THE BILINEAR BOCHNER-RIESZ PROBLEM

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ABSTRACT. Motivated by the problem of spherical summability of products of Fourier series, we study the boundedness of the bilinear Bochner-Riesz multipliers $(1-|\xi|^2-|\eta|^2)_+^\delta$ and we make some advances in this investigation. We obtain an optimal result concerning the boundedness of these means from $L^2 \times L^2$ into L^1 with minimal smoothness, i.e., any $\delta > 0$, and we obtain estimates for other pairs of spaces for larger values of δ. Our study is broad enough to encompass general bilinear multipliers $m(\xi,\eta)$ radial in ξ and η with minimal smoothness, measured in Sobolev space norms. The results obtained are based on a variety of techniques, that include Fourier series expansions, orthogonality, and bilinear restriction and extension theorems.

CONTENTS

1. INTRODUCTION

The study of the summability of the product of two n-dimensional Fourier series leads to questions concerning the norm convergence of partial sums of the form

$$\sum_{|m|^2+|k|^2\le R^2} \widehat{F}(m)e^{2\pi i m\cdot x}\,\widehat{G}(k)e^{2\pi i k\cdot x},$$

Date: March 31, 2013.
2010 *Mathematics Subject Classification.* 42B15, 42B20, 47B38.
Key words and phrases. Bilinear Bochner-Riesz problem; bilinear Fourier multipliers; bilinear restriction-extension operators.

as $R \to \infty$, or more generally, of the bilinear Bochner-Riesz means

$$\sum_{|m|^2+|k|^2 \le R^2} \left(1 - \tfrac{|m|^2+|k|^2}{R^2}\right)^{\delta} \widehat{F}(m) e^{2\pi i m\cdot x}\, \widehat{G}(k) e^{2\pi i k \cdot x} \tag{1.1}$$

for some $\delta \ge 0$. Here F, G are 1-periodic functions on the n-torus and $\widehat{F}(m), \widehat{G}(k)$ are their Fourier coefficients and $m, k \in \mathbb{Z}^n$. The bilinear Bochner-Riesz problem is the study of the norm convergence of the sum in (1.1). By basic functional analysis and transference, this problem is equivalent to the study of the $L^{p_1} \times L^{p_2} \to L^p$ boundedness of the bilinear Fourier multiplier operator

$$S^{\delta}(f,g)(x) := \iint_{|\xi|^2+|\eta|^2 \le 1} \left(1 - |\xi|^2 - |\eta|^2\right)^{\delta} \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i x\cdot(\xi+\eta)} d\xi d\eta. \tag{1.2}$$

Here $x \in \mathbb{R}^n$, f, g are functions on $\mathbb{R}^n$ and $\widehat{f}, \widehat{g}$ are their Fourier transforms.

The Bochner-Riesz summability question is a fundamental problem in mathematics. Its study has led to the development of important notions, tools, and results in Fourier analysis, and has created numerous directions of research. The Bochner-Riesz conjecture is well known to be difficult and remains unsolved for indices p near 2 in dimensions $n \ge 3$. The bilinear Bochner-Riesz problem is more difficult than its linear counterpart because of the natural complexity that arises from the mixed summability and also from the shortage of techniques to study bilinear Fourier multipliers with minimal smoothness. The present work is motivated by this problem and fits under the scope of the program to find minimal smoothness conditions for a bilinear Fourier multiplier to be bounded on products of Lebesgue spaces. We are mainly interested in theorems concerning compactly supported Fourier multipliers. The main question we address is what is the least amount of differentiability required of a generic function on $\mathbb{R}^n \times \mathbb{R}^n$ to become a bilinear Fourier multiplier on a certain product of Lebesgue spaces. For the purposes of this article, differentiability is measured in terms of Sobolev space norms which quantitatively fine-tune fractional smoothness. Our results concerning the bilinear Bochner-Riesz means fit in this general framework.

It is well known that linear multiplier operators are L^2 bounded if and only if the multiplier is a bounded function. But we know from [28] that there exist smooth functions m satisfying

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta)| \le C_{\alpha\beta} |\xi|^{-|\alpha|} |\eta|^{-|\beta|}, \qquad \xi, \eta \ne 0$$

for all multi-indices α, β and also from [2] that there exist smooth functions m satisfying

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta)| \le C_{\alpha\beta}$$

for all $(\xi,\eta) \in \mathbb{R}^{2n}$ and all multi-indices α and β, which do not give rise to bounded bilinear operators (as defined in (2.1)) from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $1/p_1 + 1/p_2 = 1/p$ and $1 \le p_1, p_2, p \le \infty$. So there is no direct analogy with the linear case where L^2 presents itself as a natural starting point of the investigation of multiplier theorems.

So we aim to focus our study on more particular bilinear operators. Suppose that a bilinear operator T, initially acting from $\mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$, admits an $L^{p_1} \times L^{p_2} \to L^p$ bounded extension, i.e., it is a bilinear Fourier multiplier for some $1 < p_1, p_2, p < \infty$ with $1/p_1 + 1/p_2 = 1/p$. Then the following properties are equivalent:

(i) *Frequency representation.* There exists a bounded function m on $\mathbb{R}^{2n}$ such that for all $f, g, h \in \mathscr{S}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} T(f,g)(x)\overline{h(x)}\, dx = \int_{\mathbb{R}^{2n}} m(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)\overline{\widehat{h}(\xi+\eta)}\, d\xi d\eta.$$

(ii) *Kernel representation.* There exists a tempered distribution K on $\mathbb{R}^{2n}$ such that for all $f, g \in \mathscr{S}(\mathbb{R}^n)$ we have

$$T(f,g)(x) = \langle K, f(x-\cdot) \otimes g(x-\cdot)\rangle,$$

where $(f(x-\cdot)\otimes g(x-\cdot))(y,z) = f(x-y)g(x-z)$ for all $x,y,z \in \mathbb{R}^n$.
(iii) *Commutativity with simultaneous translation.* For every $y \in \mathbb{R}^n$ and for every function $f, g \in \mathscr{S}(\mathbb{R}^n)$ we have

$$T(\tau_y(f), \tau_y(g)) = \tau_y(T(f,g))$$

where τ_y is the translation operator $\tau_y(f)(x) = f(x-y)$. This property takes into account the additive structure of the Euclidean space via the group of translations.

Bilinear multipliers are not invariant under rotations but the following is true: let T be a bilinear Fourier multiplier on $\mathbb{R}^n$ and m be its symbol; then the symbol is biradial, i.e., $m(\xi,\eta) = m_0(|\xi|,|\eta|)$ (for some $m_0 \in L^\infty(\mathbb{R}^2)$) if and only if for every pair of orthogonal transformations (rotations) $\mathcal{R}_1,\ \mathcal{R}_2$ of $\mathbb{R}^n$ we have

$$T(f,g)(0) = T(f\circ\mathcal{R}_1, g\circ\mathcal{R}_2)(0).$$

Such operators naturally appear in the study of scattering properties associated to quadratic PDEs involving functions of the Laplacian (see [3, Section 2.3]).

Of course, this property reduces, in some sense, a $2n$-dimensional symbol to a 2-dimensional symbol and this work aims to understand how one can take advantage of this property. We observe that for radial multipliers, differentiability is only relevant in the radial direction, and the point $L^2 \times L^2 \to L^1$ seems to be the one requiring the least smoothness. We point out that the duals of a bi-radial bilinear multiplier $m_0(|\xi|,|\eta|)$, $m_0(|\xi+\eta|,|\eta|)$ and $m_0(|\xi|,|\xi+\eta|)$, are not bi-radial functions, so certain results we obtain are not symmetric in the local L^2 triangle, i.e., the set $\{(1/p_1, 1/p_2, 1/p) \text{ with } 2 \le p_1, p_2, p' \le \infty\}$; here $p' = p/(p-1)$.

Let us give some examples of bilinear multipliers, pointing out different situations with respect to the nature of the singular space of the symbol m: we say that m is allowed to be singular a set $\Gamma \subset \mathbb{R}^{2n}$ if m is smooth in the complement Γ^c and satisfies

$$\left|\partial^\alpha_{(\xi,\eta)} m(\xi,\eta)\right| \le C_\alpha d((\xi,\eta),\Gamma)^{-|\alpha|} \tag{1.3}$$

for every $(\xi,\eta) \in \Gamma^c$ and multi-index α.

- *Singularity at one point* $\Gamma := \{0\}$ (Coifman and Meyer [16, 17, 18].) Suppose that the bounded function $m(\xi,\eta)$ on $\mathbb{R}^{2n}$ satisfies (1.3) with $\Gamma := \{0\}$ and so $d((\xi,\eta),\Gamma) \simeq |\xi| + |\eta|$. Then the operator T_m is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$ when $1/p_1 + 1/p_2 = 1/p$, $1 < p_1, p_2, p \le \infty$. This theorem was extended to the case $1/2 < p \le 1$ by Grafakos and Torres [31] and independently by Kenig and Stein [34]. This extension also includes the endpoint case $L^1 \times L^1 \to L^{1/2,\infty}$.
- *Singularity along a line* (Lacey and Thiele [36, 37].) The bilinear Hilbert transform was shown to be bounded on Lebesgue spaces by Lacey and Thiele. This corresponds to the case where Γ is a non-degenerate line of $\mathbb{R}^2$.
- *Singularity along the circle,* $\Gamma := \mathbb{S}^1$ (Grafakos and Li [29].) The characteristic function of the unit disc is a bilinear Fourier multiplier from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$ when $2 \le p_1, p_2, p' < \infty$ and $1/p_1 + 1/p_2 = 1/p$.
- *Singularity on the boundary of a disc,* $\Gamma := \mathbb{S}^1$ (Diestel and Grafakos [21].) The characteristic function of the unit disc in $\mathbb{R}^4$ is not a bilinear Fourier multiplier from $L^{p_1}(\mathbb{R}^2) \times L^{p_2}(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$ when $1/p_1 + 1/p_2 = 1/p$ and exactly one of p_1, p_2, p' is less than 2.

- *Singularity along a curve* (Bernicot-Germain [7].) In this work, certain one-dimensional bilinear operators whose symbols are singular along a curve are shown to be bounded. Taking advantage of the non-degeneracy or the non-vanishing curvature some sharp estimates in the Hölder scaling (or sub-Hölder scaling) are proved. There, the variables are uni-dimensional and Γ is a curve in $\mathbb{R}^2$ and so it has dimension 1.
- *Singularity along a subspace* (Demeter, Pramanik and Thiele [19, 20].) In [20], if Γ is a subspace, preserving the "n-coordinates structure" and of dimension $\kappa \leq \frac{3d}{2}$, then operators associated to symbols singular along such non-degenerate subspace are shown to be bounded on Lebesgue spaces [20]. However, the time-frequency analysis used for the bilinear Hilbert transform is not adapted to the multi-dimensional setting with a high-dimensional singular subspace (as observed in [19]). Indeed, it does not allow to understand how the mixing of the coordinates behave in the frequency plane. A simpler model was considered by Bernicot and Kovac to handle the "twisted paraproducts" [5, 35].
- *Boundedness on Hardy spaces* (Miyachi and Tomita [39], Tomita [47].) Suppose that $0 < p_1, p_2 \leq \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and that

$$s_1 > \max\left\{\frac{n}{2}, \ \frac{n}{p_1} - \frac{n}{2}\right\}, \quad s_2 > \max\left\{\frac{n}{2}, \ \frac{n}{p_2} - \frac{n}{2}\right\},$$

$$s_1 + s_2 > n\Big(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{2}\Big) = n\Big(\frac{1}{p} - \frac{1}{2}\Big).$$

  Assume that for some smooth bump Ψ supported in $6/7 \leq |\xi| \leq 2$ and equal to 1 on $1 \leq |\xi| \leq 12/7$ we have

$$K = \sup_{j\in\mathbb{Z}} \|m(2^j\xi_1, 2^j\xi_2)\Psi(\xi_1, \xi_2)\|_{W^{(s_1,s_2)}} < \infty,$$

  where

$$\|F\|_{W^{(s_1,s_2)}} = \Big(\int_{\mathbb{R}^{2n}} (1 + |\xi_1|^2)^{2s_1}(1 + |\xi_2|^2)^{2s_2}|\widehat{F}(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2\Big)^{1/2}.$$

  Then T_m is a bounded bilinear operator on products of Hardy spaces with norm

$$\|T_m\|_{H^{p_1}(\mathbb{R}^n)\times H^{p_2}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \leq C\,K,$$

  where $L^\infty(\mathbb{R}^n)$ should be replaced by $BMO(\mathbb{R}^n)$ when $p_1 = p_2 = \infty$.

From this quick review of existing results, it appears that high-dimensional symbols singular along hypersurfaces have not been studied, according to our understanding. Our approach of biradial bilinear Fourier multiplier will allow us to consider the bilinear counterpart S^δ (as defined in (1.2)) of the celebrated Bochner-Riesz multiplier. Here the symbol is singular along the sphere $\{(\xi, \eta) \in \mathbb{R}^{2n}, \ |\xi|^2 + |\eta|^2 = 1\}$ which has dimension $2n - 1$.

The almost optimal solution of the bilinear Bochner-Riesz problem in dimension 1 is outlined in Theorem 4.1. We end this introduction by summarizing some critical estimates obtained in this article for the bilinear Bochner-Riesz means when $n \geq 2$:

- S^δ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ when $\delta > 0$. (Theorem 4.7.)
- S^δ is bounded from $L^2(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ if $\delta > \frac{n-1}{2}$. (Theorem 4.8.)
- S^δ is bounded from $L^1(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ when $\delta > \frac{n}{2}$. (Theorem 4.9.)
- S^δ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $1 \leq p_1, p_2 < 2n/(n+1)$, $1/p = 1/p_1 + 1/p_2$, $\delta > n\alpha(p_1, p_2) - 1$, where $\alpha(p_1, p_2)$ is as in (3.12). (Theorem 4.3.)

2. Notation and preliminary results

2.1. **Notation.** We introduce the notation that will be relevant for this paper. We use $A \lesssim B$ to denote the statement that $A \le CB$ for some implicit, universal constant C, and the value of C may change from line to line. We denote by $x \cdot y = \sum_j x_j y_j$ the usual dot product of points $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in $\mathbb{R}^n$. We denote by $\mathscr{S}(\mathbb{R}^n)$ the Schwartz space of all rapidly decreasing smooth functions on $\mathbb{R}^n$. For a function f in $\mathscr{S}(\mathbb{R}^n)$, we define the Fourier transform $\mathscr{F} f$ and its inverse Fourier transform $\mathscr{F}^{-1} f$ by the formulae

$$\mathscr{F} f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)\, dx$$

and

$$\mathscr{F}^{-1} f(\xi) = \check{f}(\xi) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} f(x)\, dx.$$

For $1 \le p \le \infty$, we denote by p' its conjugate exponent, i.e., the unique number in $[1, \infty]$ such that $1/p + 1/p' = 1$. For $1 \le p \le +\infty$, we denote the norm of a function $f \in L^p(\mathbb{R}^n)$ by $\|f\|_p$. For $s \ge 0$ and $1 < p < \infty$, the Sobolev space $W^{s,p}(\mathbb{R}^n)$ is defined as the space of all functions such that

$$(I - \Delta)^{2/s}(f) = \mathscr{F}^{-1}\Big((1 + 4\pi^2|\xi|^2)^{s/2} \mathscr{F} f(\xi)\Big)$$

lies in $L^p(\mathbb{R}^n)$. In this case we set $\|f\|_{W^{s,p}} = \|(I - \Delta)^{s/2}(f)\|_{L^p}$.

The scalar product in $L^2(\mathbb{R}^n)$ is denote by

$$\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx\,.$$

Let X, Y, Z be quasi-normed spaces. If T is a bounded bilinear operator from $X \times Y$ to Z, we write $\|T\|_{X\times Y\to Z}$ for the operator norm of T. Given a subset $E \subseteq \mathbb{R}^n$, we denote by χ_E the characteristic function of E and we denote by

$$P_E f(x) = \chi_E(x) f(x)$$

the "projection" operator on E.

Given a bounded function $m(\xi, \eta)$ on $\mathbb{R}^n \times \mathbb{R}^n$, we denote by T_m the bilinear Fourier multiplier with symbol m. This operator is written in the form

$$T_m(f, g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{2\pi i x\cdot(\xi+\eta)} m(\xi, \eta) \widehat{f}(\xi)\widehat{g}(\eta) d\xi d\eta \tag{2.1}$$

for Schwartz functions f, g. Equivalently, in physical space is given as

$$T_m(f, g)(x) = \mathscr{F}^{-1}\Big[\int_{\mathbb{R}^n} m(\xi - \eta, \eta)\widehat{f}(\xi - \eta)\widehat{g}(\eta) d\eta\Big](x)$$

and also as

$$T_m(f, g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{m}(y - x, z - x) f(y) g(z) dy dz\,. \tag{2.2}$$

This is a bilinear translation invariant operator with kernel $K(y, z) = \widehat{m}(-y, -z)$, i.e., it has the form

$$T_m(f, g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x - y, x - z)\, f(y) g(z)\, dy dz\,. \tag{2.3}$$

2.2. **Criteria for boundedness of bilinear multipliers.** We begin with the following trivial situation.

Lemma 2.1. *Let* $1 \le p_1, p_2, p \le \infty$ *and* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. *If the symbol* $m(\xi, \eta)$ *satisfies*

$$A_1 = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\widehat{m}(x,y)| dx dy < \infty,$$

then T_m *maps* $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ *with*

$$\|T_m\|_{L^{p_1} \times L^{p_2} \to L^p} \le A_1.$$

The proof of Lemma 2.1 is omitted since it is an easy consequence of Minkowski's integral inequality and Hölder's inequality.

We now consider an off-diagonal case.

Lemma 2.2. (i) *If the symbol* $m(\xi, \eta)$ *satisfies*

$$A_2 = \sup_{\xi \in \mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |m(\xi - \eta, \eta)|^2 d\eta \Big)^{1/2} < \infty, \tag{2.4}$$

then T_m *maps* $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *with*

$$\|T_m(f,g)\|_2 \le A_2 \|f\|_2 \|g\|_2.$$

(ii) *If the symbol* $m(\xi, \eta)$ *is supported on a ball of radius* R, *say* $B(0, R)$, *and satisfies* (2.4), *then for all* $1 \le p, q \le 2 \le r \le \infty$, *there exists a constant* $C = C_{p,q,r}$ *such that*

$$\|T_m(f,g)\|_r \le C A_2 R^{n(\frac{1}{p} + \frac{1}{q} - \frac{1}{r} - \frac{1}{2})} \|f\|_p \|g\|_q.$$

Proof. The proof of (i) follows from an application of the Plancherel identity and the Cauchy-Schwarz inequality.

$$\begin{aligned}
\|T_m(f,g)\|_2^2 &= \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} m(\xi - \eta, \eta) \widehat{f}(\xi - \eta) \widehat{g}(\eta) d\eta \Big|^2 d\xi \\
&\le \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |m(\xi - \eta, \eta)|^2 d\eta \Big) \Big(\int_{\mathbb{R}^n} |\widehat{f}(\xi - \eta) \widehat{g}(\eta)|^2 d\eta \Big) d\xi \\
&\le A_2^2 \|f\|_2^2 \|g\|_2^2.
\end{aligned}$$

We now prove (ii). Since the symbol $m(\xi, \eta)$ is supported in the ball $B(0, R)$, we use the Cauchy-Schwarz inequality and Plancherel's identity to obtain

$$\begin{aligned}
\|T_m(f,g)\|_\infty &\le \Big\| \mathscr{F}_\xi^{-1} \Big[\int_{\mathbb{R}^n} m(\xi - \eta, \eta) \widehat{f}(\xi - \eta) \widehat{g}(\eta) d\eta \Big] \Big\|_\infty \\
&\lesssim R^{n/2} \Big\| \int_{\mathbb{R}^n} m(\xi - \eta, \eta) \widehat{f}(\xi - \eta) \widehat{g}(\eta) d\eta \Big\|_2 \\
&\lesssim R^{n/2} \|T_m(f,g)\|_2 \,.
\end{aligned}$$

In view of the support properties of m, in the expression $\|T_m(f,g)\|_2$, one may replace f and g by $(\widehat{f}\chi_{B(0,R)})^\vee$ and $(\widehat{f}\chi_{B(0,R)})^\vee$, respectively. Let $r \ge 2$. It follows by interpolation and by the result in (i) that

$$\begin{aligned}
\|T_m(f,g)\|_r &\lesssim R^{\frac{n}{2}(1 - \frac{2}{r})} \|T_m(f,g)\|_2 \\
&\lesssim A_2 R^{\frac{n}{2}(1 - \frac{2}{r})} \|\widehat{f}\|_{L^2(B(0,R))} \|\widehat{g}\|_{L^2(B(0,R))} \\
&\lesssim A_2 R^{n(\frac{1}{p} + \frac{1}{q} - \frac{1}{r} - \frac{1}{2})} \|f\|_p \|g\|_q.
\end{aligned}$$

This proves (ii), and thus completes the proof of Lemma 2.2. □

The following lemma is inspired by the result of Guillarmou, Hassell, and Sikora [32] in the linear case.

Lemma 2.3. *Let $1 \le p, q \le \infty$ and $1/r = 1/p + 1/q$ and $0 < r \le \infty$. Suppose T is a bounded bilinear operator from $L^{p_1}(\mathbb{R}^n) \times L^{q_1}(\mathbb{R}^n) \to L^s(\mathbb{R}^n)$ for some s, p_1, q_1 satisfying $0 < r \le s$, $1 \le p_1 \le p$ and $1 \le q_1 \le q$ such that the kernel K_T of T satisfies*

$$\operatorname{supp} K_T \subseteq \mathcal{D}_\rho := \big\{(x, y, z) : \ |x - y| < \rho, |x - z| < \rho\big\}$$

for some $\rho > 0$. Then there exists a constant $C = C_{r,s} > 0$ such that

$$\|T\|_{L^p \times L^q \to L^r} \le C\rho^{n(\frac{1}{p_1} + \frac{1}{q_1} - \frac{1}{s})} \|T\|_{L^{p_1} \times L^{q_1} \to L^s}. \tag{2.5}$$

Proof. We fix $\rho > 0$. Then we choose a sequence of points $(x_i)_i$ in $\mathbb{R}^n$ such that for $i \ne j$ we have $|x_i - x_j| > \rho/10$ and $\sup_{x \in \mathbb{R}^n} \inf_i |x - x_i| \le \rho/10$. Such sequence exists because $\mathbb{R}^n$ is separable. Secondly, we let $B_i = B(x_i, \rho)$ and define $\widetilde{B_i}$ by the formula

$$\widetilde{B_i} = \overline{B\Big(x_i, \frac{\rho}{10}\Big)} \setminus \bigcup_{j<i} \overline{B\Big(x_j, \frac{\rho}{10}\Big)},$$

where $\overline{B\,(x, \rho)} = \{y \in \mathbb{R}^n \colon |x - y| \le \rho\}$. Finally we set $\chi_i = \chi_{\widetilde{B}_i}$, where $\chi_{\widetilde{B}_i}$ is the characteristic function of the set $\widetilde{B}_i$. Note that for $i \ne j$, $B(x_i, \frac{\rho}{20}) \cap B(x_j, \frac{\rho}{20}) = \emptyset$. Hence

$$K = \sup_i \#\{j : \ |x_i - x_j| < 2\rho\} \le \sup_x \frac{|B(x, (2 + \frac{1}{20})\rho)|}{|B(x, \frac{\rho}{20})|} = 41^n < \infty. \tag{2.6}$$

It is not difficult to see that

$$\mathcal{D}_\rho \ \subseteq \bigcup_{\substack{i,j,k\colon |x_i - x_j| < 2\rho \\ |x_i - x_k| < 2\rho}} \widetilde{B}_i \times (\widetilde{B}_j \times \widetilde{B}_k) \subset \mathcal{D}_{4\rho} \tag{2.7}$$

and so

$$T(f, g) = \sum_i \sum_{\substack{j\colon |x_i - x_j| < 2\rho \\ k\colon |x_i - x_k| < 2\rho}} P_{\widetilde{B}_i} T(P_{\widetilde{B}_j} f, P_{\widetilde{B}_k} g).$$

Let $K_r = \max\{1, K^{2(r-1)}\}$. By Hölder's inequality we have

$$\begin{aligned}
\|T(f, g)\|_r^r &= \Bigg\| \sum_i \sum_{\substack{j\colon |x_i - x_j| < 2\rho \\ k\colon |x_i - x_k| < 2\rho}} P_{\widetilde{B}_i} T(P_{\widetilde{B}_j} f, P_{\widetilde{B}_k} g) \Bigg\|_r^r \\
&= \sum_i \Bigg\| \sum_{\substack{j\colon |x_i - x_j| < 2\rho \\ k\colon |x_i - x_k| < 2\rho}} P_{\widetilde{B}_i} T(P_{\widetilde{B}_j} f, P_{\widetilde{B}_k} g) \Bigg\|_r^r \\
&\lesssim K_r \sum_i \sum_{\substack{j\colon |x_i - x_j| < 2\rho \\ k\colon |x_i - x_k| < 2\rho}} \|P_{\widetilde{B}_i} T(P_{\widetilde{B}_j} f, P_{\widetilde{B}_k} g)\|_r^r \\
&\lesssim K_r \rho^{nr(\frac{1}{r} - \frac{1}{s})} \sum_i \sum_{\substack{j\colon |x_i - x_j| < 2\rho \\ k\colon |x_i - x_k| < 2\rho}} \|T(P_{\widetilde{B}_j} f, P_{\widetilde{B}_k} g)\|_s^r.
\end{aligned}$$

Since T is a bounded bilinear operator from $L^{p_1}(\mathbb{R}^n) \times L^{q_1}(\mathbb{R}^n) \to L^s(\mathbb{R}^n)$, we have

$$\begin{aligned} \|T(P_{\widetilde{B}_j} f, P_{\widetilde{B}_k} g)\|_s &\leq \|T\|_{L^{p_1}\times L^{q_1}\to L^s} \|P_{\widetilde{B}_j} f\|_{p_1} \|P_{\widetilde{B}_k} g\|_{q_1} \\ &\lesssim \rho^{n(\frac{1}{p_1}+\frac{1}{q_1}-\frac{1}{p}-\frac{1}{q})} \|T\|_{L^{p_1}\times L^{q_1}\to L^s} \|P_{\widetilde{B}_j} f\|_{p} \|P_{\widetilde{B}_k} g\|_{q}. \end{aligned}$$

We proceed by estimating

$$\mathscr{E}(f,g) = \sum_i \sum_{\substack{j:\,|x_i-x_j|<2\rho \\ k:\,|x_i-x_k|<2\rho}} \|P_{\widetilde{B}_j} f\|_p^r \|P_{\widetilde{B}_k} g\|_q^r .$$

Note that $1/r = 1/p + 1/q$. We use Hölder's inequality twice, together with (2.6), to bound $\mathscr{E}(f,g)$ by

$$\begin{aligned} & K \sum_i \Big\{ \sum_{j:\,|x_i-x_j|<2\rho} \|P_{\widetilde{B}_j} f\|_p^p \Big\}^{r/p} \Big\{ \sum_{k:\,|x_i-x_k|<2\rho} \|P_{\widetilde{B}_k} g\|_q^q \Big\}^{r/q} \\ &\leq K \Big\{ \sum_i \sum_{j:\,|x_i-x_j|<2\rho} \|P_{\widetilde{B}_j} f\|_p^p \Big\}^{r/p} \Big\{ \sum_i \sum_{k:\,|x_i-x_k|<2\rho} \|P_{\widetilde{B}_k} g\|_q^q \Big\}^{r/q} \\ &\leq K^2 \Big\{ \sum_j \|P_{\widetilde{B}_j} f\|_p^p \Big\}^{r/p} \Big\{ \sum_k \|P_{\widetilde{B}_k} g\|_q^q \Big\}^{r/q} \\ &\leq K^2 \|f\|_p^r \|g\|_q^r. \end{aligned}$$

This estimate combined with the previously obtained estimate for $\|T(f,g)\|_r^r$ in terms of $\mathscr{E}(f,g)$ yields (2.5). The proof is now complete. □

It will be useful to apply Lemma 2.3 for operators, which do not have such perfect localization properties. For such, we have the following version:

Lemma 2.4. *Let $1 \leq p, q \leq \infty$ and $1/r = 1/p + 1/q$ and $0 < r \leq \infty$. Suppose T is a bounded bilinear operator from $L^{p_1}(\mathbb{R}^n) \times L^{q_1}(\mathbb{R}^n) \to L^s(\mathbb{R}^n)$ for some s, p_1, q_1 satisfying $0 < r \leq s$, $1 \leq p_1 \leq p$ and $1 \leq q_1 \leq q$ such that the kernel K_T of T satisfies*

$$|K_T(x,y,z)| \lesssim \rho^{-d} \left(1 + \rho^{-1}|x-y| + \rho^{-1}|x-z|\right)^{-M}$$

for some $\rho \geq 1$, $d > 0$ and every large enough integer $M > 0$. Then for every $\epsilon > 0$ (as small as we want) and $N > 0$ (as large as we want) there exists a constant $C = C_{r,s,\epsilon} > 0$ such that

$$\|T\|_{L^p\times L^q\to L^r} \leq C \rho^{\epsilon + n(\frac{1}{p_1}+\frac{1}{q_1}-\frac{1}{s})} \|T\|_{L^{p_1}\times L^{q_1}\to L^s} + \rho^{-N}. \tag{2.8}$$

Proof. The proof is very similar to the previous one. Let us fix $\epsilon > 0$ and consider a collection of points $(x_i)_i$ in $\mathbb{R}^n$ such that for $i \neq j$ we have $|x_i - x_j| > \rho^{1+\epsilon}/10$ and $\sup_{x\in\mathbb{R}^n} \inf_i |x - x_i| \leq \rho^{1+\epsilon}/10$. Then, with the previous notation, we have

$$\begin{aligned} &\|T(f,g)\|_r \\ &\lesssim \Big\| \sum_i \sum_{\substack{j:\,|x_i-x_j|<2\rho^{1+\epsilon} \\ k:\,|x_i-x_k|<2\rho^{1+\epsilon}}} P_{\widetilde{B}_i} T(P_{\widetilde{B}_j} f, P_{\widetilde{B}_k} g) \Big\|_r + \Big\| \sum_i \sum_{\substack{j,k:\,|x_i-x_j|>2\rho^{1+\epsilon} \\ \text{or } |x_i-x_k|>2\rho^{1+\epsilon}}} P_{\widetilde{B}_i} T(P_{\widetilde{B}_j} f, P_{\widetilde{B}_k} g) \Big\|_r \\ &:= I + II. \end{aligned}$$

For the I, we repeat exactly the same reasoning as for Lemma 2.3 (since it corresponds to the diagonal part), by replacing ρ by $\rho^{1+\epsilon}$. So we obtain

$$I \lesssim \rho^{(1+\epsilon)n(\frac{1}{p_1}+\frac{1}{q_1}-\frac{1}{s})}\|T\|_{L^{p_1}\times L^{q_1}\to L^s},$$

which is as claimed since ϵ can be chosen as small as we want.

We now deal with the second quantity II. We have

$$\begin{aligned} II &\leq \sum_{2^\ell \geq \rho^\epsilon} \Big\| \sum_i \sum_{\substack{j,k: \\ |x_i-x_j|+|x_i-x_k|\simeq \rho 2^\ell}} P_{\widetilde{B}_i} T(P_{\widetilde{B}_j} f, P_{\widetilde{B}_k} g) \Big\|_r \\ &\lesssim \sum_{2^\ell \geq \rho^\epsilon} \rho^{-d} 2^{-\ell M} (\rho 2^\ell)^{3n} \Big\| \sum_j |P_{\widetilde{B}_j} f| \Big\|_p \Big\| \sum_k |P_{\widetilde{B}_k} g| \Big\|_q, \end{aligned}$$

where we used that for j,k fixed, there is at most $(\rho 2^\ell)^n$ points x_i satisfying

$$|x_i - x_j| + |x_i - x_k| \simeq \rho 2^\ell$$

and the pointwise estimate of the bilinear kernel. So we conclude that

$$\begin{aligned} II &\lesssim \|f\|_p \|g\|_q \Big(\sum_{2^\ell \geq \rho^\epsilon} \rho^{-d} 2^{-\ell M} (\rho 2^\ell)^{3n} \Big) \\ &\lesssim \rho^{-d-\epsilon M+3n(1+\epsilon)} \|f\|_p \|g\|_q, \end{aligned}$$

which is also as claimed since M can be chosen as large as we want. □

3. Compactly supported bilinear multipliers

In this section we assume $n \geq 2$ and we are concerned with the boundedness of compactly supported bilinear multipliers. We focus attention to radial such multipliers. These can be written in the form

$$\iint_{\mathbb{R}^{2n}} e^{2\pi i x\cdot(\xi+\eta)} m_0(|\xi|,|\eta|)\widehat{f}(\xi)\widehat{g}(\eta)d\xi d\eta \tag{3.1}$$

for $f, g \in \mathscr{S}(\mathbb{R}^n)$, where $m_0 \in L^\infty(\mathbb{R}^+ \cup \{0\} \times \mathbb{R}^+ \cup \{0\})$. This is exactly the bilinear multiplier operator $T_m(f,g)$, where $m_0(|\xi|,|\eta|) = m(\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_n)$, $\xi = (\xi_1,\dots,\xi_n)$ and $\eta = (\eta_1,\dots,\eta_n)$.

Lemma 3.1. *Let m_0 be an even function on $\mathbb{R}^2$ whose Fourier transform $\widehat{m_0}$ is supported in $[-L,L]^2$. Then the Fourier transform of the biradial function $m(x,y) := m_0(|x|,|y|)$ on $\mathbb{R}^{2n}$ is supported in $[-L,L]^{2n}$.* [1]

[1]We give here a proof, using the finite speed propagation property of the wave propagator. Actually in the linear framework, the claim can be rephrased as follows: a Fourier band-limited function is also a Hankel band-limited function, for the "J_0" Hankel transform. We also refer the reader to [40, 13] for another approach to this question using the Hankel transform.

Proof. We have

$$\begin{aligned}\widehat{m}(\xi,\eta) &= \iint_{\mathbb{R}^{2n}} e^{2i\pi(x\cdot\xi+y\cdot\eta)} m_0(|x|,|y|)dxdy \\ &= \iint_{[0,\infty)^2} m_0(u,v)R_u(\xi)R_v(\eta)du\,dv \\ &= m_0(\sqrt{-\Delta},\sqrt{-\Delta})(\delta_0,\delta_0)(\xi,\eta) \\ &= K_{m_0(\sqrt{-\Delta},\sqrt{-\Delta})}(\xi,\eta),\end{aligned}$$

where $K_{m_0(\sqrt{-\Delta},\sqrt{-\Delta})}$ is the bilinear kernel of the bilinear operator $m_0(\sqrt{-\Delta},\sqrt{-\Delta})$. Expressing m_0 in terms of its 2-dimensional Fourier transform yields

$$\begin{aligned}K_{m_0(\sqrt{-\Delta},\sqrt{-\Delta})}(\xi,\eta) &= \int_{\mathbb{R}^2} \widehat{m_0}(s,t)K_{e^{2i\pi s\sqrt{-\Delta}}e^{2i\pi t\sqrt{-\Delta}}}(\xi,\eta)du\,dv \\ &= \int_{[-L,L]^2} \widehat{m_0}(s,t)K_{e^{2i\pi s\sqrt{-\Delta}}}(\xi)K_{e^{2i\pi t\sqrt{-\Delta}}}(\eta)du\,dv.\end{aligned}$$

Then using finite speed propagation property of the wave propagator, we know that for every $s\in\mathbb{R}$, the kernel $K_{e^{2i\pi s\sqrt{-\Delta}}}$ is supported on $[-|s|,|s|]^n$. Hence, we conclude that $K_{m_0(\sqrt{-\Delta},\sqrt{-\Delta})}$ is supported on $[-L,L]^{2n}$. □

3.1. **Bilinear restriction-extension operators.** For $f\in\mathscr{S}(\mathbb{R}^n)$ recall the restriction-extension operator

$$\mathscr{R}_\lambda f(x) = \lambda^{n-1}\int_{\mathbb{S}^{n-1}} e^{2\pi i\lambda x\cdot\omega}\widehat{f}(\lambda\omega)d\omega \qquad \lambda>0 \tag{3.2}$$

in the linear setting. In the sequel we set

$$\mathscr{R}_{\lambda_1,\lambda_2}(f,g)(x) = \mathscr{R}_{\lambda_1}f(x)\mathscr{R}_{\lambda_2}g(x).$$

Lemma 3.2. *Let $m(\xi,\eta):=m_0(|\xi|,|\eta|)$. For $f,g\in\mathscr{S}(\mathbb{R}^n)$, we have the following formula:*

$$T_m(f,g)(x) = \int_0^\infty\int_0^\infty m_0(\lambda_1,\lambda_2)\mathscr{R}_{\lambda_1,\lambda_2}(f,g)(x)d\lambda_1 d\lambda_2.$$

Proof. The proof can be obtained by expressing $T_m(f,g)(x)$ in polar coordinates. □

To study boundedness of the bilinear restriction-extension operator $\mathscr{R}_{\lambda_1,\lambda_2}$, we first recall some properties of the operator $\mathscr{R}_1$ in the linear setting. Let $d\sigma$ denote surface measure on the unit sphere $\mathbb{S}^{n-1}$. In view of the theory of Bessel function (see page 428 of [26]),

$$\mathscr{R}_1 f(x) = \int_{\mathbb{S}^{n-1}} e^{2\pi i x\cdot\theta}\widehat{f}(\theta)d\theta = \widehat{d\sigma}*f(x), \tag{3.3}$$

where

$$\widehat{d\sigma}(x) = \int_{\mathbf{S}^{n-1}} e^{2\pi i x\cdot\omega}d\omega = \frac{2\pi}{|x|^{\frac{n-2}{2}}}J_{\frac{n-2}{2}}(2\pi|x|) \tag{3.4}$$

and

$$J_\zeta(t) \;=\; \frac{2}{\Gamma(1/2)}\frac{(t/2)^\zeta}{\Gamma(\zeta+1/2)}\int_0^1(1-u^2)^{\zeta-1/2}\cos(ut)du, \quad \mathrm{Re}(\zeta)>-\frac{1}{2}. \tag{3.5}$$

The problem of L^p-L^q boundedness of $\mathscr{R}_1$ has been studied by several authors (see for instance, [1], [9] and [33]). The first results in this direction were obtained by Tomas and Stein

[46], [42]; they showed that $\mathscr{R}_1$ is bounded from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$ for $p = (2n+2)/(n+3)$ and $p' = p/(p-1)$, which implies the sharp $L^p - L^2$ restriction theorem for the sphere $\mathbb{S}^{n-1}$. To describe all pairs (p, q) such that the operator $\mathscr{R}_1$ on $\mathbb{R}^n$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, we define vertices in the square $[0,1] \times [0,1]$ by setting

$$A(n) = \Big(\frac{n+1}{2n}, 0\Big), \qquad B(n) = \Big(\frac{n+1}{2n}, \frac{n-1}{2n} - \frac{n-1}{n^2+n}\Big),$$
$$A'(n) = \Big(1, \frac{n-1}{2n}\Big), \qquad B'(n) = \Big(\frac{n+1}{2n} + \frac{n-1}{n^2+n}, \frac{n-1}{2n}\Big).$$

Let $\Delta(n)$ be the closed pentagon with vertices $A(n), B(n), B'(n), A'(n)$, $(1, 0)$ from which closed line segments $[A(n), B(n)], [A'(n), B'(n)]$ are removed. Namely,

$$\Delta(n) = \left\{ \begin{array}{r} \Big(\frac{1}{p}, \frac{1}{q}\Big) \in [0,1] \times [0,1] : \;\; 0 \le \frac{1}{q} \le \frac{1}{p} \le 1, \;\; \frac{1}{p} - \frac{1}{q} \ge \frac{2}{n+1}, \\ \frac{1}{p} > \frac{n+1}{2n}, \;\; \frac{1}{q} < \frac{n-1}{2n} \end{array} \right\} \tag{3.6}$$

Proposition 3.3. *Let $\mathscr{R}_1$ be defined as in (3.2). There exists a constant $C = C_{p,q} > 0$, independent of f, such that*

$$\|\mathscr{R}_1(f)\|_q \le C\|f\|_p$$

if and only if $(1/p, 1/q)$ in $\Delta(n)$.

Proof. For the proof of Proposition 3.3, we refer it to Remark 1, p. 497, [33]. See also [1], and [9]. □

As a consequence of Proposition 3.3, we obtain the following result.

Proposition 3.4. *(i) Let $1/s = 1/q_1 + 1/q_2$, $0 < s \le \infty$ and let $(1/p_1, 1/q_1)$ and $(1/p_2, 1/q_2)$ be both in $\Delta(n)$ as defined as in (3.6). For every $\lambda_1, \lambda_2 > 0$, the bilinear restriction-extension operator $\mathscr{R}_{\lambda_1,\lambda_2}$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^s(\mathbb{R}^n)$ such that*

$$\|\mathscr{R}_{\lambda_1,\lambda_2}(f,g)\|_s \lesssim \lambda_1^{n(\frac{1}{p_1}-\frac{1}{q_1})-1} \lambda_2^{n(\frac{1}{p_2}-\frac{1}{q_2})-1} \|f\|_{p_1}\|g\|_{p_2}.$$

(ii) In the endpoint case $s = 2$ and $p_1 = p_2 = 1$ we have ,

$$\|\mathscr{R}_{\lambda_1,\lambda_2}(f,g)\|_2 \lesssim \lambda_1^{n-\frac{3}{2}} \lambda_2^{\frac{n-1}{2}} \|f\|_1\|g\|_1, \tag{3.7}$$

assuming $\lambda_1 << \lambda_2$. This corresponds to the result in (i) with $q_2 = \frac{2n}{n-1}$.

Proof. (i) By Hölder's inequality,

$$\|\mathscr{R}_{\lambda_1,\lambda_2}(f,g)\|_s = \|\mathscr{R}_{\lambda_1} f \mathscr{R}_{\lambda_2} g\|_s \;\; \le \;\; \|\mathscr{R}_{\lambda_1} f\|_{q_1} \|\mathscr{R}_{\lambda_2} g\|_{q_2}, \tag{3.8}$$

where $1/s = 1/q_1 + 1/q_2$. Note that by Proposition 3.3, we obtain that if both $(1/p_1, 1/q_1)$ and $(1/p_2, 1/q_2)$ are in $\Delta(n)$, then

$$\|\mathscr{R}_{\lambda_1} f\|_{q_1} \lesssim \lambda_1^{n(\frac{1}{p_1}-\frac{1}{q_1})-1} \|f\|_{p_1} \tag{3.9}$$

and

$$\|\mathscr{R}_{\lambda_2} g\|_{q_2} \lesssim \lambda_2^{n(\frac{1}{p_2}-\frac{1}{q_2})-1} \|g\|_{p_2}. \tag{3.10}$$

The desired estimate now follows from (3.8), (3.9) and (3.10).

(ii) We now prove (3.7). Let $\mathbb{B}$ be the unit ball in $\mathbb{R}^n$. First, $\mathscr{R}_{\lambda_1,\lambda_2}(f,g)$ has a spectrum included in

$$Sp := \lambda_1\mathbb{S}^{n-1} + \lambda_2\mathbb{S}^{n-1} = (\lambda_1+\lambda_2)\mathbb{B} \setminus (\lambda_2-\lambda_1)\mathbb{B}$$

which has a n-dimensional measure

$$|Sp| = |\lambda_1\mathbb{S}^{n-1} + \lambda_2\mathbb{S}^{n-1}| \lesssim \lambda_1\lambda_2^{n-1} \tag{3.11}$$

since $\lambda_1 << \lambda_2$. Moreover, since $f, g \in L^1$ then by Plancherel equality, we have

$$\begin{aligned}
\|\mathscr{R}_{\lambda_1,\lambda_2}(f,g)\|_2 &= \|\mathscr{F}\left[\mathscr{R}_{\lambda_1,\lambda_2}(f,g)\right]\|_2 \\
&= \|\widehat{\mathscr{R}_{\lambda_1}(f)} * \widehat{\mathscr{R}_{\lambda_2}(g)}\|_2 \\
&= \sup_{\substack{h\in L^2\\ \|h\|_2=1}} \iint_{\lambda_1\mathbb{S}^{n-1}\times\lambda_2\mathbb{S}^{n-1}} \widehat{\mathscr{R}_{\lambda_1}(f)}(\xi)\widehat{\mathscr{R}_{\lambda_2}(g)}(\eta)h(\xi+\eta)d\xi d\eta \\
&\le \|f\|_1\|g\|_1 \sup_{\substack{h\in L^2\\ \|h\|_2=1}} \iint_{\lambda_1\mathbb{S}^{n-1}\times\lambda_2\mathbb{S}^{n-1}} |h(\xi+\eta)|d\xi d\eta \\
&\le \|f\|_1\|g\|_1\lambda_1^{n-2} \sup_{\substack{h\in L^2\\ \|h\|_2=1}} \int_{Sp} |h(\omega)|d\omega.
\end{aligned}$$

At the last inequality, we used that for every $\omega \in Sp$

$$\text{meas}\left(\{(\xi,\eta)\in\lambda_1\mathbb{S}^{n-1}\times\lambda_2\mathbb{S}^{n-1} :\ \xi+\eta=\omega\}\right) \lesssim \lambda_1^{n-2},$$

where meas denotes $(n-2)$-dimensional Hausdorff measure. Finally, via (3.11) we obtain

$$\|\mathscr{R}_{\lambda_1,\lambda_2}(f,g)\|_2 \lesssim \|f\|_1\|g\|_1\lambda_1^{n-2}|Sp|^{\frac{1}{2}} \lesssim \|f\|_1\|g\|_1\lambda_1^{n-2}(\lambda_1\lambda_2^{n-1})^{\frac{1}{2}},$$

which yields (3.7). □

3.2. **Restriction-extension estimates imply bilinear multiplier estimates.**

For every $n \ge 2$, set

$$a_n = \frac{n+1}{2n}, \quad \text{and} \quad b_n = \frac{n+1}{2n} + \frac{n-1}{n^2+n}.$$

Let $\epsilon > 0$ and for every $1 \le p_1, p_2 < \frac{2n}{n+1}$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, define

$$\alpha(p_1,p_2,\epsilon) = \begin{cases}
\frac{4}{n+1}, & \text{if } \left(\frac{1}{p_1},\frac{1}{p_2}\right) \in \big(a_n,b_n\big)\times\big(a_n,b_n\big) \\
\frac{2}{n+1} - \frac{n-1}{2n} + \frac{1}{p_2} + \epsilon, & \text{if } \left(\frac{1}{p_1},\frac{1}{p_2}\right) \in \big(a_n,b_n\big)\times\big[b_n,1\big] \\
\frac{2}{n+1} - \frac{n-1}{2n} + \frac{1}{p_1} + \epsilon, & \text{if } \left(\frac{1}{p_1},\frac{1}{p_2}\right) \in \big[b_n,1\big]\times\big(a_n,b_n\big) \\
\frac{1}{p_1} + \frac{1}{p_2} - \frac{n-1}{n} + \epsilon, & \text{if } \left(\frac{1}{p_1},\frac{1}{p_2}\right) \in \big[b_n,1\big]\times\big[b_n,1\big].
\end{cases} \tag{3.12}$$

For simplicity, we will write $\alpha(p_1,p_2)$ instead of $\alpha(p_1,p_2,0)$.

Now we prove the following result.

Theorem 3.5. *Let $1 \le p_1, p_2 < 2n/(n+1)$ and $1/p = 1/p_1 + 1/p_2$. Suppose that m_0 is an even bounded function supported in $[-1,1]\times[-1,1]$ that lies in $W^{\beta,1}(\mathbb{R}^2)$, for some $\beta > n\alpha(p_1,p_2)$. Let $m(\xi,\eta) := m_0(|\xi|,|\eta|)$. Then T_m is a bounded operator from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ and we have*

$$\|T_m\|_{L^{p_1}\times L^{p_2}\to L^p} \le C\|m_0\|_{W^{\beta,1}(\mathbb{R}^2)}. \tag{3.13}$$

Proof. Let $\phi \in C_c^\infty(\mathbb{R})$ be an even function with $\operatorname{supp}\phi \subseteq \{t: \ 1/4 \le |t| \le 1\}$ and

$$\sum_{\ell\in\mathbb{Z}} \phi(2^{-\ell}\lambda) = 1 \qquad \forall \lambda > 0.$$

Then we set $\phi_0(\lambda) = 1 - \sum_{\ell\ge 1}\phi(2^{-\ell}\lambda)$,

$$m_0^{(0)}(\lambda_1,\lambda_2) = \iint_{\mathbb{R}^2} \phi_0(\sqrt{|t_1|^2+|t_2|^2}\,)\widehat{m_0}(t_1,t_2)e^{2\pi i(t_1\lambda_1+t_2\lambda_2)}\, dt_1dt_2 \tag{3.14}$$

and

$$m_0^{(\ell)}(\lambda_1,\lambda_2) = \iint_{\mathbb{R}^2} \phi\big(2^{-\ell}\sqrt{|t_1|^2+|t_2|^2}\,\big)\widehat{m_0}(t_1,t_2)e^{2\pi i(t_1\lambda_1+t_2\lambda_2)}\, dt_1dt_2. \tag{3.15}$$

Note that in view of Fourier inversion and of the preceding decomposition we have that

$$m_0(\lambda_1,\lambda_2) \ = \ \sum_{\ell\ge 0} m_0^{(\ell)}(\lambda_1,\lambda_2). \tag{3.16}$$

Consequently,

$$\begin{aligned} T_m(f,g)(x) \ &= \ \iint_{\mathbb{R}^{2n}} e^{2\pi i x\cdot(\xi+\eta)} m_0(|\xi|,|\eta|)\widehat{f}(\xi)\widehat{g}(\eta)d\xi d\eta \\ &= \ \sum_{\ell\ge 0} T_{m^{(\ell)}}(f,g)(x), \end{aligned} \tag{3.17}$$

where $m^{(\ell)}(\xi,\eta) = m_0^{(\ell)}(|\xi|,|\eta|)$ for $\ell \ge 0$ and

$$T_{m^{(\ell)}}(f,g)(x) = \iint_{\mathbb{R}^{2n}} e^{2\pi i x\cdot(\xi+\eta)} m_0^{(\ell)}(|\xi|,|\eta|)\widehat{f}(\xi)\widehat{g}(\eta)d\xi d\eta. \tag{3.18}$$

It follows from the support properties of ϕ and Lemma 3.1 that the kernel of $T_{m^{(\ell)}}$ is supported in

$$\mathcal{D}_{2^\ell} = \big\{(x,y,z): \ |x-y| < 2^\ell, |x-z| < 2^\ell\big\}.$$

Recall the set $\Delta(n)$ given in (3.6). We observe that if $\epsilon > 0$ is small enough, then there exist $(1/p_1, 1/q_1) \in \Delta(n), (1/p_2, 1/q_2) \in \Delta(n)$ such that

$$\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{q_1} - \frac{1}{q_2} = \alpha(p_1,p_2,\epsilon).$$

Note that $p < 1$. Let $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{s}$ and so $s > 1$. Then Lemma 2.3 yields the existence of a constant $C = C_{p,s}$ such that

$$\|T_{m^{(\ell)}}\|_{L^{p_1}\times L^{p_2}\to L^p} \ \le \ C\, 2^{\ell n(\frac{1}{p}-\frac{1}{s})}\|T_{m^{(\ell)}}\|_{L^{p_1}\times L^{p_2}\to L^s}, \tag{3.19}$$

which yields

$$\begin{aligned} \|T_m\|_{L^{p_1}\times L^{p_2}\to L^p} \ &\le \ \Big\|\sum_{\ell\ge 0} T_{m^{(\ell)}}\Big\|_{L^{p_1}\times L^{p_2}\to L^p} \\ &\lesssim \ \sum_{\ell\ge 0} 2^{\ell\theta+\ell n(\frac{1}{p}-\frac{1}{s})}\|T_{m^{(\ell)}}\|_{L^{p_1}\times L^{p_2}\to L^s} \end{aligned} \tag{3.20}$$

for some constant $\theta \in (0, (\beta - n\alpha(p_1,p_2))/2)$.

Since $m^{(\ell)}$ is not compactly supported we choose a smooth function ψ supported in $(-8,8)$ such that $\psi(\lambda) = 1$ for $\lambda \in (-4,4)$. We set $\Psi(x_1,x_2) = \psi(|x_1|+|x_2|)$ for $x_1,x_2 \in \mathbb{R}^n$ and we note that

$$\|T_{m^{(\ell)}}\|_{L^{p_1}\times L^{p_2}\to L^s} \le \|T_{\Psi m^{(\ell)}}\|_{L^{p_1}\times L^{p_2}\to L^s} + \|T_{(1-\Psi)m^{(\ell)}}\|_{L^{p_1}\times L^{p_2}\to L^s}, \tag{3.21}$$

where $(\Psi m^{(\ell)})(\xi,\eta) =: \psi(|\xi|+|\eta|)m_0^{(\ell)}(|\xi|,|\eta|)$.

To estimate $\|T_{\Psi m^{(\ell)}}\|_{L^{p_1}\times L^{p_2}\to L^s}$, we apply Lemma 3.2, together with Minkowski's inequality ($s>1$), and Proposition 3.4 to obtain

$$\begin{aligned}\|T_{\Psi m^{(\ell)}}\|_{L^{p_1}\times L^{p_2}\to L^s} &\leq \int_0^\infty\int_0^\infty |\psi(\lambda_1+\lambda_2)m_0^{(\ell)}(\lambda_1,\lambda_2)|\|\mathscr{R}_{\lambda_1,\lambda_2}\|_{L^{p_1}\times L^{p_2}\to L^s}d\lambda_1 d\lambda_2\\ &\lesssim \int_0^8\int_0^8 |m_0^{(\ell)}(\lambda_1,\lambda_2)|\lambda_1^{n(\frac{1}{p_1}-\frac{1}{q_1})-1}\lambda_2^{n(\frac{1}{p_2}-\frac{1}{q_2})-1}d\lambda_1 d\lambda_2\\ &\lesssim \|m^{(\ell)}\|_1 \qquad (3.22)\end{aligned}$$

since $\frac{1}{p_i}-\frac{1}{q_i}\geq\frac{2}{n+1}$, thus $n(\frac{1}{p_i}-\frac{1}{q_i})-1\geq\frac{n-1}{n+1}\geq 0$ in view of the fact that $(\frac{1}{p_i},\frac{1}{q_i})\in\Delta(n)$.

Notice that $\frac{1}{p}-\frac{1}{s}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{q_1}-\frac{1}{q_2}=\alpha(p_1,p_2,\epsilon)$. Then we have

$$\begin{aligned}\sum_{\ell\geq 0}2^{\ell\theta+\ell n(\frac{1}{p}-\frac{1}{s})}\|T_{\Psi m_0^{(\ell)}}\|_{L^{p_1}\times L^{p_2}\to L^s} &\lesssim \sum_{\ell\geq 0}2^{\ell(n\alpha(p_1,p_2,\epsilon)+\theta)}\|m_0^{(\ell)}\|_1\\ &\lesssim \|m_0\|_{B_{1,1}^{n\alpha(p_1,p_2,\epsilon)+\theta}}, \qquad (3.23)\end{aligned}$$

where the last inequality follows from the definition Besov space. See, e.g., [4, Chap. VI]. Recall also that if $\beta>n\alpha(p_1,p_2,\epsilon)+\theta$ then

$$W^{\beta,1}(\mathbb{R}^2)\subseteq B_{1,1}^{n\alpha(p_1,p_2,\epsilon)+\theta}(\mathbb{R}^2)$$

and $\|m_0\|_{B_{1,1}^{n\alpha(p_1,p_2,\epsilon)+\theta}(\mathbb{R}^2)}\leq C_\beta\|m_0\|_{W^{\beta,1}(\mathbb{R}^2)}$, see again [4].

Next we obtain bounds for $\|T_{(1-\Psi)m^{(\ell)}}\|_{L^{p_1}\times L^{p_2}\to L^s}$. Since the function $1-\psi$ is supported outside the interval $(-4,4)$, we can choose a function $\eta\in C_c^\infty(4,16)$ such that

$$1=\psi(t)+\sum_{k\geq 0}\eta(2^{-k}t)$$

for all $t>0$. Hence for $\lambda_1,\lambda_2>0$ we have

$$\big(1-\psi(\lambda_1+\lambda_2)\big)m_0^{(\ell)}(\lambda_1,\lambda_2)=\sum_{k\geq 0}\eta(2^{-k}(\lambda_1+\lambda_2))m_0^{(\ell)}(\lambda_1,\lambda_2).$$

We then apply an argument as in (3.22) to show that

$$\begin{aligned}&\|T_{(1-\Psi)m^{(\ell)}}\|_{L^{p_1}\times L^{p_2}\to L^s}\\ &\lesssim \sum_{k\geq 0}\int_0^{2^{k+4}}\int_0^{2^{k+4}}|\eta(2^{-k}(\lambda_1+\lambda_2))m_0^{(\ell)}(\lambda_1,\lambda_2)|\lambda_1^{n(\frac{1}{p_1}-\frac{1}{q_1})-1}\lambda_2^{n(\frac{1}{p_2}-\frac{1}{q_2})-1}d\lambda_1 d\lambda_2. \qquad (3.24)\end{aligned}$$

Observe that

$$m_0^{(\ell)}(\lambda_1,\lambda_2)=\iint_{[-1,1]^2}m_0(s_1,s_2)\Big[\iint_{\mathbb{R}^2}\phi(2^{-\ell}\sqrt{|t_1|^2+|t_2|^2})e^{2\pi i((\lambda_1-s_1)t_1+(\lambda_2-s_2)t_2)}dt_1dt_2\Big]ds_1ds_2.$$

We can integrate by parts M times to obtain

$$|\eta(2^{-k}(\lambda_1+\lambda_2))m_0^{(\ell)}(\lambda_1,\lambda_2)|\leq C_M 2^{-(\ell+k)M+2\ell}\|m_0\|_1.$$

Substituting this back into (3.24) with M sufficiently large such that

$$n\Big(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{q_1}-\frac{1}{q_2}\Big)-M+2+\theta<0,$$

we obtain

$$\begin{aligned}\|T_{(1-\Psi)m^{(\ell)}}\|_{L^{p_1}\times L^{p_2}\to L^s} &\leq C_M 2^{-\ell(M-2)}\|m_0\|_1 \sum_{k\geq 0} 2^{-kM+kn(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{q_1}-\frac{1}{q_2})} \\ &\leq C_M 2^{-\ell(M-2)}\|m_0\|_1,\end{aligned}$$

which yields

$$\sum_{\ell\geq 0} 2^{\ell\theta+\ell n(\frac{1}{p}-\frac{1}{s})}\|T_{(1-\Psi)m^{(\ell)}}\|_{L^{p_1}\times L^{p_2}\to L^s} \lesssim \|m_0\|_1. \tag{3.25}$$

Finally, (3.13) follows from (3.17), (3.20), (3.21), (3.23) and (3.25). This completes the proof of Theorem 3.5. □

Remark 3.6. *The previous proof relies on a bilinear spherical decomposition of the symbol. The bilinear restriction operator $\mathscr{R}_{\lambda_1,\lambda_2}$ is not well-defined on $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ and so one cannot use these elementary operators to obtain boundedness from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. However, even if the linear operator $\mathscr{R}_{\lambda_1}$ is not well-defined on $L^2(\mathbb{R}^n)$, it is interesting to observe that the average of such operators is well-defined. Indeed, a simple computation gives*

$$\int_\lambda^\mu \mathscr{R}_u(f)(x)du = \int_{\lambda\leq|\xi|\leq\mu} e^{2\pi i x\cdot\xi}\widehat{f}(\xi)d\xi$$

which is bounded on $L^2(\mathbb{R}^n)$ and one has

$$\sup_{\lambda<\mu}\left\|\int_\lambda^\mu \mathscr{R}_u du\right\|_{L^2\to L^2} \leq 1. \tag{3.26}$$

Moreover, in view of the celebrated result of Fefferman [23]*, this operator is unbounded on $L^p(\mathbb{R}^n)$ if $p\neq 2$ (as soon as $n\geq 2$).*

So we can obtain boundedness from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ without employing a spherical decomposition but via a decomposition along a scale of "smoother" operators.

Following the previous remark, we have the following observation concerning the $L^2\times L^2\to L^1$ boundedness of bilinear multipliers.

Lemma 3.7. *Suppose that m_0 is a bounded function with support in $[-1,1]\times[-1,1]$ which satisfies*

$$\partial_{\lambda_1}\partial_{\lambda_2} m_0(\lambda_1,\lambda_2)\in L^1(\mathbb{R}^2).$$

Let $m(\xi,\eta)=m_0(|\xi|,|\eta|)$. Then T_m is bounded from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$; Moreover,

$$\|T_m\|_{L^2\times L^2\to L^1} \lesssim \iint_{\mathbb{R}^2} |\partial_{\lambda_1}\partial_{\lambda_2} m_0(\lambda_1,\lambda_2)|\, d\lambda_1\, d\lambda_2. \tag{3.27}$$

Proof. We employ a proof via a decomposition of the symbol as an average of bilinear restriction operators. First, by modulation and dilation we may assume that m is supported on $[\frac{1}{2},1]\times[\frac{1}{2},1]$. So via an integration by parts we have

$$\begin{aligned} T_m(f,g) &= \iint_{[0,1]\times[0,1]} m_0(\lambda_1,\lambda_2)\,\mathscr{R}_{\lambda_1,\lambda_2}(f,g)\,d\lambda_1\,d\lambda_2 \\ &= \iint_{[0,1]\times[0,1]} \partial_{\lambda_1}\partial_{\lambda_2} m_0(\lambda_1,\lambda_2)\left(\int_0^{\lambda_1}\int_0^{\lambda_2}\mathscr{R}_{a,b}(f,g)da\,db\right)d\lambda_1\,d\lambda_2 \\ &= \iint_{[0,1]\times[0,1]} \partial_{\lambda_1}\partial_{\lambda_2} m_0(\lambda_1,\lambda_2)\left(\int_0^{\lambda_1}\mathscr{R}_a(f)\,da\int_0^{\lambda_2}\mathscr{R}_b(g)\,db\right)d\lambda_1\,d\lambda_2. \end{aligned}$$

Using (3.26) and the Hölder inequality, we deduce

$$\|T_m\|_{L^2\times L^2\to L^1} \leq \iint_{[0,1]\times[0,1]} |\partial_{\lambda_1}\partial_{\lambda_2} m_0(\lambda_1,\lambda_2)|\,d\lambda_1\,d\lambda_2,$$

which concludes the proof. □

Still concerning the boundedness from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, we have the following result:

Proposition 3.8. *Let m_0 be an even function supported in $[-1,1]^2$ which satisfies the regularity condition:*

$$\sup_{u\in[-1,1]} \|m_0(|u|,|\cdot|)\|_{W^{1+\alpha,1}(\mathbb{R})} < \infty$$

for some $\alpha>0$. Then the bilinear operator T_m associated with the symbol $m(\xi,\eta)=m_0(|\xi|,|\eta|)$ is bounded from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.

Proof. We begin by expressing the operator T_m as follows:

$$\begin{aligned} T_m(f,g)(x) &:= \int_{\mathbb{R}^{2n}} e^{2\pi i x\cdot(\xi+\eta)}\widehat{f}(\xi)\widehat{g}(\eta)m_0(|\xi|,|\eta|)d\xi d\eta \\ &= \iint_{[-1,1]^2} m_0(|u|,|v|)\mathscr{R}_{|u|}(f)(x)\mathscr{R}_{|v|}(f)(x). \end{aligned}$$

The idea is to express the function $m_0(|u|,|v|)$ as a tensorial product, so that a product of L^2-bounded linear operators appears. So we fix $u\in[-1,1]$ and we examine the function $v\to m_0(|u|,|v|)$ which is supported in $[-1,1]$ and vanishes at the endpoints ±1. We can expand this function in Fourier series (by considering a periodic extension on $\mathbb{R}$ of period 2) and thus we may write for $u,v\in[-1,1]$

$$m_0(|u|,|v|) = \sum_{k\in\mathbb{Z}} \gamma_k(u)e^{i\pi kv}$$

with Fourier coefficients

$$\gamma_k(u) := \frac{1}{2}\int_{-1}^{1} e^{-i\pi kv} m_0(|u|,|v|)\,dv.$$

These coefficients also satisfy the bound for $\alpha\in(0,1)$

$$|\gamma_k(u)| \lesssim (1+|k|)^{-1-\alpha}\|m_0(|u|,|\cdot|)\|_{W^{1+\alpha,1}([-1,1])} \tag{3.28}$$

for some $\alpha > 0$. Since we have

$$T_m(f,g)(x) = \frac{1}{2}\sum_{k\in\mathbb{Z}} \iint_{[-1,1]^2} \gamma_k(u)\mathscr{R}_{|u|}(f)(x)e^{i\pi kv}\mathscr{R}_{|v|}(f)(x)dudv$$
$$= \frac{1}{2}\sum_{k\in\mathbb{Z}} \left(\int_{[-1,1]} \gamma_k(u)\mathscr{R}_{|u|}(f)(x)du\right)\left(\int_{[-1,1]} e^{i\pi kv}\mathscr{R}_{|v|}(f)(x)dv\right),$$

we conclude by Hölder inequality and (3.28),

$$\|T_m(f,g)\|_1$$
$$\lesssim \sum_{k\in\mathbb{Z}}(1+|k|)^{-1-\alpha}\left\|\int_{[-1,1]}(1+|k|)^{1+\alpha}\gamma_k(u)\mathscr{R}_{|u|}(f)du\right\|_2\left\|\int_{[-1,1]} e^{i\pi kv}\mathscr{R}_{|v|}(f)(x)dv\right\|_2$$
$$\lesssim \|f\|_2\|g\|_2.$$

Here we used that $\int_{[-1,1]}(1+|k|)^{1+\alpha}\gamma_k(u)\mathscr{R}_{|u|}du$ is the linear Fourier multiplier operator associated with the symbol $(1+|k|)^{1+\alpha}\gamma_k(|\xi|)$ which is uniformly (in k) bounded on L^2, since the symbol is bounded with respect to k in view of (3.28). ☐

3.3. **An extension of Theorem 3.5.** Using Lemma 2.3, we may argue as in the proof of Theorem 3.5 to obtain the following result. The proof is similar and for brevity is omitted.

Theorem 3.9. *Let $n \geq 2$ and $1 \leq q_1, q_2 < 2n/(n+1)$ and let $q_1 \leq p_1 \leq \infty, q_2 \leq p_2 \leq \infty$ with $1/p = 1/p_1 + 1/p_2$ and $0 < p \leq \infty$. Also assume that*

$$\frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{p} \leq \alpha(q_1, q_2),$$

where $\alpha(q_1,q_2)$ is defined in (3.12). Suppose that m_0 is a bounded function supported in $[-1,1]\times[-1,1]$ such that $m_0 \in W^{\beta,1}(\mathbb{R}^2)$ for some $\beta > n\alpha(q_1,q_2)$. Let $m(\xi,\eta) := m_0(|\xi|,|\eta|)$. Then T_m is bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. In addition,

$$\|T_m\|_{L^{p_1}\times L^{p_2}\to L^p} \leq C\|m_0\|_{W^{\beta,1}(\mathbb{R}^2)}.$$

4. Bounds for bilinear Bochner-Riesz means

Consider the bilinear Bochner-Riesz means of order δ on $\mathbb{R}^n\times\mathbb{R}^n$, given by

$$S_R^\delta(f,g)(x) = \iint_{|\xi|^2+|\eta|^2\leq R^2} e^{2\pi i x\cdot(\xi+\eta)}\left(1-\frac{|\xi|^2+|\eta|^2}{R^2}\right)^\delta \widehat{f}(\xi)\widehat{g}(\eta)d\xi d\eta. \tag{4.1}$$

In this section, we investigate the range of δ for which the bilinear Bochner-Riesz means S_R^δ are bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. This boundedness holds independently of the parameter $R>0$, so we take $R=1$ in our work and for simplicity we write S^δ instead of S_1^δ.

We first describe the results in the one-dimensional setting, there Bochner-Riesz multipliers are closely related to the problem of the disc multiplier.

Theorem 4.1. *Let $n = 1$. The Bochner-Riesz operator is bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for $1/p = 1/p_1 + 1/p_2$ in the following situations:*

(i) *Strict local L^2-case: $2 < p_1, p_2, p' < \infty$ and $\delta \geq 0$.*
(ii) *Endpoint cases: $\{p_1, p_2, p'\} = \{2, 2, \infty\}$ and $\delta > 0$.*

(iii) *Banach triangle case:* $1 \leq p_1, p_2, p' \leq \infty$ *and* $\delta > 0$.

Proof. The first case is a consequence of the positive result for the disc multiplier problem in [29]. The endpoint can be obtained using a discrete spherical decomposition with [7, Proposition 6.1]. The Banach situation follows from similar arguments with [7, Proposition 6.2]. Indeed, let us check the point $L^\infty \times L^\infty \to L^\infty$. Consider a smooth decomposition of the symbol $(1-|\xi|^2-|\eta|^2)_+^\delta = \sum_{\ell \geq 0} 2^{-\delta\ell} m_\ell(\xi,\eta)$ where m_ℓ is supported in a circular neighborhood of the unit circle of approximate distance $2^{-\ell}$ from the circle. From [7, Proposition 6.2], we know that

$$\|T_{m_\ell}(f,g)\|_{L^p\times L^q \to L^\infty} \lesssim 2^{-\ell(\frac{3}{4q}+\frac{1}{2p})} \tag{4.2}$$

for any $2 \leq p, q < \infty$. However using integration by parts, it is easy to check that the bilinear kernel $K_\ell(x-y, x-z)$ of T_{m_ℓ} satisfies

$$|K_\ell(u,v)| \leq \frac{C_N\, 2^{-\ell}}{(1+2^{-l}|(u,v)|)^N}$$

for every $N > 0$. In this way, (4.2) can be improved in some off-diagonal estimates as follows: fix $x_0 \in \mathbb{R}$ and define $I := [x_0-1, x_0+1]$,

$$\begin{aligned} &|T_{m_\ell}(f,g)(x)| \\ &\lesssim 2^{-\ell(\frac{3}{4q}+\frac{1}{2p})} \left(\|f\|_{L^p(2^M I)} \|g\|_{L^q(2^M I)} + \sum_{k_1,k_2 \geq M} 2^{-\max\{k_1,k_2\}}\, 2^{\frac{k_1}{p'}+\frac{k_2}{q'}} \|f\|_{L^p(2^{k_1} I)} \|g\|_{L^q(2^{k_2} I)} \right). \end{aligned}$$

We also conclude that

$$\begin{aligned} |T_{m_\ell}(f,g)(x)| &\lesssim 2^{-\ell(\frac{3}{4q}+\frac{1}{2p})} \left(2^{\frac{M}{p}+\frac{M}{q}} + \sum_{k_1,k_2 \geq 0} 2^{-N\max\{k_1,k_2\}+(N-1)\ell}\, 2^{k_1+k_2} \right) \|f\|_\infty \|g\|_\infty \\ &\lesssim 2^{-\ell(\frac{3}{4q}+\frac{1}{2p})} \left(2^{\frac{M}{p}+\frac{M}{q}} + 2^{-M(N-2)+(N-1)\ell} \right) \|f\|_\infty \|g\|_\infty. \end{aligned}$$

If we choose M, N such that $M(N-2) = (N-1)\ell$ then we get

$$\|T_{m_\ell}\|_{L^\infty\times L^\infty \to L^\infty} \lesssim 2^{-\ell(\frac{3}{4q}+\frac{1}{2p})} \left(2^{(\frac{1}{p}+\frac{1}{q})\frac{N-1}{N-2}\ell} \right),$$

which holds for every $p, q \in [2, \infty)$. By taking p, q sufficiently large, we deduce that

$$\|T_{m_\ell}\|_{L^\infty\times L^\infty \to L^\infty} \lesssim 2^{\rho\ell}$$

for every $\rho > 0$ as small as possible, which concludes the proof by taking $\rho < \delta$. □

These results are optimal in the strict local L^2 case, in the endpoint cases, and on the boundary of the Banach triangle. It still unknown whether boundedness holds in the limiting case $\delta = 0$ in the interior of the Banach triangle minus the local L^2 triangle.

We may therefore focus on the higher-dimensional situation. First, we have the following proposition.

Proposition 4.2. *Let* $1 \leq p_1, p_2 \leq \infty$ *and* $1/p = 1/p_1 + 1/p_2$ *with* $0 < p \leq \infty$. *Then we have*

(i) *If* $\delta > n - 1/2$, *then*

$$\|S^\delta(f,g)\|_p \leq C \|f\|_{p_1} \|g\|_{p_2}.$$

(ii) *If* $\delta \leq n(1/p-1)-1/2$*, i.e.,*

$$p \leq \frac{2n}{2n+2\delta+1},$$

then S^δ *is not bounded from* $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ *into* $L^p(\mathbb{R}^n)$*.*

(iii) *If* $\delta \leq n\left|\frac{1}{p}-\frac{1}{2}\right|-\frac{1}{2}$*, then* S^δ *is unbounded from* $L^p(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ *into* $L^p(\mathbb{R}^n)$*, from* $L^\infty(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ *into* $L^p(\mathbb{R}^n)$*, and also from* $L^p(\mathbb{R}^n) \times L^{p'}(\mathbb{R}^n)$ *into* $L^1(\mathbb{R}^n)$ *for any* $1 \leq p \leq \infty$*.*

Proof. Note that the kernel of the bilinear Bochner-Riesz means S^δ is

$$K_\delta(x_1,x_2) = c\,\frac{J_{\delta+n}(2\pi|x|)}{|x|^{\delta+n}}, \qquad x=(x_1,x_2)$$

and since $\alpha > n-1/2$, we have that this satisfies an estimate of the form:

$$|K_\delta(x_1,x_2)| \lesssim \frac{1}{(1+|x|)^{\delta+n+1/2}}$$

by using properties of Bessel functions. But for such δ we have $\delta+n+1/2 > 2n$, so the kernel satisfies

$$|K_\delta(x_1,x_2)| \lesssim \frac{1}{(1+|x_1|)^{n+\epsilon}}\frac{1}{(1+|x_2|)^{n+\epsilon}},$$

for some $\epsilon > 0$. It follows that the bilinear operator is bounded by a product of two linear operators, each of which has a good integrable kernel. So, (i) follows by Hölder's inequality.

We now prove (ii) by using an argument as in the proof of Proposition 10.2.3 in [26]. Let $h \in \mathscr{S}(\mathbb{R}^n)$ be a Schwartz function of $\mathbb{R}^n$ satisfying that

$$\widehat{h}(\xi) = \begin{cases} 1, & |\xi| \leq 2 \\ 0, & |\xi| \geq 4. \end{cases}$$

This gives

$$S^\delta(h,h)(x) = \iint_{|\xi|^2+|\eta|^2\leq 1} (1-|\xi|^2-|\eta|^2)^\delta e^{2\pi i x\cdot(\xi+\eta)}\,d\xi d\eta = c\,\frac{J_{n+\delta}(2\pi|(x,x)|)}{|(x,x)|^{n+\delta}}.$$

Then $S^\delta(h,h)(x)$ is a smooth function that is equal to

$$c'\,\frac{\cos(2\pi\sqrt{2}|x| - \frac{\pi}{2}(n+\delta+\frac{1}{2}))}{(\sqrt{2}|x|)^{n+\delta+\frac{1}{2}}} + O\left(\frac{1}{|x|^{n+\delta+\frac{3}{2}}}\right)$$

as $|x| \to \infty$. Then we have

$$|S^\delta(h,h)(x)|^p \approx \frac{1}{|x|^{p(n+\delta+\frac{1}{2})}} + O\left(\frac{1}{|x|^{p(n+\delta+\frac{3}{2})}}\right) \tag{4.3}$$

for all $|x|$ satisfying

$$k + \frac{n+\delta}{4} \leq |x| \leq k + \frac{n+\delta}{4} + \frac{1}{4}$$

for positive large integers k.

Now we observe that the error term in (4.3) is of lower order than the main term at infinity and thus it does not affect the behavior of $|S^\delta(h,h)(x)|^p$. So we conclude that $|S^\delta(h,h)(x)|^p$ is not integrable when $p(n+\delta+1/2) \leq n$, i.e. when $p \leq 2n/(2n+2\delta+1)$.

To prove the first assertion in (iii), we take the L^∞ function to be 1, and then $S^\delta(f,1) = B^\delta(f)$, where f is the linear Bochner-Riesz operator. So the conclusion follows from the linear result. The second assertion in (iii) is similar. To prove the third assertion in (iii), by symmetry we may assume that $p \le 2$. It will suffice to show that the second dual $(S^\delta)^{*2}$ of S^δ is unbounded from $L^p \times L^\infty$ to L^p. Let h be the Schwartz function in case (ii). Then $(S^\delta)^{*2}(h,1)(x) = c\,|x|^{-n/2-\delta} J_{n/2+\delta}(2\pi|x|/\sqrt{2})$ which is not an L^p function if $\delta \le n(1/p-1/2)-1/2$. □

4.1. **Bilinear Bochner-Riesz means as bilinear multipliers.**

4.1.1. *Main results.* The aim of this section is to prove the following result.

Theorem 4.3. *Let $n \ge 2$ and $1 \le p_1, p_2 < 2n/(n+1)$ and $1/p = 1/p_1 + 1/p_2$ and $0 < p \le \infty$. Also let $\alpha(p_1,p_2)$ be as in (3.12). If $\delta > n\alpha(p_1,p_2) - 1$, then the bilinear Bochner-Riesz means operator S^δ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$.*

Proof. The proof is a consequence of Theorem 3.5 and of Lemma 4.4 proved below. □

Lemma 4.4. *Let $1 \le q < \infty$ and $\delta = \sigma + i\tau$. If $0 < s < \sigma + \frac{1}{q}$, then $(1-|x|^2)_+^\delta \in W^{s,q}(\mathbb{R}^n)$. Moreover, there exist constants $C, c > 0$ that depend on n, q, and s such that*

$$\left\|(1-|x|^2)_+^\delta\right\|_{W^{s,q}(\mathbb{R}^n)} \le C\, e^{c|\tau|} \tag{4.4}$$

as long as $\sigma \le c'$, where c' is a constant.

Proof. To compute the $W^{s,q}$ norm of $w(x) = (1-|x|^2)_+^\delta$ on $\mathbb{R}^n$, we argue as follows (see [4, Theorem 6.3.2]):

$$\|w\|_{W^{s,q}} \approx \|w\|_{L^q} + \|w\|_{\dot{W}^{s,q}},$$

where $\dot{W}^{s,q}$ is the homogeneous Sobolev space defined as $\|w\|_{\dot{W}^{s,q}} = \|\Delta^{s/2} w\|_{L^q}$. We have that $\Delta^{s/2} w$ is a radial function and we can write

$$\begin{aligned}
\Delta^{s/2} w(x) &= c \int_{\mathbb{R}^n} \frac{J_{n/2+\delta}(2\pi|\xi|)}{|\xi|^{n/2+\delta}} |\xi|^s e^{2\pi i x\cdot\xi} d\xi \\
&= C \int_0^\infty \frac{J_{n/2+\delta}(2\pi r)}{r^{n/2+\delta}} r^{s+n-1} \int_{S^{n-1}} e^{2\pi i x\cdot r\theta} d\theta\, dr \\
&= C' \int_0^\infty \frac{J_{n/2+\delta}(2\pi r)}{r^{n/2+\delta}} r^{s+n-1} \frac{J_{(n-2)/2}(2\pi r|x|)}{(r|x|)^{(n-2)/2}}\, dr \\
&= C'' \int_0^\infty \tilde{J}_{n/2+\delta}(r) r^{s+n-1} \tilde{J}_{(n-2)/2}(r|x|)\, dr = I,
\end{aligned}$$

where we set $\tilde{J}_\nu(t) = J_\nu(t) t^{-\nu}$ for $t > 0$. Note that we clearly have that $|\tilde{J}_\nu(t)| \le C_\nu (1+t)^{-\nu-\frac{1}{2}}$ for all ν and $t \ge 0$. Moreover, $\tilde{J}_\nu'(t) = -t\tilde{J}_{\nu+1}(t)$ for all $t > 0$. We consider the following cases:

Case 1: $|x| \le 1/2$.

In this case we introduce a smooth cut-off $\psi(r)$ such that $\psi(r)$ is equal to 1 for $r \ge 3/2$ and $\psi(r)$ vanishes when $r \le 1$. Then I is equal to the sum

$$\int_1^\infty \tilde{J}_{n/2+\delta}(r) r^{s+n-1} \tilde{J}_{(n-2)/2}(r|x|)\psi(r)\, dr$$

$$+\int_0^{3/2} \tilde{J}_{n/2+\delta}(r) r^{s+n-1} \tilde{J}_{(n-2)/2}(r|x|)(1-\psi(r))\, dr\,.$$

The second integral is clearly bounded and hence it lies in $L^q(|x| \le 1/2)$. We focus attention on the first integral. Using properties of the function $\tilde{J}_\nu$ we write

$$\int_1^\infty \tilde{J}_{n/2+\delta}(r) r^{s+n-1} \tilde{J}_{(n-2)/2}(r|x|)\psi(r)\, dr$$

$$= (-1)^k \int_1^\infty \Big(\frac{d}{rdr}\Big)^k \tilde{J}_{n/2+\delta-k}(r) r^{s+n-1} \tilde{J}_{(n-2)/2}(r|x|)\psi(r)\, dr$$

Applying a k-fold integration by parts we can write the preceding integral as

$$\int_1^\infty \tilde{J}_{n/2+\delta-k}(r) \Big(\frac{d}{dr}\frac{1}{r}\Big)^k \Big(r^{s+n-1} \tilde{J}_{(n-2)/2}(r|x|)\psi(r)\Big)\, dr.$$

If at least one derivative falls on $\psi(r)$, then the integral is easily shown to be bounded. Thus the worst term appears when no derivative falls on ψ. In this case we have

$$\int_1^\infty \psi(r) \tilde{J}_{\frac{n}{2}+\delta-k}(r) \sum_{\ell=0}^k c_\ell r^{s+n-1-2k+2\ell} \tilde{J}_{\frac{n-2}{2}+\ell}(r|x|)|x|^{2\ell}\, dr\,.$$

We examine the ℓth term of the sum when $\ell < k$. In this case we split up the integral in the two cases $r \ge |x|^{-1}$ and $1 \le r \le |x|^{-1}$. In the case where $r \ge |x|^{-1}$ the integral contains a factor of $r^{s-\sigma-1-k+\ell}$ and this is absolutely convergent since $s-\sigma < 1/q \le 1$ and $k-\ell \ge 1$. The term overall produces a factor of the form $|x|^{-s+\sigma+k-\frac{n-1}{2}}$ which is in $L^q(|x| \le 1/2)$. In the case where $1 \le r \le |x|^{-1}$ one obtains a factor of $|x|^{-s+\sigma+\frac{n-1}{2}+k}$ which is also in $L^q(|x| \le 1/2)$.

It remains to consider the case where $\ell = k$. Here we need to show that the term

$$|x|^{2k} \int_1^\infty \psi(r) \tilde{J}_{\frac{n}{2}+\delta-k}(r) r^{s+n-1} \tilde{J}_{\frac{n-2}{2}+k}(r|x|)\, dr\,.$$

is a convergent integral times a positive power of $|x|$. The part of this integral from 1 to $|x|^{-1}$ is bounded by

$$C|x|^{2k} \int_1^{|x|^{-1}} \psi(r) \frac{1}{r^{\frac{n+1}{2}+\delta-k}} r^{s+n-1}\, dr$$

which produces a factor of $|x|^{k-c}$, which lies in $L^q(|x| \le 1/2)$. The part of the integral from $|x|^{-1}$ to ∞ is

$$|x|^{2k}\Big[\int_{|x|^{-1}}^\infty \psi(r) \frac{e^{\pm ir}}{r^{\frac{n+1}{2}+\delta-k}} r^{s+n-1} \frac{e^{\pm ir|x|}}{(r|x|)^{\frac{n-1}{2}+k}}\, dr + |x|^{-\frac{n+1}{2}-k} \int_{|x|^{-1}}^\infty O\Big(\frac{r^{s+n-1}}{r^{n+2+\delta}}\Big)\, dr\Big]$$

using the asymptotic behavior of the Bessel functions. The second integral converges absolutely while the first integral contains the phase $ir(\pm 1 \pm |x|)$ which is never vanishing and so it can be integrated by parts to show that it converges, since $s-\sigma < 1/q \le 1$. At the end one obtains a factor of $|x|^{k+c}$ which is in $L^q(|x| \le 1/2)$ if k is large.

Case 2: $|x| \ge 2$.

In this case we will use again the smooth cut-off $\psi(r)$ which is equal to 1 for $r \geq 3/2$ and $\psi(r)$ vanishes when $r \leq 1$. Then I is equal to the sum

$$\begin{aligned} &\frac{1}{|x|^{n+s}} \int_1^\infty \tilde{J}_{n/2+\delta}(r/|x|) r^{s+n-1} \tilde{J}_{(n-2)/2}(r)\psi(r)\,dr \\ &+ \frac{1}{|x|^{n+s}} \int_0^{3/2} \tilde{J}_{n/2+\delta}(r/|x|) r^{s+n-1} \tilde{J}_{(n-2)/2}(r)(1-\psi(r))\,dr\,. \end{aligned} \tag{4.5}$$

The second integral in (4.5) is clearly bounded and since $|x|^{-n-s}$ lies in $L^q(|x| \geq 2)$ the second term in (4.5) lies in $L^q(|x| \geq 2)$.

We write the first term in (4.5) as

$$(-1)^k \frac{1}{|x|^{n+s}} \int_1^\infty \tilde{J}_{n/2+\delta}(r/|x|)\psi(r) r^{s+n-1} \Big(\frac{d}{rdr}\Big)^k \tilde{J}_{\frac{n-2}{2}-k}(r)\,dr$$

for any $k > 0$ and by a k-fold integration by parts this is equal to

$$\frac{1}{|x|^{n+s}} \int_1^\infty \Big(\frac{d}{dr}\frac{1}{r}\Big)^k \Big(\psi(r)\tilde{J}_{n/2+\delta}(r/|x|) r^{s+n-1}\Big) \tilde{J}_{\frac{n-2}{2}-k}(r)\,dr\,.$$

The worst term appears when no derivative falls on $\psi(r)$. In this case we obtain a term of the form

$$\frac{1}{|x|^{n+s}} \int_1^\infty \psi(r) \sum_{\ell=0}^k c_\ell \tilde{J}_{\frac{n}{2}+\delta+\ell}(r/|x|) \frac{1}{|x|^{2\ell}} r^{s+n-1-2k+2\ell} \tilde{J}_{\frac{n-2}{2}-k}(r)\,dr$$

When $\ell < k$, the ℓth term is estimated by

$$C|x|^{-n-s-2\ell} \int_1^\infty r^{s+n-1-2k+2\ell} \frac{1}{(1+r/|x|)^{\frac{n+1}{2}+\sigma+\ell}} \frac{1}{r^{\frac{n-1}{2}-k}} dr\,.$$

Considering the cases $r \leq |x|$ and $r \geq |x|$ separately, in each case we obtain a convergent integral times a factor of $|x|^{c-k}$, and since k is arbitrarily large, we deduce that this lies in L^q in the range $|x| \geq 2$. (For the convergence of the integral in the case $r \geq |x|$ we use that $s - \sigma < 1/q \leq 1$.) For the term $\ell = k$ we need the oscillation of the Bessel function to show that the integral

$$\frac{1}{|x|^{n+s+2k}} \int_1^\infty \psi(r) \tilde{J}_{\frac{n}{2}+\delta+k}(r/|x|) r^{s+n-1} \tilde{J}_{\frac{n-2}{2}-k}(r)\,dr \tag{4.6}$$

converges. We split (4.6) as the sum of the term

$$\frac{1}{|x|^{n+s+2k}} \int_1^{|x|} \psi(r) \tilde{J}_{\frac{n}{2}+\delta+k}(r/|x|) r^{s+n-1} \tilde{J}_{\frac{n-2}{2}-k}(r)\,dr,$$

which is bounded by

$$C\frac{1}{|x|^{n+s+2k}} \int_1^{|x|} \psi(r) r^{s+n-1} \frac{1}{r^{\frac{n-1}{2}-k}}\,dr \leq C'\,|x|^{-k+c},$$

plus the term

$$\frac{1}{|x|^{n+s+2k}} \Big[\int_{|x|}^\infty \psi(r) \frac{e^{\pm i|x|^{-1}r}}{(r/|x|)^{\frac{n+1}{2}+\delta+k}} r^{s+n-1} \frac{e^{\pm ir}}{r^{\frac{n-1}{2}-k}} dr + |x|^{\frac{n+3}{2}+\delta+k} \int_{|x|}^\infty O\Big(\frac{r^{s+n-1}}{r^{n+2+\delta}}\Big)\,dr \Big]\,.$$

Notice that the phase $ir(\pm 1 \pm |x|^{-1})$ never vanishes, and since $s-\sigma < 1/q \leq 1$, an integration by parts yields an absolutely convergent integral times a factor of $|x|^{-k+c}$. If k is large, these terms have rapid decay at infinity and thus they lie in $L^q(|x| \geq 2)$.

Notice that all integrations by parts have produced constants that grow at most like a multiple of $1+|s-\sigma+i\tau|^k$ so far.

Case 3: $1/2 \le |x| \le 2$.

The part of integral I over the region $r \le 2$ is easily shown to be in L^∞ and thus in L^q of the annulus $1/2 \le |x| \le 2$. It suffices to consider the part of the integral I over the region $r \ge 2$. Here both r and $r|x|$ are greater than 1 and we use the asymptotics of the Bessel function to write this part as a sum of terms of the form

$$C_1 \int_2^\infty r^{s-\delta-1} e^{ic_1(|x|+1)r} dr + C_2 \int_2^\infty r^{s-\delta-1} e^{ic_2(|x|-1)r} dr + O\Big(\int_2^\infty r^{s-\delta-2} dr\Big)$$

for some constants C_1, C_2, c_1, c_2. Of these terms the middle one contains a phase that may be vanishing while the other terms are bounded by constants that grow at most linearly in $|\tau|$. Recall that $\delta = \sigma + i\tau$. Now define an analytic function of δ by setting

$$I_x(\delta) = \int_2^\infty r^{s-\delta-1} e^{ic_2(|x|-1)r} dr\,.$$

Notice that when $s-\sigma = -\varepsilon_1 < 0$ we have

$$|I_x(\delta)| \le C'\,\varepsilon_1^{-1}.$$

Also notice that when $|x| \ne 1$ and $s-\sigma = 1-\varepsilon_2 < 1$ we have that

$$|I_x(\delta)| \le C''\Big(1+\frac{|\tau|}{\varepsilon_2}\Big)\,\big|\,|x|-1\big|^{-1}.$$

But $I_x(\delta)$ is an analytic function of δ and Hirschman's version of the 3-lines lemma (Lemma 1.3.8 in [25]) gives that

$$|I_x(\delta)| \le \; C(\delta)\big|\,|x|-1\big|^{-1/q+\varepsilon}$$

when $s-\sigma = 1/q-\varepsilon < 1/q$. Here

$$C(\sigma+i\tau) \le \exp\Big\{\frac{\sin(\pi\sigma)}{2}\int_{-\infty}^\infty \Big[\frac{\log(C'\varepsilon_1^{-1})}{\cosh(\pi t)-\cos(\pi\sigma)} + \frac{\log(C''(1+\varepsilon_2^{-1}|t+\tau|))}{\cosh(\pi t)+\cos(\pi\sigma)}\Big]dt\Big\} \le C_1\, e^{C_2|\tau|}\,,$$

where the last estimate is seen by estimating the logarithm by a linear term. But the function $\big|\,|x|-1\big|^{-1/q+\epsilon}$ lies in $L^q(1/2 \le |x| \le 2)$. So, we have proved that when

$$s < \frac{1}{q}+\sigma$$

we have that $w \in W^{s,q}(\mathbb{R}^n)$. □

4.1.2. *An extension of Theorem 4.3.* Using Theorem 3.9 and Lemma 4.4, we can obtain the following result.

Theorem 4.5. *Let $n \ge 2$ and $1 \le q_1, q_2 < 2n/(n+1)$ and let $q_1 \le p_1 \le \infty, q_2 \le p_2 \le \infty$ satisfy $1/p = 1/p_1 + 1/p_2$ and $0 < p \le \infty$. Also assume that*

$$\frac{1}{q_1}+\frac{1}{q_2}-\frac{1}{p} \le \alpha(q_1,q_2),$$

where $\alpha(q_1,q_2)$ is defined in (3.12). If $\delta > n\alpha(q_1,q_2)-1$, then the bilinear Bochner-Riesz means S^δ is bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. In addition, for some constant $C = C_\delta$ we have

$$\|S^\delta\|_{L^{p_1}\times L^{p_2}\to L^p} \le C.$$

Remark 4.6. *The preceding result is interesting for (p_1, p_2) near $(1,1)$ as for points away from $(1,1)$ we will obtain better results. Let us show this claim by the two examples $(p_1, p_2) = (1, \infty)$ and $(p_1, p_2) = (1, \frac{2n}{n+1})$.*

- *First let us examine the point $p_1 = 1$ and $p_2 = \infty$. The previous theorem yields that if $\delta > \frac{n-1}{2} + \frac{2n}{n+1}$, then the operator S^δ is bounded from $L^1(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.*
  Indeed, we take $1 \le q_2 < 2n/(n+1)$ such that $1/q_2 \in (a_n, b_n)$ (see (3.12)) and $q_1 = 1$, and so $q_2 \le p_2 = \infty$ and $q_1 \le p_1 = 1$. By (3.12), we have
  $$\alpha(q_1, q_2) = \frac{1}{2} + \frac{2}{n+1} + \frac{1}{2n}.$$
  On the other hand, $\frac{1}{q_1} + \frac{1}{q_2} - 1 \le \alpha(q_1, q_2)$. By Theorem 4.5, we have that if $\delta > n\alpha(q_1, q_2) - 1 = \frac{n-1}{2} + \frac{2n}{n+1}$, then the operator S^δ is bounded from $L^1(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.
  However, we will see in the next section, that for some particular points, such as those with $p_1 = 1$ and $p_2 = \infty$, we have a better result ($\delta > \frac{n}{2}$) using more precisely the structure of the symbol.
- *Let us now focus on the point $p_1 = 1$ and $p_2 = \frac{2n}{n+1}$. By interpolation between $(1, 1, \frac{1}{2})$ and $(1, \infty, 1)$, Theorem 4.9 below and Proposition 4.2 imply that S^δ is bounded on $L^1(\mathbb{R}^n) \times L^{\frac{2n}{n+1}}(\mathbb{R}^n)$ if $\delta > \frac{3n-2}{4} + \frac{1}{4n}$. In this situation, Theorem 4.3 proves that $\delta > n\alpha(\frac{2n}{n+1}, 1) - 1 = \frac{n-1}{2} + \frac{2n}{n+1}$ is only necessary (which is better).*

4.2. **Study of particular points.** We now focus on determining the range of δ for which the bilinear Bochner-Riesz means S^δ are bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, when $1/p_1 + 1/p_2 = 1/p$, for some specific triples of points (p_1, p_2, p).

4.2.1. *The point $(2, 2, 1)$ and its dual $(2, \infty, 2)$.* We may easily obtain that the operator S^δ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ when $\delta > 1$. Indeed, to see this we apply Lemma 3.7, so
$$\left\| S^\delta \right\|_{L^2 \times L^2 \to L^1} \lesssim \iint_{\mathbb{R}^2} |\partial_{\lambda_1} \partial_{\lambda_2} m(\lambda_1, \lambda_2)| \, d\lambda_1 d\lambda_2,$$
where $m(\lambda_1, \lambda_2) = (1 - \lambda_1^2 - \lambda_2^2)_+^\delta$. On the disc, we have
$$|\partial_{\lambda_1} \partial_{\lambda_2} m(\lambda)| \lesssim (1 - \lambda_1^2 - \lambda_2^2)_+^{\delta - 2}$$
and we can then compute the L^1-norm:
$$\iint_{\mathbb{R}^2} |\partial_{\lambda_1} \partial_{\lambda_2} m(\lambda_1, \lambda_2)| \, d\lambda_1 d\lambda_2 \lesssim \iint_{\mathbb{R}^2} (1 - \lambda_1^2 - \lambda_2^2)_+^{\delta-2} d\lambda_1 d\lambda_2 \lesssim 1 + \int_{\frac{1}{2}}^1 (1-u)^{\delta-2} du,$$
which is finite when $\delta - 2 > -1$.

The restriction $\delta > 1$ is not necessary as shown in our next result:

Theorem 4.7. *Let $n \ge 2$ and $\delta > 0$. Then the operator S^δ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. That is, for some constant $C = C_\delta$ we have*

$$\left\| S^\delta \right\|_{L^2 \times L^2 \to L^1} \le C. \tag{4.7}$$

Moreover, this result fails when $\delta = 0$.

Proof. As an application of Proposition 3.8, we have just to check that for $\delta > 0$, there exists $\alpha > 0$ with

$$\sup_{u\in[-1,1]} \|(1-u^2-\cdot^2)_+^\delta\|_{W^{1+\alpha,1}(\mathbb{R})} < \infty. \tag{4.8}$$

Denote by $\dot{W}^{s,p}(\mathbb{R}^d)$ the homogeneous Sobolev space on $\mathbb{R}^d$ with norm

$$\|h\|_{\dot{W}^{s,p}(\mathbb{R}^d)} = \|(-\Delta)^{s/2}h\|_{L^p(\mathbb{R}^d)}.$$

For any $r > 0$, we have

$$\|h(r\cdot)\|_{\dot{W}^{s,p}(\mathbb{R}^d)} = r^{-\frac{d}{p}+s}\|h(\cdot)\|_{\dot{W}^{s,p}(\mathbb{R}^d)}.$$

Now for a fixed $\delta > 0$ we pick $0 < \alpha < \delta$. By Lemma 4.4 we have,

$$\|(1-|v|^2)_+^\delta\|_{\dot{W}^{1+\alpha,1}(\mathbb{R})} \le C < \infty.$$

Let $f(v) = (1-|v|^2)_+^\delta$ and $r_u = \frac{1}{\sqrt{1-u^2}}$. Combining the preceding facts we obtain

$$\begin{aligned}
\|(1-u^2-\cdot^2)_+^\delta\|_{\dot{W}^{1+\alpha,1}(\mathbb{R})} &= (1-u^2)^\delta\|f(r_u\cdot)\|_{\dot{W}^{1+\alpha,1}(\mathbb{R})}\\
&= (1-u^2)^{\delta-\alpha}\|f(\cdot)\|_{\dot{W}^{1+\alpha,1}(\mathbb{R})}\\
&\lesssim (1-u^2)^{\delta-\alpha}\\
&\lesssim 1,
\end{aligned}$$

since $\delta-\alpha > 0$ and then $(1-u^2)^{\delta-\alpha} \lesssim 1$ for $u \in [-1,1]$.

Certainly, $\|(1-u^2-\cdot^2)_+^\delta\|_{L^1(\mathbb{R})} < \infty$. Hence (4.8) is proved and then we conclude the proof by invoking Proposition 3.8.

We now turn to the sharpness of the requirement that δ be positive. Let $\mathbb{B}'$ be the unit ball in $\mathbb{R}^{2n}$. If the ball multiplier

$$T_{\chi_{\mathbb{B}'}}(f,g)(x) = \iint_{\mathbb{R}^{2n}} \widehat{f}(\xi)\widehat{g}(\eta)\chi_{\mathbb{B}'}(\xi,\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta$$

were bounded from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ with norm C_0, then by a simple translation and dilation the multipliers $T_{\chi_{\mathbb{B}'_{v,w,\rho}}}$ would also be bounded on the same spaces with norm C_0, where

$$\mathbb{B}'_{w,v,\rho} = \{(\xi,\eta) \in \mathbb{R}^n\times\mathbb{R}^n:\ |\xi-\rho w|^2 + |\eta-\rho v|^2 \le 2\rho^2\}.$$

uniformly for all $\rho > 0$ and all unit vectors v, w in $\mathbb{R}^n$. Letting $\rho\to\infty$ we would obtain that the operators

$$T_{\chi_{P'_{w,v}}}(f,g)(x) = \iint_{\xi\cdot w+\eta\cdot v\ge 0} \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta$$

are bounded from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ with norm C_0 uniformly in $v, w \in \mathbb{S}^{n-1}$; here $P'_{v,w} = \{(\xi,\eta)\in\mathbb{R}^n\times\mathbb{R}^n:\ \xi\cdot w+\eta\cdot v \ge 0\}$ is a half-space in $\mathbb{R}^{2n}$. Let $P_v = \{\xi\in\mathbb{R}^n:\ \xi\cdot v\ge 0\}$ be a half-space in $\mathbb{R}^n$ determined by v. A simple calculation shows that

$$T_{\chi_{P'_{v,v}}}(f,g) = (\widehat{fg}\ \chi_{P_v})^\vee,$$

and this operator is unbounded from $L^2\times L^2 \to L^1$ by taking $f = g = \chi_U$, where U is the unit cube in $\mathbb{R}^n$ and $v = (1,0,\dots,0)$. This produces a contradiction. □

We note that a modification of the preceding counterexample also proves that S^0 does not map $L^2(\mathbb{R}^n)\times L^\infty(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Indeed, we take v in $\mathbb{S}^{n-1}$ and define balls

$$\mathbb{B}''_{v,\rho} = \{(\xi,\eta)\in\mathbb{R}^n\times\mathbb{R}^n:\ |\xi|^2 + |\eta-\rho v|^2 \le \rho^2\},$$

which converge to $\{(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n : \ \xi \cdot v \geq 0\}$ when $\rho \to \infty$. Then one obtains the operator $f(\widehat{g}\,\chi_{P_v})^\vee$ in the limit which is unbounded from $L^2(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

In the positive direction we show that for $\delta > \frac{n-1}{2}$ boundedness holds in this case. As of this writing we are uncertain as to whether boundedness holds for the intermediate δ.

Theorem 4.8. *If $\delta > \frac{n-1}{2}$, then the operator S^δ is bounded from $L^2(\mathbb{R}^n)\times L^\infty(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Moreover, for some constant $C = C_\delta$ we have*

$$\big\|S^\delta\big\|_{L^2\times L^\infty\to L^2} \leq C. \tag{4.9}$$

Proof. Recall that S^δ is a bilinear multiplier with the symbol $m_0(|\xi|,|\eta|) = \big(1-|\xi|^2 - |\eta|^2\big)_+^\delta$. We now perform a "spherical decomposition". To this end, we choose a smooth function χ supported on $[1/2,2]$, which is equal to 1 on $[3/4,5/4]$, and which satisfies

$$\sum_{j\geq 0} \chi(2^j t) = 1$$

for every $t \in (0,1]$. For $j \geq 0$ we introduce the functions

$$m_0^j(s,t) = (1-s^2-t^2)_+^\delta\, \chi\left(2^j(1-s^2-t^2)\right).$$

These symbols give us a spherical decomposition of our initial symbol such that

$$\left|\partial_s^\alpha \partial_t^\beta m_0^j(s,t)\right| \lesssim 2^{-\delta j} 2^{(\alpha+\beta)j}. \tag{4.10}$$

For each such function m_0^j, we have the bilinear symbol $m^j(\xi,\eta) := m_0^j(|\xi|,|\eta|)$ on $\mathbb{R}^{2n}$. From (4.10), the bilinear operator T_{m^j} has a bilinear kernel $K_{m^j}(x-y,x-z)$ satisfying

$$|K_{m^j}(x-y,x-z)| \lesssim 2^{-\delta j} 2^{-j}\left(1+2^{-j}|x-y|+2^{-j}|x-z|\right)^{-M}$$

for every large enough integer $M > 0$.

We may also apply Lemma 2.4 and from part (i) of Lemma 2.2, we get there exists a constant $C > 0$ such that

$$\begin{aligned}\|T_{m^j}\|_{L^2\times L^\infty\to L^2} &\leq C2^{j(n/2+\epsilon)}\|T_{m^j}\|_{L^2\times L^2\to L^2} + 2^{-Nj}\\ &\leq C2^{j(n/2+\epsilon)} \sup_{\xi\in\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} |m_0^j(|\xi-\eta|,|\eta|)|^2 d\eta\Big)^{1/2} + 2^{-Nj},\end{aligned}$$

where $\epsilon > 0$ (resp. $N > 0$) can be chosen as small (resp. large) as we want. Moreover, m_0^j is supported in the set

$$\{(s,t) : \ 1 - 2\cdot 2^{-j} \leq |(s,t)|^2 \leq 1 - \tfrac{1}{2}2^{-j}\}.$$

So from (4.10), we have

$$\Big(\int_{\mathbb{R}^n} |m_0^j(|\xi-\eta|,|\eta|)|^2 d\eta\Big)^{1/2} \lesssim 2^{-\delta j} 2^{-\frac{j}{2}},$$

where we used standard estimates for sub-level sets in dimension $n \geq 2$. So finally, we conclude that

$$\begin{aligned}\|S^\delta\|_{L^2\times L^\infty\to L^2} &\lesssim \sum_{j\geq 0} \|T_{m^j}\|_{L^2\times L^\infty \to L^2} \\ &\lesssim \sum_{j\geq 0}\left[2^{j(n/2+\epsilon)}2^{-\delta j}2^{-\frac{j}{2}} + 2^{-Nj}\right] \\ &\lesssim 1 + \sum_{j\geq 0} 2^{-j(\delta - \frac{n-1}{2} - \epsilon)}.\end{aligned}$$

The proof is then finished since for every $\delta > \frac{n-1}{2}$, we may choose a small enough $\epsilon > 0$ such that $\delta > \frac{n-1}{2} + \epsilon$ in order that the previous sum be finite. □

4.2.2. *The point* $(1, \infty, 1)$. Related to this point we have the following result which should be contrasted with the unboundedness known in this case when $\delta \leq \frac{n-1}{2}$ (cf. Proposition 4.2 (iii)).

Theorem 4.9. *Suppose $n \geq 2$. If $\delta > \frac{n}{2}$, then the operator S^δ is bounded from $L^1(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Moreover, for some constant $C = C_\delta$ we have*

$$\left\|S^\delta\right\|_{L^1\times L^\infty \to L^1} \leq C.$$

Proof: The proof relies on a mixture of arguments involving in Lemma 2.4 and the optimal result Theorem 4.7.

As previously, express m_0 in terms of its spherical decomposition

$$m_0 = \sum_{j\geq 0} m_0^j.$$

We have seen (in the proof of Theorem 4.8), that we can apply Lemma 2.4, which gives

$$\begin{aligned}\|S^\delta\|_{L^1\times L^\infty\to L^1} &\lesssim \sum_{j\geq 0}\|T_{m^j}\|_{L^1\times L^\infty\to L^1} \\ &\lesssim 1 + \sum_{j\geq 0} 2^{j(\frac{n}{2}+\epsilon)}\|T_{m^j}\|_{L^1\times L^2\to L^1}.\end{aligned}$$

By Bernstein's inequality (since we only deal with bounded frequencies), it follows that

$$\|S^\delta\|_{L^1\times L^\infty\to L^1} \lesssim 1 + \sum_{j\geq 0} 2^{j(\frac{n}{2}+\epsilon)}\|T_{m^j}\|_{L^2\times L^2\to L^1}.$$

According to Theorem 4.7 and Proposition 3.8, we have

$$\|T_{m^j}\|_{L^2\times L^2\to L^1} \lesssim \sup_{u\in[-1,1]} \|m_0^j(|u|,\cdot)\|_{W^{1+\alpha,1}(\mathbb{R})}$$

for $\alpha > 0$ (as small as we wish). Since

$$m_0^j(s,t) = (1-s^2-t^2)_+^\delta\, \chi\left(2^j(1-s^2-t^2)\right),$$

we have

$$\|m_0^j(|u|,\cdot)\|_{W^{1+\alpha,1}(\mathbb{R})} \lesssim 2^{j(1+\alpha-\delta)}2^{-j} \lesssim 2^{j(\alpha-\delta)}.$$

Consequently, we deduce that

$$\|S^\delta\|_{L^1\times L^\infty\to L^1} \lesssim 1+\sum_{j\geq 0} 2^{j(\frac{n}{2}+\epsilon+\alpha-\delta)} < \infty,$$

since ϵ, α can be chosen arbitrarily small and $\delta > \frac{n}{2}$. □

4.3. **Interpolation between the different results.** Interpolation for S^δ can be achieved using the bilinear complex method adapted to the setting of analytic families or via the an alternative argument, which is based on bilinear interpolation using the real method [30]. The latter argument is outlined as follows: We fix $j \geq 0$ and obtain intermediate estimates for each T_{m^j} (depending on j) starting from the existing estimates for given points. Since we are still working with "open conditions", i.e., a strict inequality of the type $\delta > \delta_0$, we may obtain intermediate boundedness for S^δ by interpolating the boundary conditions.

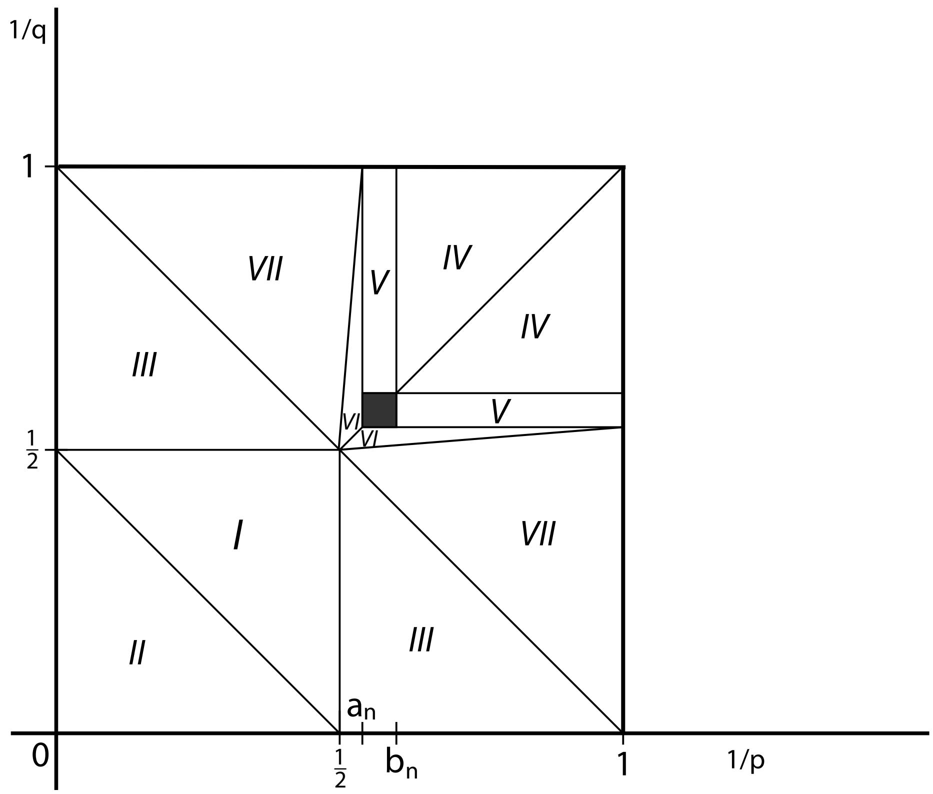

Figure : Exponents $(\frac{1}{p}, \frac{1}{q})$ for $p, q \geq 1$. Here $a_n = \frac{n+1}{2n}$, $b_n = \frac{n+1}{2n} + \frac{n-1}{n^2+n}$

We consider a spherical decomposition, as in Theorem 4.8 splitting the symbol $m_\delta(\xi,\eta) = (1-|(\xi,\eta)|)_+^\delta$ as

$$m_\delta = \sum_{j\geq 0} 2^{-j\delta} m^{j,\delta} \tag{4.11}$$

with bi-radial symbols $m^{j,\delta}(\xi,\eta) = m_0^{j,\delta}(|\xi|,|\eta|)$, where

$$m_0^{j,\delta}(t,s) = \left[2^j(1-t^2-s^2)\right]^\delta \chi\big(2^j(1-t^2-s^2)\big),$$

and thus $m^{j,\delta}$ are supported in the annulus $1-|(\xi,\eta)| \simeq 2^{-j}$ and are regular at the scale 2^{-j}.

In the proof of boundedness of T_{m_δ} at the points

$$(p_1,p_2,p)\in\mathcal{S}=\{(2,2,1),(1,\infty,1),(\infty,1,1),(2,\infty,2),(\infty,2,2),\dots\},$$

we actually obtained estimates of the form

$$\|T_{m^{j,\delta}}\|_{L^{p_1}\times L^{p_2}\to L^p}\lesssim 2^{j\delta(p_1,p_2,p)} \tag{4.12}$$

for some $\delta(p_1,p_2,p)$ depending only on the points p_1,p_2,p. In proving (4.12), we only used the biradial nature of $m^{j,\delta}$, its support properties, and the bounds

$$|\partial_t^\alpha\partial_s^\beta m_0^{j,\delta}(t,s)|\lesssim 2^{(\alpha+\beta)j}$$

that are independent of δ; thus estimate (4.12) also holds for any other $m^{j,\delta'}$, i.e.,

$$\|T_{m^{j,\delta'}}\|_{L^{p_1}\times L^{p_2}\to L^p}\lesssim 2^{j\delta(p_1,p_2,p)}. \tag{4.13}$$

We now fix j and δ' and apply estimate (4.13) and bilinear real interpolation (as in [30]) on $T_{m^{p,\delta'}}$ between the points (p_1^0,p_2^0,p^0) and (p_1^1,p_2^1,p^1) to obtain a bound

$$\|T_{m^{j,\delta'}}\|_{L^{p_1}\times L^{p_2}\to L^p}\lesssim (2^j)^{(1-\theta)\delta(p_1^0,p_2^0,p^0)+\theta\delta(p_1^1,p_2^1,p^1)}.$$

Define $\delta(p_1,p_2,p)=(1-\theta)\delta(p_1^0,p_2^0,p^0)+\theta\delta(p_1^1,p_2^1,p^1)$ whenever $(p_1^0,p_2^0,p^0),(p_1^1,p_2^1,p^1)$ are in $\mathcal{S}$ and

$$\Big(\frac{1}{p_1},\frac{1}{p_2},\frac{1}{p}\Big)=(1-\theta)\Big(\frac{1}{p_1^0},\frac{1}{p_2^0},\frac{1}{p^0}\Big)+\theta\Big(\frac{1}{p_1^1},\frac{1}{p_2^1},\frac{1}{p^1}\Big).$$

Then we obtain the bound (4.13) for $T_{m^{j,\delta'}}$ and for any triple of points (p_1,p_2,p) and any δ'. Picking $\delta'=\delta$ and summing over j yields a bound for S^δ from $L^{p_1}\times L^{p_2}\to L^p$ when $\delta>\delta(p_1,p_2,p)$. The summation over j is straightforward when $p\geq 1$. In the case where $p\leq 1$ we sum the series as follows:

$$\Big\|\sum_{j\geq 0}2^{-j\delta}T_{m^{j,\delta}}\Big\|^p_{L^{p_1}\times L^{p_2}\to L^p}\leq\sum_{j\geq 0}2^{-j\delta p}\big\|T_{m^{j,\delta}}\big\|^p_{L^{p_1}\times L^{p_2}\to L^p}\lesssim\sum_{j\geq 0}2^{-j\delta p}(2^j)^{p\delta(p_1,p_2,p)}<\infty\,,$$

which also converges as long as $\delta>\delta(p_1,p_2,p)$.

Via this method we obtain the following results:

Proposition 4.10 (Local-L^2 case)**.** *Let $p,q,r'\in[2,\infty)$ with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r'}=1$ (region I). If $\delta>\frac{n-1}{r'}$, then S^δ is bounded from $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$.*

Proposition 4.11 (Banach case)**.**
(a) *Let $p,q\in[2,\infty)$ and $r'<2$ with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r'}=1$ (region II). If $\delta>\frac{n-1}{2}+n(\frac{1}{r'}-\frac{1}{2})$, then S^δ is bounded from $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$.*
(b) *Let $q,r'\in[2,\infty)$ and $p<2$ with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r'}=1$ (region III). If $\delta>n(\frac{1}{2}-\frac{1}{q})-\frac{1}{r'}$, then S^δ is bounded from $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$.*
(c) *Let $p,r'\in[2,\infty)$ and $q<2$ with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r'}=1$. If $\delta>n(\frac{1}{2}-\frac{1}{p})-\frac{1}{r'}$, then S^δ is bounded from $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$.*

We now address the non-Banach case situation which is more complicated: if $q\geq p$, then interpolating between the point $(1,1,\frac{1}{2})$ and Theorems 4.7 and 4.9 yields

$$\delta>\delta_1:=n\Big(\frac{1}{p}-\frac{1}{2}\Big)-\frac{n-1}{2r'}.$$

(Here we recall that $\frac{1}{r'} = 1 - \frac{1}{r} \leq 0$). But Theorem 4.3 (for $a_n \leq \frac{1}{q} \leq \frac{1}{p} \leq 1$) gives the condition

$$\delta > \delta_2 := n\alpha(p,q) - 1.$$

If $a_n \leq \frac{1}{q} \leq \frac{1}{p} \leq b_n$, then $\alpha(p,q) = \frac{4}{n+1}$ and we check that $\delta_2 \geq \delta_1$, so Theorem 4.3 does not improve the exponent δ_1 (and the same if p, q are bigger).

If $a_n \leq \frac{1}{q} \leq b_n \leq \frac{1}{p} \leq 1$, then we see that $\delta_2 \leq \delta_1$ if an only if

$$\frac{1}{r} \geq \frac{3n-1}{n^2-1} + 1.$$

If $b_n \leq \frac{1}{q} \leq \frac{1}{p} \leq 1$, then we have $\delta_2 \leq \delta_1$ if and only if

$$\frac{1}{q} \leq \frac{1}{p} + \frac{1}{nr'}.$$

Collecting this information together, we deduce the following result.

Proposition 4.12 (Non-Banach case, part 1). *Let $p \leq \min\{2, q\}$ and $\frac{1}{r} := \frac{1}{p} + \frac{1}{q} > 1$. Then S^δ is bounded from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$*

- *if $b_n \leq \frac{1}{q} \leq \frac{1}{p} \leq 1$ (region IV): $\delta > \delta_2$ for $\frac{1}{q} \leq \frac{1}{p} + \frac{1}{nr'}$, and $\delta > \delta_1$ for $\frac{1}{q} > \frac{1}{p} + \frac{1}{nr'}$;*
- *or if $a_n \leq \frac{1}{q} \leq b_n \leq \frac{1}{p} \leq 1$ (region V): $\delta > \delta_2$ for $\frac{1}{r} \geq \frac{3n-1}{n^2-1} + 1$, and $\delta > \delta_1$ for $\frac{1}{r} < \frac{3n-1}{n^2-1} + 1$;*
- *or if $a_n \leq \frac{1}{q} \leq \frac{1}{p} \leq b_n$ (shaded region) and $\delta > \delta_1$.*

We are left with regions VI and VII. For region VI we interpolate between the point $(2,2,1)$ and the line segments $\{(a_n, 1/q) : 1/q \in (a_n, 1]\}$ and $\{(1/p, b_n) : 1/p \in (a_n, 1]\}$ to obtain the following result:

Proposition 4.13 (Non-Banach case, part 2). *For a point $(1/p^0, 1/q^0)$ in the part of region VI above the diagonal find $\theta \in (0,1)$ and find $1/q \in (a_n, 1]$ such that $1/p^0 = (1-\theta)/2 + \theta a_n$ and $1/q^0 = (1-\theta)/2 + \theta/q$. Then S^δ is bounded from $L^{p^0} \times L^{q^0}$ to L^{r^0} where $1/r^0 = 1/p^0 + 1/q^0$ whenever $\delta > \theta n\alpha(\frac{2n}{n+1}, q) - 1$.*

An analogous result holds for the part of region VI below the diagonal.

Finally in region VII we apply a similar interpolation between the point $(2,2,1)$ and the line segments joining the points $(1,\infty,1)$ with $(1, a_n, a_n/(a_n+1))$ and $(\infty, 1, 1)$ with $(a_n, 1, a_n/(a_n+1))$ to obtain $L^{p_1} \times L^{p_2} \to L^p$ boundedness for δ bigger than some critical value $\delta(p_1, p_2, p)$.

5. Concluding remarks

The linear Bochner-Riesz problem has been studied by several authors; we refer readers to [8, 10, 11, 12, 14, 22, 23, 24, 26, 38, 41, 42, 43, 44, 45] and the references therein for further relevant literature. We are not sure how to adapt the techniques in these articles to the bilinear setting but we hope to investigate whether the bilinear approach to the restriction and Kakeya conjectures in [45] could potentially shed some new light in this problem.

Acknowledgements: F. Bernicot is supported by the ANR under the project AFoMEN no. 2011-JS01-001-01. L. Grafakos is supported by the National Science Foundation (USA) under grant number 0900946. L. Song is supported by NNSF of China (No. 11001276). L.

Yan is supported by NNSF of China (Grant No. 10925106). L. Yan would like to thank A. Seeger and C. Sogge for fruitful discussions.

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