ON ROUGH OSCILLATORY SINGULAR INTEGRAL OPERATORS

AHMAD AL SALMAN AND LOUKAS GRAFAKOS

Abstract. Let $P : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a polynomial mapping where $\mathbb{R}^n$ is the $n-$dimensional euclidean space (with $n \geq 2$). Let $h : (0, \infty) \to \mathbb{R}$ be a measurable function, and let $\Omega \in L^1\left(\mathbb{S}^{n-1}\right)$ be a homogeneous function of degree zero on $\mathbb{R}^n$ with mean value zero over the unit sphere $\mathbb{S}^{n-1}$. The aim of this paper is to investigate the long-standing problem concerning the L^p boundedness of the oscillatory singular integral operator

$$T_{P,\Omega,h}f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} h(|x-y|) \, |x-y|^{-n} \, \Omega(x-y) f(y) dy$$

under the assumption that $\Omega \in L \log L(\mathbb{S}^{n-1})$ and the radial function h is rough in the sense that it satisfies an integrability condition in the form

$$\sup_{R>0} \frac{1}{R} \int_0^R |h(t)|^{\gamma} \, dt < \infty$$

for some $\gamma > 1$.

1. Introduction And Statement of Results

This paper deals with the long standing problem of the L^p boundedness of oscillatory singular integral operators with homogeneous kernels that are rough in the radial direction. Oscillatory singular integrals have become an active research topic since their appearance in the works of Ricci and Stein in [RS]. They play an important role in the study of singular integrals on lower-dimensional varieties, in relation to twisted convolution on Heisenberg group, and in the theory of singular Radon transform. For a thorough discussion concerning oscillatory singular integral operators and recent results, we refer the readers to consult [AlS], [DLY], [FY], [MWW], [P1], [P], [FP], [JL], [LZ], and [RS], among others.

Let $n \geq 2$ and $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$. Let $h : (0, \infty) \to \mathbb{R}$ be a measurable function and let $\Omega \in L^1\left(\mathbb{S}^{n-1}\right)$ be a homogeneous function of degree zero on $\mathbb{R}^n$ that satisfies

$$\int_{\mathbb{S}^{n-1}} \Omega(x') \, d\sigma(x') = 0. \tag{1}$$

We let $K_{\Omega,h} : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ be the singular kernel of Calderón-Zygmund type defined by

$$K_{\Omega,h}(y) = h(|y|) \, |y|^{-n} \, \Omega(y).$$

Let $\mathcal{P}$ be the class of all polynomial mappings $P : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. The oscillatory singular integral operator associated to $K_{\Omega,h}$ and P is defined by

$$T_{P,\Omega,h}f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} K_{\Omega,h}(x-y) f(y) dy. \tag{2}$$

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A considerably difficult problem is whether the operators $T_{P,\Omega,h}$ map $L^p \to L^p$ for some $1 < p < \infty$ provided that the radial function h is rough. It is the main aim of this note to discuss such problem. In the following series of remarks, we shed some light on a number of related historical results:

(i) When $h = 1$, the operator $T_{P,\Omega} = T_{P,\Omega,1}$ was introduced by Ricci and Stein in [RS]. Since then, various authors have investigated the L^p mapping properties of the class of operators $T_{P,\Omega}$. It is worth mentioning that the the special operator $T_{0,\Omega,1}$ is the classical Calderón-Zygmund singular integral operator, often denoted by T_Ω. In [CZ1] and [CZ2], Calderón and Zygmund introduced the method of rotations and proved that the operator T_Ω is bounded on L^p for $1 < p < \infty$, given that Ω is either an odd function in $L^1(\mathbb{S}^{n-1})$ or an even function in $L\log^+ L(\mathbb{S}^{n-1})$. Here, $\Omega \in L\log^+ L(\mathbb{S}^{n-1})$ means

$$\int_{\mathbb{S}^{n-1}} |\Omega(y')| \log(2 + |\Omega(y')|) d\sigma(y') < \infty \tag{3}$$

Moreover, they showed that the condition $\Omega \in L\log^+ L(\mathbb{S}^{n-1})$ is optimal in the sense that T_Ω may fail to be bounded on L^p for any p if the condition $\Omega \in L\log^+ L(\mathbb{S}^{n-1})$ is replaced by any condition $\Omega \in L(\log^+ L)^{1-\varepsilon}(\mathbb{S}^{n-1})$, $\varepsilon > 0$ [CZ1], [CZ2].

(ii) When $h = 1$, Ricci and Stein proved in [RS] that the operator $T_{P,\Omega}$ maps $L^p(\mathbb{R}^n)$ into itself for some $1 < p < \infty$ provided that Ω is smooth in $\mathcal{C}^1(\mathbb{S}^{n-1})$. Later, Lu-Zhang [LZ] showed that the operator $T_{P,\Omega,h}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ provided that $\Omega \in L^q(\mathbb{S}^{n-1})$ for some $q > 1$ and that h is of bounded variation. Subsequently, the condition $\Omega \in L^q(\mathbb{S}^{n-1})$ for some $q > 1$ was relaxed by Jiang and Lu to the condition $\Omega \in L\log^+ L(\mathbb{S}^{n-1})$ [JL]. In fact, Jiang and Lu proved that $T_{P,\Omega,h}$ is bounded on L^p, $1 < p < \infty$ provided that $\Omega \in L\log^+ L(\mathbb{S}^{n-1})$ and h is of bounded variation. Here, it is worth pointing out that $L\log L(\mathbb{S}^{n-1})$ contains the space $L^q(\mathbb{S}^{n-1})$ (for any $q > 1$) properly.

(iii) When $P = 0$, the operator $T_{\Omega,h} = T_{0,\Omega,h}$ was introduced by R. Fefferman [Fe]. In 1979, R. Fefferman introduced the class of operators

$$T_{\Omega,h}(f)(x) = \text{p.v.} \int_{R^n} h(|x-y|)\, |x-y|^{-n}\, \Omega(x-y) f(y) dy.$$

He showed that $T_{\Omega,h}$ is bounded on L^p for all $1 < p < \infty$, provided that $\Omega \in Lip_\alpha(\mathbb{S}^{n-1})$ for some $\alpha > 0$ and that $h \in L^\infty(0,\infty)$. In 1986, J. Namazi showed that Fefferman's result still holds under the weaker condition $\Omega \in L^q(\mathbb{S}^{n-1})$ for some $q > 1$ [NA]. Subsequently, in 1986, Duoandikoetxea and Rubio de Francia improved Namazi's result by considering weaker condition on the function h. In fact, they showed that the operator $T_{\Omega,h}$ is bounded on L^p for all $1 < p < \infty$, provided that $\Omega \in L^q(\mathbb{S}^{n-1})$ for some $q > 1$ and that h satisfies the integrability condition

$$\sup_{R>0} \frac{1}{R} \int_0^R |h(t)|^2 \, dt < \infty. \tag{4}$$

In [AlSP], Al-Salman and Pan improved the result of Duoandikoetxea and Rubio de Francia by showing that $T_{\Omega,h}$ is bounded on L^p for all $1 < p < \infty$, provided that $\Omega \in L\log^+ L(\mathbb{S}^{n-1})$ and that h lies in the general class Δ_γ for some $\gamma > 1$. Here, Δ_γ is the class of all measurable functions $h:(0,\infty)\to\mathbb{R}$ that satisfy (4) with 2 replaced by γ. For $\gamma>1$, we then define

$$\|h\|_{\Delta_\gamma}=\left(\sup_{R>0}\frac{1}{R}\int_0^R |h(t)|^\gamma\,dt\right)^{\frac{1}{\gamma}}.$$

It can be easily shown that the following inclusions hold

$$L^\infty(\mathbb{R})=\Delta_\infty\subset\Delta_{\gamma_2}\subset\Delta_{\gamma_1}\text{ for }\gamma_1<\gamma_2. \tag{5}$$

(iv) In general, if $T_{P,\Omega,h}$ is of convolution type (i.e., $P(x,y)=P(x-y)$), the L^p boundedness properties of $T_{P,\Omega,h}$ are well understood for various functions Ω and h. For background information for such case, we advise readers to consult [AlS], [AlSP], [FP], and references therein.

However, regarding rough h, there is still very little known about the L^p boundedness (even for $p=2$) of the general operator $T_{P,\Omega,h}$. In the following, we will tackle this problem, starting with introducing the following class of mappings:

Let $\alpha\ge 1$. A mapping $\Phi:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is called $\alpha-$separable if there exist mappings $\varphi,g_1,g_2:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that $\varphi=\varphi(\cdot,\cdot,\cdot,\cdot)$ is smooth in the third variable and that

$$\Phi((r+v)y+z,z+uy)=\varphi(y,z,r,v)u^\alpha+g_1(y,z,r,v)+g_2(y,z,u,v) \tag{6}$$

for all $(y,z,u,v,r)\in\mathbb{S}^{n-1}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ with $y\cdot z=0$. We let $\mathfrak{S}^\alpha$ be the class of all $\alpha-$separable mappings. For $q\ge 1$ and $0\le\delta<1$, we let $\mathfrak{S}^{\alpha,\delta,q}$ be the class of mappings $\Phi\in\mathfrak{S}^\alpha$ that satisfy

$$\|I_{j,\delta,\Phi}(\cdot)\|_{L^q(\mathbb{S}^{n-1})}\le 2^{\varepsilon_{q,\delta}j}C_\Phi \tag{7}$$

for some $0<\varepsilon_{q,\delta}<2\alpha-1$, where

$$I_{j,\delta,\Phi}(y)=\sup_{z\in\mathbb{R}^n}\int_{2^{-j-1}}^{2^j}\int_1^2\left|\frac{\partial}{\partial r}\varphi(y,z,2^jr,\pm v)\right|^{-(1-\delta)}drdv. \tag{8}$$

Here, j is a positive integer.

It is clear that $\mathfrak{S}^{\alpha,\delta_1,q_1}\subset\mathfrak{S}^{\alpha,\delta_2,q_2}$ whenever $\delta_1\le\delta_2$ and $q_2\le q_1$. Examples of mappings in $\mathfrak{S}^{\alpha,\delta,q}$ are widely available. A particular example is the mapping $\Phi(x,y)=|x|^2|y|^2$ which lies in $\mathfrak{S}^{2,\delta,q}$ for $1/2<\delta<1$. Furthermore, we can show that the mapping $\Phi(x,y)=|x|^{\frac{5}{4}}|y|^2$ is in $\mathfrak{S}^{2,0,q}$. In section 2, we shall present a class of of polynomial mappings that is contained in $\mathfrak{S}^{\alpha,\delta,q}$. Our results are the following:

Theorem A. *Let $\Omega\in L\log^+L(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero on $\mathbb{R}^n$ and satisfying* (1). *Let $\alpha\ge 1$, $0\le\beta<1$, and $q\ge 1$. Suppose that $\Phi\in\mathfrak{S}^{\alpha,\beta,q}$ and that $h\in\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}$. Let*

$$T^{\infty,\beta}_{\Phi,\Omega,h}f(x)=\int_{|x-y|\ge 1}e^{i\Phi(x,y)}|x-y|^{-n}h(|x-y|)\Omega(x-y)(1+|y|^2)^{-\beta}f(y)dy. \tag{9}$$

Then for $1<p<\infty$, there exists a constant $C_p>0$ independent of Ω and h such that

$$\left\|T^{\infty,\beta}_{\Phi,\Omega,h}f\right\|_p\le C_p\|h\|_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}}\|\Omega\|_{L\log L}\|f\|_p \tag{10}$$

for $f\in L^p(\mathbb{R}^n)$.

Theorem B. *Let $\Omega \in L\log^+ L(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero on $\mathbb{R}^n$ and satisfying* (1). *Suppose that $P_Q(x,y) = Q(x)\,|y|^2$ or $P_Q(x,y) = Q(y)\,|x|^2$, where Q is a real valued polynomial on $\mathbb{R}^n$. Suppose that $1 - 1/dq < \beta \leq 1$ and that $h \in \Delta_{\frac{8}{2+\beta}}$ where $d = \deg(Q)$. Then for $1 < p < \infty$, there exists a constant $C_p > 0$ independent of Ω and h such that the operator*

$$T^{\beta}_{P_Q,\Omega,h} f(x) = \int_{\mathbb{R}^n} e^{iP(x,y)}\,|x-y|^{-n}\,h(|x-y|)\Omega(x-y)(1+|y|^2)^{-\beta} f(y)dy \tag{11}$$

satisfies

$$\left\| T^{\beta}_{P_Q,\Omega,h} f \right\|_p \leq C_p \,\|h\|_{\Delta_{\frac{8}{2+\beta}}}\, \|\Omega\|_{L\log L}\, \|f\|_p \tag{12}$$

for $f \in L^p(\mathbb{R}^n)$. The constant C_p may depend on the degree of the polynomial mapping Q. But it is independent of the coefficients.

Theorem C. *Let $\Omega \in L\log^+ L(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero on $\mathbb{R}^n$ and satisfying* (1). *Suppose that $P(x,y) = Q(x)(a\cdot y)$ or $P(x,y) = Q(y)(a\cdot x)$, where Q is a real valued polynomial on $\mathbb{R}^n$ of degree d and a is a point in $\mathbb{R}^n$. Suppose that $1 - 1/(d+1)q < \beta \leq 1$ and that $h \in \Delta_{\frac{4}{\beta}}$. Then for $1 < p < \infty$, there exists a constant $C_p > 0$ independent of Ω and h such that the operator*

$$T^{\beta}_{P,\Omega,h} f(x) = \int_{\mathbb{R}^n} e^{iP(x,y)}\,|x-y|^{-n}\,h(|x-y|)\Omega(x-y)(1+|y|^2)^{-\beta} f(y)dy.$$

satisfies

$$\left\| T^{\beta}_{P,\Omega,h} f \right\|_p \leq C_p \,\|h\|_{\Delta_{\frac{4}{\beta}}}\, \|\Omega\|_{L\log L}\, \|f\|_p$$

for $f \in L^p(\mathbb{R}^n)$. The constant C_p may depends on the degree of the polynomial mapping Q. But it is independent of the coefficients of Q the point a.

It should be noted that since the class L^∞ is contained in $\Delta_{\frac{4}{\beta}}$ and $\Delta_{\frac{8}{2+\beta}}$, the results of Theorems A, B, and C hold for $h \in L^\infty$. This demonstrates that Theorems B and C represent substantial improvements over the corresponding result in [JL] for the discussed classes of polynomials. One can easily observe that Theorem C implies that the operator $T_{P,\Omega,h}$ is bounded on the subclass $L^p_{-\beta}(\mathbb{R}^n) = (1+|y|^2)^{-\beta} L^p(\mathbb{R}^n)$ where $1 - 1/(d+1)q < \beta \leq 1$.

Finally, we shall prove the following result:

Theorem D. *Let $\Omega \in L\log^+ L(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero on $\mathbb{R}^n$ and satisfy* (1) *and let a be a point in $\mathbb{R}^n$. Suppose that for some $0 \leq \gamma \leq 1$, the mapping satisfies any of the following conditions*
(i) $\Phi(x,y) = |x|^\gamma\,(a\cdot y)$
(ii) $\Phi(x,y) = |y|^\gamma\,(a\cdot x)$
(iii) $\Phi(x,y) = |x|^\gamma\,|y|^2$
(iv) $\Phi(x,y) = |y|^\gamma\,|x|^2$.
If $h \in \Delta_4$, then for $1 < p < \infty$, there exists a constant $C_p > 0$ independent of Ω and h such that the operator $T_{\Phi,\Omega,h}$ satisfies

$$\|T_{\Phi,\Omega,h} f\|_p \leq C_p\,\|h\|_{\Delta_4}\,\|\Omega\|_{L\log L}\,\|f\|_p$$

for $f \in L^p(\mathbb{R}^n)$. The constant C_p is independent of the point a in case the mapping Φ satisfies (i) or (ii).

It should be noted that Theorem D is not previously known, even in the case of smooth radial functions h. It represents a substantial development in the effort to resolve the L^p boundedness problem for the class of operators in (2) when the radial function h is rough. To prove the results in this paper, we rely on orthogonality arguments as in [RS]. However, due to the presence of the rough radial function h in the one-dimensional oscillatory integrals, it is not possible to apply Van der Corput's lemma. Therefore, we will prove an alternative lemma in the next section, enabling us to obtain good L^2 estimates. We believe that this lemma may have applications beyond those discussed in this paper.

2. A Lemma

As highlighted in the introduction, the aim of this section is to introduce an alternative to Van der Corput's lemma.

Lemma 2.1. *Suppose that φ is a continuously differentiable function on (a,b) whose derivative has N_0 zeros on (a,b). Then for all $0 \leq \beta < 1$ and $\alpha \geq 1$, we have that*

(13)
$$\int_{-\infty}^{\infty}\left|\int_a^b e^{2\pi i|\lambda|^\alpha\phi(t)}h(t)\,dt\right|^2\frac{d\lambda}{(1+|\lambda|)^\beta} \leq C_{\beta,\alpha}\,N_0\,\|h\|^2_{L^{\frac{2\alpha}{(2\alpha+\beta-2)}}[a,b]}\left(\int_a^b|\varphi'|^{-(1-\beta)}\,dt\right)^{\frac{1}{\alpha}}$$

Proof. We shall prove (13) for $\alpha > 1$. The case $\alpha = 1$ follows by minor modifications. Since φ' has only a finite number of zeros on (a,b), there exist finitely many subintervals of (a,b) on which φ' has constant sign. Let (a',b') be such a subinterval of (a,b) on which $\varphi' > 0$ or $\varphi' < 0$. Then a change of variables gives

$$\int_{a'}^{b'} e^{2\pi i|\lambda|^\alpha\phi(t)}h(t)\,dt = \int_{\varphi(a')}^{\varphi(b')} e^{2\pi i|\lambda|^\alpha u}\frac{h(\varphi^{-1}(u))}{\varphi'(\varphi^{-1}(u))}\,du = \widehat{H}(-|\lambda|^\alpha),$$

where

$$H(u) = \chi_{[\varphi(a'),\varphi(b')]}\frac{h(\varphi^{-1}(u))}{\varphi'(\varphi^{-1}(u))}\,.$$

Now $(1+|\lambda|)^{\beta/2} \approx (1+4\pi^2|\lambda|^{2\alpha})^{(\beta/2\alpha)/2}$, and the inverse Fourier transform in λ of this function is the Bessel potential $G_{\beta/2\alpha}$ acting on $|\lambda|^\alpha$. Then the left hand side of (13) is comparable to

$$\begin{aligned}
\int_{-\infty}^{\infty}\left|\frac{\widehat{H}(-|\lambda|^\alpha)}{(1+4\pi^2|\lambda|^{2\alpha})^{(\beta/2\alpha)/2}}\right|^2 d\lambda &= \int_{-\infty}^{\infty}\left|\widehat{H}(-|\lambda|^\alpha)\widehat{G_{\beta/2\alpha}}(|\lambda|^\alpha)\right|^2 d\lambda \\
&= 2\int_0^{\infty}\left|\widehat{H}(-\lambda^\alpha)\widehat{G_{\beta/2\alpha}}(\lambda^\alpha)\right|^2 d\lambda \\
&= \frac{2\,(2\pi)^{\frac{\alpha-1}{\alpha}}}{\alpha}\int_0^{\infty}\left|\widehat{H}(-\lambda)\widehat{G_{\beta/2\alpha}}(\lambda)\,|2\pi\lambda|^{-\frac{\alpha-1}{2\alpha}}\right|^2 d\lambda \\
&= \frac{(2\pi)^{\frac{\alpha-1}{\alpha}}}{\alpha}\int_{-\infty}^{\infty}\left|\widehat{H}(-|\lambda|)\widehat{G_{\beta/2\alpha}}(|\lambda|)\,|2\pi\lambda|^{-\frac{\alpha-1}{2\alpha}}\right|^2 d\lambda \\
&\leq \frac{1}{\alpha}\int_{-\infty}^{\infty}\left|I_{\frac{\alpha-1}{2\alpha}}(H*G_{\beta/2\alpha})(\lambda)\right|^2 d\lambda
\end{aligned}$$

by Plancherels' theorem. Where $I_{\frac{\alpha-1}{2\alpha}}$ is Riesz potential of order $\frac{\alpha-1}{2\alpha}$. Since the Riesz potential maps L^p to L^2 when $1/p - 1/2 = \frac{\alpha-1}{2\alpha}$ we get

$$\int_{-\infty}^{\infty} \left| \frac{\widehat{H}(-\lambda^m)}{(1+4\pi^2|\lambda|^{2m})^{(\beta/2m)/2}} \right|^2 d\lambda \leq \left(\int_{-\infty}^{\infty} \left| H * G_{\beta/2m})(\lambda) \right|^{\frac{2m}{2m-1}} d\lambda \right)^{\frac{2m-1}{m}} .$$

But the Bessel potential operator acts like the Riesz potential and maps L^p to $L^{\frac{2\alpha}{2\alpha-1}}$ when $1/p - \frac{2\alpha-1}{2\alpha} = \beta/2\alpha$. Thus the last expression is bounded by a constant multiple of

$$\begin{aligned} C_{\beta,\alpha} \|H\|^2_{L^{2\alpha/(2\alpha+\beta-1)}} &= \left(\int_{\varphi(a')}^{\varphi(b')} \left| \frac{h(\varphi^{-1}(u))}{\varphi'(\varphi^{-1}(u))} \right|^{\frac{2\alpha}{2\alpha+\beta-1}} du \right)^{\frac{2\alpha+\beta-1}{\alpha}} \\ &= \left(\int_{a'}^{b'} \left| \frac{h(t)}{\varphi'(t)} \right|^{\frac{2\alpha}{2\alpha+\beta-1}} |\varphi'(t)| \, dt \right)^{\frac{2\alpha+\beta-1}{\alpha}} \\ &= \left(\int_{a'}^{b'} |h(t)|^{\frac{2\alpha}{2\alpha+\beta-1}} |\varphi'(t)|^{\frac{\beta-1}{2\alpha+\beta-1}} \, dt \right)^{\frac{2\alpha+\beta-1}{\alpha}} . \end{aligned}$$

Apply Hölder's inequality with exponents $(2\alpha+\beta-1)/(2\alpha+\beta-2)$ and $2\alpha+\beta-1$ to bound the above by a constant multiple of

$$\left(\int_{a'}^{b'} |h(t)|^{\frac{2\alpha}{(2\alpha+\beta-2)}} dt \right)^{\frac{(2\alpha+\beta-2)}{\alpha}} \left(\int_{a'}^{b'} |\varphi'|^{-(1-\beta)} \, dt \right)^{\frac{1}{\alpha}} .$$

We now split up the interval (a,b) as a union of intervals (a',b') in such a way that the endpoints of these intervals are exactly the zeros of φ' on (a,b) union the points a and b. Then the Cauchy-Schwarz inequality yields the required inequality (13) with the extra factor of N_0.

3. Certain Class of Polynomials

The purpose of this section is to present two classes of polynomials that lie in the class $\mathfrak{S}^{\alpha,\delta,q}$. We start by recalling the following lemma in [RS]:

Lemma 3.1. ([RS]). *Let $P(x) = \sum_{|\alpha|\leq d} a_\alpha x^\alpha$ be a polynomial mapping in $\mathbb{R}^n$ of degree at most d. Suppose that $\varepsilon < 1/d$. Then*

$$\int_{|x|<1} |P(x)|^{-\varepsilon} \, dx \leq A_\varepsilon \left(\sum_{|\alpha|\leq d} |a_\alpha| \right)^{-\varepsilon} .$$

The bound A_ε depends on ε (and the dimension), but not on the coefficients $\{a_\alpha\}$.

Now, we prove the following proposition:

Proposition 3.2. *Let $Q(x) = \sum_{|\beta|\leq d} a_\beta x^\beta : \mathbb{R}^n \to \mathbb{R}$ be a polynomial mapping of degree $d \geq 1$. Then the mapping $\Phi_Q(x,y) = Q(x)\,|y|^2$ belongs to the class $\mathfrak{S}^{2,\delta,q}$ where $q \geq 1$* and *$1 - 1/dq < \delta \leq 1$. Moreover, the following inequality holds*

$$\left\| I_{j,\delta,\Phi_Q}(\cdot) \right\|_{L^q(\mathbb{S}^{n-1})} \leq C 2^{-(1-\delta)dj+j} \left(\sum_{|\beta|=d} |a_\beta| \right)^{-(1-\delta)} . \tag{14}$$

Proof. We begin by observing that

$$\Phi_Q((r+v)y+z,z+uy)=\varphi(y,z,r,v)u^2+g_1(y,z,r,v)$$

where $\varphi(y,z,r,v)=Q((r+v)y+z)$ and $g_1(y,z,r,v)=|z|^2\,Q((r+v)y+z)$. Notice that

$$\begin{aligned}\varphi(y,z,r,v) &= \sum\nolimits_{|\beta|\le d} a_\beta((r+v)y+z)^\beta\\ &= \sum\nolimits_{|\beta|=d} a_\beta((r+v)y+z)^\beta+\sum\nolimits_{|\beta|<d} a_\beta((r+v)y+z)^\beta\\ &= \sum\nolimits_{|\beta|=d} a_\beta\prod_{l=1}^{n}((r+v)y_l+z_l)^{\beta_l}+\sum\nolimits_{|\beta|<d} a_\beta((r+v)y+z)^\beta\\ &= \left(\sum\nolimits_{|\beta|=d} a_\beta y^\beta\right)r^d+\sum_{l=1}^{d-1}a_l(y,z,v)r^l \qquad (15)\end{aligned}$$

where $a_l(y,z,v)$ is a polynomial in the variables y, z, and v. Thus, the equation (15), combined with an application of Lemma 2.1 implies that

$$\begin{aligned}I_{j,\delta,\Phi_Q}(y) &= \sup_{z\in\mathbb{R}^n}\int_{2^{-j-1}}^{2^j}\int_1^2\left|\frac{\partial}{\partial r}\varphi(y,z,2^j r,v)\right|^{-(1-\delta)}drdv\\ &= \sup_{z\in\mathbb{R}^n}\int_{2^{-j-1}}^{2^j}\int_1^2\left|d\left(\sum\nolimits_{|\beta|=d} a_\beta y^\beta\right)2^{dj}r^{d-1}+\sum_{l=1}^{d-1}2^{lj}la_l(y,\pm z,v)r^{l-1}\right|^{-(1-\delta)}drdv\\ &\le C\sup_{z\in\mathbb{R}^n}\int_{2^{-j-1}}^{2^j}\left|\left|d\left(\sum\nolimits_{|\beta|=d} a_\beta y^\beta\right)2^{dj}\right|+\sum_{l=1}^{d-1}\left|2^{lj}la_l(y,\pm z,v)\right|\right|^{-(1-\delta)}dv\\ &\le C2^{-(1-\delta)dj+j}\left|\sum\nolimits_{|\beta|=d} a_\beta y^\beta\right|^{-(1-\delta)}. \qquad (16)\end{aligned}$$

By applying Lemma 2.1 one more time, we get

$$\begin{aligned}\left\|I_{j,\delta,\Phi_Q}(y)\right\|_q &\le C2^{-(1-\delta)dj+j}\left(\int_{\mathbb{S}^{n-1}}\left|\sum\nolimits_{|\beta|=d} a_\beta y^\beta\right|^{-(1-\delta)q}\right)^{\frac{1}{q}}\\ &\le C2^{-(1-\delta)dj+j}\left(\sum\nolimits_{|\beta|=d}|a_\beta|\right)^{-(1-\delta)}.\end{aligned}$$

This completes the proof of Proposition 3.2.

Our second example of polynomials that lie in the class $\mathfrak{S}^{\alpha,\delta,q}$ is the following, whose verification follows a similar argument to that in Proposition 3.2:

Proposition 3.3. *Let* $Q(x)=\sum\nolimits_{|\beta|\le d} a_\beta x^\beta:\mathbb{R}^n\to\mathbb{R}$ *be a polynomial mapping of degree* $d\ge 1$ *and let* a *be a point in* $\mathbb{R}^n$*. Then the mapping* $\Phi_Q(x,y)=Q(x)(a\cdot y)$ *belongs to the class* $\mathfrak{S}^{1,\delta,q}$ *where* $q\ge 1$ and $1-1/(d+1)q<\delta\le 1$*. Moreover, the following inequality holds*

$$\left\|I_{j,\delta,\Phi_Q}(\cdot)\right\|_{L^q(\mathbb{S}^{n-1})}\le C2^{-(1-\delta)dj+j}\left(\|a\|\sum\nolimits_{|\beta|=d}|a_\beta|\right)^{-(1-\delta)}. \qquad (17)$$

4. An Extrapolation Theorem

In order to prove results in this paper, we shall make use of the decomposition of Ω in [AlSP]. To this end, it suffices to prove the following version of Theorem A:

Theorem 4.1. *Let $\Omega \in L^q(\mathbb{S}^{n-1}), q > 1$, be a homogeneous function of degree zero on $\mathbb{R}^n$ and satisfy (1). Let $\alpha \geq 1$ and $0 \leq \beta < 1$. Suppose that $\Phi \in \mathfrak{S}^{\alpha,\beta,q'}$ and that $h \in \Delta_{\frac{4\alpha}{2\alpha+\beta-2}}$. Then for $1 < p < \infty$, there exists a constant $C_p > 0$ independent of Ω and h such that*

$$\left\| T^{\infty,\beta}_{\Phi,\Omega,h} \right\|_p \leq \ln(e + \|\Omega\|_q) \|h\|_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}} \|\Omega\|_1^{\frac{\ln(e+\|\Omega\|_q)-1}{\ln(e+\|\Omega\|_q)}} C_p(1 + C_\Phi) \|f\|_{L^p}$$

for $f \in L^p(\mathbb{R}^n), 1 < p < \infty$.

Proof. For $j \in \mathbb{N}$, let I_j be the interval $I_j = (2^{j-1}, 2^j]$. Let

$$K^\beta_{\Omega,h}(x, y) = |x - y|^{-n} \Omega(x - y)h(|x - y|)(1 + |y|^2)^{-\beta} \tag{18}$$

and

$$K^\beta_{\Omega,h,j}(x, y) = K^\beta_{\Omega,h}(x, y)\chi_{I_j}(|x - y|), \tag{19}$$

where χ_{I_j} is the characteristic function of the interval I_j. Then,

$$T^{\infty,\beta}_{\Phi,\Omega,h}(f)(x) = \sum_{j=1}^{\infty} T^{\infty,\beta}_{\Phi,\Omega,h,j}(f)(x). \tag{20}$$

where

$$T^{\infty,\beta}_{\Phi,\Omega,h,j}(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{i\Phi(x,y)} K^\beta_{\Omega,h,j}(x, y)f(y)dy.$$

By similar argument as in [LZ], for fixed $y' \in \mathbb{S}^{n-1}$, let Y be the hyperplane through the origin orthogonal to y'. Then for $x \in \mathbb{R}^n$, there exist $s \in \mathbb{R}$ and $z \in Y$ such that $x = z + sy'$. Therefore,

$$T^{\infty,\beta}_{\Phi,\Omega,h,j}(f)(x) = \int_{\mathbb{S}^{n-1}} \Omega(y')N_{j,y',z}(f(z + \cdot y')(t)dtd\sigma(y'), \tag{21}$$

where $N_{j,y',z}$ is the operator defined on $L^2(\mathbb{R})$ by

$$N_{j,y',z}(g)(s) = \int_{2^{j-1} \leq s-t < 2^j} e^{i\Phi(z+sy',z+ty')} \frac{h(s - t)}{(s - t)(1 + |z|^2 + t^2)^\beta} g(t)dt.$$

In order to estimate $\left\|N_{j,y',z}\right\|_2$, we use orthogonality argument. We consider the operator $(N_{j,y',z})^* N_{j,y',z}$ which has the kernel

$$M_j(u, v) = \frac{1}{(1 + |z|^2 + v^2)^{\frac{\beta}{2}} (1 + |z|^2 + u^2)^{\frac{\beta}{2}}} \int_{\frac{1}{2}}^{1} e^{iE_j(y',z,u,v,r)} b_j(r, v - u)dr, \tag{22}$$

where

$$E_j(y', z, u, v, r) = \Phi((2^j r + v)y' + z, z + vy') - \Phi((2^j r + v)y' + z, z + uy')$$

and

$$b_j(r, v) = h(2^j r)h(2^j r + v)r^{-1}(2^j r + v)^{-1}\chi_{[2^{j-1},2^j]}(2^j r + v). \tag{23}$$

Notice that

$$M_j(u,v) = 0 \text{ if } |v-u| > 2^{j-1}. \tag{24}$$

Now, we write $(N_{j,y',z})^* N_{j,y',z}$ as

$$(N_{j,y',z})^* N_{j,y',z} = L_{j,y',z} + R_{j,y',z} \tag{25}$$

where $L_{j,y',z}$ is the operator with kernel $M_j(u,v)\chi_{[0,2^{-j}]}(|v-u|)$ and $R_{j,y',z}$ is the operator with kernel $M_j(u,v)\chi_{[2^{-j},2^{j-1}]}(|v-u|)$.

First, we observe that

$$\begin{aligned}
&\left| M_j(u,v)\chi_{[0,2^{-j}]}(|v-u|)\right| \\
\leq\ & 2^{-2j+1}\left(\int_{2^{j-1}}^{2^j} h(r)h(r+v)\chi_{[2^{j-1},2^j]}(r+v)dr\right)\chi_{[0,2^{-j}]}(|v-u|) \\
\leq\ & 2^{-2j+1}\left(\int_{2^{j-1}}^{2^j} |h(r)|^{\frac{4\alpha}{2\alpha+\beta-2}}\, dr\right)^{\frac{2\alpha+\beta-2}{4\alpha}}\left(\int_{2^{j-1}}^{2^j} |h(r)|^{\frac{4\alpha}{2\alpha-\beta+2}}\, dr\right)^{\frac{2\alpha-\beta+2}{4\alpha}}\chi_{[0,2^{-j}]}(|v-u|) \\
\leq\ & 2^{-2j+1}2^{(\frac{2-\beta}{2\alpha})j}\left(\int_{2^{j-1}}^{2^j} |h(r)|^{\frac{4\alpha}{2\alpha+\beta-2}}\, dr\right)^{\frac{2\alpha+\beta-2}{2\alpha}}\chi_{[0,2^{-j}]}(|v-u|) \\
\leq\ & 2^{-j+1}\,\|h\|^2_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}}\chi_{[0,2^{-j}]}(|v-u|).
\end{aligned}$$

Thus, we immediately obtain

$$\sup_u \int_{u-2^{-j}}^{u+2^{-j}} \left|M_j(u,v)\chi_{[0,2^{-j}]}(|v-u|)\right| dv \leq 2^{-2j+1}\,\|h\|^2_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}} \tag{26}$$

and

$$\sup_v \int_{v-2^{-j}}^{v+2^{-j}} \left|M_j(u,v)\chi_{[0,2^{-j}]}(|v-u|)\right| du \leq 2^{-2j+1}\,\|h\|^2_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}}. \tag{27}$$

Therefore, we get the following L^2-norm of the operator $L_{j,y',z}$:

$$\left\|L_{j,y',z}\right\|_{L^2\to L^2} \leq 2^{-2j+1}\,\|h\|^2_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}}. \tag{28}$$

Next, we estimate the L^2 norm of the operator $N_{j,y',z}$. Notice that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|M_j(u,v)\chi_{[2^{-j},2^{j-1}]}(|v-u|)\right|^2 dvdu\\
=&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|M_j(u,v+u)\chi_{[2^{-j},2^{j-1}]}(|v|)\right|^2 dvdu\\
=&\int_{-\infty}^{\infty}\int_{2^{-j}}^{2^{j-1}}|M_j(u,v+u)|^2\,dvdu+\int_{-\infty}^{\infty}\int_{-2^{j-1}}^{-2^{-j}}|M_j(u,v+u)|^2\,dvdu\\
=&\int_{2^{-j}}^{2^{j-1}}\int_{-\infty}^{\infty}|M_j(u,v+u)|^2\,dudv+\int_{-2^{j-1}}^{-2^{-j}}\int_{-\infty}^{\infty}|M_j(u,v+u)|^2\,dudv.
\end{aligned}$$

Now by the decomposition (6), we have

$$\begin{aligned}
&\left|\int_{\frac{1}{2}}^{1}e^{iE_j(y',z,u,v+u,r)}b_j(r,v)dr\right|\\
=&\left|\int_{\frac{1}{2}}^{1}e^{i\varphi(y,z,2^jr,v)u^\alpha}e^{i\left(g_1(y,z,2^jr,v)+g_2(y,z,u,v)\right)}b_j(r,v)dr\right|\\
=&\left|\int_{\frac{1}{2}}^{1}e^{i\varphi(y,z,2^jr,v)u^\alpha}e^{ig_1(y,z,2^jr,v)}b_j(r,v)dr\right|. \qquad (29)
\end{aligned}$$

Therefore, (29) and the definition of the kernel M_j imply that

$$|M_j(u,v+u)|=\frac{1}{(1+|z|^2+(u+v)^2)^{\frac{\beta}{2}}(1+|z|^2+u^2)^{\frac{\beta}{2}}}\left|\int_{\frac{1}{2}}^{1}e^{i\varphi(y,z,2^jr,v)u^\alpha}e^{ig_1(y,z,2^jr,v)}b_j(r,v)dr\right|.$$

Thus, by Lemma 2.1 along with the observation that

$$
\begin{aligned}
&\left\| e^{ig_1(j,v,\cdot,z)} b_j(\cdot,v)\right\|^2_{L^{\frac{2\alpha}{2\alpha+\beta-2}}[\frac{1}{2},1]} \\
\leq\;& \left(\int_{\frac{1}{2}}^{1} \left| h(2^j r) h(2^j r+v) r^{-1} (2^j r+v)^{-1} \chi_{[2^{j-1},2^j]}(2^j r+v)\right|^{\frac{2\alpha}{2\alpha+\beta-2}} \right)^{\frac{2\alpha+\beta-2}{\alpha}} \\
\leq\;& 2^{-2j+4} \left(\int_{\frac{1}{2}}^{1} \left| h(2^j r) h(2^j r+v) \chi_{[2^{j-1},2^j]}(2^j r+v)\right|^{\frac{2\alpha}{2\alpha+\beta-2}} \right)^{\frac{2\alpha+\beta-2}{\alpha}} \\
\leq\;& 2^{-2j+4} \left(\frac{1}{2^j} \int_{2^{j-1}}^{2^j} |h(r)|^{\frac{4\alpha}{2\alpha+\beta-2}} \right)^{\frac{2\alpha+\beta-2}{\alpha}} \leq 2^{-2j+4} \|h\|^4_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}},
\end{aligned}
$$

we get

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| M_j(u,v)\chi_{[2^{-j},2^{j-1}]}(|v-u|)\right|^2 dv du \\
\leq\;& 2^{-2j+4}\|h\|^4_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}} \left\{ \begin{array}{l} \int_{2^{-j}}^{2^{j-1}} \left(\int_{\frac{1}{2}}^{1} \left| \frac{d}{dr}\varphi(y,z,2^j r,v)\right|^{-(1-\beta)} dr \right)^{\frac{1}{\alpha}} dv + \\ \int_{-2^{j-1}}^{-2^{-j}} \left(\int_{\frac{1}{2}}^{1} \left| \frac{d}{dr}\varphi(y,z,2^j r,-v)\right|^{-(1-\beta)} dr \right)^{\frac{1}{\alpha}} dv \end{array} \right\} \\
\leq\;& 2^{-2j+4}\|h\|^4_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}} 2^{\frac{1}{\alpha}j+1} \left\{ \begin{array}{l} \left(\int_{2^{-j}}^{2^{j-1}} \int_{\frac{1}{2}}^{1} \left| \frac{d}{dr}\varphi(y,z,2^j r,v)\right|^{-(1-\beta)} dr dv \right)^{\frac{1}{\alpha}} + \\ \left(\int_{-2^{j-1}}^{-2^{-j}} \int_{\frac{1}{2}}^{1} \left| \frac{d}{dr}\varphi(y,z,2^j r,-v)\right|^{-(1-\beta)} dr dv \right)^{\frac{1}{\alpha}} \end{array} \right\} \\
\leq\;& 2^{(-2+\frac{1}{\alpha})j}\|h\|^4_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}} C \left(I_{j,\delta,\Phi}(y)\right)^{\frac{1}{\alpha}}.
\end{aligned}
$$

In the above step, we used the observation that $|M_j(u,v)| \leq C(1+|u|)^{-\beta}$. Thus, we have

$$
\left\| N_{j,y',z}\right\|_{L^2\to L^2} \leq 2^{(-1+\frac{1}{2\alpha})j} \|h\|^2_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}} C\left(I_{j,\delta}(y)\right)^{\frac{1}{2\alpha}};
$$

when combined with (25) and (28) imply

$$
\left\| (N_{j,y',z})^* N_{j,y',z}\right\|_{L^2\to L^2} \leq 2C \max\left(2^{(-1+\frac{1}{2\alpha})j} \left(I_{j,\delta}(y)\right)^{\frac{1}{2\alpha}}, 2^{-j}\right) \|h\|^2_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}}.
$$

Therefore, we arrive at the following L^2-norm of $N_{j,y',z}$:

$$
\left\| N_{j,y',z}\right\|_{L^2\to L^2} \leq C \max\left(2^{(-\frac{1}{2}+\frac{1}{4\alpha})j} \left(I_{j,\delta}(y)\right)^{\frac{1}{4\alpha}}, 2^{-\frac{1}{2}j}\right) \|h\|_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}}. \tag{30}
$$

Now, by the boundedness of the Hardy Littlewood maximal function on L^p for all $1 < p < \infty$, the following inequality holds

$$\left\| N_{j,y',z} \right\|_{L^p \to L^p} \le C \left\| h \right\|_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}} \tag{31}$$

for $p > \frac{4\alpha}{2\alpha-\beta+2}$.

By interpolation between (30) and (31), we have

$$\begin{aligned} \left\| N_{j,y',z} \right\|_{L^p \to L^p} &\le C \max\left(2^{(-\frac{1}{2}+\frac{1}{4\alpha})\theta_p j} \left(I_{j,\delta}(y) \right)^{\frac{\theta_p}{4\alpha}}, 2^{-\frac{\theta_p}{2} j} \right) \left\| h \right\|_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}} \\ &\le C \max\left(2^{(-\frac{1}{2}+\frac{1}{4\alpha})\theta_p j} \left(1 + I_{j,\delta}(y) \right), 2^{-\frac{\theta_p}{2} j} \right) \left\| h \right\|_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}} \end{aligned} \tag{32}$$

for some $0 < \theta_p < 1$ and $p > \frac{4\alpha}{2\alpha-\beta+2}$. Notice here that $\frac{4\alpha}{2\alpha-\beta+2}$ is less than 2.

By (21), (32), and Minkowaski inequality, we obtain

$$\begin{aligned} \left\| T^{\infty,\beta}_{\Phi,\Omega,h,j}(f) \right\|_p &\le 2^{(-\frac{1}{2}+\frac{1}{4\alpha})\theta_p j} \left(\int\limits_{\mathbb{S}^{n-1}} \left| \Omega(y') \right| \left(1 + I_{j,\delta,\Phi}(y) \right) d\sigma(y') \right) \left\| f \right\|_p \\ &\le 2^{(-\frac{1}{2}+\frac{1}{4\alpha})\theta_p j} \left(\left\| \Omega \right\|_q \left\| I_{j,\delta,\Phi} \right\|_{q'} + \left\| \Omega \right\|_{L^1} \right) \left\| f \right\|_p \\ &\le 2^{(-\frac{1}{2}+\frac{1+\varepsilon_{q'}}{4\alpha})\theta_p j} \left\| \Omega \right\|_q \left\| h \right\|_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}} \left\| f \right\|_p C C_\Phi \end{aligned} \tag{33}$$

for $p > \frac{4\alpha}{2\alpha-\beta+2}$.

On the other hand, by (21), (31), and Minkowaski inequality, we have

$$\left\| T^{\infty,\beta}_{\Phi,\Omega,h,j}(f) \right\|_p \le \left\| \Omega \right\|_1 \left\| h \right\|_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}} \left\| f \right\|_p C. \tag{34}$$

Thus, by interpolation between (33) and (34), we get

$$\begin{aligned} &\left\| T^{\infty,\beta}_{\Phi,\Omega,h,j}(f) \right\|_p \\ \le\ & 2^{(-\frac{1}{2}+\frac{1}{4\alpha})\frac{\theta_p}{\ln(e+\|\Omega\|_q)} j} \left\| \Omega \right\|_q^{\frac{1}{\ln(e+\|\Omega\|_q)}} \left\| \Omega \right\|_1^{\frac{\ln(e+\|\Omega\|_q)-1}{\ln(e+\|\Omega\|_q)}} \left\| h \right\|_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}} \left(1 + C_\Phi \right) \left\| (1+|y|^2)^{-\beta} f \right\|_p \\ \le\ & 2^{(-\frac{1}{2}+\frac{1}{4\alpha})\frac{\theta_p}{\ln(e+\|\Omega\|_q)} j} \left\| \Omega \right\|_q^{\frac{1}{\ln(e+\|\Omega\|_q)}} \left\| \Omega \right\|_1^{\frac{\ln(e+\|\Omega\|_q)-1}{\ln(e+\|\Omega\|_q)}} \left\| h \right\|_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}} \left(1 + C_\Phi \right) \left\| f \right\|_p \\ \le\ & 2^{(-\frac{1}{2}+\frac{1}{4\alpha})\frac{\theta_p}{\ln(e+\|\Omega\|_q)} j} \left\| \Omega \right\|_1^{\frac{\ln(e+\|\Omega\|_q)-1}{\ln(e+\|\Omega\|_q)}} \left\| h \right\|_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}} \left(1 + C_\Phi \right) \left\| f \right\|_p . \end{aligned}$$

Thus,

$$\begin{aligned}
&\left\|T^{\infty,\beta}_{\Phi,\Omega,h}(f)\right\|_p\\
\leq\;& \sum_{j=1}^{\infty}\left\|T^{\infty,\beta}_{\Phi,\Omega,h,j}(f)\right\|_{L^p}\\
\leq\;& \|\Omega\|_1^{\frac{\ln(e+\|\Omega\|_q)-1}{\ln(e+\|\Omega\|_q)}}\|h\|_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}}(1+C_\Phi)\|f\|_p\sum_{j=1}^{\infty}2^{(-\frac{1}{2}+\frac{1}{4\alpha})\frac{\theta p}{\ln(e+\|\Omega\|_q)}j}\\
\leq\;& \ln(e+\|\Omega\|_q)\|h\|_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}}\|\Omega\|_1^{\frac{\ln(e+\|\Omega\|_q)-1}{\ln(e+\|\Omega\|_q)}}C_p(1+C_\Phi)\|f\|_{L^p} \qquad (35)
\end{aligned}$$

for $p>\frac{4\alpha}{2\alpha-\beta+2}$. Since $\frac{4\alpha}{2\alpha-\beta+2}<2$, duality implies that the L^p bounds hold for all $1<p<\infty$. This completes the proof.

5. Some Reduction Lemmas

Lemma 5.1. *Let $\Omega\in L^\infty(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying (1), $\|\Omega\|_1\leq 1$, and $\|\Omega\|_\infty\leq 2^A$ for some $A>1$. Suppose that $h\in\Delta_{\frac{4}{\beta}}$ for some $1/2<\beta\leq 1$. Let $a,b\in\mathbb{R}^n$ and let $K^\beta_{\Omega,h}$ be given by (18). Then the operator*

$$S^\beta_{a,b,\Omega,h}f(x)=\int_{\mathbb{R}^n}e^{i(b\cdot x)(a\cdot y)}K^\beta_{\Omega,h}(x,y)f(y)dy$$

satisfies

$$\left\|S^\beta_{a,b,\Omega,h}f\right\|_p\leq C_p\|h\|_{\Delta_{\frac{4}{\beta}}}A\|f\|_p \qquad (36)$$

for $f\in L^p(\mathbb{R}^n)$, $1<p<\infty$ with constant C_p independent of A and the vectors a and b.

Proof. By dilation invariance, we may assume that $\|a\|\,\|b\|=1$. We decompose $S^\beta_{a,b,\Omega,h}$ as follows

$$S^\beta_{a,b,\Omega,h}=S^{\beta,0}_{a,b,\Omega,h}+S^{\beta,\infty}_{a,b,\Omega,h} \qquad (37)$$

where

$$S^{\beta,0}_{a,b,\Omega,h}(f)(x)=\int_{|x-y|<1}e^{i(b\cdot x)(a\cdot y)}K^\beta_{\Omega,h}(x,y)f(y)dy \qquad (38)$$

and

$$S^{\beta,\infty}_{a,b,\Omega,h}(f)(x)=\int_{|x-y|\geq 1}e^{i(b\cdot x)(a\cdot y)}K^\beta_{\Omega,h}(x,y)f(y)dy. \qquad (39)$$

Now, since the function $g(y)=(1+|y|^2)^{-\beta}f(y)$ satisfies $\|g\|_p\leq\|f\|_p$, it follows by Theorem 1.3 in [AlSP] that the operator

$$S^\beta_{\Omega,h}(f)(x)=\int_{\mathbb{R}^n}K^\beta_{\Omega,h}(x,y)f(y)dy \qquad (40)$$

satisfies

$$\left\|S^\beta_{\Omega,h}(f)\right\|_p\leq C_p\|h\|_{\Delta_{\frac{4}{\beta}}}\|\Omega\|_{L\log L}\leq C_p\|h\|_{\Delta_{\frac{4}{\beta}}}A\|f\|_p \qquad (41)$$

for all $1 < p < \infty$. Here, we used the fact that $\|\Omega\|_{L\log L} \leq CA$. By replacing h in (40) with $h(t)\chi_{|t|<1}$, the inequality (41) implies that the local operator

$$S^{\beta,0}_{\Omega,h}(f)(x) = \int_{\mathbb{R}^n} K^{\beta}_{\Omega,h}(x,y)f(y)dy \tag{42}$$

satisfies

$$\left\|S^{\beta,0}_{\Omega,h}(f)\right\|_p \leq C_p \|h\|_{\Delta_{\frac{4}{\beta}}} \|\Omega\|_{L\log L} \|f\|_p \leq C_p \|h\|_{\Delta_{\frac{4}{\beta}}} A \|f\|_p \tag{43}$$

for all $1 < p < \infty$.

Now, for any given $w \in \mathbb{R}^n$, we have

$$(b\cdot x)(a\cdot y) = (b\cdot(x-w))(a\cdot(y-x)) + (b\cdot(x-w))(a\cdot x) + (b\cdot w)(a\cdot y). \tag{44}$$

Let

$$R^{\beta,0}_{a,b,\Omega,h}(f)(x) = S^{\beta,0}_{a,b,\Omega,h}(f)(x) - e^{i(b\cdot(x-w))(a\cdot x)} S^{\beta,0}_{\Omega,h}(e^{-(b\cdot w)(a\cdot y)} f)(x). \tag{45}$$

Thus, the operator $S^{\beta,0}_{a,b,\Omega,h}$ can be dominated as

$$\left|S^{\beta,0}_{a,b,\Omega,h}(f)(x)\right| \leq \left|R^{\beta,0}_{a,b,\Omega,h}(f)(x)\right| + \left|e^{i(b\cdot(x-w))(a\cdot x)} S^{\beta,0}_{\Omega,h}(e^{-(b\cdot w)(a\cdot y)} f)(x)\right|. \tag{46}$$

Now, $|x-y| < 1$ and $|x-w| < 1/4$, we have

$$\begin{aligned}
&\left|R^{\beta,0}_{a,b,\Omega,h}(f)(x)\right| \\
\leq & \int_{|x-y|<1} |(b\cdot(x-w))(a\cdot(y-x))| \left|K^{\beta}_{\Omega,h}(x,y)\right| |f(y)|\, dy \\
\leq & \int_{|x-y|<1} |x-w|\, |x-y|^{-n+1} |h(|x-y|)|\, |\Omega(x-y)|\, |f(y)|\, dy.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{|x-w|<\frac{1}{4}} \left|R^{\beta,0}_{a,b,\Omega,h}(f)(x)\right|^p dx \\
\leq\ & C\{(\int_{|z|<1} |h(|z|)|\, |z|^{1-n} |\Omega(z)|\, dz)^p\} \int_{|y-w|<\frac{5}{4}} |f(y)|^p\, dy \\
\leq\ & C \|\Omega\|^p_{L^1} \int_{|y-w|<\frac{5}{4}} |f(y)|^p\, dy. \qquad (47)
\end{aligned}$$

By (45), (42), and (47) along with $\|\Omega\|_{L^1} \leq 1$, we have

$$\int_{|x-w|<\frac{1}{4}} \left|S^{\beta,0}_{a,b,\Omega,h} f(x)\right|^p dx \leq C_p \|h\|_{\Delta_{\frac{4}{\beta}}} A \|f\|^p_p + C_p \int_{|y-w|<\frac{5}{4}} |f(y)|^p\, dy. \tag{48}$$

Since $w \in \mathbb{R}^n$ is arbitrary, (48) implies

$$\left\|S^{\beta,0}_{a,b,\Omega,h}(f)\right\|_p \leq C_p \|h\|_{\Delta_{\frac{4}{\beta}}} A \|f\|_p \tag{49}$$

for all $1 < p < \infty$.

Next, we move to estimate $\left\|S^{\beta,\infty}_{a,b,\Omega,h}\right\|$. However, this is a consequence of Theorem 4.1 since the mapping $P(x,y)=(b\cdot x)(a\cdot y)$ belongs to $\mathfrak{S}^{1,\delta,1}$ for $1/2<\delta\le 1$. Thus, by Theorem 4.1, (17), and the assumption $\|a\|\,\|b\|=1$, we get

$$\left\|S^{\beta,\infty}_{a,b,\Omega,h}(f)\right\|_p \le C_p A\,\|h\|_{\Delta_{\frac{4}{\beta}}}\,\|f\|_p \tag{50}$$

for all $1<p<\infty$ with constant C_p independent of the essential variables. Hence, by (49), (50), (37), Minkowaski inequality, we obtain (36). This completes the proof.

Lemma 5.2. *Let $\Omega\in L^\infty(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying (1), $\|\Omega\|\le 1$, and $\|\Omega\|_\infty\le A$ for some $A>1$. Suppose that $d\ge 1$ and that $h\in\Delta_{\frac{4}{\beta}}$ for some $1-1/(d+1)q<\beta\le 1$. Then for any polynomial mappings P with degree d and any vector $a\in\mathbb{R}^n$, the operator*

$$S^{\beta}_{a,P,\Omega,h}f(x)=\int\limits_{\mathbb{R}^n} e^{iP(x)(a\cdot y)}K^{\beta}_{\Omega,h}(x,y)f(y)dy \tag{51}$$

satisfies

$$\left\|S^{\beta}_{a,P,\Omega,h}f\right\|_p \le C_p\,\|h\|_{\Delta_{\frac{4}{\beta}}}\,A\,\|f\|_p \tag{52}$$

for $f\in L^p(\mathbb{R}^n)$, $1<p<\infty$ with constant C_p independent of A, the of the coefficients of the polynomial mapping P, and the vector a.

Proof. We argue by induction on d. If $d=1$, then for some constant c, we have

$$S^{\beta}_{a,P,\Omega,h}f(x)=\int\limits_{\mathbb{R}^n} e^{i(b\cdot x)(a\cdot y)}K^{\beta}_{\Omega,h}(x,y)e^{ic(a\cdot y)}f(y)dy. \tag{53}$$

Thus, (52) follows by Lemma 5.1. Next, assume that (52) holds for all polynomial mappings P of degree at most $d-1$ and vectors $a\in\mathbb{R}^n$. Let $P(x)=\sum\limits_{|\alpha|=d}a_\alpha x^\alpha$ be a polynomial of degree d and let a be a vector in $\mathbb{R}^n$. By dilation invariance, we may assume that $\left(\sum_{|\alpha|=d}|a_\alpha|\right)\|a\|=1$. We let

$$S^{\beta,\infty}_{a,P,\Omega,h}f(x)=\int\limits_{|x-y|\ge 1} e^{iP(x)(a\cdot y)}K^{\beta}_{\Omega,h}(x,y)f(y)dy \tag{54}$$

and

$$S^{\beta,0}_{a,P,\Omega,h}f(x)=\int\limits_{|x-y|<1} e^{iP(x)(a\cdot y)}K^{\beta}_{\Omega,h}(x,y)f(y)dy. \tag{55}$$

Then

$$S^{\beta}_{a,P,Q,\Omega,h}f(x)=S^{\beta,\infty}_{a,P,\Omega,h}f(x)+S^{\beta,0}_{a,P,\Omega,h}f(x). \tag{56}$$

By assumptions and Theorem 4.1, we have

$$\left\|S^{\beta,\infty}_{a,P,\Omega,h}f\right\|_p \le C_p\,\|h\|_{\Delta_{\frac{4}{\beta}}}\,A\,\|f\|_p \tag{57}$$

for all $1<p<\infty$ with constant C_p independent of A, the of the coefficients of the polynomial mapping P, and the vector a.

Now, we turn to estimate $\left\| S^{\beta,0}_{a,P,\Omega,h} \right\|_p$. For any given $w \in \mathbb{R}^n$, we have

$$P(x)(a\cdot y) = \left(\sum_{|\alpha|=d} a_\alpha (x-w)^\alpha \right)(a\cdot(y-x)) + \left(\sum_{|\alpha|=d} a_\alpha (x-w)^\alpha \right)(a\cdot x) + \tilde{P}_w(x)(a\cdot y) \tag{58}$$

where $\tilde{P}_w$ is a polynomial of degree at most $d-1$. Thus,

$$\begin{aligned} \left| S^{\beta,0}_{a,P,Q,\Omega,h}(f)(x) \right| &\leq \left| S^{\beta,0}_{a,P,\Omega,h}(f)(x) - e^{i\left(\sum_{|\alpha|=d} a_\alpha (x-w)^\alpha\right)(a\cdot x)} S^{\beta,0}_{a,\tilde{P},\Omega,h}(f)(x) \right| + \left| S^{\beta,0}_{a,\tilde{P},\Omega,h}(f)(x) \right| \\ &= \left| R^{\beta,0}_{a,P,\Omega,h}(f)(x) \right| + \left| S^{\beta,0}_{a,\tilde{P},\Omega,h}(f)(x) \right| \end{aligned} \tag{59}$$

By induction assumption, we have

$$\left\| S^{\beta,0}_{a,\tilde{P},\Omega,h}(f) \right\|_p \leq C_p \left\| h \right\|_{\Delta_{\frac{4}{\beta}}} A \left\| f \right\|_p \tag{60}$$

for all $1 < p < \infty$. The constant C_p is independent of A, the of the coefficients of the polynomial mapping $\tilde{P}$, and the vector a. Hence, it is independent of w.

Next, for $|x-y| < 1$ and $|x-w| < \frac{1}{4}$, we have

$$\begin{aligned} &\left| P(x)(a\cdot y) - \left(\sum_{|\alpha|=d} a_\alpha (x-w)^\alpha \right)(a\cdot x) - \tilde{P}_w(x)(a\cdot y) \right| \\ \leq\ & \left(\sum_{|\alpha|=d} |a_\alpha| \left| x-w \right|^d \right) \|a\| \, |y-x| \leq \left((\frac{1}{4})^d \|a\| \sum_{|\alpha|=d} |a_\alpha| \right) |y-x| \\ \leq\ & C\, |y-x| . \end{aligned}$$

Thus, by similar argument as in the steps (46) to (49), we obtain

$$\left\| R^{\beta,0}_{a,P,\Omega,h}(f) \right\|_p \leq C_p \left\| h \right\|_{\Delta_{\frac{4}{\beta}}} A \left\| f \right\|_p \tag{61}$$

for all $1 < p < \infty$. Hence, (52) follows by (56), (57), (59), (60), (61), and Minkowaski inequality. This competes the proof.

We end this section by proving the following lemma, which will be used to prove Theorem D.

Lemma 5.3. *Let $\Omega \in L^\infty(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying (1), $\|\Omega\| \leq 1$, and $\|\Omega\|_\infty \leq A$ for some $A > 1$, and let a be a point in $\mathbb{R}^n$. Suppose that for some $0 \leq \gamma \leq 1$, the mapping satisfies any of the following conditions*
(i) $\Phi(x,y) = |x|^\gamma (a\cdot y)$
(ii) $\Phi(x,y) = |y|^\gamma (a\cdot x)$
(iii) $\Phi(x,y) = |x|^\gamma |y|^2$
(iv) $\Phi(x,y) = |y|^\gamma |x|^2$.
If $h \in \Delta_4$, then for $1 < p < \infty$, there exists a constant $C_p > 0$ independent of Ω and h such that the operator $T_{\Phi,\Omega,h}$ satisfies

$$\left\| T_{\Phi,\Omega,h} f \right\|_p \leq C_p \left\| h \right\|_{\Delta_4} \left\| \Omega \right\|_{L\log L} \left\| f \right\|_p$$

for $f \in L^p(\mathbb{R}^n)$. The constant C_p is independent of the point a in case the mapping Φ satisfies (i) or (ii).

Proof (of Lemma 5.3). We shall present the proof for the cases (i) and (iii). The other cases follow by considering the adjoint operator. We start by the case $\Phi(x,y) = |x|^\gamma (a\cdot y)$. By dilation invariance, we may assume that $\|a\| = 1$. First, we observe that $\Phi \in \mathfrak{S}^{1,0,\infty}$. Thus, by Theorem 4.1, we have that the operator

$$T^{\infty}_{\Phi,\Omega,h} f(x) = \int\limits_{|x-y|>1} e^{i\Phi(x,y)} |x-y|^{-n} h(|x-y|)\Omega(x-y) f(y) dy$$

satisfies

$$\left\|T^{\infty}_{\Phi,\Omega,h}\right\|_p \le C_p A \|f\|_{L^p} \tag{62}$$

for $f \in L^p(\mathbb{R}^n), 1 < p < \infty$, with constant C_p independent of the point a.

Next, by similar argument as in the proof of Lemma 5.2 along with the observation that $|t^\gamma - s^\gamma| \le C|t-s|$, we can show that the local operator

$$T^{0}_{\Phi,\Omega,h} f(x) = \int\limits_{|x-y|\le 1} e^{i\Phi(x,y)} |x-y|^{-n} h(|x-y|)\Omega(x-y) f(y) dy$$

satisfies

$$\left\|T^{0}_{\Phi,\Omega,h}\right\|_p \le C_p A \|f\|_{L^p} \tag{63}$$

for $f \in L^p(\mathbb{R}^n), 1 < p < \infty$, with constant C_p independent of the point a. Hence, by (62), (63), and Minkowaski inequality, we obtain

$$\left\|T_{\Phi,\Omega,h}\right\|_p \le C_p A \|f\|_{L^p} \tag{64}$$

for $f \in L^p(\mathbb{R}^n), 1 < p < \infty$, with constant C_p independent of the point a.

Now, we turn to the case $\Phi(x,y) = |x|^\gamma |y|^2$ Since $\Phi \in \mathfrak{S}^{2,0,\infty}$, we get that the corresponding global operator $T^{\infty}_{\Phi,\Omega,h}$ satisfies (62). Thus, we only need to prove that the corresponding local operator $T^{0}_{\Phi,\Omega,h}$ satisfies (63). For any given $w \in \mathbb{R}^n$, we have

$$\begin{aligned} \Phi(x,y) &= (|x|^\gamma - |w|^\gamma)|y-x|^2 + |x|^{\gamma+2} \\ &\quad +2(|x|^\gamma - |w|^\gamma)(y-x)\cdot x + |w|^\gamma |y|^2 - |w|^\gamma |x|^2 . \end{aligned}$$

Let

$$\begin{aligned} H_w(x,y) &= |x|^{\gamma+2} + 2(|x|^\gamma - |w|^\gamma)(y-x)\cdot w + |w|^\gamma |y|^2 - |w|^\gamma |x|^2 \\ &= \left(|x|^{\gamma+2} - 2|w|^\gamma (y\cdot w) - 2(|x|^\gamma - |w|^\gamma)(x\cdot w) + |w|^\gamma |y|^2 - |w|^\gamma |x|^2\right) + 2|x|^\gamma (y\cdot w). \end{aligned}$$

Notice that

$$T_{H_w,\Omega,h} f(x) = e^{ik(x)} \int\limits_{\mathbb{R}^n} e^{i2|x|^\gamma (w\cdot y)} |x-y|^{-n} \left(h(|x-y|)\right) \Omega(x-y) e^{-i\left(2|w|^\gamma (w\cdot y) - |w|^\gamma |y|^2\right)} f(y) dy$$

where

$$k(x) = |x|^{\gamma+2} - 2(|x|^\gamma - |w|^\gamma)(x\cdot w) - |w|^\gamma |x|^2 .$$

Thus, by result in case (i), we have

$$\left\|T_{H_w,\Omega,h}\right\|_p \le C_p A \|f\|_{L^p} \tag{65}$$

for $f \in L^p(\mathbb{R}^n), 1 < p < \infty,$ with constant C_p independent of the point w. The estimate (65) implies that the local operator corresponding to H_w satisfies

$$\left\|T^0_{H_w,\Omega,h}\right\|_p \le C_p A \left\|f\right\|_{L^p} \tag{66}$$

for $f \in L^p(\mathbb{R}^n), 1 < p < \infty,$ with constant C_p independent of the point w. By noticing that

$$\begin{aligned} & |\Phi(x,y) - H_w(x,y)| \\ \le \;& ||x|^\gamma - |w|^\gamma| \, |y-x|^2 + 2\, ||x|^\gamma - |w|^\gamma| \, |y-x| \, |x-w| \le c\, |y-x| \end{aligned}$$

for $|x-y| < 1$ and $|x-w| < \frac{1}{4}$, an argument similar to that in the proof of Lemma 5.2 implies that the operator $T^0_{\Phi,\Omega,h}$ satisfies (63). This completes the proof of the lemma.

Finally, we end this section by proving the following lemma:

Lemma 5.4. *Let $\Omega \in L^\infty(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying (1), $\|\Omega\| \le 1$, and $\|\Omega\|_\infty \le A$ for some $A > 1$. Suppose that $0 \le \beta < 1$. Let $K^\beta_{\Omega,h}$ be given by (18). Suppose also that for all polynomial mappings L with degree at most $d-1$ and $h \in \Delta_{\frac{8}{2+\beta}}$, the operator*

$$T^\beta_{L,\Omega,h} f(x) = \int\limits_{\mathbb{R}^n} e^{iL(x)|y|^2} K^\beta_{\Omega,h}(x,y) f(y) dy \tag{67}$$

satisfies

$$\left\|T^\beta_{L,\Omega,h} f\right\|_p \le C_p \left\|h\right\|_{\Delta_{\frac{8}{2+\beta}}} A \left\|f\right\|_p \tag{68}$$

for $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$ with constant C_p independent of A and the of the coefficients of the polynomial mapping L. Then for any polynomial mapping $P : \mathbb{R}^n \to \mathbb{R}^n$ of degree at most $d \ge 1$ and any polynomial mapping Q with degree at most $d-1$, the local operator

$$S^{\beta,0}_{a,P,Q,\Omega,h} f(x) = \int\limits_{|x-y|<1} e^{iP(x)\cdot y + Q(x)|y|^2} K^\beta_{\Omega,h}(x,y) f(y) dy \tag{69}$$

satisfies

$$\left\|S^{\beta,0}_{a,P,Q,\Omega,h} f\right\|_p \le C_p \left\|h\right\|_{\Delta_{\frac{8}{2+\beta}}} A \left\|f\right\|_p \tag{70}$$

for $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$ with constant C_p independent of A, the of the coefficients of the polynomial mappings L and P.

Proof. We argue by induction on d. If $d = 1$, then for some constant c and vectors a, b, and $\tilde{b}$ in $\mathbb{R}^n$, we have

$$S^\beta_{a,P,Q,\Omega,h} f(x) = \int\limits_{\mathbb{R}^n} e^{i(b\cdot x)(a\cdot y)} K^\beta_{\Omega,h}(x,y) e^{i(\tilde{b}\cdot y) + c_2|y|^2)} f(y) dy.$$

Thus, (52) follows by Lemma 5.2 and the observation that $h(t)\chi_{|t|<1} \in \Delta_{\frac{8}{2+\beta}}$. Next, assume that (68) and (70) hold for polynomial mappings P of degree at most d and polynomials L and Q of degrees at most $d-1$. Let $P(x) = (\sum_{|\alpha| \le d+1} a_{\alpha,j} x^\alpha)_{1 \le j \le n}$ be a polynomial mapping of degree $d+1$ and and let $Q(x) = \sum_{|\alpha|=d} b_\alpha x^\alpha$ be a polynomial mapping of degree at most d. By dilation invariance, we may assume that

$$\left(\sum_{j=1}^{n}\left|\sum_{|\alpha|=d+1} a_{\alpha,j}\right|^2\right)^{\frac{1}{2}} + \sum_{|\alpha|=d} |b_\alpha| = 1.$$

For any given $w \in \mathbb{R}^n$, we have

$$\begin{aligned}
& P(x)\cdot y + Q(x)\,|y|^2 \\
= & \left(\sum_{|\alpha|=d+1} a_{\alpha,j}(x-w)^\alpha\right)_{1\le j\le n} \cdot y + \tilde{P}(x,w)\cdot y + Q(x)\,|y|^2 \\
= & \left(\sum_{|\alpha|=d+1} a_{\alpha,j}(x-w)^\alpha\right)_{1\le j\le n} \cdot y + \left(\sum_{|\alpha|=d} b_\alpha (x-w)^\alpha\right)|y-w|^2 + 2\left(\sum_{|\alpha|=d} b_\alpha (x-w)^\alpha\right)(w\cdot y) \\
& + \left(\sum_{|\alpha|=d} b_\alpha (x-w)^\alpha\right)|w|^2 + \tilde{P}(x,w)\cdot y + \tilde{Q}(x)\,|y|^2 \\
= & \left(\sum_{|\alpha|=d+1} a_{\alpha,j}(x-w)^\alpha\right)_{1\le j\le n} \cdot (y-x) + \left(\sum_{|\alpha|=d} b_\alpha (x-w)^\alpha\right)|y-w|^2 + O(x,w)\cdot y + \tilde{Q}(x)\,|y|^2 \\
(71)\quad & + \left(\sum_{|\alpha|=d} b_\alpha (x-w)^\alpha\right)|w|^2
\end{aligned}$$

where $\tilde{P}(x,w) : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial of degree at most d in the x variable, $\tilde{Q}(x)$ is a polynomial of degree at most $d-1$, and $O(x,w) : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial of degree at most d in the x variable.

Thus,

$$\begin{aligned}
\left|S^{\beta,0}_{a,P,Q,\Omega,h}(f)(x)\right| & \le \left|S^{\beta,0}_{a,P,Q,\Omega,h}(f)(x) - e^{i\left(\sum\limits_{|\alpha|=d} b_\alpha (x-w)^\alpha\right)|w|^2} S^{\beta,0}_{a,O,\tilde{Q},\Omega,h}(\tilde{f})(x)\right| + \left|S^{\beta,0}_{a,\tilde{P},Q,\Omega,h}(f)(x)\right| \\
(72) \qquad & = \left|R^{\beta,0}_{a,P,Q,\Omega,h}(f)(x)\right| + \left|S^{\beta,0}_{a,\tilde{P},Q,\Omega,h}(f)(x)\right|
\end{aligned}$$

where

$$\tilde{f}(y) = e^{ig(y)} f(y)$$

and

$$g(y) = \exp\left(i\left(\sum_{|\alpha|=d} b_\alpha (y-w)^\alpha\right)|y-w|^2\right).$$

By induction assumption, we have

$$\left\|S^{\beta,0}_{a,\tilde{P},Q,\Omega,h}(f)\right\|_p \le C_p \,\|h\|_{\Delta_{\frac{8}{2+\beta}}} A \,\|f\|_p \tag{73}$$

for all $1 < p < \infty$.

Now, notice that

$$\left|P(x)\cdot y+Q(x)\,|y|^2-\left(\sum_{|\alpha|=d}b_\alpha(y-w)^\alpha\,|y-w|^2+O(x,w)\cdot y+\tilde{Q}(x)\,|y|^2+\left(\sum_{|\alpha|=d}b_\alpha(x-w)^\alpha\right)|w|^2\right)\right|$$

$$\leq\left|\left(\sum_{|\alpha|=d+1}a_{\alpha,j}(x-w)^\alpha\right)_{1\leq j\leq n}\cdot(y-x)+\left(\sum_{|\alpha|=d}b_\alpha(x-w)^\alpha\right)|y-w|^2\right|$$

$$\leq\left(\sum_{j=1}^{n}\left|\sum_{|\alpha|=d+1}a_{\alpha,j}(x-w)^\alpha\right|^2\right)^{\frac{1}{2}}|x-y|+\frac{25}{16}\sum_{|\alpha|=d}|b_\alpha(x-w)^\alpha-b_\alpha(y-w)^\alpha|$$

$$\leq\left((\frac{1}{4})^d\left(\sum_{j=1}^{n}\left|(74)\sum_{|\alpha|=d+1}a_{\alpha,j}\right|^2\right)^{\frac{1}{2}}+C\sum_{|\alpha|=d}|b_\alpha|\right)|x-y|\leq C\,|x-y|\,.$$

Thus, by (71), (74), and similar argument as in the steps (46) to (49), we obtain

$$\left\|R^{\beta,0}_{a,P,Q,\Omega,h}(f)\right\|_p\leq C_p\,\|h\|_{\Delta_{\frac{8}{2+\beta}}}\,A\,\|f\|_p \tag{75}$$

for all $p>\frac{8}{6-\beta}$. Hence, (52) follows by (75), (73), (73), Minkowaski inequality, and duality. This competes the proof.

6. Proof of Main Results

We start by the proof of Theorem A:

Proof (of Theorem A). By the same argument in [AlSP], we construct a sequence $\{A_m : m\in\mathbb{N}\}$ of functions on $\mathbb{S}^{n-1}$ and a sequence $\{\lambda_m : m\in\mathbb{N}\}\subset\mathbb{R}$ such that

$$\int_{\mathbb{S}^{n-1}}A_m(u')d\sigma(u')=0 \tag{76}$$

$$A_m(ru')=A_m(u'),r>0, \tag{77}$$

$$\|A_m\|_1\leq 4,\ \ \|A_m\|_\infty\leq 2^{2m+2}, \tag{78}$$

$$\Omega(x)=\sum_{m=1}^{\infty}\lambda_m A_m(x), \tag{79}$$

and

$$\sum_{m=1}^{\infty}(m+2)\,\lambda_m\leq\|\Omega\|_{L(\log L)(\mathbb{S}^{n-1})}. \tag{80}$$

By the identity (79), it follows that

$$T^{\infty,\beta}_{\Phi,\Omega,h}f=\sum_{m=1}^{\infty}\lambda_m T^{\infty,\beta}_{\Phi,A_m,h}f \tag{81}$$

where $T^{\infty,\beta}_{\Phi,A_m,h}$ is the operator given by (9) with Ω is replaced by A_m. By Theorem 4.1 with Ω is replaced by A_m and $q=\infty$, we get

$$\begin{aligned} \left\|T^{\infty,\beta}_{\Phi,A_m,h}f\right\|_p &\leq \ln(e+\|A_m\|_\infty)\,\|h\|_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}}\,\|\Omega\|_1^{\frac{\ln(e+\|\Omega\|_\infty)-1}{\ln(e+\|\Omega\|_\infty)}}\,C_p(1+C_\Phi)\,\|f\|_{L^p} \\ &\leq (2m+2)\,\|h\|_{\Delta_{\frac{4\alpha}{2\alpha+\beta-2}}}\,C_p(1+C_\Phi)\,\|f\|_{L^p} \end{aligned} \tag{82}$$

for all $1<p<\infty$. Hence, by (81), Minkowski inequality, (80), and (82), we obtain (10). This completes the proof of Theorem A.

Next, we prove Theorem B:

Proof (of Theorem B). We shall prove the case where $P_Q(x,y)=Q(x)\,|y|^2$. The case $P_Q(x,y)=Q(y)\,|x|^2$ follows by considering the adjoint operator. Assume that $P_Q(x,y)=Q(x)\,|y|^2$. We let $\{A_m : m\in\mathbb{N}\}$ and $\{\lambda_m : m\in\mathbb{N}\}$ be as in the proof of Theorem A. Then by (79), we get

$$T^{\beta}_{P_Q,\Omega,h}f(x)=\sum_{m=1}^{\infty}\lambda_m T^{\beta}_{P,A_m,h}f \tag{83}$$

where $T^{\beta}_{P_Q,A_m,h}$ has the same definition as $T^{\beta}_{P_Q,\Omega,h}$ with Ω is replaced by A_m. By Minkowski inequality, (80), and (81), we only need to prove the following

$$\left\|T^{\beta}_{P_Q,A_m,h}f\right\|_p\leq m\,\|h\|_{\Delta_{\frac{8}{2+\beta}}}\,C_p\,\|f\|_p \tag{84}$$

for all $1<p<\infty$ and $h\in\Delta_{\frac{8}{2+\beta}}$ with constants C_p independent of m and the coefficients of the polynomial mapping Q.

We argue by induction on the degree $d=\deg(Q)$. If $d=0$, then

$$T^{\beta}_{P_Q,A_m,h}f(x)=T_{A_m,h}g(x)$$

where

$$g(x)=e^{ic|x|^2}(1+|x|^2)^{-\beta}f(x).$$

Thus, by Theorem 1.3 in [AlSP] and (78), it follows that

$$\left\|T^{\beta}_{P_Q,A_m,h}f\right\|_p\leq C_p\,\|A_m\|_{L\log^+L}\,\|h\|_{\Delta_{\frac{8}{2+\beta}}}\,\|g\|_p\leq mC_p\,\|f\|_p$$

for all $1<p<\infty$ and $h\in\Delta_{\frac{8}{2+\beta}}$ with constants C_p independent of m.

Next, assume that the inequality (12) holds for all polynomials Q with degree less than d and $h\in\Delta_{\frac{8}{2+\beta}}$. Let $Q(x)=\sum_{|\gamma|\leq d}a_\gamma x^\gamma$ be of degree d and let $h\in\Delta_{\frac{8}{2+\beta}}$. By dilations invariance, we may assume that

$$\sum_{|\gamma|=d}|a_\gamma|=1. \tag{85}$$

We let $T^{\beta,0}_{P,A_m,h}$ be the local operator

$$T^{\beta,0}_{P_Q,A_m,h}f(x)=\int_{|x-y|<1}e^{iP_Q(x,y)}K^{\beta}_{A_m,h}(x,y)f(y)dy. \tag{86}$$

where $K^{\beta}_{A_m,h}$ is given by (18) with Ω is replaced by A_m. Thus,

$$T^{\beta}_{P_Q,A_m,h}f(x) = T^{\beta,0}_{P_Q,A_m,h}f(x) + T^{\beta,\infty}_{P_Q,A_m,h}f(x). \tag{87}$$

Let

$$Q_{d-1}(x) = \sum\nolimits_{|\gamma|\le d-1} a_\gamma x^\gamma$$

and

$$P_{d-1}(x,y) = Q_{d-1}(x)\,|y|^2\,.$$

Let $T^{\beta}_{P_{d-1},\Omega,h}$ and $T^{\beta,0}_{P-1,\Omega,h}$be the operators given by (11) and (86) respectively with P_Q is replaced by P_{d-1}. By induction assumption, we have

$$\left\|T^{\beta}_{P_{d-1},A_m,b}f\right\|_p \le m\,\|b\|_{\Delta_{\frac{8}{2+\beta}}}\,C_p\,\|f\|_p \tag{88}$$

for all $1<p<\infty$ and $b\in\Delta_{\frac{8}{2+\beta}}$ with constants C_p independent of m and the coefficients of the polynomial mapping Q_{d-1}. By noting that $\overline{h}(t) = h(t)\chi_{|t|<1}(t)$ is in $\Delta_{\frac{8}{2+\beta}}$, it follows from the induction assumption that

$$\left\|T^{\beta,0}_{P_Q,A_m,h}f\right\|_p = \left\|T^{\beta}_{P_Q,A_m,\overline{h}}f\right\|_p \le m\,\|h\|_{\Delta_{\frac{8}{2+\beta}}}\,C_p\,\|f\|_p \tag{89}$$

for all $1<p<\infty$, $h\in\Delta_{\frac{8}{2+\beta}}$, and polynomials Q with degree less than d, where C_p is independent of m and the coefficients of the polynomial mapping Q In particular, we have

$$\left\|T^{\beta,0}_{P_{d-1},A_m,h}f\right\|_p \le m\,\|h\|_{\Delta_{\frac{8}{2+\beta}}}\,C_p\,\|f\|_p \tag{90}$$

for all $1<p<\infty$ with constants C_p independent of m and the coefficients of the polynomial mapping Q_{d-1}. By Theorem 4.1 and (85), we get

$$\begin{aligned}\left\|T^{\beta,\infty}_{P_Q,A_m,h}f(x)\right\|_p &\le \ln(e+\|A_m\|_\infty)\,\|h\|_{\Delta_{\frac{8}{2+\beta}}}\,\|A_m\|_1^{\frac{\ln(e+\|A_m\|_\infty)-1}{\ln(e+\|A_m\|_\infty)}}\,C_p\,\|f\|_{L^p}\\ &\le m\,\|h\|_{\Delta_{\frac{8}{2+\beta}}}\,C_p\,\|f\|_{L^p}\end{aligned} \tag{91}$$

for all $1<p<\infty$ with constants C_p independent of m the coefficients of the polynomial mapping Q.

Next, we move to estimate $\left\|T^{\beta,0}_{P_Q,A_m,h}f\right\|_p$. For $w\in\mathbb{R}^n$, we have

$$\begin{aligned}P_Q(x,y) &= \left(\sum\nolimits_{|\gamma|=d} a_\gamma(x-w)^\gamma\right)|y|^2 + \tilde{Q}(x,w)\,|y|^2\\ &= \left(\sum\nolimits_{|\gamma|=d} a_\gamma(x-w)^\gamma\right)|y-w|^2 - \left(\sum\nolimits_{|\gamma|=d} a_\gamma(x-w)^\gamma\right)|w|^2 + \tilde{Q}(x,w)\,|y|^2\\ &\quad + \left(\sum\nolimits_{|\gamma|\le d} a_\gamma(x-w)^\gamma\right)(2w\cdot y)\end{aligned} \tag{92}$$

where $\tilde{Q}$ is a polynomial of degree less than or equal $d-1$ in the $x-$variable. Set

$$H_{d,w}(x) = \left(\sum\nolimits_{|\gamma|=d} a_\gamma(x-w)^\gamma\right).$$

Let

$$R^{(w,0)}_{\tilde{Q},A_m,h}(f)(x)=\int\limits_{|x-y|<1} e^{iH_{d,w}(x)(2w\cdot y)+\tilde{Q}(x,w)|y|^2}K^{\beta}_{\Omega,h}(x,y)f(y)dy$$

Thus,by induction assumption and Lemma 5.4, we get

$$\left\|R^{(w,0)}_{\tilde{Q},A_m,h}f\right\|_p\leq C_p\left\|h\right\|_{\Delta_{\frac{8}{2+\beta}}} m\left\|f\right\|_p \tag{93}$$

for $f\in L^p(\mathbb{R}^n)$, $1<p<\infty$ with constant C_p independent of m, the of the coefficients of the polynomial mappings $H_{d,w}$, $\tilde{Q}$, and the point w.

Now

$$\left|T^{\beta,0}_{P_Q,A_m,h}(f)(x)\right|\leq\left|T^{\beta,0}_{P_Q,A_m,h}(f)(x)-e^{ig(x)}R^{(w,0)}_{\tilde{Q},A_m,h}(\tilde{f})(x)\right|+\left|R^{(w,0)}_{\tilde{Q},A_m,h}(f)(x)\right| \tag{94}$$

where

$$g(x)=\left(\sum\nolimits_{|\gamma|=d}a_\gamma(x-w)^\gamma\right)|w|^2$$

and

$$\tilde{f}(y)=\exp\left(i\left(\sum\nolimits_{|\gamma|=d}a_\gamma(y-w)^\gamma\right)|y-w|^2\right).$$

Notice that for $|x-y|<1$ and $|x-w|<\frac{1}{4}$, we have

$$\begin{aligned}&\left|P_Q(x,y)-H_{d,w}(x)-\tilde{Q}(x,w)\left|y\right|^2-g(x)-\left(\sum\nolimits_{|\gamma|=d}a_\gamma(x-w)^\gamma\right)\left|y-w\right|^2\right|\\ \leq\;&\frac{16}{25}\sum\nolimits_{|\gamma|=d}\left|a_\gamma\right|\left|(x-w)^\gamma-(y-w)^\gamma\right|\\ \leq\;&C\left(\sum\nolimits_{|\beta|=d}\left|a_\beta\right|\right)\left|x-y\right|=C\left|x-y\right|.\end{aligned} \tag{95}$$

Thus by (92), (93), (94), (95), Minkowaski inequality, and similar argument as that led to (49) in the proof of Lemma 5.1, we obtain

$$\left\|T^{\beta,0}_{P_Q,A_m,h}f\right\|_p\leq C_p\left\|h\right\|_{\Delta_{\frac{8}{2+\beta}}} m\left\|f\right\|_p \tag{96}$$

for $f\in L^p(\mathbb{R}^n)$, $1<p<\infty$ with constant C_p independent of m, the of the coefficients of the polynomial mapping Q. By (87), (91), (96), Minkowaski inequality, and (80), we get the (12). This completes the proof.

Proof (of Theorem C). Theorem C is an immediate consequence of Lemma 5.2 and similar argument as in the proof of Theorem B. We omit details.

Finally, we move to the proof of Theorem D:

Proof (of Theorem D**).** Theorem D is an immediate consequence of Lemma 5.3 and similar argument as in the proof of Theorem B. We omit details.

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Ahmad Al Salman[a,b],, [a]Sultan Qaboos University, College of Science, Department Mathematics, Muscat, Sultanate of Oman

Email address: `E-mail: alsalman.squ.edu.om`

[b]Department of Mathematics, Yarmouk University, Irbid, Jordan.

Email address: `E-mail: alsalman@yu.edu.jo`

Loukas Grafakos, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

Email address: `grafakosl@missouri.edu`