ON AN INEQUALITY OF SAGHER AND ZHOU CONCERNING STEIN'S LEMMA

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Abstract. We provide two alternative proofs of the following formulation of Stein's lemma obtained by Sagher and Zhou [6]: there exists a constant $A > 0$ such that for any measurable set $E \subset [0,1]$, $|E| \neq 0$, there is an integer N that depends only on E such that for any square-summable real-valued sequence $\{c_k\}_{k=0}^{\infty}$ we have:

$$A \cdot \sum_{k>N} |c_k|^2 \leq \sup_I \inf_{a\in\mathbb{R}} \frac{1}{|I|} \int_{I\cap E} |f(t)-a|^2\, dt\,, \tag{1}$$

where the supremum is taken over all dyadic intervals I and $f(t) = \sum_{k=0}^{\infty} c_k r_k(t)$, where r_k denotes the kth Rademacher function. The first proof does not rely on Khintchine's inequality while the second is succinct and applies to general lacunary Walsh series.

1. Introduction

The jth Rademacher function r_j on $[0,1)$, $j = 0,1,2,\dots$, is defined as follows: $r_0 = 1$, $r_1 = 1$ on $[0,1/2)$ and $r_1 = -1$ on $[1/2,1)$, $r_2 = 1$ on $[0,1/4) \cup [1/2,3/4)$ and $r_2 = -1$ on $[1/4,1/2) \cup [3/4,1)$, etc.

The following is a classical result that can be found in Zygmund [10] (page 213): For every subset E of $[0,1]$ and every $\lambda > 1$, there is a positive integer N such that for all complex-valued square-summable sequences $\{a_j\}$ we have

$$\sum_{j\geq N} |a_j|^2 \leq \lambda \sup_{t\in E} \Big| \sum_{j\geq N} a_j r_j(t) \Big|^2 . \tag{2}$$

A related version of this inequality is contained in Lemma 2 of Stein [9] (page 147): For every subset E of $[0,1]$ there is a positive integer N_E

1991 *Mathematics Subject Classification.* Primary 42B20. Secondary 42E30.

Key words and phrases. Rademacher functions, BMO, lacunary sequences.

Annoni's research was supported by the University of Missouri Research Council. Grafakos' research was supported by the NSF under grant DMS0400387 and by the University of Missouri Research Council. Honzík was supported was supported by the Institutional Research Plan no. AV0Z10190503 of the Academy of Sciences of the Czech Republic (AS CR).

and a constant C_E such that for all complex-valued square-summable sequences $\{a_j\}$ we have

$$\sum_{j\geq N_E} |a_j|^2 \leq C_E \sup_{t\in E} \Big| \sum_{j\geq 0} a_j r_j(t) \Big|^2 . \tag{3}$$

Estimate (3) has been referred to in the literature as Stein's lemma and has been found to be a useful tool in applications concerning almost everywhere convergence, see for instance [1], [9], [7]. Unpublished versions of Stein's lemma have been independently obtained by several authors, including D. Burkholder A. M. Garsia, R. F. Gundy, P. A. Meyer, S. Sawyer, and G. Weiss (c.f. [2], [3]). A version of this lemma in the context of independent sequences of random variables with very good control of the constants has been published by Burkholder [2]. Other authors have published related results. Sagher and Zhou [4] published a version of inequality (2) in which the supremum is replaced by the L^p average over E. In [5] the same authors proved analogous inequalities for lacunary series. Carefoot and Flett [3] have obtained a version of inequality (3) in which the ℓ^2 norm on the left is replaced by a supremum of truncated ℓ^1 norms. Recently, Slavin and Volberg [8] have obtained a profound local version of the Chang-Wilson-Wolff inequality which may be thought as analogous to the aforementioned local versions of Khintchine's inequality.

The crux of Stein's lemma is beautifully captured by the following local inequality of Sagher and Zhou [6]: there exists a constant $A > 0$ such that for any measurable set $E \subset [0, 1]$, $|E| \neq 0$, and any q-lacunary sequence K, $1 < q < \infty$, there is an integer N depending only on E and q such that for any real numbers $\{c_k\}_{k\in K}$ with $\sum_{k\in K} |c_k|^2 < \infty$, we have:

$$A \cdot \sum_{k\in K_N} |c_k|^2 \leq \sup_{I} \inf_{a\in\mathbb{R}} \frac{1}{|I|} \int_{I\cap E} |f(t) - a|^2 \, dt , \tag{4}$$

where I is a dyadic interval, $f(t) = \sum_{k\in K} c_k w_k(t)$, $\sum_{k=0}^{\infty} |c_k|^2 < \infty$, $K_N = \{k \in K : k \geq 2^N\}$, and w_k's are the Walsh functions in Paley's order. Note that Rademacher series are 2-lacunary Walsh series.

In this article we focus attention on (1) and more generally on (4). In Section 2 we prove a stronger variant of (1) (without making use of Khintchine's inequality). In Section 3 we provide an alternative formulation of (4). This is proved in a quick and efficient way that yields the optimal constant $A = 1 - \delta$ for any $\delta > 0$; a careful examination of the proof in [6] also yields $A = 1 - \delta$ for any $\delta > 0$.

2. First formulation

The following formulation slightly strengthens the inequality in (1):

Theorem 2.1. *For every measurable subset* E *of* $[0,1]$ *with* $|E|>0$ *and each* $\lambda>1$ *there exists a dyadic interval* $I\subset[0,1]$ *(depending on* E *and* λ*) such that for any real-valued square-summable sequence* $\{a_j\}_{j=0}^{\infty}$ *there is a partition* J_1, J_2 *of* I *that only depends on* $\{a_j\}_{j=N}^{\infty}$ *such that* $|J_1|=|J_2|=\frac{1}{2}|I|$ *and*

$$\begin{aligned}&\sum_{j\geq N+1}|a_j|^2\leq\\ &\max\left\{\frac{\lambda}{|J_1|}\int_{J_1\cap E}\Big|\sum_{j\geq 0}a_jr_j(t)\Big|^2dt,\frac{\lambda}{|J_2|}\int_{J_2\cap E}\Big|\sum_{j\geq 0}a_jr_j(t)\Big|^2dt\right\},\end{aligned}\tag{5}$$

where $N=-\log_2|I|$.

Naturally, estimate (5) implies (3) for real-valued sequences. It also yields (3) with a constant C_E independent of the set E; in fact, it follows from (5) that the constant C_E in (3) can be taken to be $1+\delta$ for real-valued sequences and $C_E=2+2\delta$ for complex-valued sequences, for any $\delta>0$. Estimate (5) also implies (1). Indeed we have

$$\begin{aligned}&\max\left\{\frac{\lambda}{|J_1|}\int_{J_1\cap E}\Big|\sum_{j\geq 0}a_jr_j(t)\Big|^2dt,\frac{\lambda}{|J_2|}\int_{J_2\cap E}\Big|\sum_{j\geq 0}a_jr_j(t)\Big|^2dt\right\}\\ &\quad=\frac{2\lambda}{|I|}\max\left\{\int_{J_1\cap E}\Big|\sum_{j\geq 0}a_jr_j(t)\Big|^2dt,\int_{J_2\cap E}\Big|\sum_{j\geq 0}a_jr_j(t)\Big|^2dt\right\}\\ &\quad\leq\frac{2\lambda}{|I|}\int_{I\cap E}\Big|\sum_{j\geq 0}a_jr_j(t)\Big|^2dt\,.\end{aligned}$$

Since the interval I doesn't depend on a_0, replacing a_0 by a_0-a yields

$$\sum_{j\geq N+1}|a_j|^2\leq\inf_a\frac{2\lambda}{|I|}\int_{I\cap E}\Big|\sum_{j\geq 0}a_jr_j(t)-a\Big|^2dt\,,$$

thus obtaining (4) with $A=(2\lambda)^{-1}$.

To prove Theorem 2.1 we need the following two auxiliary results:

Lemma 2.2. *For every square-summable complex sequence* $\{a_j\}_{j=0}^{\infty}$ *and every measurable subset* $E\subseteq[0,1]$ *with positive measure, we have:*

$$\int_E\Big|\sum_{j\geq 0}a_jr_j\Big|^2\leq\left(|E|+\sqrt{|E|}\right)\int_0^1\Big|\sum_{j\geq 0}a_jr_j\Big|^2.$$

Proof. Expanding out the square on the left we obtain

$$\begin{aligned}
\int_E \Big|\sum_{j\ge 0} a_j r_j\Big|^2 &\le |E| \sum_{j=0}^\infty |a_j|^2 + \sum_{j\neq k} a_j \overline{a_k} \int_E r_j r_k \, dt \\
&\le |E| \sum_{j=0}^\infty |a_j|^2 + \big(\sum_{j\neq k} |a_j a_k|^2\big)^{\frac12} \Big(\sum_{j\neq k} \Big|\int_E r_j r_k \, dt\Big|^2\Big)^{\frac12} \\
&\le |E| \sum_{j=0}^\infty |a_j|^2 + \big(\sum_{j=0}^\infty |a_j|^2\big) \Big(\sum_{j\neq k} \Big|\int_E r_j r_k \, dt\Big|^2\Big)^{\frac12} \\
&\le (|E| + \sqrt{|E|}\,) \sum_{j=0}^\infty |a_j|^2 \,,
\end{aligned}$$

making use of the inequality

$$\sum_{\substack{k,\ell\ge 0\\ k\neq \ell}} |\langle f, r_k r_\ell\rangle|^2 \le \|f\|_{L^2}^2$$

for all f in $L^2[0,1]$. This completes the proof of the lemma since

$$\int_0^1 \Big|\sum_{j\ge 0} a_j r_j\Big|^2 = \sum_{j\ge 0} |a_j|^2.$$

$\square$

For a dyadic subinterval $I_N = [m2^{-N}, (m+1)2^{-N})$ of $[0,1)$ and a real sequence $\{a_j\}_{j\in\mathbb{N}}$ define sets depending on $\{a_j\}$

$$\begin{aligned}
I_N^{++} &= \Big\{t \in I_N : \sum_{j\ge N+1} a_j r_j(t) > 0\Big\}, \\
I_N^{--} &= \Big\{t \in I_N : \sum_{j\ge N+1} a_j r_j(t) < 0\Big\}, \\
I_N^{0} &= \Big\{t \in I_N : \sum_{j\ge N+1} a_j r_j(t) = 0\Big\}.
\end{aligned}$$

It is straightforward to check that the disjoint sets I_N^{++} and I_N^{--} have equal measure but it may not be the case that their union is equal to I_N. To arrange for this to happen, we find disjoint subsets $I_N^{0,+}$ and $I_N^{0,-}$ of I_N^0 of equal measure whose union is I_N^0 and we define $I_N^+ = I_N^{++} \cup I_N^{0,+}$ and $I_N^- = I_N^{--} \cup I_N^{0,-}$ Then we have $I_N^+ \cup I_N^- = I_N$ and by construction we have $|I_N^+| = |I_N^-| = |I_N|/2$. Moreover we have that $\sum_{j\ge N+1} a_j r_j \ge 0$ on I_N^+ and $\sum_{j\ge N+1} a_j r_j \le 0$ on I_N^-. Next we have the following:

Lemma 2.3. *For any real-valued square-summable sequence* $\{a_j\}$, *for any positive integer* N, *for every dyadic interval* $I_N \subseteq [0,1)$ *with* $|I_N| = 2^{-N}$, *and any measurable subset* $E \subseteq [0,1]$ *satisfying*

$$\frac{|E^c \cap I_N|}{|I_N|} + \sqrt{\frac{|E^c \cap I_N|}{|I_N|}} < \frac{1}{2},$$

we have

$$\int_{I_N} \Big| \sum_{j \geq N+1} a_j r_j \Big|^2 \leq \frac{1}{\left(\frac{1}{2} - \frac{|E^c \cap I_N|}{|I_N|} - \sqrt{\frac{|E^c \cap I_N|}{|I_N|}}\right)} \int_{I'_N \cap E} \Big| \sum_{j \geq N+1} a_j r_j \Big|^2$$

where $I'_N = I^+_N$ *or* $I'_N = I^-_N$.

Proof. First take $I'_N = I^+_N$. We write

$$\begin{aligned} &\int_{I_N} \Big| \sum_{j \geq N+1} a_j r_j \Big|^2 = \\ &\int_{I^+_N \cap E} \Big| \sum_{j \geq N+1} a_j r_j \Big|^2 + \int_{I^-_N \cap E} \Big| \sum_{j \geq N+1} a_j r_j \Big|^2 + \int_{I_N \cap E^c} \Big| \sum_{j \geq N+1} a_j r_j \Big|^2 \end{aligned} \tag{6}$$

and obviously we have

$$\int_{I^-_N \cap E} \Big| \sum_{j \geq N+1} a_j r_j \Big|^2 \leq \int_{I^-_N} \Big| \sum_{j \geq N+1} a_j r_j \Big|^2. \tag{7}$$

By the definition of I^-_N it follows that

$$\int_{I^-_N} \Big| \sum_{j \geq N+1} a_j r_j \Big|^2 = \frac{1}{2} \int_{I_N} \Big| \sum_{j \geq N+1} a_j r_j \Big|^2. \tag{8}$$

On the other hand, by a simple change of variables we get

$$\int_{I_N \cap E^c} \Big| \sum_{j \geq N+1} a_j r_j \Big|^2 = |I_N| \int_F \Big| \sum_{j \geq 1} r_j a_{j+N} \Big|^2 \tag{9}$$

for some measurable subset $F \subseteq [0,1]$ with measure

$$|F| = \frac{|I_N \cap E^c|}{|I_N|}. \tag{10}$$

By Lemma 2.2 we obtain

$$\begin{aligned} \int_F \left| \sum_{j \geq 1} r_j a_{j+N} \right|^2 &\leq \left(|F| + \sqrt{|F|}\right) \int_0^1 \Big| \sum_{j \geq 1} r_j a_{j+N} \Big|^2 \\ &= \left(|F| + \sqrt{|F|}\right) \frac{1}{|I_N|} \int_{I_N} \Big| \sum_{j \geq N+1} r_j a_j \Big|^2. \end{aligned} \tag{11}$$

Combining (6), (7), and (8) we deduce

$$\frac{1}{2}\int_{I_N}\Big|\sum_{j\geq N+1} a_jr_j\Big|^2 \leq \int_{I_N^+\cap E}\Big|\sum_{j\geq N+1} a_jr_j\Big|^2 + \int_{I_N\cap E^c}\Big|\sum_{j\geq N+1} a_jr_j\Big|^2 .$$

This estimate together with (9), (11), and (10) yields

$$\left(\frac{1}{2}-\frac{|I_N\cap E^c|}{|I_N|}-\sqrt{\frac{|I_N\cap E^c|}{|I_N|}}\right)\int_{I_N}\Big|\sum_{j\geq N+1} a_jr_j\Big|^2 \leq \int_{I_N^+\cap E}\Big|\sum_{j\geq N+1} a_jr_j\Big|^2$$

proving the required estimate with $I_N' = I_N^+$. Obviously, we may interchange the roles of I_N^+ and I_N^+ and the claimed result follows. □

Having completed all the preliminary material, we now give the proof of Theorem 2.1

Proof. Given $\lambda > 1$, pick an $\epsilon > 0$ small enough such that

$$0 < \frac{1}{1/2-\epsilon-\sqrt{\epsilon}} < 2\lambda .$$

By standard measure theory, for every measurable subset $E \subseteq [0,1]$ there exists a dyadic subinterval I_N of $[0,1]$ of size 2^{-N} such that

$$\frac{|I_N\cap E^c|}{|I_N|} < \epsilon .$$

Since $\{r_j\}_{j\in\mathbb{N}}$ is an orthogonal system in $L^2([0,1])$, by a change of variables we obtain

$$\sum_{j\geq N+1}|a_j|^2 = \frac{1}{|I_N|}\int_{I_N}\Big|\sum_{j\geq N+1} a_jr_j\Big|^2$$

and an application of Lemma 2.3 gives

$$\sum_{j\geq N+1}|a_j|^2 \leq \frac{1}{|I_N|}\frac{1}{(1/2-\epsilon-\sqrt{\epsilon})}\int_{I_N'\cap E}\Big|\sum_{j\geq N+1} a_jr_j\Big|^2 \tag{12}$$

where $I_N' = I_N^+$ or $I_N' = I_N^-$.

The important observation is that the functions r_j, $j = 0, 1, \ldots, N$ are constant on I_N. This implies that for any choice of $a_0, \ldots, a_N$, the sum $\sum_{j=0}^N a_jr_j$ is a real-valued constant on I_N. We may first assume that

$$\sum_{j=0}^N a_jr_j > 0 \qquad \text{on } I_N.$$

Then we have

$$\Big|\sum_{j=N+1}^{\infty} a_j r_j\Big| = \sum_{j=N+1}^{\infty} a_j r_j \leq \sum_{j=0}^{\infty} a_j r_j = \Big|\sum_{j=0}^{\infty} a_j r_j\Big| \qquad \text{on } I_N^+.$$

Choosing $I_N' = I_N^+$ in (12) we write

$$\begin{aligned}
\sum_{j\geq N+1} |a_j|^2 &\leq \frac{1}{|I_N|}\frac{1}{(1/2-\epsilon-\sqrt{\epsilon})}\int_{I_N^+\cap E}\Big|\sum_{j\geq N+1} a_j r_j\Big|^2 \\
&\leq \frac{1}{|I_N|}\frac{1}{(1/2-\epsilon-\sqrt{\epsilon})}\int_{I_N^+\cap E}\Big|\sum_{j\geq 0} a_j r_j\Big|^2 \\
&\leq \frac{2\lambda}{|I_N|}\int_{I_N^+\cap E}\Big|\sum_{j\geq 0} a_j r_j\Big|^2 \\
&= \frac{\lambda}{|J_1|}\int_{J_1\cap E}\Big|\sum_{j\geq 0} a_j r_j\Big|^2
\end{aligned}$$

where $J_1 = I_N^+$. We argue likewise when $\sum_{j=0}^N a_j r_j$ is a negative constant on I_N, in which case we pick $J_2 = I_N^-$. The claim of the theorem is proved with $I = I_N$, $J_1 = I_N^+$, and $J_2 = I_N^-$. □

3. Second formulation

Given a dyadic interval $I \subset [0,1]$, there is an integer $N \geq 0$ and $m \in \{0, 1, ..., 2^N - 1\}$ such that $I = [m\cdot 2^{-N}, (m+1)\cdot 2^{-N})$. In particular, $|I| = 2^{-N}$. Define a function $f \in L^2([0,1])$ via the Rademacher series:

$$f(t) = \sum_{k=0}^{\infty} a_k r_k(t)$$

for some sequence $\{a_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$. For every $k \leq N$, r_k is constant on I; we denote this constant by $r_k(I)$. Furthermore, as $\{r_k\}_{k=N}^{\infty}$ is an orthonormal system on $L^2\big(I, \frac{dt}{|I|}\big)$, we have that

$$\frac{1}{|I|}\int_I \Big|\sum_{k=N}^{\infty} b_k r_k(t)\Big|^2 dt = \sum_{k=N}^{\infty} |b_k|^2 .$$

So, we have the following identities:

$$\begin{aligned}
\frac{1}{|I|}\int_I f(t)\,dt &= \frac{1}{|I|}\int_I \sum_{k=0}^{\infty} a_k r_k(t)\,dt \\
&= \frac{1}{|I|}\int_I \sum_{k=0}^{N} a_k r_k(t)\,dt + \frac{1}{|I|}\int_I \sum_{k=N+1}^{\infty} a_k r_k(t)\,dt \\
&= \sum_{k=0}^{N} a_k r_k(I) + \frac{1}{|I|}\sum_{k=N+1}^{\infty} a_k \int_I r_k(t)\,dt \\
&= \sum_{k=0}^{N} a_k r_k(I)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{|I|}\int_I |f(t)|^2\,dt &= \frac{1}{|I|}\int_I \left|\sum_{k=0}^{N} a_k r_k(t) + \sum_{k=N+1}^{\infty} a_k r_k(t)\right|^2 dt \\
&= \frac{1}{|I|}\int_I \left|\sum_{k=0}^{N} a_k r_k(I) + \sum_{k=N+1}^{\infty} a_k r_k(t)\right|^2 dt \\
&= \Big(\sum_{k=0}^{N} a_k r_k(I)\Big)^2 + \sum_{k=N+1}^{\infty} |a_k|^2 \\
&= \left(\frac{1}{|I|}\int_I f(t)\,dt\right)^2 + \sum_{k=N+1}^{\infty} |a_k|^2 .
\end{aligned}$$

Thus, one obtains

$$\begin{aligned}
\sum_{k=N+1}^{\infty} |a_k|^2 &= \frac{1}{|I|}\int_I |f(t)|^2\,dt - \left(\frac{1}{|I|}\int_I f(t)\,dt\right)^2 \\
&= \frac{1}{|I|}\int_I \left|f(t) - \frac{1}{|I|}\int_I f(s)\,ds\right|^2 dt\,.
\end{aligned} \tag{13}$$

We now state another general formulation of the inequality in (1).

Theorem 3.1. *Given constants $A > 1$, $B \geq 1$, a measurable set $E \subset [0,1)$ with $|E| > 0$, and given a point $x \in E$ of Lebesgue density for the characteristic function χ_E, there is a dyadic subinterval I of $[0,1]$ containing x (and depending on A, B, and E) such that for* *any function* f *in* $L^2([0,1])$ *satisfying*

$$\left(\frac{1}{|J|}\int_J |f(t)|^4\,dt\right)^{1/4} \le B\left(\frac{1}{|J|}\int_J |f(t)|^2\,dt\right)^{1/2} \tag{14}$$

for every dyadic subinterval J *of* $[0,1]$*, we have:*

$$\frac{1}{|I|}\int_I |f(t)|^2\,dt \le \frac{A}{|I|}\int_{I\cap E} |f(t)|^2\,dt\,. \tag{15}$$

Proof. The condition $|E|>0$ guarantees that there exists a point $x\in E$ of Lebesgue density for the characteristic function χ_E. For any such point x, the Lebesgue differentiation theorem yields

$$\lim_{n\to\infty}\frac{|E^c\cap I_n|}{|I_n|} = \lim_{n\to\infty} 1-\frac{|E\cap I_n|}{|I_n|} = 1-\lim_{n\to\infty}\frac{1}{|I_n|}\int_{I_n}\chi_E(t)dt$$
$$= 1-\chi_E(x) = 0\,,$$

where each dyadic interval I_n is uniquely determined by the condition that it has measure equal to 2^{-n} and contains x; such intervals shrink to x and the Lebesgue differentiation theorem applies. As $A>1$, there exists an $n_0\in\mathbb{N}$ such that:

$$\frac{|I_{n_0}\cap E^c|}{|I_{n_0}|} < \left(\frac{A-1}{A\cdot B^2}\right)^2. \tag{16}$$

Now we set $I = I_{n_0}$. We have:

$$\begin{aligned}
\frac{1}{|I|}\int_I |f(t)|^2 dt &= \frac{1}{|I|}\int_{I\cap E} |f(t)|^2\,dt + \frac{1}{|I|}\int_{I\cap E^c} |f(t)|^2\,dt\\
&\le \frac{1}{|I|}\int_{I\cap E} |f(t)|^2\,dt + \sqrt{\frac{|E^c\cap I|}{|I|}}\left(\frac{1}{|I|}\int_I |f(t)|^4 dt\right)^{\frac{1}{2}}\\
&\le \frac{1}{|I|}\int_{I\cap E} |f(t)|^2\,dt + \sqrt{\frac{|E^c\cap I|}{|I|}}\,\frac{B^2}{|I|}\int_I |f(t)|^2\,dt,
\end{aligned}$$

where we used the Cauchy-Schwarz inequality and the assumption on f. Solving for $\frac{1}{|I|}\int_I |f(t)|^2 dt$ and recalling (16), we obtain:

$$\begin{aligned}
\frac{1}{|I|}\int_I |f(t)|^2\,dt &\le \frac{1}{1-\sqrt{\frac{|I\cap E^c|}{|I|}}\,B^2}\,\frac{1}{|I|}\int_{I\cap E} |f(t)|^2\,dt\\
&\le \frac{A}{|I|}\int_{I\cap E} |f(t)|^2\,dt.
\end{aligned}$$

□

We end with some remarks. If f is a real-valued function, equation (15) obviously implies:

$$\frac{1}{|I|}\int_I |f(t)|^2\,dt - \left(\frac{1}{|I|}\int_I f(t)dt\right)^2 \leq \frac{A}{|I|}\int_{I\cap E} |f(t)|^2\,dt\,.$$

Thus, if $f(t)=\sum_{k=0}^{\infty} a_k r_k(t)$ for some real-valued, square-summable sequence $\{a_k\}_{k\in\mathbb{N}}$, we use identity (13) to express the previous inequality as:

$$\sum_{k=N+1}^{\infty} |a_k|^2 \leq \frac{A}{|I|}\int_{I\cap E} |f(t)|^2\,dt\,,$$

where $N=-\log_2|I|$. Since the left-hand side of the preceding inequality doesn't depend on the coefficient a_0 of the constant function r_0, we may also write:

$$\sum_{k=N+1}^{\infty} |a_k|^2 \leq \inf_{a_0\in\mathbb{R}} \frac{A}{|I|}\int_{I\cap E} |f(t)-a_0|^2\,dt\,.$$

This implies estimate (4) for the Rademacher series.

Next we indicate why Theorem 3.1 applies to lacunary Walsh series as well. Indeed, the crucial point is to verify that (14) holds for a lacunary Walsh series f. Sagher and Zhou [6] (page 58) proved that

$$\left(\frac{1}{|J|}\int_J |f(t)-f_J|^p dt\right)^{1/p} \leq B(p,q)\left(\sum_{k\in K_N} |c_k|^2\right)^{1/2}, \tag{17}$$

where $f(t)=\sum_{k\in K} c_k w_k(t)$ is a q-lacunary Walsh series, $\{w_k\}_{k=0}^{\infty}$ is the Walsh system in the Paley order, K is a q-lacunary sequence of natural numbers, $N\in\mathbb{N}$, $K_N=\{k\in K : k\geq 2^N\}$, J is a dyadic interval of length 2^{-N}, $\sum_{k\in K}|c_k|^2<\infty$, $f_J=\frac{1}{|J|}\int_J f(t)\,dt$, $0<p<\infty$, and $1<q<\infty$. A version of (13) is easily shown to hold for (q-lacunary or not) Walsh series f, i.e.,

$$\left(\sum_{k\in K_N} |c_k|^2\right) = \frac{1}{|J|}\int_J |f(t)-f_J|^2\ dt\,. \tag{18}$$

Combining (18) and (17) one obtains

$$\left(\frac{1}{|J|}\int_J |f(t)|^p\,dt\right)^{1/p} \leq B(p,q)\left(\frac{1}{|J|}\int_J |f(t)|^2\,dt\right)^{1/2} \tag{19}$$

for every q-lacunary Walsh series f with mean value zero on J. Via the splitting $f=(f-f_J)+f_J$, estimate (19) easily extends to all f, with some other constant $B'(p,q)$. Thus (14) holds for q-lacunary Walsh series and Theorem 3.1 also applies for them.

The second author would like to thank Nigel Kalton for some useful discussions on this topic.

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