

# Fractional Fourier transforms on $L^p$ and applications<sup>☆</sup>

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## Abstract

This paper is devoted to the  $L^p(\mathbb{R})$  theory of the fractional Fourier transform (FRFT) for  $1 \leq p < 2$ . In view of the special structure of the FRFT, we study FRFT properties of  $L^1$  functions, via the introduction of a suitable chirp operator. However, in the  $L^1(\mathbb{R})$  setting, problems of convergence arise even when basic manipulations of functions are performed. We overcome such issues and study the FRFT inversion problem via approximation by suitable means, such as the fractional Gauss and Abel means. We also obtain the regularity of fractional convolution and results on pointwise convergence of FRFT means. Finally we discuss  $L^p$  multiplier results and a Littlewood-Paley theorem associated with FRFT.

*Keywords:* Fractional Fourier transform, fractional approximate identities,  $L^p$  multipliers, Littlewood-Paley theorem

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## 1. Introduction

In classical Fourier analysis three important classes of operators arise: maximal averages, singular integrals, and oscillatory integrals. The Hardy-Littlewood maximal operator, the Hilbert transform and the Fourier transform, respectively, are prime examples of these classes of operators. In recent decades fractional versions of the first two types of operators have been widely studied, but less attention has been paid to the mathematical theory of the fractional Fourier transform. In this paper we undertake this task, which is strongly motivated by the important role it plays in practical applications.

The Fourier transform is one of the most important and powerful tools in theoretical and applied mathematics. Mainly driven by the need to analyze and process non-stationary signals, the Fourier transform of fractional order has been proposed and developed by several scholars. At present, the fractional Fourier transform (FRFT for short) has found applications in many aspects of scientific research and engineering technology, such as swept filter, artificial neural network, wavelet transform, time-frequency analysis, time-varying filtering, complex transmission and so on (see, e.g., [3, 14, 18, 22, 23, 25]). In addition, it was also used widely in fields of solving partial differential equations (cf., [11, 17]), quantum mechanics (cf., [17, 21]), diffraction theory and optical transmission (cf., [20]), optical system and optical signal processing (cf., [1, 10, 19]), optical image processing (cf., [10, 11]), etc.

The FRFT is a fairly old mathematical tool. It dates back to work of Wiener [26] in 1929, but it was not until the past three decades that significant attention was paid to this object starting with Namias' work [17] in 1980. The approach used by Namias relies primarily on eigenfunction expansions. For suitable functions  $f$  on the line, the classical Fourier transform  $\mathcal{F}$  is defined as follows

$$(\mathcal{F}f)(x) = \int_{-\infty}^{+\infty} f(t)e^{-2\pi ixt} dt. \quad (1.1)$$

It is known that  $\mathcal{F}$  is a homeomorphism on  $L^2(\mathbb{R})$  and has eigenvalues

$$\lambda_n = e^{-in\pi/2}, \quad n = 0, 1, 2, \dots$$

with corresponding eigenfunctions

$$\psi_n(x) = e^{-x^2/2} H_n(x),$$

where  $H_n$  is the Hermite polynomial of degree  $n$  (see [5]). Since  $\{\psi_n\}$  is an orthonormal basis of  $L^2(\mathbb{R})$ , it follows that

$$\mathcal{F}f = \sum_n e^{-in\pi/2} (f, \psi_n)_2 \psi_n, \quad \forall f \in L^2(\mathbb{R}).$$

1 This naturally leads to the definition of the fractional order operators  $\{\mathcal{F}_\alpha\}$  for  $\alpha \in \mathbb{R}$   
 2 via

$$3 \quad \mathcal{F}_\alpha f = \sum_n e^{-in\alpha} (f, \psi_n)_2 \psi_n, \quad \forall f \in L^2(\mathbb{R}). \quad (1.2)$$

4 It is clear that  $\mathcal{F}_\alpha = \mathcal{F}$  when  $\alpha = \pi/2$ .

5 In 1987, McBride and Kerr [12] provided a rigorous definition on the Schwartz  
 6 space  $S(\mathbb{R})$  of the FRFT in integral form, based on a modification of Namias' fractional  
 7 operators. For  $|\alpha| \in (0, \pi)$ , McBride and Kerr defined the FRFT by

$$8 \quad (\mathcal{F}_\alpha f)(x) = \frac{e^{i(\hat{\alpha}\pi/4 - \alpha/2)}}{\sqrt{|\sin \alpha|}} e^{i\pi x^2 \cot \alpha} \int_{-\infty}^{+\infty} f(t) e^{-\pi i(2xt \csc \alpha - t^2 \cot \alpha)} dt, \quad (1.3)$$

9 where  $\hat{\alpha} = \text{sgn}(\sin \alpha)$ . The definition extends to all  $\alpha \in \mathbb{R}$  by periodicity. The authors  
 10 in [12] derive the equivalence between the definition of FRFT in terms of Hermite  
 11 functions (1.2) and that of (1.3). Moreover, they proved that

12 **Theorem 1.1** ([12]). *For all  $f \in S(\mathbb{R})$  and all  $\alpha, \beta \in \mathbb{R}$  we have*

- 13 (i)  $\mathcal{F}_\alpha : S(\mathbb{R}) \rightarrow S(\mathbb{R})$  is a homeomorphism;  
 14 (ii)  $\mathcal{F}_\alpha \mathcal{F}_\beta f = \mathcal{F}_{\alpha+\beta} f$ .

15 Later, Kerr [9] studied the  $L^2(\mathbb{R})$  theory of  $\mathcal{F}_\alpha$ . He gave the definition of FRFT on  
 16  $L^2(\mathbb{R})$  by interpreting (1.3) as follows:

$$17 \quad (\mathcal{F}_\alpha f)(x) = \frac{e^{i(\hat{\alpha}\pi/4 - \alpha/2)}}{\sqrt{|\sin \alpha|}} e^{i\pi x^2 \cot \alpha} \lim_{R \rightarrow \infty} \int_{-R}^R f(t) e^{-\pi i(2xt \csc \alpha - t^2 \cot \alpha)} dt \quad (1.4)$$

18 and proved the following result.

19 **Theorem 1.2** ([9]). *For all  $f, g \in L^2(\mathbb{R})$  and all  $\alpha, \beta \in \mathbb{R}$  we have*

- 20 (i)  $\mathcal{F}_\alpha : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  is a homeomorphism;  
 21 (ii)  $\|\mathcal{F}_\alpha f\|_2 = \|f\|_2$ ;  
 22 (iii)  $\mathcal{F}_\alpha \mathcal{F}_\beta f = \mathcal{F}_{\alpha+\beta} f$ ;  
 23 (iv)  $\int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) g(x) dx = \int_{-\infty}^{+\infty} f(x) (\mathcal{F}_\alpha g)(x) dx$ ;  
 24 (v)  $\{\mathcal{F}_\alpha : \alpha \in \mathbb{R}\}$  is a strongly continuous unitary group of operators on  $L^2(\mathbb{R})$ .

25 More recently in [30], Zayed used a method similar to Namias' to produce frac-  
 26 tional versions of a wider class of transforms.

27 In [28], Zayed extended the FRFT to different spaces of generalized functions by  
 28 two approaches. The first is analytic and uses the so-called embedding method to  
 29 define the FrFT on the space  $\mathcal{E}'$  of distributions with compact support. The second is  
 30 algebraic, and involves the theory of Boehmians. Prior to this, Kerr [6] extended the  
 31 action of the FRFT on the space  $\mathcal{S}'$  of tempered distributions via duality. For additional

1 work in this area, refer to [16]. Very recently, Kamalakkannan and Roopkumar [8]  
 2 proved an inversion theorem and Parseval's identity for the multidimensional fractional  
 3 Fourier transform. In analogy with the existing fractional convolutions on functions of  
 4 a single variable, they introduced a generalized fractional convolution on functions  
 5 of several variables and derived several properties that relate to the multidimensional  
 6 fractional Fourier transform.

7 In an attempt to take the theory of FRFT beyond  $S(\mathbb{R})$  or  $L^2(\mathbb{R})$ , we discuss in  
 8 this paper (Section 4) the behavior of FRFT on  $L^p(\mathbb{R})$  for  $1 \leq p < 2$ . In Section 2,  
 9 we discuss the elementary properties of FRFT on  $L^1(\mathbb{R})$ . Section 3 is devoted to the  
 10 problem of FRFT inversion, which is established via an approximation in terms of  
 11 FRFT integral means. In Section 5, we discuss  $L^p$  multiplier results and a Littlewood-  
 12 Paley theorem associated with FRFT. Using the language of time-frequency analysis,  
 13 this means that an  $L^1$  chirp signal, whose FRFT is non-integrable, is recovered from the  
 14 frequency domain as a limit of the inverted Abel means of its FRFT; this is discussed  
 15 in the last section.

## 16 2. FRFT on $L^1(\mathbb{R})$

17 It is natural to begin our exposition by defining the FRFT on  $L^1(\mathbb{R})$ ; our definition is  
 18 like that in [12]. In  $L^1(\mathbb{R})$ , problems of convergence arise when certain manipulations  
 19 of functions are performed and FRFT inversion is not possible.

20 **Definition 2.1.** For  $f \in L^1(\mathbb{R})$  and  $\alpha \in \mathbb{R}$ , the fractional Fourier transform of order  $\alpha$   
 21 of  $f$  is defined by

$$22 \quad (\mathcal{F}_\alpha f)(x) = \begin{cases} \int_{-\infty}^{+\infty} K_\alpha(x, t) f(t) dt, & \alpha \neq n\pi, \quad n \in \mathbb{N}, \\ f(x), & \alpha = 2n\pi, \\ f(-x), & \alpha = (2n + 1)\pi, \end{cases} \quad (2.1)$$

23 where

$$24 \quad K_\alpha(x, t) = A_\alpha \exp \left[ 2\pi i \left( \frac{t^2}{2} \cot \alpha - xt \csc \alpha + \frac{x^2}{2} \cot \alpha \right) \right]$$

25 is the kernel of FRFT and

$$26 \quad A_\alpha = \sqrt{1 - i \cot \alpha}. \quad (2.2)$$

27 As the parameter  $\alpha$  only appears as an argument of trigonometric functions (see  
 28 (2.1)), it follows that  $\mathcal{F}_\alpha$  is  $2\pi$ -periodic with respect to  $\alpha$ . Hence, throughout this paper  
 29 we shall always assume  $\alpha \in [0, 2\pi)$ .

30 Notice now that when  $n \in \mathbb{Z}$ ,  $\mathcal{F}_{n\pi/2} f = \mathcal{F}^n f$ , where  $\mathcal{F}^n$  is the  $n$ th power of the  
 31 classical Fourier operator (1.1). Therefore,  $\mathcal{F}_\alpha$  can be regarded as the  $s$ th power of the  
 32 Fourier transform, where  $s = 2\alpha/\pi$ , that is,

$$33 \quad \mathcal{F}^s f = \mathcal{F}_{s\pi/2} f.$$

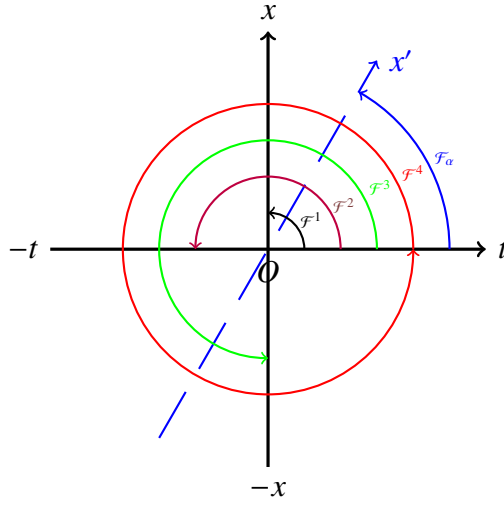


Figure 2.1: rotation of time-frequency domain

1 Denote by  $\mathcal{I}$  the identity operator and  $\mathcal{P}$  the reflection operator defined by  $\mathcal{P}g(x) =$   
 2  $g(-x)$ . We can easily see that (Figure 2.1)

$$\begin{aligned}
 3 \quad & \mathcal{F}^0 = \mathcal{F}_0 = \mathcal{I}; \\
 4 \quad & \mathcal{F}^1 = \mathcal{F}_{\pi/2} = \mathcal{F}; \\
 5 \quad & \mathcal{F}^2 = \mathcal{F}_\pi = \mathcal{P}; \\
 6 \quad & \mathcal{F}^3 = \mathcal{F}_{3\pi/2} = \mathcal{F}\mathcal{P} = \mathcal{P}\mathcal{F}; \\
 7 \quad & \mathcal{F}^4 = \mathcal{F}_{2\pi} = \mathcal{F}^0 = \mathcal{I}; \\
 8 \quad & \mathcal{F}^{4n\pm s} = \mathcal{F}_{2n\pi\pm\alpha} = \mathcal{F}_{\pm\alpha} = \mathcal{F}^{\pm s}.
 \end{aligned}$$

10 **Example 2.1.** Define the following function on the line:

$$11 \quad f(t) = \sum_{n=1}^{\infty} n e^{-i\pi t^2 \cot \alpha} \chi_{[n, n+\frac{1}{n^3})}(t).$$

12 Using (2.1), we can easily calculate the FRFT of this function:

$$13 \quad (\mathcal{F}_\alpha f)(x) = \frac{A_\alpha e^{i\pi x^2 \cot \alpha}}{2\pi i x \csc \alpha} \sum_{n=1}^{\infty} n e^{-2\pi i x \csc \alpha} \left( 1 - e^{-\frac{2\pi i x \csc \alpha}{n^3}} \right),$$

14 where  $A_\alpha$  is as in (2.2). This function lies in  $L^1(\mathbb{R})$  but not in  $L^2(\mathbb{R})$  as

$$15 \quad \int_{-\infty}^{+\infty} |f(t)| dt = \sum_{n=1}^{\infty} \int_n^{n+\frac{1}{n^3}} n dt = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$$

1 and

$$2 \int_{-\infty}^{+\infty} |f(t)|^2 dt = \sum_{n=1}^{\infty} \int_n^{n+\frac{1}{n^3}} n^2 dt = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

3 **Remark 2.1.** Define the chirp operator  $\mathcal{M}_\alpha$  acting on functions  $\phi$  in  $L^1(\mathbb{R})$  as follows:

$$4 \mathcal{M}_\alpha \phi(x) = e^{\pi i x^2 \cot \alpha} \phi(x).$$

5 Then for  $\alpha \neq n\pi$ , let  $A_\alpha$  be as in (2.2). Then the FRFT of  $f \in L^1(\mathbb{R})$  can be written as

$$6 (\mathcal{F}_\alpha f)(x) = A_\alpha e^{i\pi x^2 \cot \alpha} \mathcal{F}[e^{i\pi t^2 \cot \alpha} f(t)](x \csc \alpha) \\ 7 = A_\alpha \mathcal{M}_\alpha \mathcal{F}[\mathcal{M}_\alpha f(t)](x \csc \alpha). \quad (2.3)$$

9 In view of (2.3), we see that the FRFT of a function (or signal)  $u(t)$  can be decom-  
10 posed into four simpler operators, according to the diagram of Figure 2.2:

- 11 (i) multiplication by a chirp signal,  $g(t) = e^{\pi i t^2 \cot \alpha} u(t)$ ;
- 12 (ii) Fourier transform,  $\hat{g}(x) = (\mathcal{F} g)(x)$ ;
- 13 (iii) scaling,  $\tilde{g}(x) = \hat{g}(x \csc \alpha)$ ;
- 14 (iv) multiplication by a chirp signal,  $(\mathcal{F}_\alpha u)(x) = A_\alpha e^{i\pi x^2 \cot \alpha} \tilde{g}(x)$ .

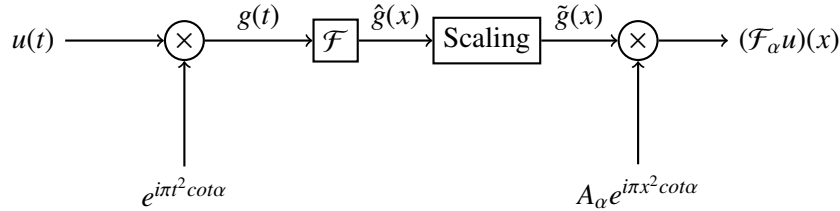


Figure 2.2: the decomposition of the FRFT

15 In view of the decomposition (2.3) of the FRFT, the boundedness properties of the  
16 fractional Fourier operator  $\mathcal{F}_\alpha$  is largely the same of the classical Fourier operator  $\mathcal{F}$ .  
17 However, due to the factors  $e^{i\pi x^2 \cot \alpha}$  and  $e^{i\pi t^2 \cot \alpha}$ , the convergence properties are not  
18 trivial. We now discuss some basic properties of the FRFT on  $L^1(\mathbb{R})$ .

19 Firstly, we consider the behavior of FRFT at infinity. The following is the fractional  
20 version of the Riemann-Lebesgue lemma.

21 **Lemma 2.2** (Riemann-Lebesgue lemma). *For  $f \in L^1(\mathbb{R})$ , we have that*

$$22 |(\mathcal{F}_\alpha f)(x)| \rightarrow 0$$

23 as  $|x| \rightarrow \infty$ .

1 *Proof.* Since  $\mathcal{M}_\alpha f \in L^1(\mathbb{R})$ , then  $|\mathcal{F}(\mathcal{M}_\alpha f)(x)| \rightarrow 0$  as  $x \rightarrow \infty$  by the Riemann-  
2 Lebesgue lemma for the classical Fourier transform. Hence, it follows from (2.3) and  
3 the boundedness of  $\mathcal{M}_\alpha$  that

$$4 \quad |(\mathcal{F}_\alpha f)(x)| = |A_\alpha \mathcal{M}_\alpha \mathcal{F}(\mathcal{M}_\alpha f)(x \csc \alpha)| \rightarrow 0$$

5 as  $|x| \rightarrow \infty$ . □

6 **Proposition 2.3.** *The following statements are valid:*

- 7 (i) *The FRFT  $\mathcal{F}_\alpha$  is a bounded linear operator from  $L^1(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$ .*  
8 (ii) *For  $f \in L^1(\mathbb{R})$ ,  $\mathcal{F}_\alpha f$  is uniformly continuous on  $\mathbb{R}$ .*

9 *Proof.* (i) It is obvious that  $\mathcal{F}_\alpha$  is linear. Moreover the claimed boundedness holds as

$$10 \quad \|\mathcal{F}_\alpha f\|_\infty = |A_\alpha| \|\mathcal{F}(\mathcal{M}_\alpha f)\|_\infty \leq |A_\alpha| \|\mathcal{M}_\alpha f\|_1 = |A_\alpha| \|f\|_1.$$

11 (ii) For an arbitrary  $\varepsilon > 0$ , it follows from Lemma 2.2 that there exists  $\eta > 0$  such that  
12 for every  $x_i \in \mathbb{R} \setminus [-\eta, \eta]$ ,  $|(\mathcal{F}_\alpha f)(x_i)| < \varepsilon/2$ ,  $i = 1, 2$ . Thus

$$13 \quad |(\mathcal{F}_\alpha f)(x_1) - (\mathcal{F}_\alpha f)(x_2)| < \varepsilon.$$

14 For every  $x_1, x_2 \in [-\eta - 1, \eta + 1]$ , by the Lagrange mean value theorem, there exists  
15  $\xi$  between  $x_1$  and  $x_2$  such that

$$16 \quad K_\alpha(x_1, t) - K_\alpha(x_2, t) = \frac{\partial}{\partial x} K_\alpha(\xi, t) (x_1 - x_2)$$

$$17 \quad = 2\pi i A_\alpha (\xi \cot \alpha - t \csc \alpha) K_\alpha(\xi, t) (x_1 - x_2).$$

19 There exist  $N > 0$  such that,

$$20 \quad \int_{|t| \geq N} |f(t)| dt < \frac{\varepsilon}{4}.$$

21 Hence,

$$22 \quad |(\mathcal{F}_\alpha f)(x_1) - (\mathcal{F}_\alpha f)(x_2)| = \left| \int_{-\infty}^{+\infty} (K_\alpha(x_1, t)f(t) - K_\alpha(x_2, t)f(t)) dt \right|$$

$$23 \quad \leq 2 \int_{|t| \geq N} |f(t)| dt + \left| \int_{|t| \leq N} f(t) (K_\alpha(x_1, t) - K_\alpha(x_2, t)) dt \right|$$

$$24 \quad < \frac{\varepsilon}{2} + 2\pi |A_\alpha| \int_{|t| \leq N} |f(t)| |\xi \cot \alpha - t \csc \alpha| |x_1 - x_2| dt$$

$$25 \quad < \frac{\varepsilon}{2} + C |x_1 - x_2| \int_{|t| \leq N} |f(t)| dt$$

$$26 \quad \leq \frac{\varepsilon}{2} + C |x_1 - x_2| \|f\|_1,$$

1

2 where  $C$  is a constant independent of  $x_1, x_2$ . Then for

3 
$$|x_1 - x_2| < \frac{\varepsilon}{2C \|f\|_1}.$$

4 we obtain

5 
$$|(\mathcal{F}_\alpha f)(x_1) - (\mathcal{F}_\alpha f)(x_2)| < \varepsilon.$$

6 So, we conclude that  $\mathcal{F}_\alpha f$  is uniformly continuous on  $\mathbb{R}$ . □

7 Lemma 2.2 and Proposition 2.3 imply that

8 
$$f \in L^1(\mathbb{R}) \Rightarrow \mathcal{F}_\alpha f \in C_0(\mathbb{R}). \quad (2.4)$$

9 A natural question is whether the reverse implication to (2.4) holds, precisely,

10 **Question.** Given  $g \in C_0(\mathbb{R})$ , is there a  $L^1$ -function  $f$  such that  $\mathcal{F}_\alpha f = g$ ?

11 The answer to this question is negative as the following example illustrates.

12 **Example 2.2.** Let

13 
$$g(x) = \begin{cases} (\ln x)^{-1} e^{\pi i x^2 \cot \alpha}, & x \geq e, \\ x e^{\pi i x^2 \cot \alpha - 1}, & -e < x < e, \\ -(\ln(-x))^{-1} e^{\pi i x^2 \cot \alpha}, & x \leq -e. \end{cases}$$

14 Then  $g \in C_0(\mathbb{R})$  and  $g$  is not the FRFT of any  $L^1$ -function. To show this, we need first  
15 to prove the following.16 **Claim.** If  $f \in L^1(\mathbb{R})$  and  $\mathcal{F}_\alpha f$  is odd, then

17 
$$\left| \int_\varepsilon^N \frac{(\mathcal{F}_\alpha f)(x)}{x} e^{-\pi i x^2 \cot \alpha} dx \right| \leq 4 |A_\alpha| \|f\|_1$$

18 for all  $N > \varepsilon > 0$ .19 Indeed, since  $\mathcal{F}_\alpha f$  is odd, we have

20 
$$\begin{aligned} (\mathcal{F}_\alpha f)(x) &= \frac{1}{2} ((\mathcal{F}_\alpha f)(x) - (\mathcal{F}_\alpha f)(-x)) \\ &= \frac{1}{2} \int_{-\infty}^{+\infty} f(t) (K_\alpha(x, t) - K_\alpha(-x, t)) dt \\ &= \frac{A_\alpha}{2} \int_{-\infty}^{+\infty} f(t) e^{\pi i (t^2 + x^2) \cot \alpha} (e^{-2\pi i x t \csc \alpha} - e^{2\pi i x t \csc \alpha}) dt \end{aligned}$$
21  
22



$$= -iA_\alpha e^{\pi i x^2 \cot \alpha} \int_{-\infty}^{+\infty} f(t) e^{\pi i t^2 \cot \alpha} \sin(2\pi x t \csc \alpha) dt.$$

Then

$$\begin{aligned} \int_{\varepsilon}^N \frac{(\mathcal{F}_\alpha f)(x)}{x} e^{-\pi i x^2 \cot \alpha} dx &= -iA_\alpha \int_{\varepsilon}^N \frac{1}{x} \left( \int_{-\infty}^{+\infty} f(t) e^{\pi i t^2 \cot \alpha} \sin(2\pi x t \csc \alpha) dt \right) dx \\ &= -iA_\alpha \int_{-\infty}^{+\infty} f(t) e^{\pi i t^2 \cot \alpha} \left( \int_{2\pi \varepsilon t \csc \alpha}^{2\pi N t \csc \alpha} \frac{\sin x}{x} dx \right) dt. \end{aligned}$$

Note that

$$\left| \int_{2\pi \varepsilon t \csc \alpha}^{2\pi N t \csc \alpha} \frac{\sin x}{x} dx \right| \leq 4, \quad \forall 0 < \varepsilon < N < +\infty.$$

Consequently,

$$\begin{aligned} \left| \int_{\varepsilon}^N \frac{(\mathcal{F}_\alpha f)(x)}{x} e^{-\pi i x^2 \cot \alpha} dx \right| &\leq |A_\alpha| \int_{-\infty}^{+\infty} |f(t)| \left| \int_{2\pi \varepsilon t \csc \alpha}^{2\pi N t \csc \alpha} \frac{\sin x}{x} dx \right| dt \\ &\leq 4 |A_\alpha| \|f\|_1. \end{aligned}$$

So the claim holds. Since  $g \in C_0(\mathbb{R})$  is an odd function and

$$\lim_{\substack{t \rightarrow \infty \\ \varepsilon \rightarrow 0^+}} \left| \int_{\varepsilon}^t \frac{g(x)}{x} e^{-\pi i x^2 \cot \alpha} dx \right| = \infty,$$

the above claim implies that  $g$  is not the FRFT of any  $L^1$ -function.

We conclude this section with a useful identity. We should point out that this formula has already been proved in the  $L^2$  setting (see [9, Theorem 2.1 (ii)]).

**Theorem 2.4** (Multiplication formula). *For every  $f, g \in L^1(\mathbb{R})$  and  $\alpha \in \mathbb{R}$  we have*

$$\int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) g(x) dx = \int_{-\infty}^{+\infty} f(x) (\mathcal{F}_\alpha g)(x) dx. \quad (2.5)$$

*Proof.* The identity (2.5) is an immediate consequence of Fubini's theorem. Indeed,

$$\begin{aligned} \int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) g(x) dx &= \int_{-\infty}^{+\infty} g(x) \left( \int_{-\infty}^{+\infty} f(t) K_\alpha(x, t) dt \right) dx \\ &= \int_{-\infty}^{+\infty} f(t) \left( \int_{-\infty}^{+\infty} g(x) K_\alpha(x, t) dx \right) dt \\ &= \int_{-\infty}^{+\infty} f(x) (\mathcal{F}_\alpha g)(x) dx, \end{aligned}$$

noting that  $K_\alpha$  is a bounded function and  $K_\alpha(x, t) = K_\alpha(t, x)$  for all  $x$  and  $t$ .  $\square$

### 3. Fractional approximate identities and FRFT inversion on $L^1(\mathbb{R})$

In this section, we study FRFT inversion. Namely, given the FRFT of an  $L^1$ -function, how to recover the original function? We naturally hope that the integral

$$\int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) K_{-\alpha}(x, t) dx \quad (3.1)$$

equals  $f(t)$ . Unfortunately, when  $f$  is integrable, one may not necessarily have that  $\mathcal{F}_\alpha f$  is integrable, so the integral (3.1) may not make sense. In fact,  $\mathcal{F}_{\frac{\pi}{2}} f$  is nonintegrable in general (cf., [4, pp. 12]).

**Example 3.1.** Let

$$f(t) = \begin{cases} e^{-\pi(2t+it^2 \cot \alpha)}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Then  $f \in L^1(\mathbb{R})$  but

$$(\mathcal{F}_\alpha f)(x) = \frac{A_\alpha e^{\pi i x^2 \cot \alpha}}{2\pi(1 + ix \csc \alpha)} \notin L^1(\mathbb{R}).$$

To overcome this difficulty, we employ integral summability methods. We introduce the fractional convolution and we establish the approximate identities in the fractional setting. Then we study the  $\Phi_\alpha$  means of the fractional Fourier integral, especially Abel means and Gauss means. Based on the regularity of the fractional convolution and the results of pointwise convergence, we can approximate  $f$  by the  $\Phi_\alpha$  means of the integral (3.1).

#### 3.1. Fractional convolution and approximate identities

In order to establish the approximate identities and fractional Fourier integral means required in this work, we need to introduce a kind of fractional convolution. Similar definitions were introduced by D. Mustard and A. Zayed; see [15, 27].

**Definition 3.1.** Let  $f, g$  be in  $L^1(\mathbb{R})$ . Define the fractional convolution of order  $\alpha$  by

$$(f \overset{\alpha}{*} g)(x) = e^{-\pi i x^2 \cot \alpha} \int_{-\infty}^{+\infty} e^{\pi i t^2 \cot \alpha} f(t) g(x-t) dt = \mathcal{M}_{-\alpha}(\mathcal{M}_\alpha f * g)(x).$$

We reserve the following notation for the  $L^1$  dilation of a function  $\phi$

$$\phi_\varepsilon(x) := \frac{1}{\varepsilon} \phi\left(\frac{x}{\varepsilon}\right), \quad \forall \varepsilon > 0.$$

The following is a fundamental result concerning fractional convolution and approximate identities.

1 **Theorem 3.2.** Let  $\phi \in L^1(\mathbb{R})$  and  $\int_{-\infty}^{+\infty} \phi(x) dx = 1$ . If  $f \in L^p(\mathbb{R})$ ,  $1 \leq p < \infty$ , then

$$2 \quad \lim_{\varepsilon \rightarrow 0} \left\| (f \overset{\alpha}{*} \phi_\varepsilon) - f \right\|_p = 0.$$

3 *Proof.* Note that

$$4 \quad (f \overset{\alpha}{*} \phi_\varepsilon)(x) - f(x) = e^{-\pi i x^2 \cot \alpha} \int_{-\infty}^{+\infty} e^{\pi i t^2 \cot \alpha} f(t) \phi_\varepsilon(x-t) dt - \int_{-\infty}^{+\infty} \phi_\varepsilon(t) f(x) dt$$

$$5 \quad = \int_{-\infty}^{+\infty} (e^{\pi i ((x-t)^2 - x^2) \cot \alpha} f(x-t) - f(x)) \phi_\varepsilon(t) dt.$$

7 By Minkowski's integral inequality, we obtain

$$8 \quad \left\| (f \overset{\alpha}{*} \phi_\varepsilon) - f \right\|_p = \left( \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} (e^{\pi i ((x-t)^2 - x^2) \cot \alpha} f(x-t) - f(x)) \phi_\varepsilon(t) dt \right|^p dx \right)^{\frac{1}{p}}$$

$$9 \quad \leq \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} |e^{\pi i ((x-t)^2 - x^2) \cot \alpha} f(x-t) - f(x)|^p dx \right)^{\frac{1}{p}} |\phi_\varepsilon(t)| dt$$

$$10 \quad = \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} |e^{\pi i ((x-\varepsilon t)^2 - x^2) \cot \alpha} f(x-\varepsilon t) - f(x)|^p dx \right)^{\frac{1}{p}} |\phi(t)| dt.$$

12 We first prove that

$$13 \quad J_\varepsilon := \left( \int_{-\infty}^{+\infty} |e^{\pi i ((x-\varepsilon t)^2 - x^2) \cot \alpha} f(x-\varepsilon t) - f(x)|^p dx \right)^{\frac{1}{p}} \rightarrow 0 \quad (3.2)$$

14 as  $\varepsilon \rightarrow 0$ .

15 In fact, for an arbitrary  $\eta > 0$ , since the space of continuous functions with compact support  $C_c(\mathbb{R})$  is dense in  $L^p(\mathbb{R})$ , there exists  $g \in C_c(\mathbb{R})$  such that

$$17 \quad \|f - g\|_p < \frac{\eta}{2}.$$

18 Since  $g$  is uniformly continuous,

$$19 \quad \lim_{\varepsilon \rightarrow 0} |g(x - \varepsilon t) - g(x)| = 0.$$

20 Note that

$$21 \quad |J_\varepsilon| \leq \left\| e^{\pi i ((\cdot) - \varepsilon t)^2 - (\cdot)^2} \cot \alpha f((\cdot) - \varepsilon t) - e^{\pi i ((\cdot) - \varepsilon t)^2 - (\cdot)^2} \cot \alpha g((\cdot) - \varepsilon t) \right\|_p$$

$$22 \quad + \left\| e^{\pi i ((\cdot) - \varepsilon t)^2 - (\cdot)^2} \cot \alpha g((\cdot) - \varepsilon t) - g((\cdot) - \varepsilon t) \right\|_p$$

$$\begin{aligned}
& + \|g(\cdot - \varepsilon t) - g\|_p + \|f - g\|_p \\
& \leq 2\|f - g\|_p + \|g\|_\infty \left\| e^{\pi i((\cdot - \varepsilon t)^2 - (\cdot)^2) \cot \alpha} - 1 \right\|_{L^p(\text{supp } g)} + \|g(\cdot - \varepsilon t) - g\|_p.
\end{aligned}$$

Consequently, it follows from Lebesgue's dominated convergence theorem that

$$\begin{aligned}
\overline{\lim}_{\varepsilon \rightarrow 0} |J_\varepsilon| & \leq \eta + \overline{\lim}_{\varepsilon \rightarrow 0} \|g(\cdot - \varepsilon t) - g\|_p \\
& + \|g\|_\infty \overline{\lim}_{\varepsilon \rightarrow 0} \left\| e^{\pi i((\cdot - \varepsilon t)^2 - (\cdot)^2) \cot \alpha} - 1 \right\|_{L^p(\text{supp } g)} = \eta.
\end{aligned}$$

Therefore (3.2) holds. In view of

$$\left( \int_{-\infty}^{+\infty} \left| e^{\pi i((x - \varepsilon t)^2 - x^2) \cot \alpha} f(x - \varepsilon t) - f(x) \right|^p dx \right)^{\frac{1}{p}} \leq 2\|f\|_p < \infty,$$

and using Lebesgue's dominated convergence theorem again, we deduce that

$$\lim_{\varepsilon \rightarrow 0} \left\| (f * \phi_\varepsilon) - f \right\|_p = 0.$$

□

Next, we discuss the pointwise convergence of approximate identities with respect to fractional convolution.

**Theorem 3.3.** *Let  $\phi \in L^1(\mathbb{R})$  and  $\int_{-\infty}^{+\infty} \phi(x) dx = 1$ . Denote the decreasing radial dominant functions of  $\phi$  by  $\psi(x) = \sup_{|t| \geq |x|} |\phi(t)|$ . If  $\psi \in L^1(\mathbb{R})$  and  $f \in L^p(\mathbb{R})$ ,  $1 \leq p < \infty$ ,*

*then*

$$\lim_{\varepsilon \rightarrow 0} (f * \phi_\varepsilon)(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}.$$

*Proof.* Since  $\psi$  is decreasing and nonnegative, we have

$$|x\psi(x)| \leq 2 \left| \int_{x/2}^x \psi(s) ds \right| \rightarrow 0$$

as  $x \rightarrow 0$  or  $x \rightarrow \infty$ . Moreover, there is a constant  $A > 0$  such that

$$|x\psi(x)| \leq A, \quad \forall x \in \mathbb{R}.$$

As  $\mathcal{M}_\alpha f \in L^p(\mathbb{R})$ , it follows from Lebesgue's differentiation theorem that, for almost all  $x \in \mathbb{R}$  we have

$$\lim_{r \rightarrow 0} \frac{1}{r} \int_{-r}^r \left| e^{\pi i(x-t)^2 \cot \alpha} f(x-t) - e^{\pi i x^2 \cot \alpha} f(x) \right| dt = 0.$$

1 Let

$$2 \quad \Omega = \left\{ x : \lim_{r \rightarrow 0} \frac{1}{r} \int_{-r}^r \left| e^{\pi i((x-t)^2 - x^2) \cot \alpha} f(x-t) - f(x) \right| dt = 0 \right\},$$

3 and

$$4 \quad G_x(t) := \int_0^t \left| e^{\pi i((x-\tau)^2 - x^2) \cot \alpha} f(x-\tau) - f(x) \right| d\tau.$$

5 Given  $x \in \Omega$  and  $\delta > 0$ , there exists  $\eta > 0$  such that

$$6 \quad \left| \frac{1}{t} G_x(t) \right| < \delta$$

7 whenever  $0 < |t| \leq \eta$ . Consider

$$\begin{aligned} 8 \quad (f * \phi_\varepsilon)(x) - f(x) &= \int_{-\infty}^{+\infty} \left( e^{\pi i((x-t)^2 - x^2) \cot \alpha} f(x-t) - f(x) \right) \phi_\varepsilon(t) dt \\ 9 &= \left( \int_{|t| \leq \eta} + \int_{|t| \geq \eta} \right) \left( e^{\pi i((x-t)^2 - x^2) \cot \alpha} f(x-t) - f(x) \right) \phi_\varepsilon(t) dt \\ 10 &=: I_1 + I_2. \end{aligned}$$

12 For  $I_1$  an integration by parts yields

$$\begin{aligned} 13 \quad I_1 &\leq \int_{-\eta}^{\eta} \left| e^{\pi i((x-t)^2 - x^2) \cot \alpha} f(x-t) - f(x) \right| \left| \frac{1}{\varepsilon} \phi\left(\frac{t}{\varepsilon}\right) \right| dt \\ 14 &\leq \int_{-\eta}^{\eta} \left| e^{\pi i((x-t)^2 - x^2) \cot \alpha} f(x-t) - f(x) \right| \frac{1}{\varepsilon} \psi\left(\frac{t}{\varepsilon}\right) dt \\ 15 &= \frac{1}{t} G(t) \frac{t}{\varepsilon} \psi\left(\frac{t}{\varepsilon}\right) \Big|_{-\eta}^{\eta} - \int_{-\eta/\varepsilon}^{\eta/\varepsilon} \frac{1}{\varepsilon} G(\varepsilon s) d\psi(s) \\ 16 &\leq A\delta - \int_{-\eta/\varepsilon}^{\eta/\varepsilon} \frac{1}{\varepsilon s} G(\varepsilon s) s d\psi(s) \\ 17 &\leq A\delta + 2\delta \int_0^{+\infty} s d\psi(s) \\ 18 &\leq A\delta + 2\delta s\psi(s) \Big|_0^{+\infty} + 2\delta \int_0^{+\infty} \psi(s) ds \\ 19 &= \delta \left( A + 2 \int_0^{+\infty} \psi(s) ds \right) =: \delta A_1. \end{aligned}$$

21 Here, we used that fact that  $\psi(x) \geq |\phi(x)|$  and  $x\psi(x) \rightarrow 0$  as  $x \rightarrow 0$  or  $x \rightarrow \infty$

22 On the other hand, it follows from Hölder's inequality that

$$23 \quad I_2 \leq \int_{|t| \geq \eta} \left| e^{\pi i((x-t)^2 - x^2) \cot \alpha} f(x-t) - f(x) \right| |\psi_\varepsilon(t)| dt$$

$$\begin{aligned}
&\leq \int_{|t|\geq\eta} |f(x-t)\psi_\varepsilon(t)| dt + |f(x)| \int_{|t|\geq\eta} \psi_\varepsilon(t) dt \\
&\leq \|f\|_p \|\chi_\eta \psi_\varepsilon\|_{p'} + |f(x)| \int_{|t|\geq\frac{\eta}{\varepsilon}} \psi(t) dt \rightarrow 0.
\end{aligned}$$

as  $\varepsilon \rightarrow 0$ , where  $\chi_\eta$  is the characteristic function of the set  $\{x : |x| \geq \eta\}$ . As  $\delta$  is arbitrary, the theorem is proved.  $\square$

**Remark 3.1.** Using the maximal function approach as in Duoandikoetxea in [4, Proposition 2.7 and Theorem 2.2], we can obtain another proof of Theorem 3.3 that extends to functions in weighted spaces with Muckenhoupt weights.

Let  $\phi$  be a function which is positive, radial, decreasing and integrable. Then

$$\sup_{\varepsilon>0} |(\phi_\varepsilon * f)(x)| \leq \|\phi\|_1 Mf(x), \quad (3.3)$$

where  $Mf(x)$  is the Hardy-Littlewood maximal function of  $f$ , defined by

$$Mf(x) = \sup_{r>0} \frac{1}{2r} \int_{-r}^r |f(x-y)| dy.$$

- (i) If  $|\phi(x)| \leq \psi(x)$  almost everywhere, where  $\psi$  is positive, radial, decreasing and integrable, it follows from (3.3) and the weighted boundedness of  $M$  that the maximal function  $\sup_{\varepsilon>0} |(\phi_\varepsilon * f)(x)|$  is weighted weak  $(1, 1)$  and weighted strong  $(p, p)$  for  $1 < p \leq \infty$ . We refer to [4, Chap. 7] for this direction.
- (ii) Combining the result in (i) and Theorem 2.2 in [4], we can get the pointwise convergence result Theorem 3.3.

### 3.2. Fractional Fourier integral means

**Definition 3.4.** Given  $\Phi \in C_0(\mathbb{R})$  and  $\Phi(0) = 1$ , a function  $f$ , and  $\varepsilon > 0$  we define

$$M_{\varepsilon, \Phi_\alpha}(f)(t) := \int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) K_{-\alpha}(x, t) \Phi_\alpha(\varepsilon x) dx,$$

where

$$\Phi_\alpha(x) := \Phi(x \csc \alpha).$$

The expressions  $M_{\varepsilon, \Phi_\alpha}(f)$  (with varying  $\varepsilon$ ) are called the  $\Phi_\alpha$  means of the fractional Fourier integral of  $f$ .

**Theorem 3.5.** Let  $f, \Phi \in L^1(\mathbb{R})$ . Then for any  $\varepsilon > 0$  and  $t \in \mathbb{R}$  we have

$$M_{\varepsilon, \Phi_\alpha}(f)(t) = (f * \tilde{\varphi}_\varepsilon)(t),$$

where  $\varphi := \mathcal{F}\Phi$  and  $\tilde{\varphi}(x) = \varphi(-x)$

1 *Proof.* Taking advantage of identity (2.3) and the multiplication formular of the clas-  
 2 sical Fourier transform, we write

$$\begin{aligned}
 3 \quad M_{\varepsilon, \Phi_\alpha}(f)(t) &= \int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) K_{-\alpha}(x, t) \Phi_\alpha(\varepsilon x) dx \\
 4 \quad &= A_{-\alpha} e^{-i\pi t^2 \cot \alpha} \int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) e^{-i\pi x^2 \cot \alpha} e^{2\pi i t x \csc \alpha} \Phi_\alpha(\varepsilon x) dx \\
 5 \quad &= A_{-\alpha} A_\alpha e^{-i\pi t^2 \cot \alpha} \int_{-\infty}^{+\infty} \mathcal{F} \left[ e^{i\pi t^2 \cot \alpha} f(t) \right] (x \csc \alpha) e^{2\pi i t x \csc \alpha} \Phi(\varepsilon x \csc \alpha) dx \\
 6 \quad &= e^{-i\pi t^2 \cot \alpha} \int_{-\infty}^{+\infty} \mathcal{F} \left[ e^{i\pi t^2 \cot \alpha} f(t) \right] (x) e^{2\pi i t x} \Phi(\varepsilon x) dx \\
 7 \quad &= e^{-i\pi t^2 \cot \alpha} \int_{-\infty}^{+\infty} e^{i\pi x^2 \cot \alpha} f(x) \mathcal{F} \left[ e^{2\pi i t(\cdot)} \Phi(\varepsilon(\cdot)) \right] (x) dx \\
 8 \quad &= e^{-i\pi t^2 \cot \alpha} \int_{-\infty}^{+\infty} e^{i\pi x^2 \cot \alpha} f(x) \varphi_\varepsilon(x - t) dx \\
 9 \quad &= \left( f *^\alpha \tilde{\varphi}_\varepsilon \right)(t). \\
 10
 \end{aligned}$$

11 The desired result is proved. □

12 In the sequel we will make use of the following well-known results.

13 **Proposition 3.6** ([24]). *Let  $\varepsilon > 0$ . Then*

- 14 (a)  $\mathcal{F} \left[ e^{-2\pi\varepsilon|\cdot|} \right] (x) = \frac{1}{\pi} \frac{\varepsilon}{\varepsilon^2 + x^2} =: P_\varepsilon(x)$  (Poisson kernel);  
 15 (b)  $\mathcal{F} \left[ e^{-4\pi^2\varepsilon|\cdot|^2} \right] (x) = \frac{1}{(4\pi\varepsilon)^{1/2}} e^{-x^2/4\varepsilon} =: W_\varepsilon(x)$  (Gauss–Weierstrass kernel).

16 **Lemma 3.7** ([4]). *For every  $\varepsilon > 0$ , the Weierstrass and Poisson kernels satisfy*

- 17 (i)  $W_\varepsilon, P_\varepsilon \in L^1(\mathbb{R})$ ;  
 18 (ii)  $\int_{-\infty}^{+\infty} W_\varepsilon(x) dx = \int_{-\infty}^{+\infty} P_\varepsilon(x) dx = 1$ .

19 **Definition 3.8.** *For  $f \in L^p(\mathbb{R})$ ,  $1 \leq p < \infty$ , and  $\varepsilon > 0$ , the expressions*

$$20 \quad u_\alpha(t, \varepsilon) := \left( f *^\alpha \tilde{P}_\varepsilon \right)(t) = \mathcal{M}_\alpha \left[ \int_{-\infty}^{+\infty} \mathcal{M}_\alpha f(x) P_\varepsilon((\cdot) - x) dx \right] (t)$$

21 *are called the fractional Poisson integrals of  $f$ . The expressions*

$$22 \quad S_\alpha(t, \varepsilon) := \left( f *^\alpha \tilde{W}_\varepsilon \right)(t) = \mathcal{M}_\alpha \left[ \int_{-\infty}^{+\infty} \mathcal{M}_\alpha f(x) W_\varepsilon((\cdot) - x) dx \right] (t)$$

23 *are called and fractional Gauss-Weierstrass integrals of  $f$ .*

1 We now focus on two functions that give rise to special  $\Phi_\alpha$  means. Denote by

$$2 \quad p_\alpha(x) = e^{-2\pi\varepsilon|\csc\alpha||x|} \quad \text{and} \quad w_\alpha(x) = e^{-4\pi^2\varepsilon x^2 \csc^2\alpha}.$$

3 **Definition 3.9.** *The  $\Phi_\alpha$  means*

$$4 \quad M_{\varepsilon, p_\alpha}(f) = \int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) K_{-\alpha}(x, \cdot) e^{-2\pi\varepsilon|\csc\alpha||x|} dx$$

5 *are called the Abel means of the fractional Fourier integral of  $f$ , while*

$$6 \quad M_{\varepsilon, w_\alpha}(f) = \int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) K_{-\alpha}(x, \cdot) e^{-4\pi^2\varepsilon x^2 \csc^2\alpha} dx$$

7 *are called the Gauss means of the fractional Fourier integral of  $f$ .*

8 By Theorem 3.5 and Proposition 3.6, the Poisson integrals and Gauss-Weierstrass  
9 integrals of  $f$  are the Abel and Gauss means, respectively. It is straightforward to verify  
10 the following identities.

11 **Proposition 3.10.** *If  $f \in L^1(\mathbb{R})$ , then for any  $\varepsilon > 0$ , the following identities are valid*

- 12 (a)  $u_\alpha(t, \varepsilon) = M_{\varepsilon, p_\alpha}(f)(t);$   
13 (b)  $S_\alpha(t, \varepsilon^2) = M_{\varepsilon, w_\alpha}(f)(t).$

### 14 3.3. FRFT inversion

15 We now address the FRFT inversion problem. In view of Theorems 3.2, 3.3 and  
16 3.5, we can derive the following conclusions.

17 **Theorem 3.11.** *If  $\Phi, \varphi := \mathcal{F}\Phi \in L^1(\mathbb{R})$  and  $\int_{-\infty}^{+\infty} \varphi(x) dx = 1$ , then the  $\Phi_\alpha$  means of the  
18 Fourier integral of  $f$  are convergent to  $f$  in the sense of  $L^1$  norm, that is,*

$$19 \quad \lim_{\varepsilon \rightarrow 0} \left\| \int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) K_{-\alpha}(\cdot, x) \Phi_\alpha(\varepsilon x) dx - f(\cdot) \right\|_1 = 0.$$

20 **Theorem 3.12.** *If  $\Phi, \varphi := \mathcal{F}\Phi \in L^1(\mathbb{R})$ ,  $\psi = \sup_{|t| \geq |x|} |\varphi(t)| \in L^1(\mathbb{R})$  and  $\int_{-\infty}^{+\infty} \varphi(x) dx = 1$ ,  
21 then the  $\Phi_\alpha$  means of the Fourier integral of  $f$  are a.e. convergent to  $f$ , that is,*

$$22 \quad \int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) K_{-\alpha}(t, x) \Phi_\alpha(\varepsilon x) dx \rightarrow f(t)$$

23 *as  $\varepsilon \rightarrow 0$  for almost all  $t \in \mathbb{R}$ .*



1 In particular, in view of Theorem 3.11-3.12, Proposition 3.10 and the properties of  
2 Weierstrass kernel and Poisson kernel (Lemma 3.7), we deduce the following result.

3 **Corollary 3.13.** *If  $f \in L^1(\mathbb{R})$ , then the Gauss and Abel means of the fractional Fourier  
4 integral of  $f$  converge to  $f$  in  $L^1$  and a.e., that is,*

$$5 \quad \lim_{\varepsilon \rightarrow 0} \|M_{\varepsilon, p_\alpha}(f) - f\|_1 = 0, \quad \lim_{\varepsilon \rightarrow 0} \|M_{\varepsilon, w_\alpha}(f) - f\|_1 = 0,$$

6 and

$$7 \quad M_{\varepsilon, p_\alpha}(f)(t) \rightarrow f(t), \quad M_{\varepsilon, w_\alpha}(f)(t) \rightarrow f(t)$$

8 for almost all  $t \in \mathbb{R}$  as  $\varepsilon \rightarrow 0$ .

9 **Remark 3.2.** We now understand Proposition 3.10 and Corollary 3.13 from the per-  
10 spective of partial differential equations as in [4, pp. 19]. The Abel–Poisson and  
11 Gauss–Weierstrass summability methods arise from solving the Laplace and heat equa-  
12 tions, respectively.

13 (i) Consider the Dirichlet boundary value problem on the upper half plane  $\mathbb{R}_+^2$ :

$$14 \quad \begin{cases} \Delta(e^{i\pi x^2 \cot \alpha} u(x, t)) = \mathcal{L}_1 u(x, t) = 0, & (x, t) \in \mathbb{R}_+^2, \\ u(x, 0) = f(x), & x \in \mathbb{R}. \end{cases} \quad (3.4)$$

15 where  $\mathcal{L}_1 = -\Delta + b(x)\frac{\partial}{\partial x} + c(x)$ ,  $b(x) = -2\pi x \cot \alpha$  and  $c(x) = 4\pi^2 x^2 \cot^2 \alpha -$   
16  $2\pi i \cot \alpha$ . If  $f \in L^p(\mathbb{R})$  ( $1 \leq p < \infty$ ), the fractional Poisson integral of  $f$ ,  $u_\alpha(x, t)$ ,  
17 is a solution of the Dirichlet problem (3.4). When the boundary value  $f$  satisfies  
18 different conditions, Proposition 3.10 and Corollary 3.13 show the relationship  
19 between the limit of solution and the boundary value  $f$ , in the sense of  $L^p$  norm  
20 and the sense of a.e..

21 (ii) Similarly, consider the initial value problem on the upper half plane  $\mathbb{R}_+^2$ :

$$22 \quad \begin{cases} \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)(e^{i\pi x^2 \cot \alpha} u(x, t)) = \mathcal{L}_2 u(x, t) = 0, & (x, t) \in \mathbb{R}_+^2, \\ u(x, 0) = f(x), & x \in \mathbb{R}. \end{cases} \quad (3.5)$$

23 where  $\mathcal{L}_2 = -\frac{\partial^2}{x^2} + \frac{\partial}{\partial t} + b(x)\frac{\partial}{\partial x} + c(x)$ ,  $b(x) = -2\pi x \cot \alpha$  and  $c(x) = 4\pi^2 x^2 \cot^2 \alpha -$   
24  $2\pi i \cot \alpha$ . If  $f \in L^p(\mathbb{R})$  ( $1 \leq p < \infty$ ), the fractional Gauss–Weierstrass inte-  
25 gral of  $f$ ,  $S_\alpha(x, t)$ , is a solution of the Cauchy problem (3.5). Proposition 3.10  
26 and Corollary 3.13 show the relationship between the limit of solution and the  
27 boundary value  $f$  in different senses.

28 **Corollary 3.14.** *If  $f, \mathcal{F}_\alpha f \in L^1$ , then for almost all  $x \in \mathbb{R}$ , we have*

$$29 \quad f(t) = \int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) K_{-\alpha}(x, t) dx.$$

1 *Proof.* Consider the Gauss mean of the fractional Fourier integral  $\mathcal{F}_\alpha f$ . On one hand,  
 2 it follows from Corollary 3.13 that

$$3 \quad M_{\varepsilon, w_\alpha}(f)(t) = \int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) K_{-\alpha}(x, t) e^{-4\pi^2 \varepsilon x^2 \csc^2 \alpha} dx \rightarrow f(t)$$

4 for almost all  $t \in \mathbb{R}$ , as  $\varepsilon \rightarrow 0$ . On the other hand, as  $\mathcal{F}_\alpha f \in L^1(\mathbb{R})$ , by the Lebesgue  
 5 dominated convergence theorem we obtain that

$$6 \quad \int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) K_{-\alpha}(x, t) e^{-4\pi^2 \varepsilon x^2 \csc^2 \alpha} dx \rightarrow \int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) K_{-\alpha}(x, t) dx$$

7 as  $\varepsilon \rightarrow 0$ . This proves the desired result.  $\square$

8 **Corollary 3.15.** *Let  $f \in L^1(\mathbb{R})$ . If  $\mathcal{F}_\alpha f \geq 0$  and  $f$  is continuous at  $t = 0$ , then*  
 9  *$\mathcal{F}_\alpha f \in L^1(\mathbb{R})$ . Furthermore,*

$$10 \quad f(t) = \int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) K_{-\alpha}(x, t) dx, \quad \text{for almost all } t \in \mathbb{R}.$$

11 *In particular,  $\int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) dx = f(0)$ .*

12 **Remark 3.3.** (i) Even if  $\mathcal{F}_\alpha f \notin L^1(\mathbb{R})$ , the Gauss and Abel means of the integral

$$13 \quad \int_{-\infty}^{+\infty} (\mathcal{F}_\alpha f)(x) K_{-\alpha}(x, t) dx$$

14 may make sense. For example, if  $\mathcal{F}_\alpha f \notin L^1(\mathbb{R})$  and  $\mathcal{F}_\alpha f$  is bounded, then

$$15 \quad M_{\varepsilon, p_\alpha}(f)(t), M_{\varepsilon, w_\alpha}(f)(t) < \infty \quad \forall \varepsilon > 0.$$

16 (ii) Even if  $\mathcal{F}_\alpha f \notin L^1(\mathbb{R})$ , the limits  $\lim_{\varepsilon \rightarrow 0} u_\alpha(t, \varepsilon)$  and  $\lim_{\varepsilon \rightarrow 0} S_\alpha(t, \varepsilon^2)$  may exist. For  
 17 example, this is the case when  $(\mathcal{F}_\alpha f)(x) = \sin x/x$ .

18 **Theorem 3.16** (Uniqueness of FRFT on  $L^1(\mathbb{R})$ ). *If  $f_1, f_2 \in L^1(\mathbb{R})$  and  $(\mathcal{F}_\alpha f_1)(x) =$   
 19  $(\mathcal{F}_\alpha f_2)(x)$  for all  $x \in \mathbb{R}$ , then*

$$20 \quad f_1(t) = f_2(t), \quad \text{a.e. } t \in \mathbb{R}. \quad (3.6)$$

21 *Proof.* Let  $g = f_1 - f_2$ . Then

$$22 \quad \mathcal{F}_\alpha g = \mathcal{F}_\alpha f_1 - \mathcal{F}_\alpha f_2.$$

23 It follows from Corollary 3.14 that

$$24 \quad g(x) = \int_{-\infty}^{+\infty} (\mathcal{F}_\alpha g)(t) K_{-\alpha}(x, t) dt = 0$$

1 a.e. on  $\mathbb{R}$ , which implies (3.6). □

## 2 4. FRFT on $L^p(\mathbb{R})$ ( $1 < p < 2$ )

3 Having set down the basic facts concerning the action of the FRFT on  $L^1(\mathbb{R})$  and  
4  $L^2(\mathbb{R})$ , we now extend its definition on  $L^p(\mathbb{R})$  for  $1 < p < 2$ . Note that  $L^p(\mathbb{R})$  is  
5 contained in  $L^1(\mathbb{R}) + L^2(\mathbb{R})$  for  $1 < p < 2$ , where

$$6 \quad L^1(\mathbb{R}) + L^2(\mathbb{R}) = \{f_1 + f_2 : f_1 \in L^1(\mathbb{R}), f_2 \in L^2(\mathbb{R})\}.$$

7 **Definition 4.1.** For  $f \in L^p(\mathbb{R})$ ,  $1 < p < 2$ , with

$$8 \quad f = f_1 + f_2, \quad f_1 \in L^1(\mathbb{R}), f_2 \in L^2(\mathbb{R}),$$

9 the FRFT of order  $\alpha$  of  $f$  defined by  $\mathcal{F}_\alpha f = \mathcal{F}_\alpha f_1 + \mathcal{F}_\alpha f_2$ .

10 **Remark 4.1.** The decomposition of  $f$  as  $f_1 + f_2$  is not unique. However, the definition  
11 of  $\mathcal{F}_\alpha f$  is independent on the choice of  $f_1$  and  $f_2$ . If  $f_1 + f_2 = g_1 + g_2$  for  $f_1, g_1 \in L^1(\mathbb{R})$   
12 and  $f_2, g_2 \in L^2(\mathbb{R})$ , we have  $f_1 - g_1 = g_2 - f_2 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ . Since those functions  
13 are equal, their FRFT are also equal, and we obtain  $\mathcal{F}_\alpha f_1 - \mathcal{F}_\alpha g_1 = \mathcal{F}_\alpha g_2 - \mathcal{F}_\alpha f_2$ , using  
14 the linearity of the FRFT, which yields  $\mathcal{F}_\alpha(f_1 + f_2) = \mathcal{F}_\alpha(g_1 + g_2)$ .

15 We have the following result concerning the action of the FRFT on  $L^p(\mathbb{R})$ .

16 **Theorem 4.2** (Hausdorff-Young inequality). Let  $1 < p \leq 2$ ,  $p' = p/(p - 1)$ . Then  $\mathcal{F}_\alpha$   
17 are bounded linear operators from  $L^p(\mathbb{R})$  to  $L^{p'}(\mathbb{R})$ . Moreover,

$$18 \quad \|\mathcal{F}_\alpha f\|_{p'} \leq A_\alpha^{p-1} \|f\|_p. \quad (4.1)$$

19 *Proof.* By Proposition 2.3 (i)  $\mathcal{F}_\alpha$  maps  $L^1$  to  $L^\infty$  (with norm bounded by  $A_\alpha$ ) and The-  
20 orem 1.2 (ii), it maps  $L^2$  to  $L^2$  with (with norm 1). It follows from the Riesz-Thorin  
21 interpolation theorem Hausdorff-Young inequality (4.1) holds. □

22 FRFT inversion also holds on  $L^p(\mathbb{R})$  ( $1 < p < 2$ ) and this can be proved by an  
23 argument similar to that for  $L^1(\mathbb{R})$  via the use of Theorems 3.2-3.3. We won't go into  
24 much detail here.

## 25 5. Multiplier theory and Littlewood-Paley theorem associated with the FRFT

### 26 5.1. Fractional Fourier transform multipliers

27 Fourier multipliers play an important role in operator theory, partial differential  
28 equations, and harmonic analysis. In this section, we study some basic multiplier  
29 theory results in the FRFT context.

1 **Definition 5.1.** Let  $1 \leq p \leq \infty$  and  $m_\alpha \in L^\infty(\mathbb{R})$ . Define the operator  $T_{m_\alpha}$  as

2 
$$\mathcal{F}_\alpha(T_{m_\alpha}f)(x) = m_\alpha(x) (\mathcal{F}_\alpha f)(x), \quad \forall f \in L^2(\mathbb{R}) \cap L^p(\mathbb{R}).$$

3 The function  $m_\alpha$  is called the  $L^p$  Fourier multiplier of order  $\alpha$ , if there exist a constant  
4  $C_{p,\alpha} > 0$  such that

5 
$$\|T_{m_\alpha}f\|_p \leq C_{p,\alpha} \|f\|_p, \quad \forall f \in L^2(\mathbb{R}) \cap L^p(\mathbb{R}). \quad (5.1)$$

6 As  $L^2(\mathbb{R}) \cap L^p(\mathbb{R})$  is dense in  $L^p(\mathbb{R})$ , there is a unique bounded extension of  $T_{m_\alpha}$  in  
7  $L^p(\mathbb{R})$  satisfying (5.1). This extension is also denoted by  $T_{m_\alpha}$ .

8 In view of Definition 5.1, many important fractional integral operators can be ex-  
9 pressed in terms of fractional  $L^p$  multiplier.

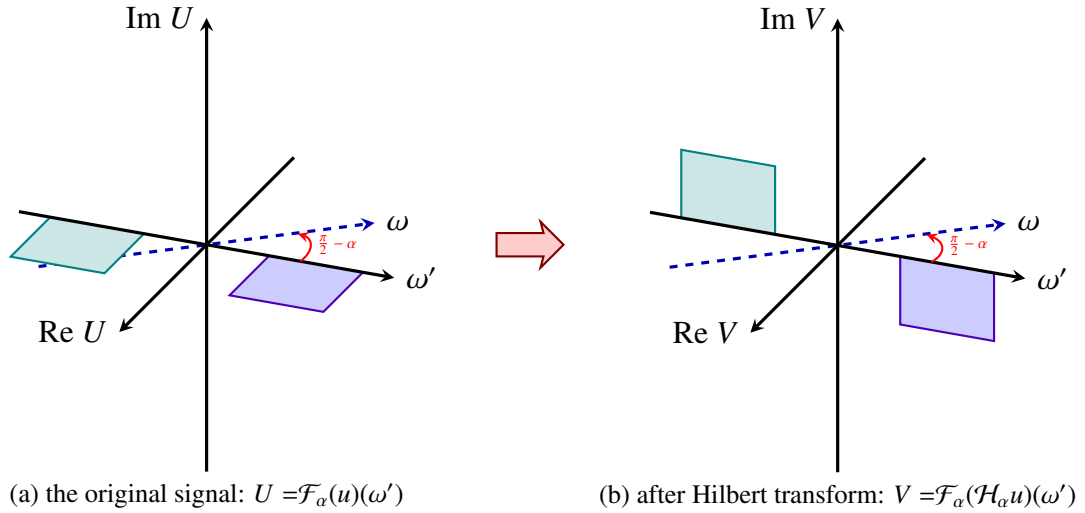


Figure 5.1: Hilbert transform of order  $\alpha$  in frequency domain

10 **Example 5.1.** Recall that the classical Hilbert transform is defined as

11 
$$(\mathcal{H}u)(\omega) = \text{p.v.} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{u(t)}{\omega - t} dt. \quad (5.2)$$

12 The Hilbert transform of order  $\alpha$  is defined as (cf., [29])

13 
$$(\mathcal{H}_\alpha u)(\omega') = \text{p.v.} e^{-i\pi\omega'^2 \cot \alpha} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{u(t)e^{i\pi t^2 \cot \alpha}}{\omega' - t} dt. \quad (5.3)$$

14 For  $1 < p < \infty$ , the operator  $\mathcal{H}_\alpha$  is bounded from  $L^p(\mathbb{R})$  to  $L^p(\mathbb{R})$ . By [29, Theorem  
15 4], we see that  $m_\alpha = -i \text{sgn}((\pi - \alpha)\omega')$  is a fractional  $L^p$  multiplier and the associated

1 operator  $T_{m_\alpha}$  is the fractional Hilbert transform, that is,

$$2 \quad (\mathcal{F}_\alpha \mathcal{H}_\alpha u)(\omega') = -i \operatorname{sgn}((\pi - \alpha)\omega') (\mathcal{F}_\alpha u)(\omega'). \quad (5.4)$$

3 Without loss of generality, assume that  $\alpha \in (0, \pi)$ . It can be seen from (5.4) that  
 4 the Hilbert transform of order  $\alpha$  is a phase-shift converter that multiplies the positive  
 5 frequency portion of the original signal by  $-i$ , that is, maintaining the same amplitude,  
 6 shifts the phase by  $-\pi/2$ , while the negative frequency portion is shifted by  $\pi/2$ . As  
 7 shown in Fig. 5.1.

8 **Example 5.2.** Let  $m_\alpha = e^{-2\pi\epsilon|\csc\alpha||x|}$ . Then the corresponding operator  $T_{m_\alpha}$  is the frac-  
 9 tional Poisson integral (see Definition 3.8). In view of the Young's inequality and  
 10 Lemma 3.7, we know that  $m_\alpha$  is a fractional  $L^p$  multiplier for  $1 \leq p < \infty$ . Simi-  
 11 larly, the fractional Gauss-Weierstrass integral is the operator  $T_{m_\alpha}$  associated with the  
 12 fractional  $L^p$  multiplier  $m_\alpha = e^{-4\pi^2\epsilon x^2 \csc^2\alpha}$ .

13 **Example 5.3.** Let  $a, b \in \mathbb{R}$  and  $a < b$ . Denote the characteristic function of the interval  
 14  $[a, b]$  by  $\chi_{[a,b]}$ . Later, in the proof of Littlewood-Paley theorem (Theorem 5.5), equality  
 15 (5.8) will show that  $\chi_{[a,b]}$  is a  $L^p$  ( $1 < p < \infty$ ) multiplier in the FRFT context. The  
 16 associated operator  $T_{\chi_{[a,b]}}$  acting on a signal  $u$  is equivalent to making a truncation in  
 17 the frequency domain of the original signal.

18 The following theorem provides a sufficient condition for judging  $L^p$  multiplier,  
 19 which is the Hörmander-Mikhlin multiplier theorem in the fractional setting.

20 **Theorem 5.2.** Let  $m_\alpha$  be a bounded function. If there exists a constant  $B > 0$  such that  
 21 one of the following condition holds:

22 (a) (Mikhlin's condition)

$$23 \quad \left| \frac{d}{dx} m_\alpha(x) \right| \leq B |x|^{-1}; \quad (5.5)$$

24 (b) (Hörmander's condition)

$$25 \quad \sup_{R>0} \frac{1}{R} \int_{R<|x|<2R} \left| \frac{d}{dx} m_\alpha(x) \right|^2 dx \leq B^2. \quad (5.6)$$

26 Then  $m_\alpha$  is a fractional  $L^p$  multiplier for  $1 < p < \infty$ , that is, there exist a constant  
 27  $C > 0$  such that

$$28 \quad \|T_{m_\alpha} f\|_p = \|\mathcal{F}_{-\alpha} [m_\alpha (\mathcal{F}_\alpha f)]\|_p \leq C \|f\|_p, \quad \forall f \in L^p(\mathbb{R}).$$

29 *Proof.* In view of the decomposition (2.3), we have

$$30 \quad \mathcal{F}_\alpha (T_{m_\alpha} f)(x) = A_\alpha e^{i\pi x^2 \cot \alpha} \mathcal{F} [e^{i\pi(\cdot)^2 \cot \alpha} (T_{m_\alpha} f)](x \csc \alpha)$$

1 and

$$2 \quad m_\alpha(x) (\mathcal{F}_\alpha f)(x) = m_\alpha(x) A_\alpha e^{i\pi x^2 \cot \alpha} \mathcal{F}[e^{i\pi(\cdot)^2 \cot \alpha} f](x \csc \alpha).$$

3 Then

$$4 \quad \mathcal{F}[e^{i\pi(\cdot)^2 \cot \alpha} (T_{m_\alpha} f)](x) = \tilde{m}_\alpha(x) \mathcal{F}[e^{i\pi(\cdot)^2 \cot \alpha} f](x),$$

5 where  $\tilde{m}_\alpha(x) = m_\alpha(x \sin \alpha)$ . Namely,

$$6 \quad T_{m_\alpha} f = e^{-i\pi(\cdot)^2 \cot \alpha} \mathcal{F}^{-1} \left[ \tilde{m}_\alpha \mathcal{F} \left( e^{i\pi(\cdot)^2 \cot \alpha} f \right) \right], \quad \forall f \in L^p(\mathbb{R}).$$

7 It is obvious that  $\tilde{m}_\alpha$  satisfies (5.5) or (5.6) and  $g := e^{i\pi(\cdot)^2 \cot \alpha} f \in L^p(\mathbb{R})$ . Therefore,  
8 it follows from the classical Hörmander-Mihlin multiplier theorem (cf., [5, 7, 13]) that  
9  $\tilde{m}_\alpha$  is an  $L^p$  Fourier multiplier. Consequently,

$$10 \quad \begin{aligned} \|T_{m_\alpha} f\|_p &= \left\| e^{-i\pi(\cdot)^2 \cot \alpha} \mathcal{F}^{-1} \left[ \tilde{m}_\alpha \mathcal{F} \left( e^{i\pi(\cdot)^2 \cot \alpha} f \right) \right] \right\|_p \\ 11 \quad &= \left\| \mathcal{F}^{-1} [\tilde{m}_\alpha (\mathcal{F} g)] \right\|_p \\ 12 \quad &\leq C \|g\|_p = C \|f\|_p \end{aligned}$$

14 for some positive constant  $C$ . □

15 The proof of the following two FRFT multiplier theorems is obtained in a similar  
16 fashion.

17 **Theorem 5.3** (Bernstein multiplier theorem). *Let  $m_\alpha \in C^1(\mathbb{R} \setminus \{0\})$  be bounded. If*  
18  *$\|m_\alpha\|_2, \|m'_\alpha\|_2 < \infty$ , then there exists a constant  $C > 0$  such that*

$$19 \quad \left\| \mathcal{F}_{-\alpha} [m_\alpha (\mathcal{F}_\alpha f)] \right\|_p \leq C \|m_\alpha\|_2^{\frac{1}{2}} \|m'_\alpha\|_2^{\frac{1}{2}} \|f\|_p,$$

20 for  $f \in L^p(\mathbb{R})$  ( $1 \leq p < \infty$ ).

21 **Theorem 5.4** (Marcinkiewicz multiplier theorem). *Let  $m_\alpha \in L^\infty(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ . If*  
22 *there exists a constant  $B > 0$  such that*

$$23 \quad \sup_{I \in \mathcal{I}} \int_I \left| \frac{d}{dx} m_\alpha(x) \right| dx \leq B,$$

24 where  $\mathcal{I} := \{[2^j, 2^{j+1}], [-2^{j+1}, -2^j]\}_{j \in \mathbb{Z}}$  is the set of binary intervals in  $\mathbb{R}$ , then, for  
25  $f \in L^p(\mathbb{R})$  ( $1 < p < \infty$ ), there exist a constant  $C > 0$  such that

$$26 \quad \left\| \mathcal{F}_{-\alpha} [m_\alpha (\mathcal{F}_\alpha f)] \right\|_p \leq C \|f\|_p.$$

## 27 5.2. Littlewood-Paley theorem in the FRFT context

28 In this subsection we study the Littlewood-Paley theorem in the FRFT context.  
29 The Littlewood-Paley is not only a powerful tool in Fourier analysis, but also plays an  
30 important role in other areas, such as partial differential equations.

1 Let  $j \in \mathbb{Z}$ . Define the binary intervals in  $\mathbb{R}$  as

$$2 \quad \begin{cases} I_j^\alpha := [2^j \sin \alpha, 2^{j+1} \sin \alpha], & -I_j^\alpha := [-2^{j+1} \sin \alpha, -2^j \sin \alpha], & \alpha \in (0, \pi), \\ I_j^\alpha := [2^{j+1} \sin \alpha, 2^j \sin \alpha], & -I_j^\alpha := [-2^j \sin \alpha, -2^{j+1} \sin \alpha], & \alpha \in (\pi, 2\pi). \end{cases}$$

3 Then those binary intervals internally disjoint and

$$4 \quad \mathbb{R} \setminus \{0\} = \bigcup_{j \in \mathbb{Z}} (-I_j^\alpha \cup I_j^\alpha).$$

5 Let  $\mathcal{I}_\alpha := \{I_j^\alpha, -I_j^\alpha\}_{j \in \mathbb{Z}}$ . Define the partial summation operator  $S_{\rho_\alpha}$  corresponding to  
6  $\rho_\alpha \in \mathcal{I}_\alpha$  by

$$7 \quad \mathcal{F}_\alpha(S_{\rho_\alpha} f)(x) = \chi_{\rho_\alpha}(x) (\mathcal{F}_\alpha f)(x), \quad \forall f \in L^2(\mathbb{R}) \cap L^p(\mathbb{R}),$$

8 where  $\chi_{\rho_\alpha}$  denote the characteristic function of the interval  $\rho_\alpha$ . It is obvious that

$$9 \quad \sum_{\rho_\alpha \in \mathcal{I}_\alpha} \|S_{\rho_\alpha}(f)\|_2^2 = \|f\|_2^2, \quad \forall f \in L^2(\mathbb{R}). \quad (5.7)$$

10 For general  $L^p(\mathbb{R})$  functions, we have the following result, which is the Littlewood-  
11 Paley theorem in the fractional setting.

12 **Theorem 5.5.** *Let  $f \in L^p(\mathbb{R})$ ,  $1 < p < \infty$ . Then*

$$13 \quad \left( \sum_{\rho_\alpha \in \mathcal{I}_\alpha} |S_{\rho_\alpha}(f)|^2 \right)^{1/2} \in L^p(\mathbb{R})$$

14 *and there exists constants  $C_1, C_2 > 0$  independent of  $f$  such that*

$$15 \quad C_1 \|f\|_p \leq \left\| \left( \sum_{\rho_\alpha \in \mathcal{I}_\alpha} |S_{\rho_\alpha}(f)|^2 \right)^{1/2} \right\|_p \leq C_2 \|f\|_p.$$

16 *Proof.* Without loss of generality, suppose that  $\alpha \in (0, \pi)$  and  $\rho_\alpha = [a_\alpha, b_\alpha]$ , where  
17  $a_\alpha = a \sin \alpha$ ,  $b_\alpha = b \sin \alpha$  and  $a < b$ . Then

$$18 \quad \chi_{\rho_\alpha}(x) = \frac{\operatorname{sgn}(x - a_\alpha) - \operatorname{sgn}(x - b_\alpha)}{2}.$$

19 For  $f \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ , by (5.7) we have

$$20 \quad \begin{aligned} \mathcal{F}_\alpha(S_{\rho_\alpha} f)(x) &= \chi_{\rho_\alpha}(x) (\mathcal{F}_\alpha f)(x) \\ 21 \quad &= \frac{i}{2} [(-i \operatorname{sgn}(x - a_\alpha)) - (-i \operatorname{sgn}(x - b_\alpha))] (\mathcal{F}_\alpha f)(x) \end{aligned}$$

$$\begin{aligned}
&= \frac{i}{2} [(-i\operatorname{sgn}(x - a_\alpha) (\mathcal{F}_\alpha f)(x)) - (-i\operatorname{sgn}(x - b_\alpha) (\mathcal{F}_\alpha f)(x))] \\
&= \frac{i}{2} [\tau_{a_\alpha}(-i\operatorname{sgn}x \cdot \tau_{-a_\alpha}(\mathcal{F}_\alpha f)(x)) - \tau_{b_\alpha}(-i\operatorname{sgn}x \cdot \tau_{-b_\alpha}(\mathcal{F}_\alpha f)(x))],
\end{aligned}$$

where  $\tau_s f(x) = f(x - s)$ . In view of the decomposition (2.3), we have

$$\begin{aligned}
\tau_{-a_\alpha}(\mathcal{F}_\alpha f)(x) &= A_\alpha \tau_{-a_\alpha} \left[ e^{i\pi x^2 \cot \alpha} \mathcal{F} [e^{i\pi t^2 \cot \alpha} f(t)](x \csc \alpha) \right] \\
&= A_\alpha \sin \alpha \cdot \tau_{-a_\alpha} \left[ e^{i\pi x^2 \cot \alpha} \mathcal{F} [e^{i\pi t^2 \sin \alpha \cos \alpha} f(t \sin \alpha)](x) \right] \\
&= A_\alpha \sin \alpha \cdot e^{i\pi(x+a_\alpha)^2 \cot \alpha} \cdot \tau_{-a_\alpha} \mathcal{F} [e^{i\pi t^2 \sin \alpha \cos \alpha} f(t \sin \alpha)](x) \\
&= A_\alpha \sin \alpha \cdot e^{i\pi(x+a_\alpha)^2 \cot \alpha} \mathcal{F} [e^{2\pi i(-a_\alpha)t} e^{i\pi t^2 \sin \alpha \cos \alpha} f(t \sin \alpha)](x).
\end{aligned}$$

Recall that  $\mathcal{F}(\mathcal{H}f)(x) = -i\operatorname{sgn}x \mathcal{F}(f)(x)$ . Hence,

$$\begin{aligned}
&-i\operatorname{sgn}x \cdot \tau_{-a_\alpha}(\mathcal{F}_\alpha f)(x) \\
&= A_\alpha \sin \alpha \cdot e^{i\pi(x+a_\alpha)^2 \cot \alpha} \left( -i\operatorname{sgn}x \mathcal{F} [e^{2\pi i(-a_\alpha)t} e^{i\pi t^2 \sin \alpha \cos \alpha} f(t \sin \alpha)](x) \right) \\
&= A_\alpha \sin \alpha \cdot e^{i\pi(x+a_\alpha)^2 \cot \alpha} \mathcal{F} \left[ \mathcal{H} \left( e^{2\pi i(-a_\alpha)(\cdot)} e^{i\pi(\cdot)^2 \sin \alpha \cos \alpha} f((\cdot) \sin \alpha) \right) (t) \right] (x).
\end{aligned}$$

Thus

$$\begin{aligned}
&\tau_{a_\alpha}(-i\operatorname{sgn}x \cdot \tau_{-a_\alpha}(\mathcal{F}_\alpha f)(x)) \\
&= A_\alpha \sin \alpha \cdot e^{i\pi x^2 \cot \alpha} \tau_{a_\alpha} \mathcal{F} \left[ \mathcal{H} \left( e^{2\pi i(-a_\alpha)(\cdot)} e^{i\pi(\cdot)^2 \sin \alpha \cos \alpha} f((\cdot) \sin \alpha) \right) (t) \right] (x) \\
&= A_\alpha \sin \alpha \cdot e^{i\pi x^2 \cot \alpha} \mathcal{F} \left[ e^{2\pi i a_\alpha t} \mathcal{H} \left( e^{2\pi i(-a_\alpha)(\cdot)} e^{i\pi(\cdot)^2 \sin \alpha \cos \alpha} f((\cdot) \sin \alpha) \right) (t) \right] (x) \\
&= A_\alpha e^{i\pi x^2 \cot \alpha} \mathcal{F} \left[ e^{2\pi i a_\alpha(\cdot) \csc \alpha} \mathcal{H} \left( e^{-2\pi i a_\alpha(\cdot) \csc \alpha} e^{i\pi(\cdot)^2 \cot \alpha} f \right) \right] (x \csc \alpha) \\
&= \mathcal{F}_\alpha \left[ e^{-i\pi(\cdot)^2 \cot \alpha} e^{2\pi i a_\alpha(\cdot)} \mathcal{H} \left( e^{-2\pi i a_\alpha(\cdot)} e^{i\pi(\cdot)^2 \cot \alpha} f \right) \right] (x).
\end{aligned}$$

The definition of the fractional Hilbert transform (5.3) implies that

$$\mathcal{H}_\alpha f = e^{-i\pi(\cdot)^2 \cot \alpha} \mathcal{H} \left( e^{i\pi(\cdot)^2 \cot \alpha} f \right).$$

Therefore,

$$\tau_{a_\alpha}(-i\operatorname{sgn}x \cdot \tau_{-a_\alpha}(\mathcal{F}_\alpha f)(x)) = \mathcal{F}_\alpha \left[ e^{2\pi i a_\alpha(\cdot)} \mathcal{H}_\alpha \left( e^{-2\pi i a_\alpha(\cdot)} f \right) \right] (x),$$

and similarly,

$$\tau_{b_\alpha}(-i\operatorname{sgn}x \cdot \tau_{-b_\alpha}(\mathcal{F}_\alpha f)(x)) = \mathcal{F}_\alpha \left[ e^{2\pi i b_\alpha(\cdot)} \mathcal{H}_\alpha \left( e^{-2\pi i b_\alpha(\cdot)} f \right) \right] (x).$$



1 Consequently,

$$\begin{aligned}
2 \quad \mathcal{F}_\alpha(S_{\rho_\alpha}f)(x) &= \frac{i}{2} \left[ \mathcal{F}_\alpha \left[ e^{2\pi ia(\cdot)} \mathcal{H}_\alpha \left( e^{-2\pi ia(\cdot)} f \right) \right] (x) - \mathcal{F}_\alpha \left[ e^{2\pi ib(\cdot)} \mathcal{H}_\alpha \left( e^{-2\pi ib(\cdot)} f \right) \right] (x) \right] \\
3 \quad &= \frac{i}{2} \mathcal{F}_\alpha \left[ e^{2\pi ia(\cdot)} \mathcal{H}_\alpha \left( e^{-2\pi ia(\cdot)} f \right) - e^{2\pi ib(\cdot)} \mathcal{H}_\alpha \left( e^{-2\pi ib(\cdot)} f \right) \right] (x). \\
4
\end{aligned}$$

5 Namely,

$$6 \quad S_{\rho_\alpha}(f)(x) = \frac{i}{2} \left[ e^{2\pi iax} \mathcal{H}_\alpha \left( e^{-2\pi ia(\cdot)} f \right) (x) - e^{2\pi ibx} \mathcal{H}_\alpha \left( e^{-2\pi ib(\cdot)} f \right) (x) \right]. \quad (5.8)$$

7 Since  $\mathcal{H}_\alpha$  is bounded from  $L^p(\mathbb{R})$  to  $L^p(\mathbb{R})$ ,  $S_{\rho_\alpha}$  can be extended to be a bounded operator on  $L^p(\mathbb{R})$ .

9 Finally, the classical partial summation operator  $S_\rho$  corresponding to  $\rho = [a, b]$  is defined by

$$11 \quad \mathcal{F}(S_\rho f)(x) = \chi_\rho(x) (\mathcal{F}f)(x),$$

12 and  $S_\rho$  can be expressed as (refer to [2, Example 5.4.6])

$$13 \quad S_\rho(f)(x) = \frac{i}{2} \left[ e^{2\pi iax} \mathcal{H} \left( e^{-2\pi ia(\cdot)} f \right) (x) - e^{2\pi ibx} \mathcal{H} \left( e^{-2\pi ib(\cdot)} f \right) (x) \right]. \quad (5.9)$$

14 Comparing (5.8) and (5.9) and applying the classical Littlewood-Paley theorem to  
15 (5.8), we easily conclude that Theorem 5.5 holds.  $\square$

## 16 6. Applications to chirps

17 A chirp is a non-stationary signal in which the frequency increases (up-chirp) or  
18 decreases (down-chirp) with time. As the FRFT reflects information about the signal  
19 in the time and frequency domains simultaneously, the FRFT is more effective than  
20 the classical Fourier transform in the spectrum analysis of non-stationary signals, especially of chirp signals. In this section, we will demonstrate the use of Theorems  
21 3.11-3.12 and Corollary 3.13 in the signal recovery of  $L^1 \setminus L^2$ -signals. For example, let  
22

$$23 \quad u(t) = \begin{cases} \frac{e^{-\pi i t^2}}{\sqrt{|t|}}, & 0 < |t| < 1, \\ \frac{e^{-\pi i t^2}}{t^2}, & |t| \geq 1. \end{cases}$$

24 Then  $u$  is a chirp signal and  $u \in L^1(\mathbb{R})$  but  $u \notin L^2(\mathbb{R})$ . The real and imaginary part  
25 graphs of  $u(t)$  in time domain are shown in Fig. 6.1.

26 Consider the FRFT of  $u$  of order  $\pi/4$ :

$$27 \quad (\mathcal{F}_{\pi/4}u)(w) = 2e^{i\pi w^2} \left( \frac{C(2^{5/4} \sqrt{|w|})}{2^{1/4} \sqrt{|w|}} - \sqrt{2}\pi^2 |w| + 2\sqrt{2}\pi w \operatorname{Si}(2\sqrt{2}\pi w) + \cos(2\sqrt{2}\pi w) \right),$$

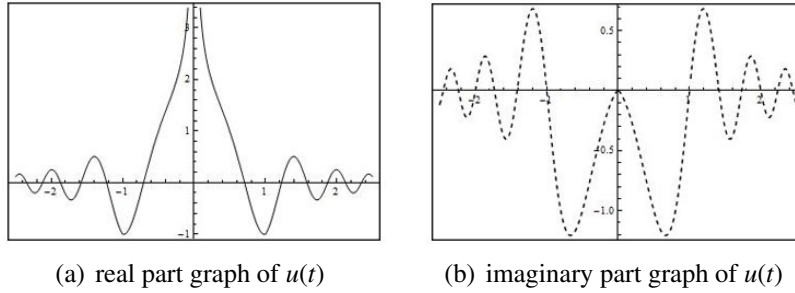


Figure 6.1: real and imaginary part graphs of  $u(t)$  in time domain.

1 where  $C(x)$  and  $\text{Si}(x)$  denote the Fresnel integral and sine integral, respectively. Namely,

$$2 \quad C(x) = \int_0^x \sin t^2 dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n)!(4n+1)},$$

$$3 \quad \text{Si}(x) = \int_0^x \frac{\sin t}{t} dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!(2n+1)}.$$

4 The real and imaginary part graphs of  $(\mathcal{F}_{\pi/4}u)(w)$  in frequency domain are shown in  
5 Fig 6.2.

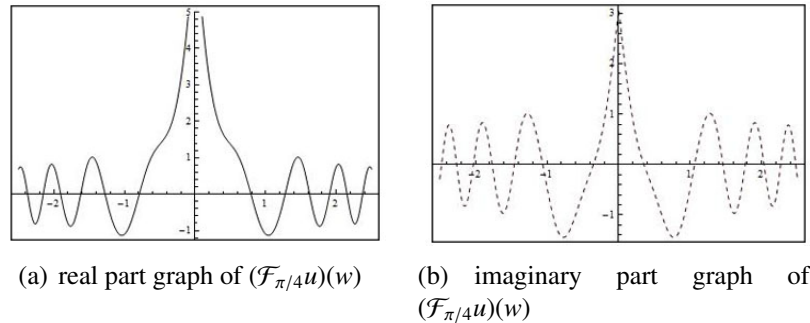


Figure 6.2: real and imaginary part graphs of  $(\mathcal{F}_{\pi/4}u)(w)$  in frequency domain.

6 It is obvious that  $\mathcal{F}_{\pi/4}u \notin L^1(\mathbb{R})$ . The inverse FRFT

$$7 \quad \int_{-\infty}^{+\infty} (\mathcal{F}_{\pi/4}u)(w) K_{-\pi/4}(x, t) dw \quad (6.1)$$

8 do not make sense. In order to recover the original signal  $u(t)$ , we should use the  
9 approximating method. Fig. 6.3 shows the Abel means of the integral (6.1) with

1  $\varepsilon = 1, 0.1, 0.01$ , that is,

$$2 \quad u_\varepsilon(t) = \int_{-\infty}^{+\infty} (\mathcal{F}_{\pi/4}u)(w)K_{-\pi/4}(x, t)e^{-2\pi\varepsilon|\csc \alpha||w|}dw.$$

3 By Theorems 3.11-3.12 and Corollary 3.13 we know that  $u_\varepsilon(t) \rightarrow u(t)$  for a.e.  $t \in \mathbb{R}$  as  
 4  $\varepsilon \rightarrow 0$ .

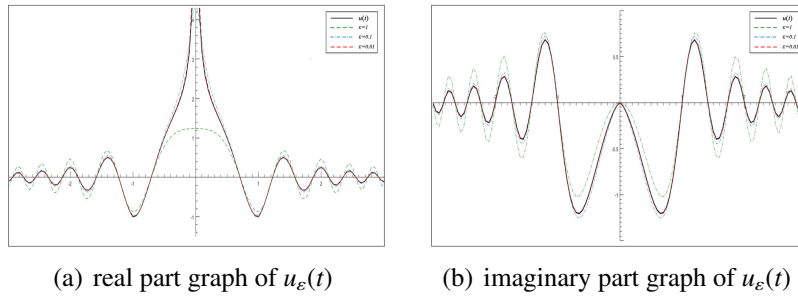


Figure 6.3: real and imaginary part graphs of  $u_\varepsilon(t)$

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