved [5, Corollary 8] to the following sharp version which will in turn lead to an improvement of Proposition 3.

Proposition 4 ([7, Theorem 1.3.]). *Suppose that $m(\xi,\eta)$ is a function in $L^q(\mathbb{R}^{2n})$ with $1 < q < 4$ such that $m \in \mathcal{C}^{M_q}$ with $M_q = \left[\frac{2n}{4-q}\right] + 1$, and*

$$\|\partial^\alpha m\|_{L^\infty} \leq C_0 < \infty \quad \text{for all } \alpha \text{ with } |\alpha| \leq M_q.$$

Then there is a constant C depending on n and q such that the bilinear operator T_m with multiplier m satisfies

$$\|T_m\|_{L^2 \times L^2 \to L^1} \leq C C_0^{1-\frac{q}{4}} \|m\|_{L^q}^{\frac{q}{4}}. \tag{2}$$

A key step towards lowering the dimension restriction in Proposition 3 is the following lemma.

Lemma 5. *Suppose that $\sigma_1(\xi,\eta)$ is defined on $\mathbb{R}^{2n}$ and for some $\delta > 0$ it satisfies:*

(i) supp $\sigma_1 \subset \{(\xi,\eta) \in \mathbb{R}^{2n} : |(\xi,\eta)| \sim 2^j, c_1 2^{-j} \leq \frac{|\xi|}{|\eta|} \leq c_2 2^j\}$ for some $j \in \mathbb{N}$,

(ii) for any multiindex $|\alpha| \leq M = 4n$, there exists a positive constant C_α independent of j such that $\|\partial^\alpha(\sigma_1(\xi,\eta))\|_{L^\infty} \leq C_\alpha 2^{-j\delta}$.

Then $T(f,g)(x) := \int_0^\infty |T_{\sigma_t}(f,g)(x)| \frac{dt}{t}$ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ with bound at most a multiple of $j\|\sigma_1\|_{L^2}^{1/2} 2^{-j\delta/2}$, where $\sigma_t(\xi,\eta) = \sigma_1(t\xi, t\eta)$.

Let us sketch the proof to see how Proposition 4 plays a role, while an interested reader may find more related details in [2].

Proof. Using Proposition 4 with $q=2$, setting $\widehat{f^j}=\widehat{f}\chi_{\{c_1\le|\xi|\le c_2 2^{j+1}\}}$, we have that

$$\|T_{\sigma_1}(f,g)\|_{L^1}\le C\|\sigma_1\|_{L^2}^{1/2}2^{-j\delta/2}\|f^j\|_{L^2}\|g^j\|_{L^2}.$$

Notice that $T_{\sigma_t}(f,g)(x)=t^{-2n}T_{\sigma_1}(f_t,g_t)(\frac{x}{t})$, where $\widehat{f_t}(\xi)=\widehat{f}(\xi/t)$. Then

$$\|T_{\sigma_t}(f,g)\|_{L^1}\ \le\ C\|\sigma_1\|_{L^2}^{1/2}2^{-j\delta/2}\|\widehat{f}\chi_{E_{j,t}}\|_{L^2}\|\widehat{g}\chi_{E_{j,t}}\|_{L^2},$$

where $E_{j,t}=\{\xi\in\mathbb{R}^n:\frac{c_1}{t}\le|\xi|\le\frac{2^jc_2}{t}\}$. This combined with the Hölder inequality implies that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_0^\infty|T_{\sigma_t}(f,g)|\frac{dt}{t}dx\\
\le&C\|\sigma_1\|_{L^2}^{1/2}2^{-j\delta/2}\left(\int_0^\infty\int_{\mathbb{R}^n}|\widehat{f}\chi_{E_{j,t}}|^2d\xi\frac{dt}{t}\right)^{\frac12}\left(\int_0^\infty\int_{\mathbb{R}^n}|\widehat{g}\chi_{E_{j,t}}|^2d\xi\frac{dt}{t}\right)^{\frac12}\\
\le&Cj\|\sigma_1\|_{L^2}^{1/2}2^{-j\delta/2}\|f\|_{L^2}\|g\|_{L^2},
\end{aligned}$$

where in the last step we use the estimate

$$\int_0^\infty\int_{\mathbb{R}^n}|\widehat{f}\chi_{E_{j,t}}|^2d\xi\frac{dt}{t}\le Cj\|f\|_{L^2}^2.$$

This completes the proof. □

2. The proof of the main result

We prove Theorem 2 in this section.

For

$$\mathcal{M}_j^1(f,g)(x):=\sup_{t>0}\Big|\int_{\mathbb{R}^{2n}}\widehat{f}(\xi)\widehat{g}(\eta)m_j^1(t\xi,t\eta)e^{2\pi ix\cdot(\xi+\eta)}d\xi d\eta\Big|$$

we have the control

$$\mathcal{M}_j^1(f,g)(x)\le\int_0^\infty|\widetilde{T}_{j,s}^1(f,g)(x)|\frac{ds}{s},$$

where

$$\widetilde{T}_{j,s}^1(f,g)(x)=\int_{\mathbb{R}^{2n}}\widehat{f}(\xi)\widehat{g}(\eta)\widetilde{m}_j^1(s\xi,s\eta)e^{2\pi ix\cdot(\xi+\eta)}d\xi d\eta$$

with $\widetilde{m}_j^1(\xi,\eta)=(\xi,\eta)\cdot(\nabla m_j^1)(\xi,\eta)$. An argument in [2, Section 4] combined with Lemma 1 implies that $|\partial^\alpha\widetilde{m}_j^1|\le C2^{-j(k-2)/2}$ for all α and that $\|\widetilde{m}_j^1\|_{L^2}\le C2^{-\frac{k-2n-2}{2}j}$. Applying Lemma 5 with $\widetilde{m}_j^1=\sigma_1$, we obtain that

$$\|\mathcal{M}_j^1\|_{L^2\times L^2\to L^1}\le j2^{-j\frac{k-n-2}{2}}.$$

It follows from [2, Lemma 6] that the the off-diagonal part satisfies the estimate

$$\|\mathcal{M}_j^2(f,g)\|_{L^1} \leq C2^{-j(\frac{k-1}{2}-\varepsilon)}\|f\|_{L^2}\|g\|_{L^2},$$

where $\mathcal{M}_j^2$ is the maximal operator associated with the bilinear multiplier m_j^2, and ε is the parameter used in the decomposition of m_j which could be arbitrarily small.

We therefore have (1) with $\delta_n = \frac{k-n-2}{2}$ this time. This implies that the $(2,2,1)$ boundedness of $\mathcal{M}_S$ holds when $\delta_n > 0$, i.e., when $k > n+2$.

3. Final remarks

Theorem 2 is the first result concerning maximal bilinear spherical averages over general surfaces. In this work we focused on the natural initial point of boundedness $L^2 \times L^2 \to L^1$. This condition $k > n+2$ fails to be sharp and furthermore cannot hold in low dimensions as $k \leq 2n-1$. In fact, in low dimensions, unboundedness holds. For instance, we recall the following one-dimensional example.

Proposition 6 ([2, Proposition 7])**.** *When $S = \mathbb{S}^1$, the bilinear spherical maximal operator $\mathcal{M}_S$ is unbounded from $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ to $L^1(\mathbb{R})$.*

We are also aware of other surfaces S with k nonzero principal curvatures such that $\mathcal{M}_S$ is unbounded on $L^2 \times L^2 \to L^1$ when $k \leq 2n-2$ is sufficiently small.

Questions related to this work include the extension of Theorem 2 to general Lebesgue spaces. In this case one should not expect the range of indices to be symmetric as the non symmetric example $S = \{(r,r^2) : 0 \leq r \leq 1\}$ indicates. Additionally, we hope to address the extension of Theorem 2 to the variable coefficient case.

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